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ABSTRACT

We present analytically and experimentally determined cost distributions for *all* euclidean two-dimensional symmetric instances of the Traveling Salesman Problem in the TSPLIB library. Results obtained show characteristic cost distributions in all cases with and a high stability against degeneration.

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Keywords and Phrases: Traveling Salesman Problem, Cost distributions.
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1. Introduction

A combinatorial optimization problem can be described by a search space which contains all valid solutions and an objective function that assigns every element in the search space a numeric value. Without loss of generality, we assume that the problem is a minimization problem, i.e., the solution assigned the least value is optimal. Given a particular solution, we refer to the value of the objective function as *costs*. We use the term *cost distribution* for the distribution of costs occurring in the search space.

In this paper, we investigate cost distributions of the two-dimensional Euclidean Symmetric Traveling Salesman Problem [GJ79], where instances are given by the two-dimensional coordinates of the cities. The optimization goal is to find the shortest possible tour visiting all cities. The length of a tour is referred to as its cost in the following. The problem considered is not only one of the most prominent test grounds for combinatorial optimization algorithms, it is also of eminent practical importance. Throughout this paper, we will refer to this particular facet of the general Traveling Salesman Problem as *TSP*.

Cost distributions often show distinct characteristics for the underlying optimization problem. These features—best visible for what can be seen as the standard or average case—depend on the problem's parameters, the most influential of which is the problem size. Distributions of problems of very small sizes usually contain a large portion of noise; in other words, it does not make much sense to speak of a cost distribution if there are only a couple of different values. But also independent of the problem instance's size, the problem may degenerate to a pathological case where for example all solutions have the same cost value, or only very few different cost values are possible. Furthermore, different optimization problems are affected differently by this degeneration, i.e.,

for some problems pathological cases are very difficult to construct whereas it can be simple with other problems [WGL99].

The questions we want to address in this paper include

- *What features characterize the cost distributions of the TSP?*
- *How susceptible to degeneration are the observed features?*

Using randomly generated problem instances for the analysis of a combinatorial optimization problem may lead to analyzing the average case only. It is more appropriate to use a collection of both synthetic and practical problems, for the analysis of the problem. In case of the TSP the TSPLIB emerged as the standard test bed widely used in its field [Rei91]. In the sequel, we determine the cost distributions of all symmetric two-dimensional euclidean instances of the Traveling Salesman Problem in the TSPLIB and contrast them with analytically derived approximations. The results provide strong evidence as to what the characteristic features of the cost distribution are and how to approximate those distributions. Moreover, results suggest that these features are highly resistant against degeneration.

Cost distributions elicit an intuitive measure of the problem's difficulty which can be used to decide what optimization algorithm to deploy.

2. Model for the Problem

An instance of the TSP is given by the coordinates of the single cities. Every city is connected to every other city by an edge the length of the euclidean distance of the two cities.

In more formal terms, we can state the problem as finding a permutation $v_1, \ldots, v_n$ of cities given as two-dimensional vectors such that

$$l = |v_n - v_1| + \sum_{i=0}^{n} |v_i - v_{i-1}|$$

is minimal. The first part of the sum, which results from the condition to connect first and last point of the tour, may be omitted depending on the interpretation of the problem: For a salesman, returning to the starting point is usually necessary, with a drilling problem, completing the tour is only necessary if the drilling is to be repeated.

In order to approximate the cost distribution of a problem instance we first determine mean $\mu^{(C)}$ and deviation $\sigma^{(C)}$ of the pairwise distances of the cities. We model the length of a tour through n cities as the sum of n normally distributed random variables with parameters $\mu^{(C)}$ and $\sigma^{(C)}$. According to the Central Limit Theorem of statistics, we can approximate the cost distribution of all tours through n cities by a normal distribution with mean $\mu^{(A)} = n \cdot \mu^{(C)}$ and $\sigma^{(A)} = \sqrt{n} \cdot \sigma^{(C)}$.

Observation. The left most quantile of the cost distribution contains the optimal solution. Figure 1 shows a cost distribution for the problem *u1817* of TSPLIB obtained from a sample of 10^7 (see also Section A.70, p. 80). The

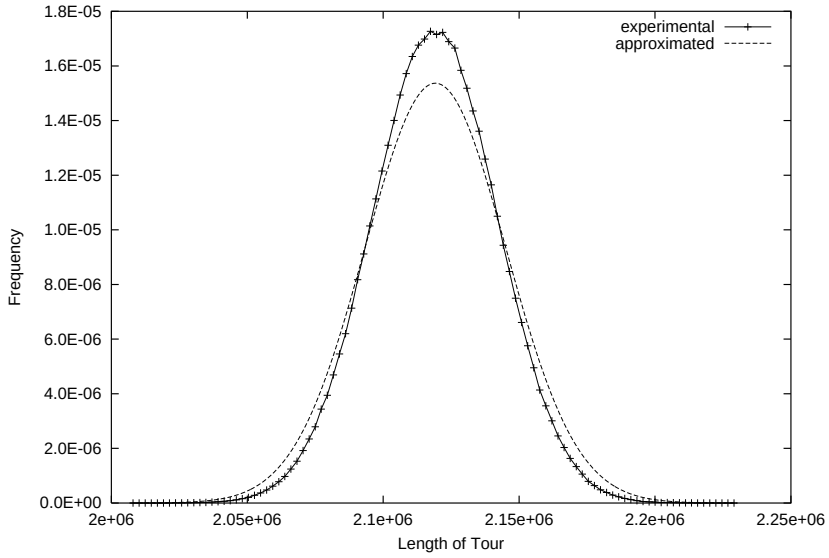

Figure 1: Comparison of analytical approximation and actual cost distribution.

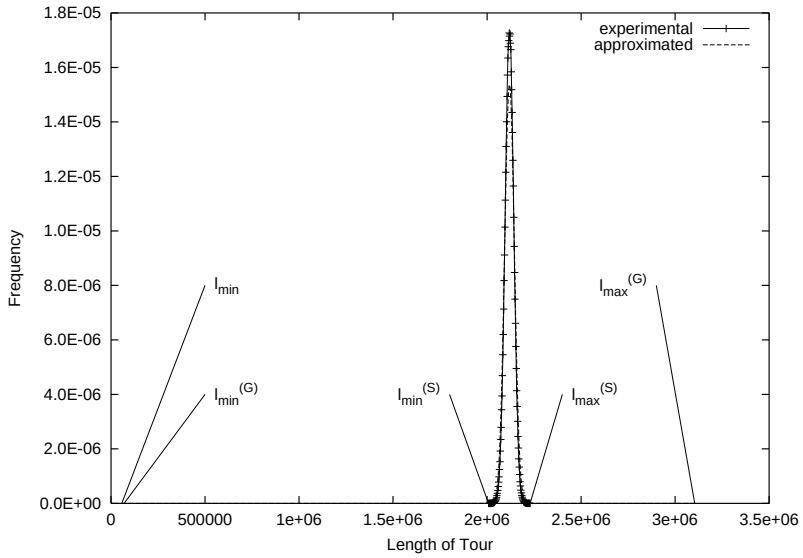

Figure 2: Comparison of approximated and actual cost distribution including optimum and longest tour found with greedy heuristic. Distributions coincide almost completely.

n	problem size
l_{min}	length of optimal tour
$l_{min}^{(G)}$	length of shortest tour found by greedy heuristic
$l_{max}^{(G)}$	length of longest tour found by greedy heuristic
$l_{min}^{(S)}$	length of shortest tour found by sampling
$l_{max}^{(S)}$	length of longest tour found by sampling
$\mu^{(S)}$	mean of sampled distribution
$\mu^{(A)}$	mean of approximated distribution
$\sigma^{(S)}$	deviation of sampled distribution
$\sigma^{(A)}$	deviation of approximated distribution
κ	Kulback-Leibler Divergence of sampled and approximated distribution centered on resp. means

Table 1: Identifiers used in the experiments.

sample can be somewhat misleading since it does not unveil how far to the left the optimal tour is. In general, the shortest tour is indeed far off and the probability to find the optimum or any tour close to it in a random sample is almost zero. Note, this problem does not affect the shape of the density, as our approximation is defined on R having positive values, but requires a rectification. To correct the histogram in this respect we need to adjust the range to cover minimal and maximal solution. For nearly all TSP problems in TSPLIB, the optimum is known. Otherwise very tight boundaries for the optimum are known. For the longest possible tour, we use an approximation generated with a greedy heuristic. In Figure 2 the rectified histogram is shown where characteristic points are marked up.

3. Experiments

With respect to the questions raised in the introduction it is important to use real world examples rather than randomly generated instances as the latter bears the danger to be always a kind of "average" case. The TSPLIB contains 77 problems of the two-dimensional euclidean type, which are either practical examples of real world application, or instances widely used in related work. For each of them we present a map and a comparison of analytical and actual cost distribution in the Appendix (see pp. 11). Having a graphical map of the problem is often useful to understand why analytical and actual distribution differ. The actual distributions are obtained from samples of size 10^7. The left plot below the map shows the sampled histogram, the right shows the rectified distribution as detailed above.

As it became clear from the abovementioned example, a comparison of the actual and the approximated distribution has to take heavy tails into account, that is, we need an appropriate measure to compute a distance. The integral

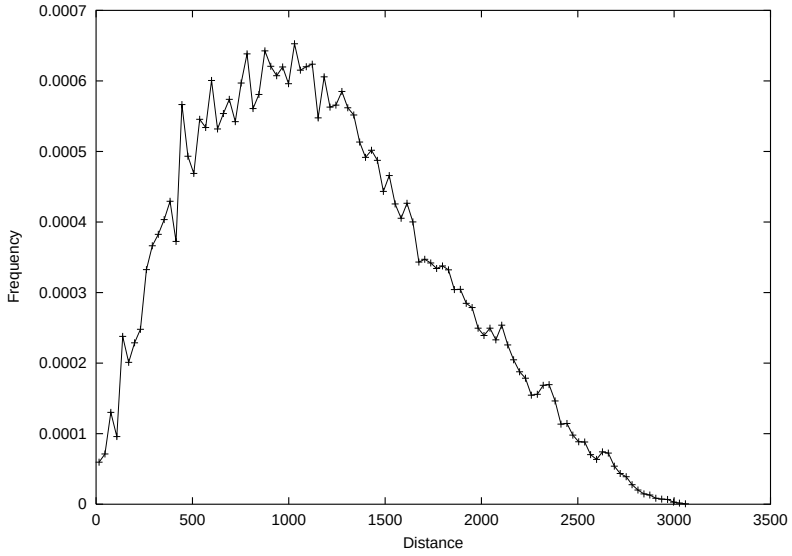

Figure 3: Distribution of pairwise distances for problem *u1817*.

over the difference is not suitable as the distributions may for example differ by this means by a value close to 2—the maximum distance of two densities under this metric—although both distributions have almost all their weight in the same quantile of one mere per cent. To overcome this problem we take two separate measures, the overlap and the similarity of shape. The first one is expressed by the difference of the means, the second by the Kullback-Leibler Divergence after centering the distributions on their means. In all experiments, the sampled tours were generated with uniform probability. In Table 1 the identifiers we use to characterize the experiments are explained.

All instances show distributions that are symmetric and largely coinciding with the normal distributions used for approximation as the small values of κ indicate.

Another effect deserving special attention is the fact that the means sampled as well as approximated are in the right half of the total range. At first glance, one might attribute this skew to the fact that the left edge of the distributions is the absolute minimum whereas the right is only approximated with a greedy heuristic. However also the minimum obtained by greedy optimization is distinctly further away from the mean than its counterpart, the maximum. The explanation requires a look at the distribution of the pairwise distances between cities. This distribution is skewed as shown in Figure 3. Though not important for the Central Limit Theorem and the shape of the approximation, the skew determines the length of the distribution's tails. For very small problem sizes, this effect is almost negligible but gains importance with increasing problem size.

4. Summary

We determined experimentally the cost distributions for *all* two-dimensional symmetric euclidean instances of the Traveling Salesman Problems in the

TSPLIB, the standard problem library in this field.

In all cases the cost distributions obtained by sampling coincide largely with normal distributions. However, the optimal solutions is significantly shorter than the shortest tour found in a uniform sample. Also, a simple greedy heuristic produces results better than the sampled ones in all cases.

As a consequence in the context of optimization with branch and bound algorithms, we can find easily upper bounds that cut off more than 99% of the search space so that the segments that actually have to be enumerated are of very small size relative to the size of the total search space.

Cost distributions have furthermore direct application of selecting a solution at random, a task that is a central element of practically every probabilistic optimization algorithm. Our further research is geared towards a better understanding of the interplay of cost distributions and the application of branch and bound algorithms. Specifically the transfer of the obtained knowledge to other areas of combinatorial optimization like database query optimization is subject of our ongoing research.

References

[GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. W. H. Freeman and Co., New York, 1979.

[Rei91] G. Reinelt. TSPLIB—A Traveling Salesman Problem Library. *ORSA Journal on Computing*, 3(4):376–384, 1991.

[WGL99] F. Waas and C. A. Galindo-Legaria. Of Packing and Traveling — An Experimental Study of Cost Distributions in Combinatorial Optimization Problems. Technical report, CWI, Amsterdam, The Netherlands, October 1999.

Name	n	l_{min}	$l_{min}^{(G)}$	$l_{min}^{(S)}$	$l_{max}^{(S)}$	$l_{max}^{(G)}$	$\mu^{(S)}$	$\mu^{(A)}$	$\frac{\lvert\mu^{(A)}-\mu^{(S)}\rvert}{l_{max}^{(G)}-l_{min}}$	$\sigma^{(S)}$	$\sigma^{(A)}$	κ
a280	280	2579	3073	29792	38699	49989	34108	34231	0.00340	954	1050	0.016
berlin52	52	7542	9535	21966	35890	38810	29915	30489	0.02020	1578	2474	0.184
bier127	127	118282	123924	527298	710918	819758	628965	633917	0.00834	19562	34869	0.247
brd14051	14051	469445	580252	41137975	42714386	58252876	41970005	41973171	0.00007	165101	196408	0.027
ch130	130	6110	7103	38589	53959	65933	46314	46673	0.00749	1665	1946	0.039
ch150	150	6528	8115	44806	62231	77463	53898	54255	0.00640	1818	2081	0.032
d1291	1291	50801	61636	1630162	1824664	2478844	1728997	1730401	0.00079	19699	24335	0.041
d15112	15112	1573152	1949461	131835902	136275361	192411817	134002559	134011265	0.00006	476535	554347	0.021
d1655	1655	62128	74144	2058776	2276637	3166964	2173698	2175032	0.00060	22670	25984	0.019
d18512	18512	645300	797063	58583658	60380569	86367860	59440418	59443677	0.00005	190628	218662	0.018
d198	198	15780	18870	152668	223090	256990	190629	191597	0.00466	7228	11362	0.161
d2103	2103	80450	90910	3106223	3390343	4788346	3250905	3252446	0.00047	30771	34461	0.013
d493	493	35002	42619	402055	490633	605848	449543	450464	0.00202	9137	13121	0.107
d657	657	48912	61904	784874	920925	1238686	854992	856287	0.00148	14082	16536	0.027
eil101	101	-	773	2776	4042	4862	3429	3463	0.00840	138	165	0.051
eil51	51	426	493	1218	2053	2297	1654	1686	0.01961	89	109	0.079
eil76	76	538	676	1971	3061	3532	2526	2559	0.01305	113	138	0.064
fl1400	1400	20127	27889	1568110	1806705	2678912	1688549	1689711	0.00065	26000	26542	0.002
fl1577	1577	22249	27313	1284181	1423188	1857672	1355243	1356110	0.00062	14224	18575	0.061
fl3795	3795	28772	38762	3449644	3711546	4997885	3581017	3581960	0.00026	25612	31333	0.036
fl417	417	11861	16417	435166	559970	776919	496043	497251	0.00220	13573	14007	0.005
fnl4461	4461	182566	226480	8056531	8624332	12073014	8329901	8331829	0.00023	55446	63707	0.018
gil262	262	2378	2987	23130	30114	38492	26708	26809	0.00364	687	782	0.025
kroA100	100	21282	27989	127681	207109	250472	171083	172781	0.00912	8227	9206	0.029
kroA150	150	26524	33975	208127	302748	376559	257597	259321	0.00623	10112	11293	0.023
kroA200	200	29368	37099	284326	396217	505065	340233	341936	0.00464	11771	13006	0.018

Table 2: Characteristics of cost distributions *(continued on next page)*.

Name	n	l_{min}	$l_{min}^{(G)}$	$l_{min}^{(S)}$	$l_{max}^{(S)}$	$l_{max}^{(G)}$	$\mu^{(S)}$	$\mu^{(A)}$	$\frac{\lvert\mu^{(A)}-\mu^{(S)}\rvert}{l_{max}^{(G)}-l_{min}}$	$\sigma^{(S)}$	$\sigma^{(A)}$	κ
kroB100	100	22141	27239	130139	205081	242863	168753	170443	0.00922	8105	9175	0.032
kroB150	150	26130	34543	203177	306801	381542	256735	258454	0.00611	10438	11335	0.018
kroB200	200	29437	39987	277412	387161	487956	332842	334502	0.00463	11255	12651	0.022
kroC100	100	20749	25661	132248	209288	251237	170055	171756	0.00900	8326	9153	0.026
kroD100	100	21294	26698	127024	196329	234766	163098	164741	0.00937	7582	8715	0.036
kroE100	100	22068	30031	133132	212399	254996	173220	174948	0.00906	8443	9382	0.028
lin105	105	14379	18916	96983	149702	176524	123632	124800	0.00861	5934	6907	0.036
lin318	318	42029	53473	518552	655152	852712	588007	589846	0.00300	14140	16108	0.022
nrw1379	1379	56638	70757	1332697	1505378	2086720	1423610	1424631	0.00070	17677	20196	0.018
p654	654	34643	45660	1818206	2250019	3256192	2038190	2041258	0.00138	46250	46983	0.003
pcb1173	1173	56892	71992	1324473	1494986	2058498	1410065	1411257	0.00083	17927	20579	0.019
pcb3038	3038	137694	175711	5218574	5601936	7938318	5402873	5404622	0.00032	41907	47520	0.015
pcb442	442	50778	61612	693579	845863	1117968	772598	774363	0.00222	15382	17484	0.021
pr1002	1002	259045	325813	5986098	6849660	9423294	6448497	6454922	0.00097	88896	100105	0.015
pr107	107	44303	58968	435595	728866	931825	578235	583657	0.00790	31608	32271	0.015
pr124	124	59030	78901	558196	827611	997362	697324	702920	0.00727	28730	31847	0.026
pr136	136	96772	121920	656621	980237	1236896	826064	832136	0.00686	31925	34475	0.022
pr144	144	58537	70353	664554	948041	1180602	812114	817729	0.00630	30374	33878	0.025
pr152	152	73682	83684	854451	1230053	1483775	1051075	1057971	0.00595	38266	45376	0.038
pr226	226	80369	99020	1450761	1921309	2518346	1695558	1703184	0.00414	51798	55880	0.015
pr2392	2392	378032	470864	14602816	15863781	22386371	15248726	15255202	0.00042	135562	152893	0.015
pr264	264	49135	59396	925167	1327881	1811672	1121643	1125842	0.00328	40276	41640	0.007
pr299	299	48191	60780	656131	865329	1118841	759636	762187	0.00312	22459	25360	0.019
pr439	439	107217	133172	1709054	2085076	2613077	1904525	1908850	0.00218	40606	54489	0.075
pr76	76	108159	140909	444163	685919	793497	574449	582020	0.01308	26396	34346	0.084
rat195	195	2323	2942	18524	26641	33363	22725	22841	0.00476	789	895	0.024

Table 3: Characteristics of cost distributions *(continued on next page)*.

Name	n	l_{min}	$l_{min}^{(G)}$	$l_{min}^{(S)}$	$l_{max}^{(S)}$	$l_{max}^{(G)}$	$\mu^{(S)}$	$\mu^{(A)}$	$\frac{\lvert\mu^{(A)}-\mu^{(S)}\rvert}{l_{max}^{(G)}-l_{min}}$	$\sigma^{(S)}$	$\sigma^{(A)}$	κ
rat575	575	6773	8627	102524	124019	167079	113682	113879	0.00168	2280	2586	0.018
rat783	783	8806	11245	164437	193213	263754	179488	179717	0.00124	3072	3488	0.017
rat99	99	1211	1683	6373	10208	12267	8418	8502	0.00932	408	468	0.034
rd100	100	7910	10088	43677	65219	80216	55565	56121	0.00968	2353	2641	0.035
rd400	400	15281	18590	191040	232083	309171	211549	212074	0.00242	4419	5017	0.021
rl11849	11849	923368	1117193	85272354	88777964	125989180	86974953	86982309	0.00008	337890	401324	0.027
rl1304	1304	252948	331303	8789654	9967772	13597353	9375867	9383113	0.00075	111987	132611	0.027
rl1323	1323	270199	332641	9239271	10336528	14180847	9794948	9802338	0.00073	118083	135540	0.019
rl1889	1889	316536	394917	14100157	15559321	21815113	14799720	14807773	0.00053	154295	174440	0.015
rl5915	5915	565530	703619	41426538	43676590	61553339	42555941	42563423	0.00017	237524	277761	0.023
rl5934	5934	556045	698687	41055206	43277072	61083560	42199834	42206752	0.00016	237706	284577	0.029
st70	70	675	738	2807	4431	5209	3659	3711	0.01381	180	205	0.048
ts225	225	126643	160490	1387805	1784325	2319813	1592992	1600083	0.00427	43939	49937	0.026
tsp225	225	3916	4565	35814	46813	58637	41289	41473	0.00428	1185	1431	0.040
u1060	1060	224094	289197	6214669	7229040	9920865	6758720	6765098	0.00091	104811	118150	0.015
u1432	1432	152970	187720	3703266	4178419	5808347	3945913	3948693	0.00069	45467	50556	0.012
u159	159	42080	50995	367606	531962	651442	449561	452419	0.00582	16962	18689	0.021
u1817	1817	57201	69758	1998739	2233071	3104248	2118919	2120044	0.00052	22904	25910	0.015
u2152	2152	64253	79201	2406398	2645804	3720928	2532525	2533721	0.00046	25217	28431	0.014
u2319	2319	234256	277770	5743509	6263032	8776096	5998335	6000964	0.00044	52194	59538	0.017
u574	574	36905	46849	620324	743549	989276	680711	681926	0.00172	13102	15086	0.022
u724	724	41910	53169	800118	946353	1276491	872580	873792	0.00134	15003	16988	0.017
usa13509	13509	19982859	25165110	2109840757	2193395599	2978881603	2153455284	2153617677	0.00007	8829631	12707734	0.106
vm1084	1084	239297	297389	7966555	9257099	12617221	8572002	8579949	0.00088	120903	136676	0.016
vm1748	1748	336556	420377	14166028	15659443	21802682	14942406	14950840	0.00055	157175	177119	0.015

Table 4: Characteristics of cost distributions.

A. Euclidean Symmetric TSP Instances

This appendix contains a comparison of actual and approximated cost distributions for all TSP instances in TSPLIB. We present (a) the problem instance as two-dimensional map, (b) comparison of sampled and analytically derived cost distribution within the range of the sampled distribution, and (c) this comparison including minimum and maximum. It is interesting to see how patterns and other structural properties of the map translate to differences between the actual and predicted distribution. Finally, the plot containing the extrema gives a good impression as to how solutions found in random samples compare to the optimum.

A.1 a280

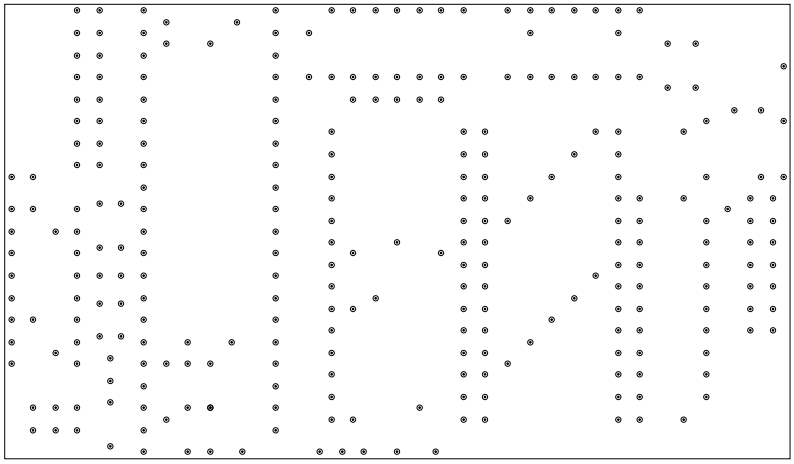

Figure 4: Problem *a280*.

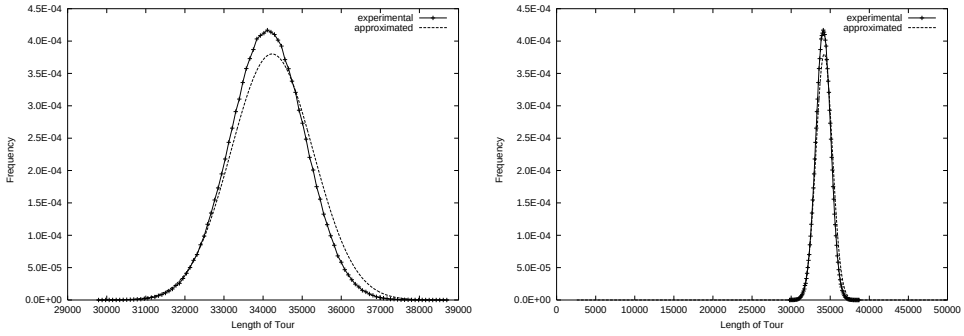

Figure 5: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.2 berlin52

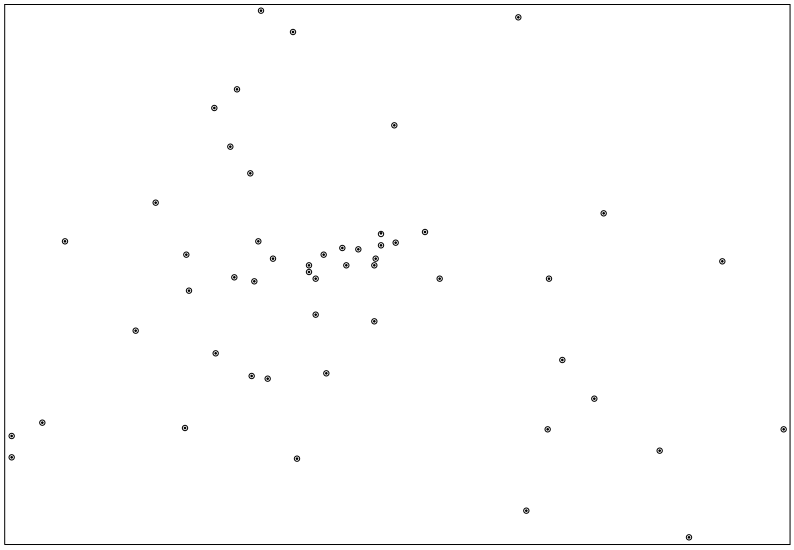

Figure 6: Problem *berlin52*.

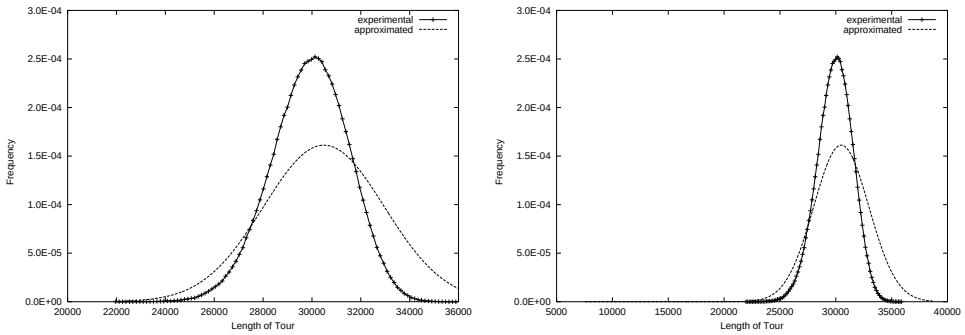

Figure 7: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.3 bier127

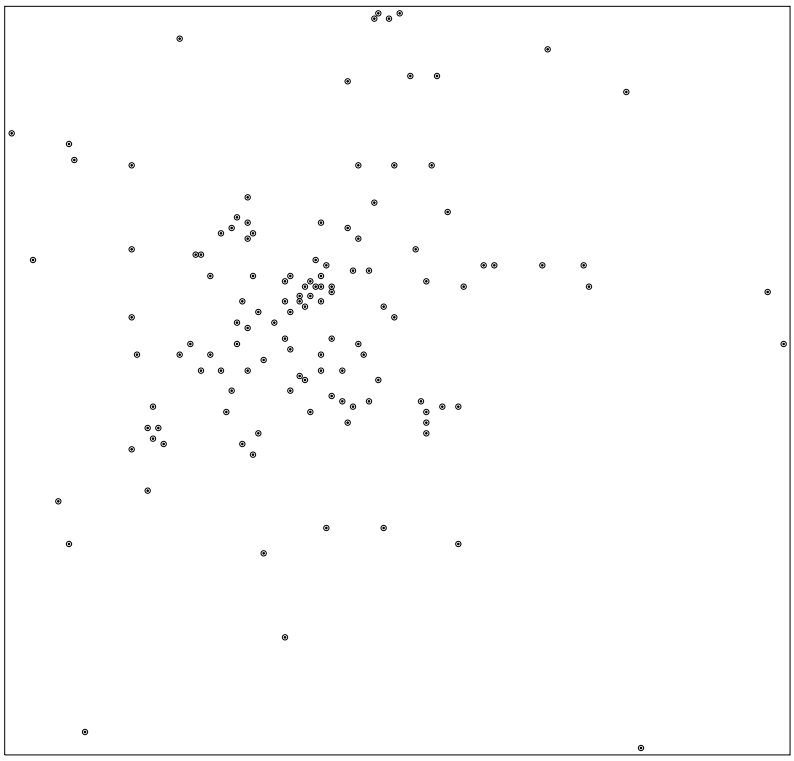

Figure 8: Problem *bier127*.

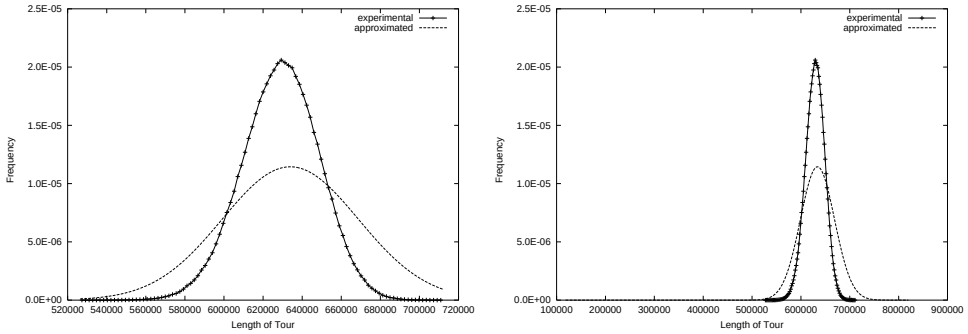

Figure 9: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.4 brd14051

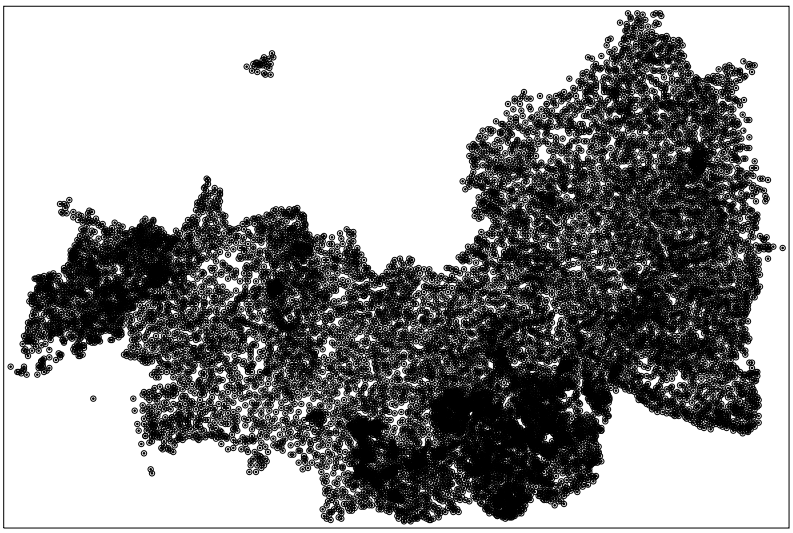

Figure 10: Problem *brd14051*.

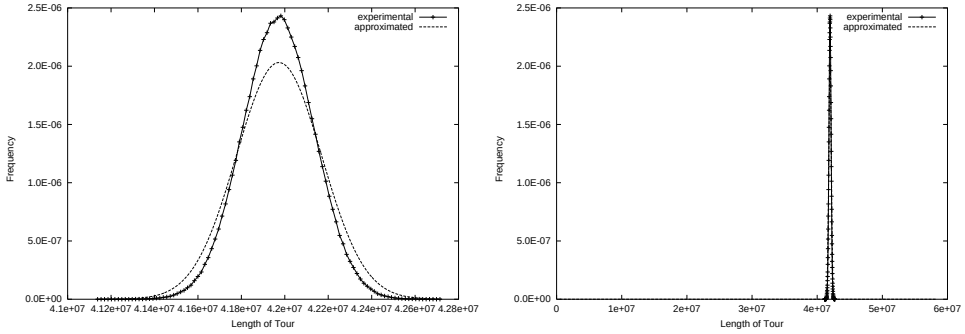

Figure 11: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.5 ch130

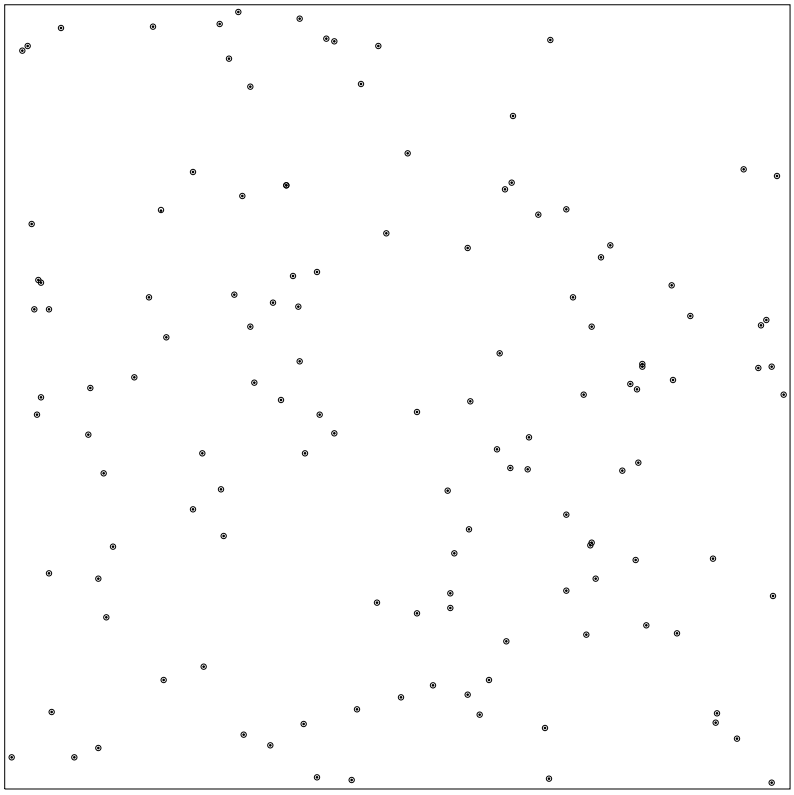

Figure 12: Problem *ch130*.

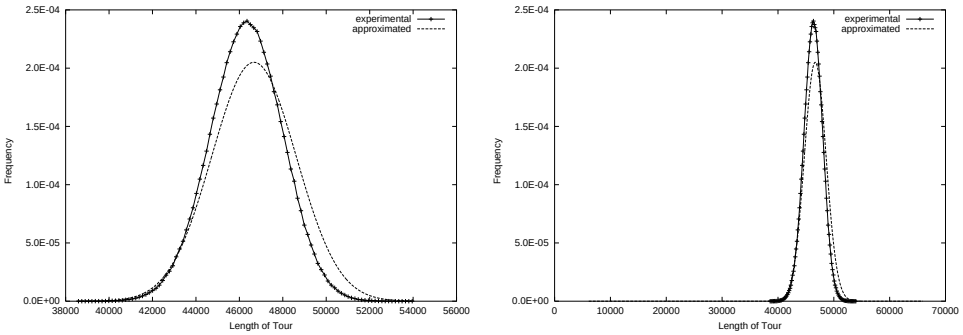

Figure 13: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.6 ch150

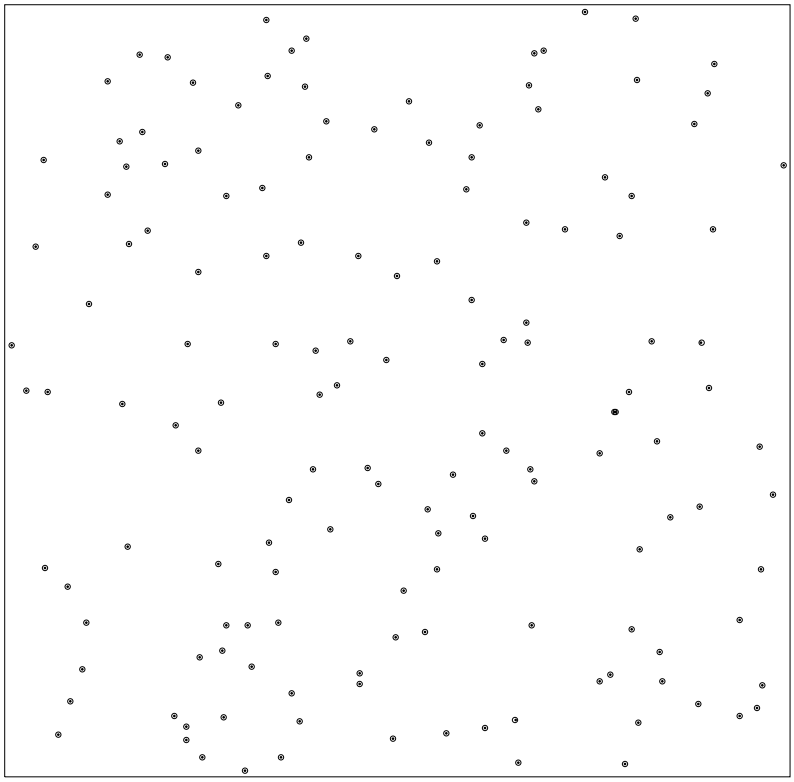

Figure 14: Problem *ch150*.

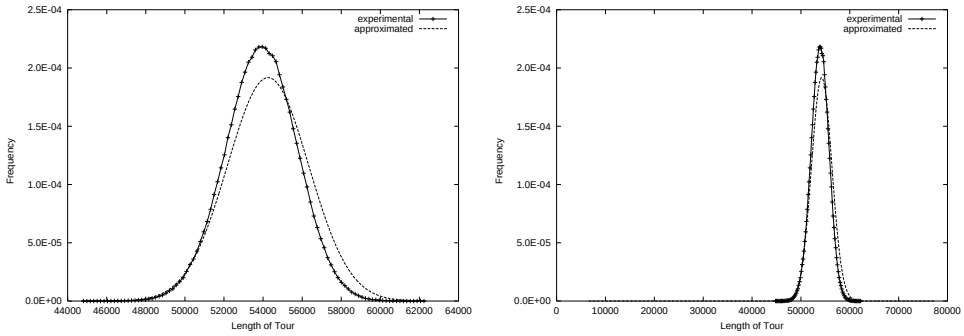

Figure 15: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.7 d1291

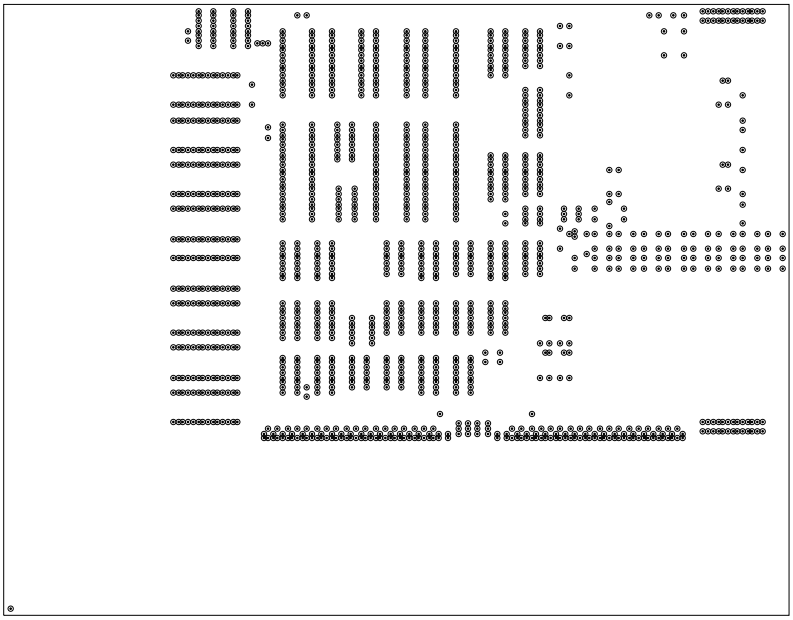

Figure 16: Problem *d1291*.

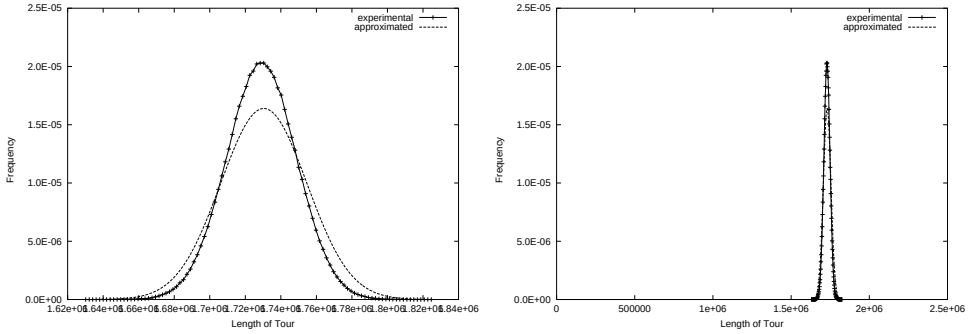

Figure 17: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.8 d15112

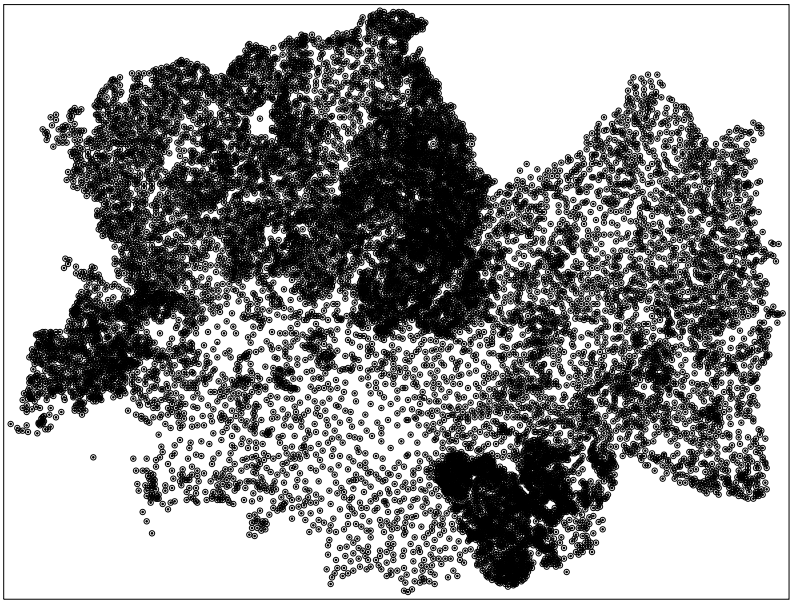

Figure 18: Problem *d15112*.

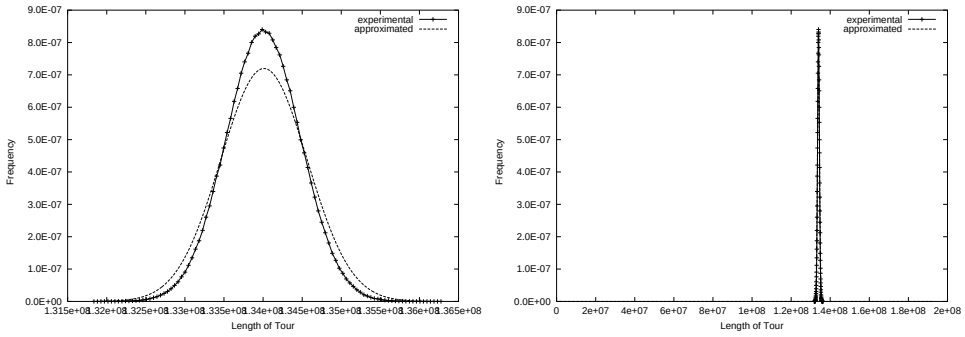

Figure 19: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.9 d1655

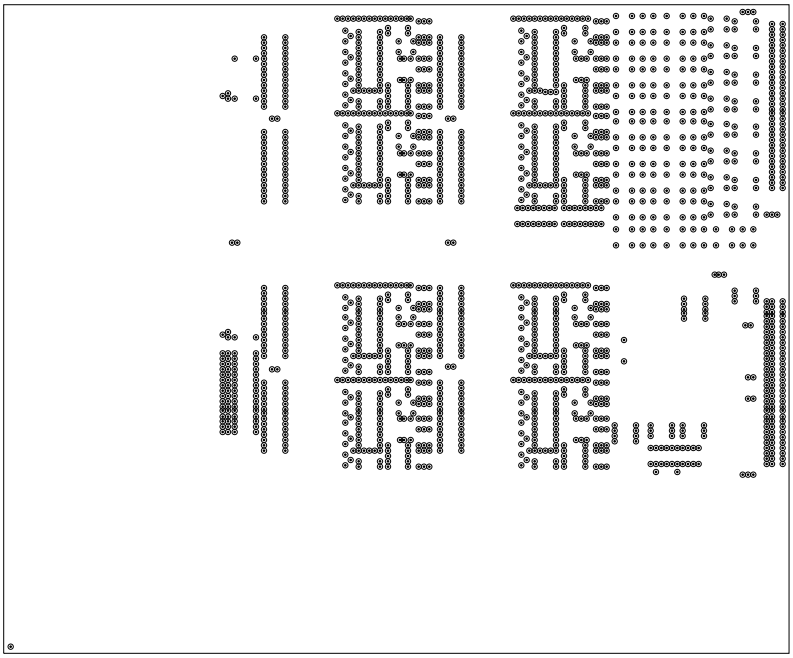

Figure 20: Problem *d1655*.

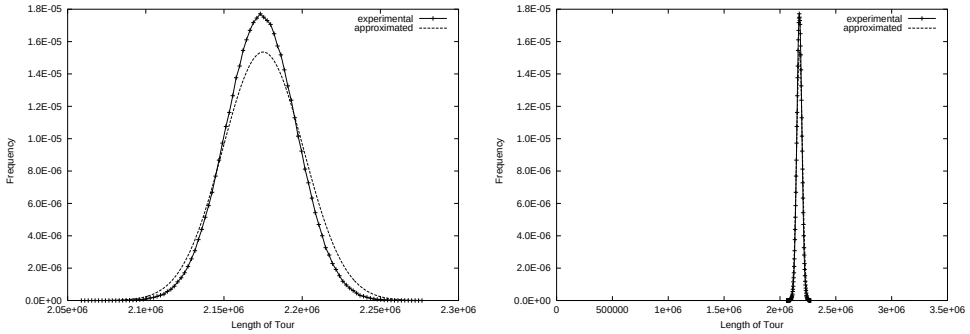

Figure 21: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.10 d18512

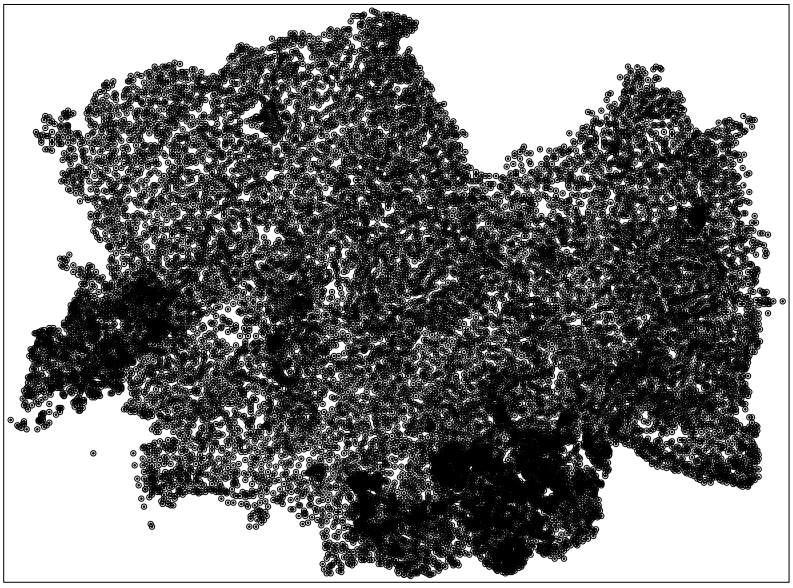

Figure 22: Problem *d18512*.

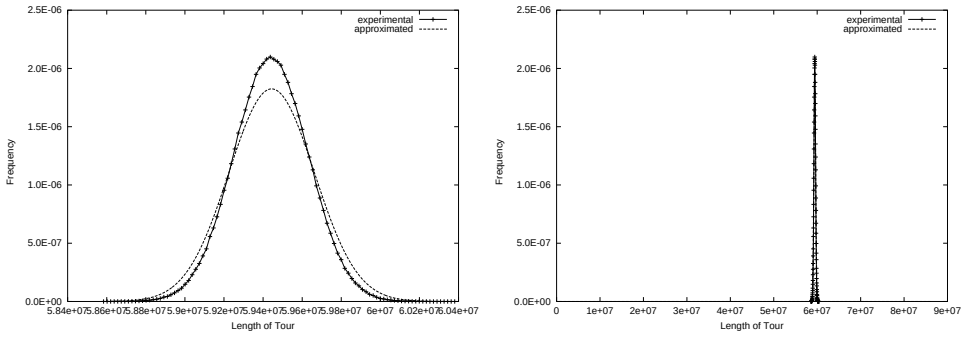

Figure 23: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.11 d198

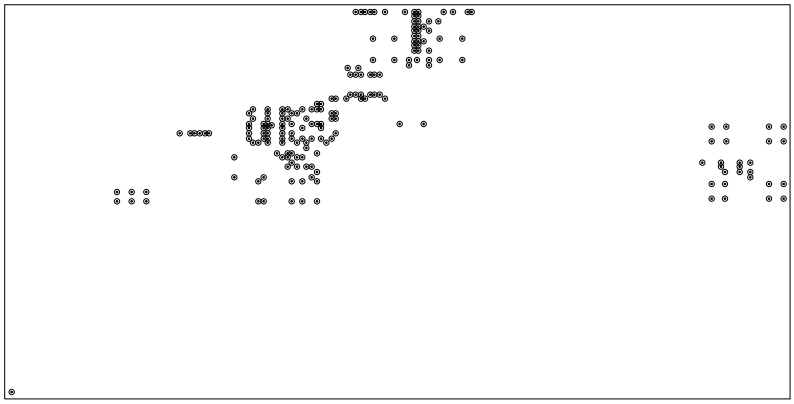

Figure 24: Problem *d198*.

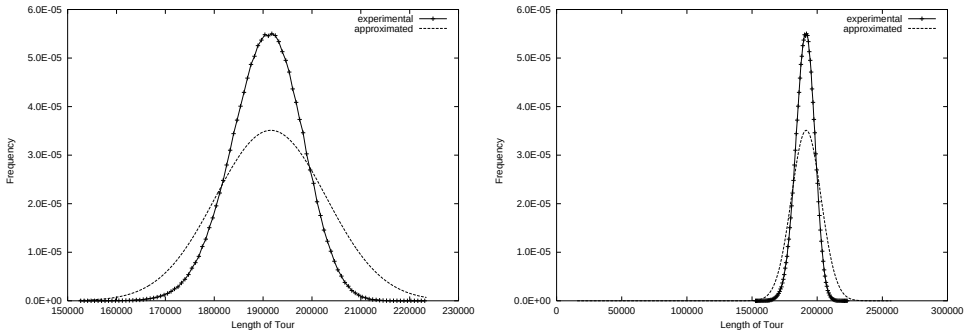

Figure 25: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.12 d2103

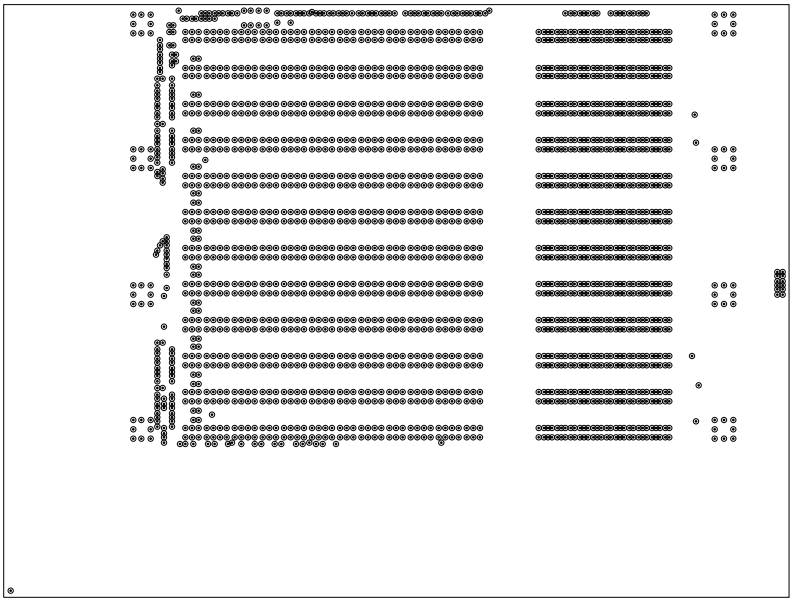

Figure 26: Problem *d2103*.

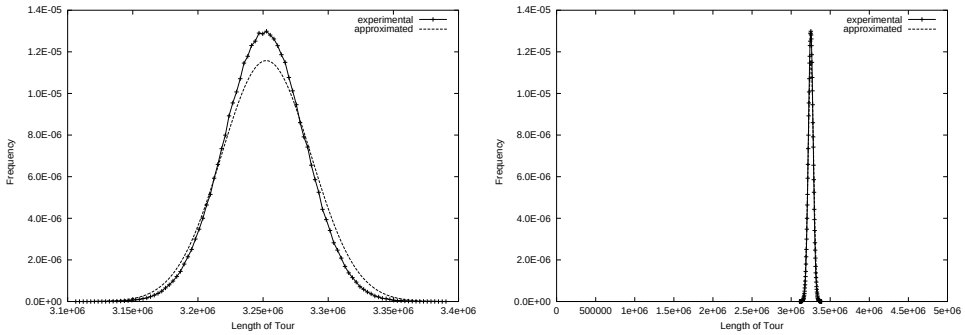

Figure 27: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.13 d493

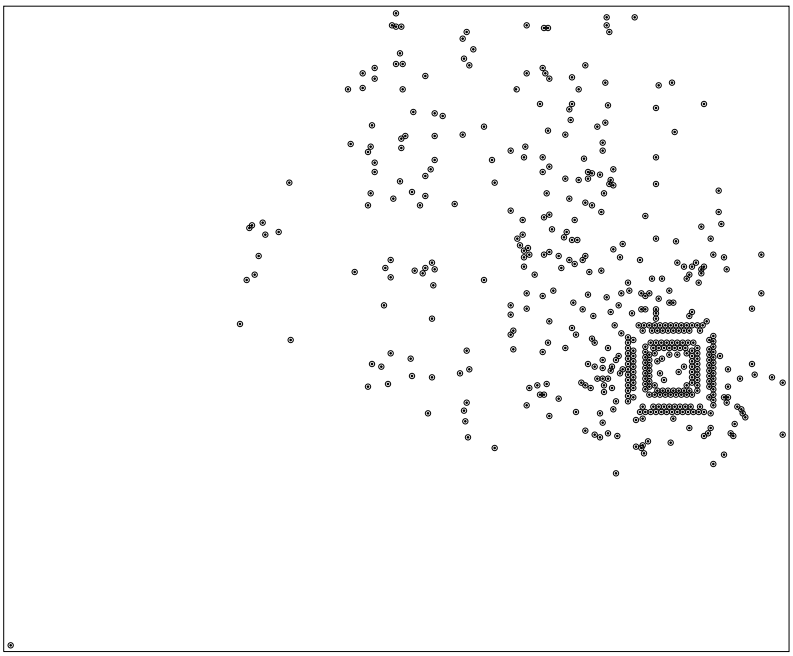

Figure 28: Problem *d493*.

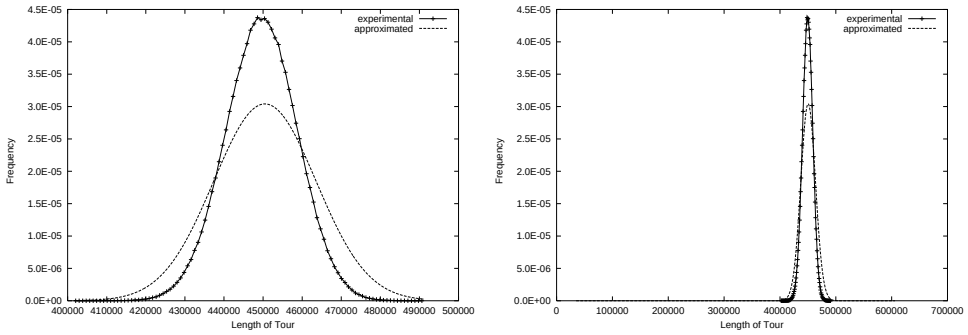

Figure 29: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.14 d657

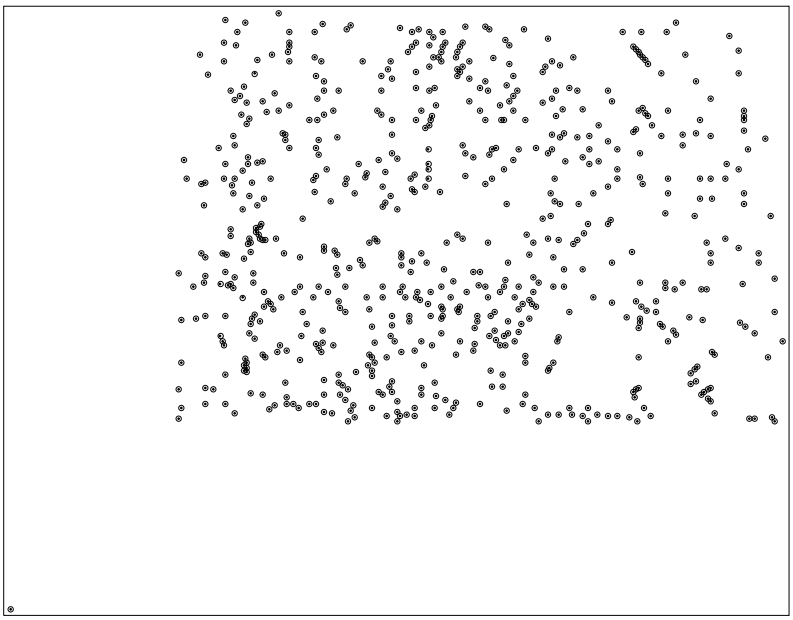

Figure 30: Problem *d657*.

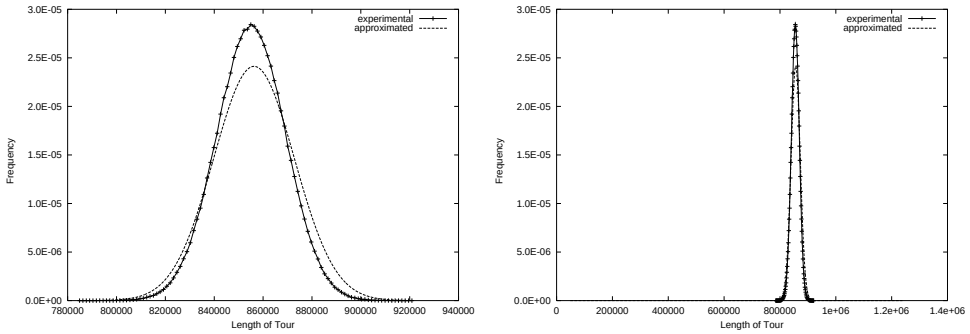

Figure 31: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.15 eil101

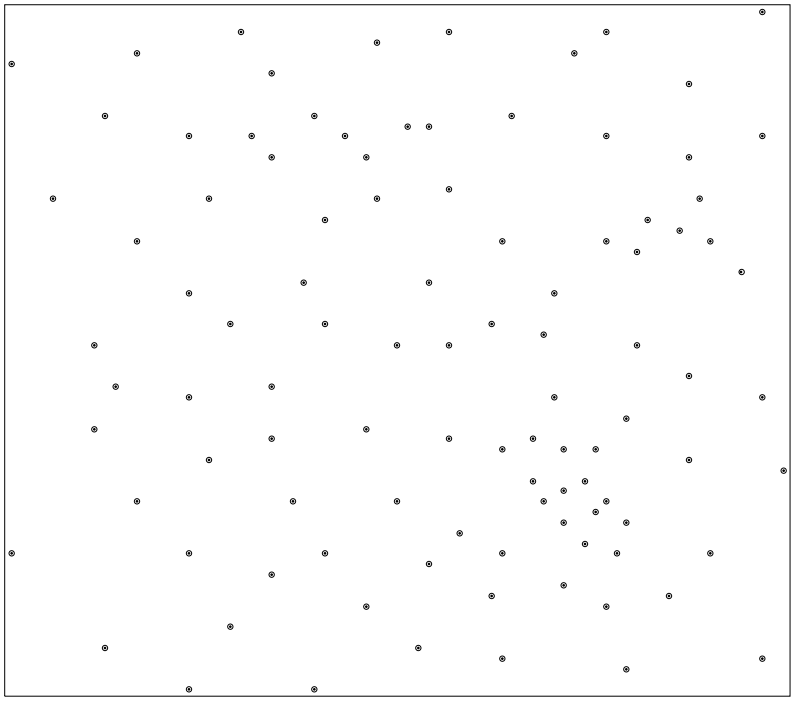

Figure 32: Problem *eil101*.

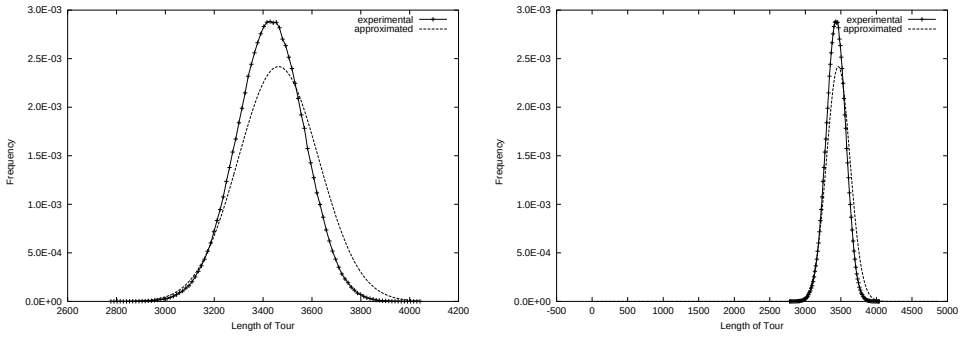

Figure 33: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.16 eil51

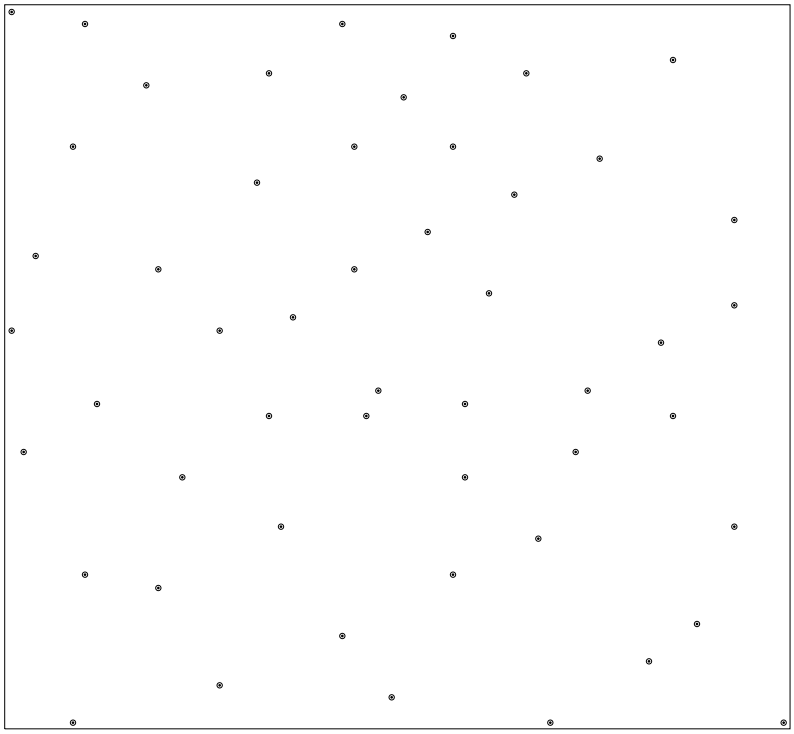

Figure 34: Problem *eil51*.

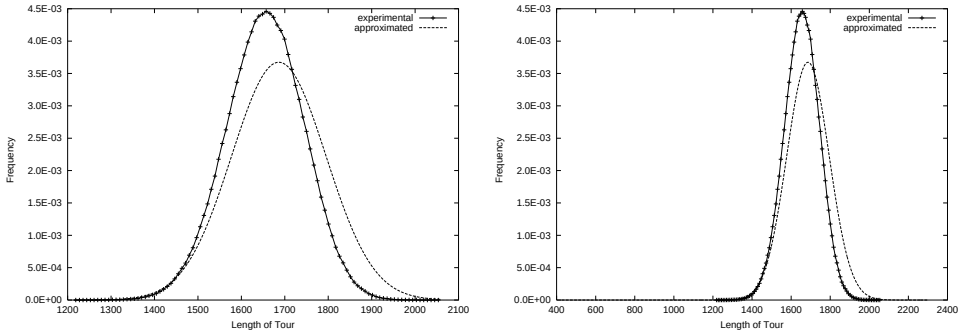

Figure 35: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.17 eil76

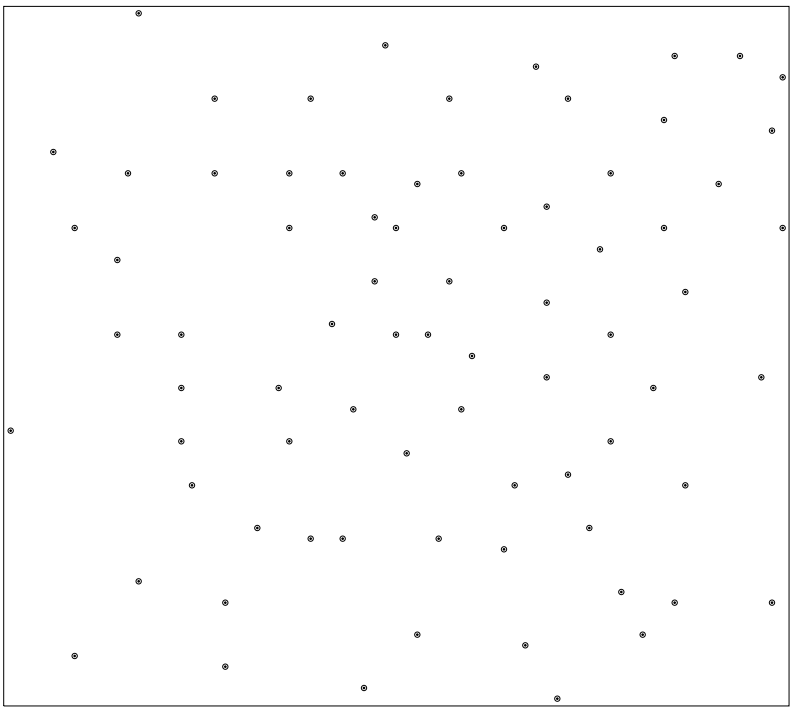

Figure 36: Problem *eil76*.

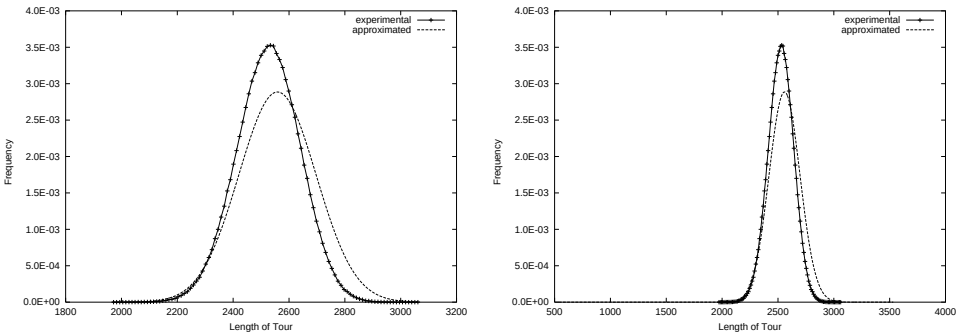

Figure 37: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.18 fl1400

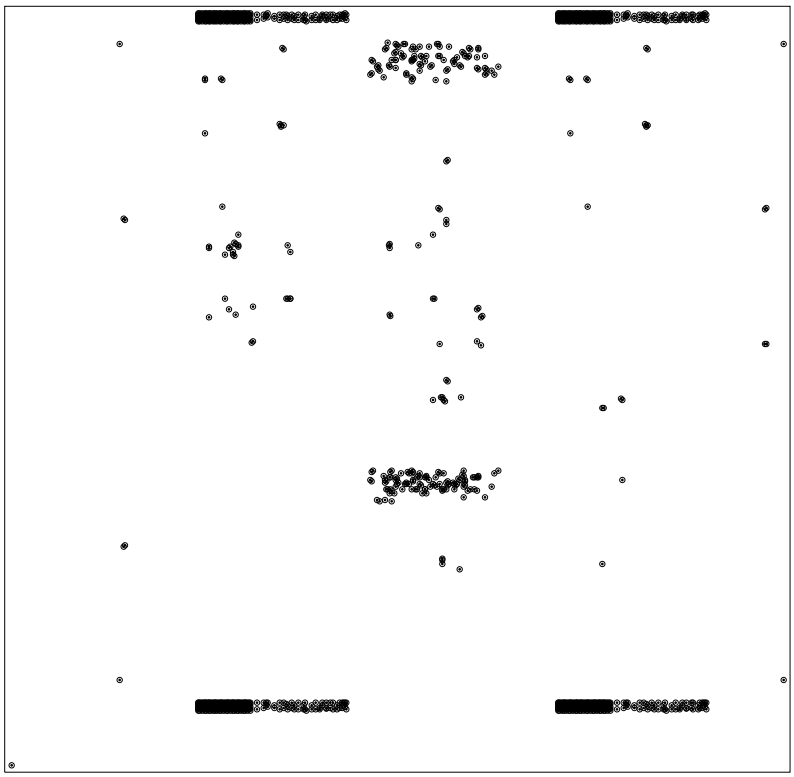

Figure 38: Problem *fl1400*.

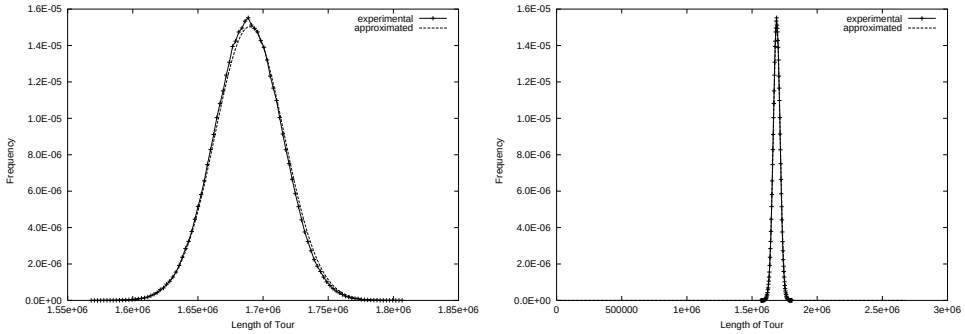

Figure 39: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.19 fl1577

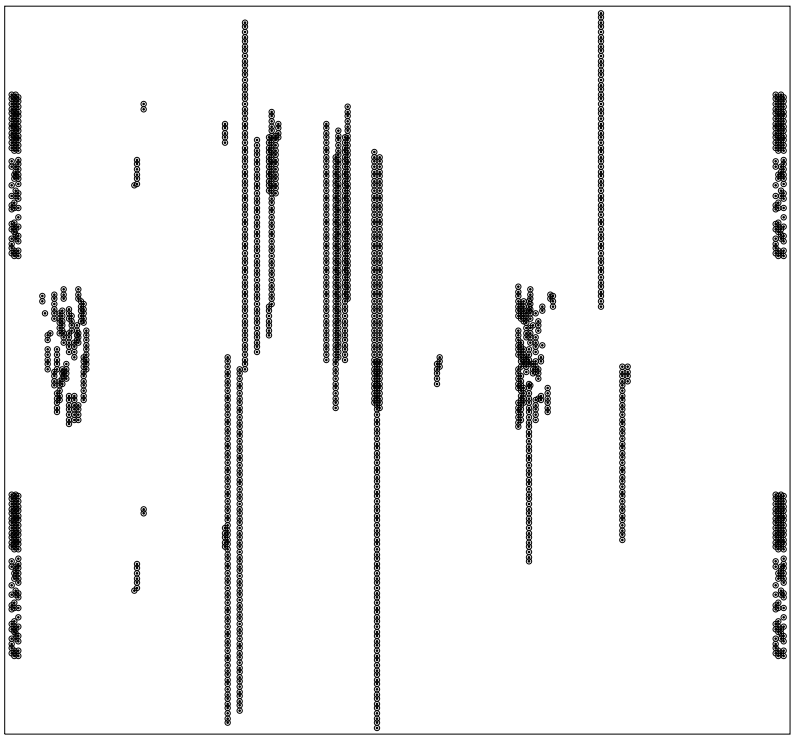

Figure 40: Problem *fl1577*.

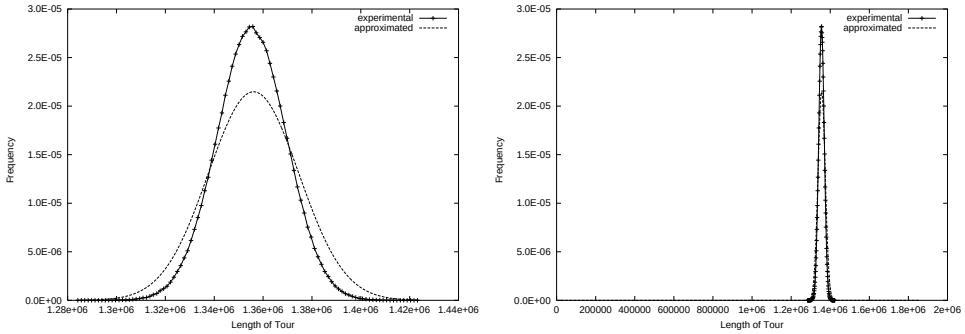

Figure 41: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.20 fl3795

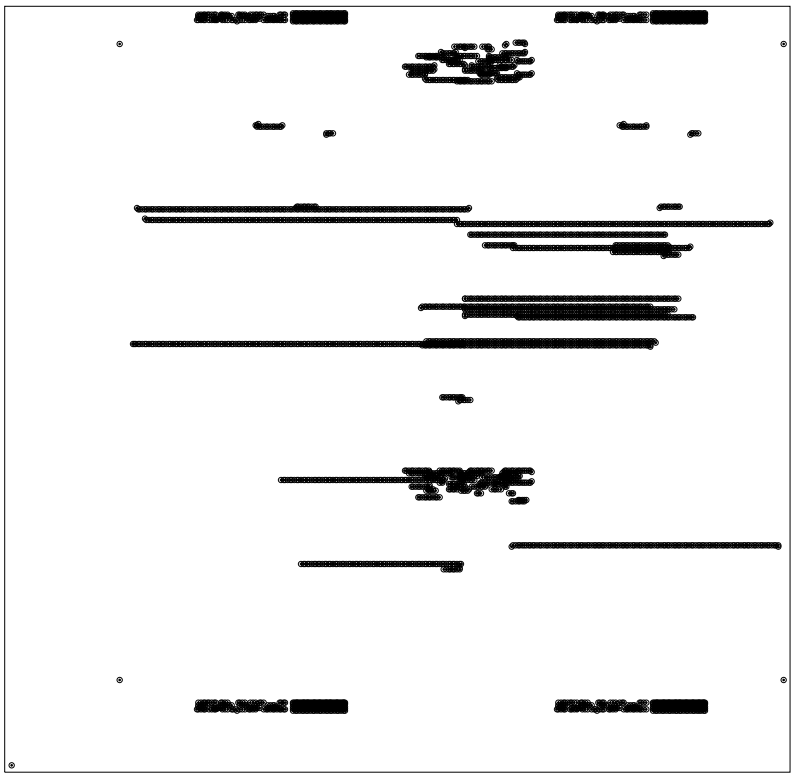

Figure 42: Problem *fl3795*.

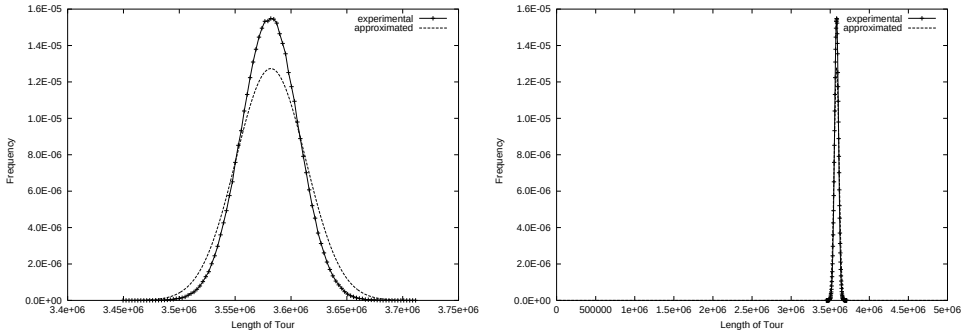

Figure 43: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.21 fl417

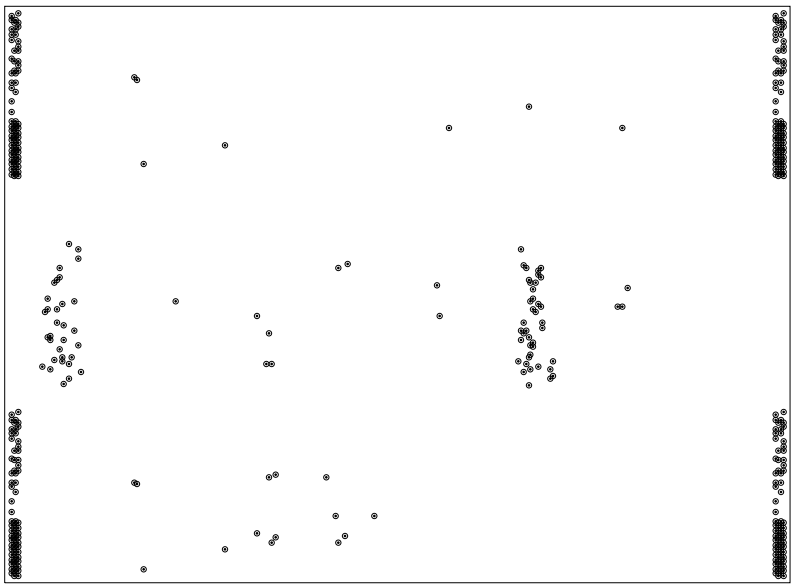

Figure 44: Problem *fl417*.

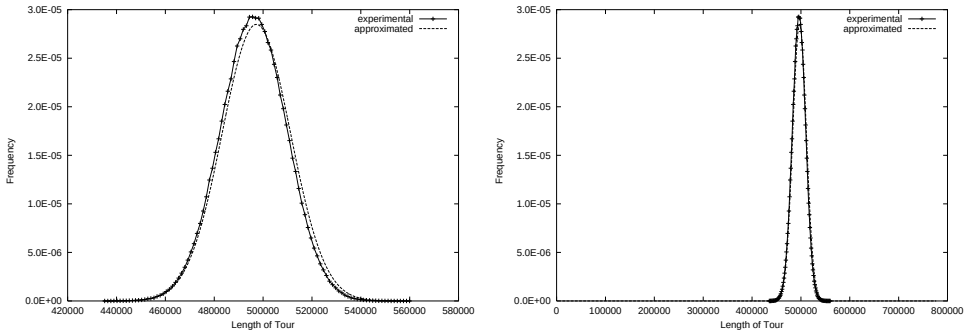

Figure 45: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.22 fnl4461

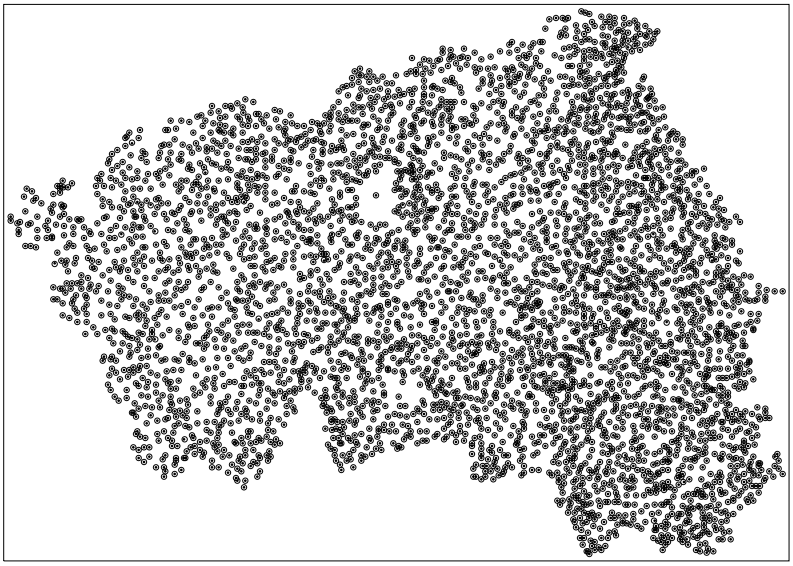

Figure 46: Problem *fnl4461*.

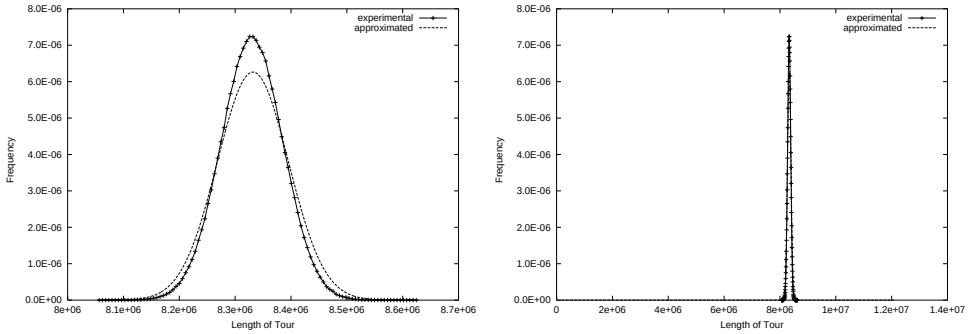

Figure 47: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.23 gil262

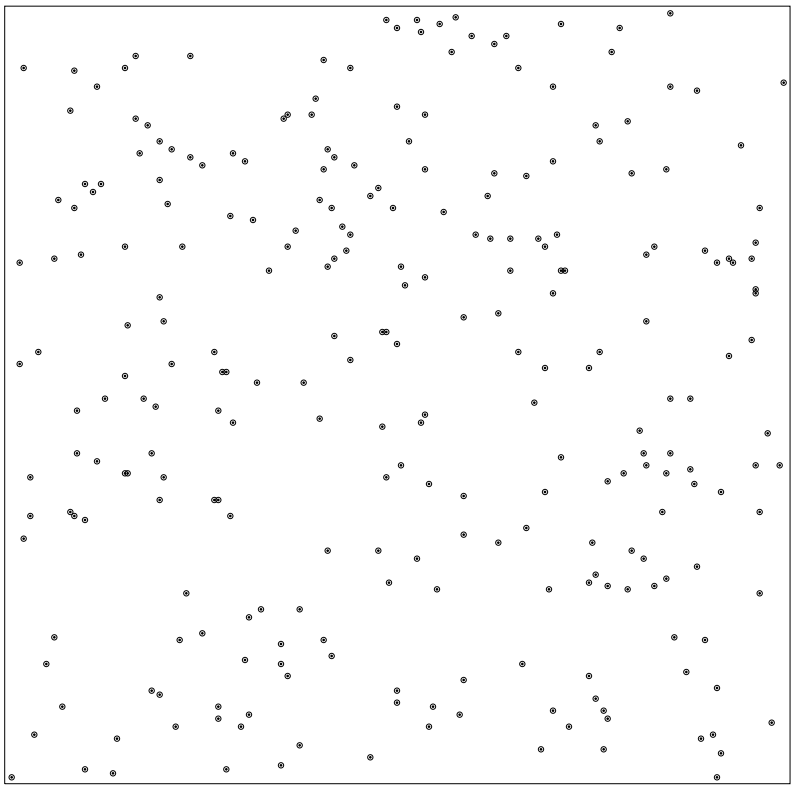

Figure 48: Problem *gil262*.

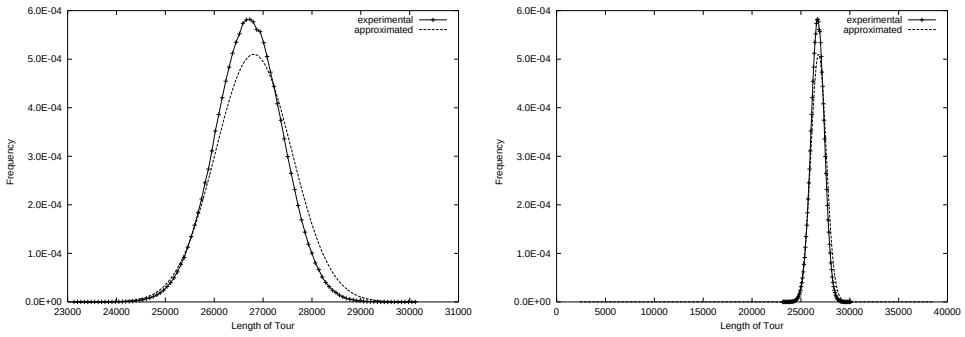

Figure 49: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.24 kroA100

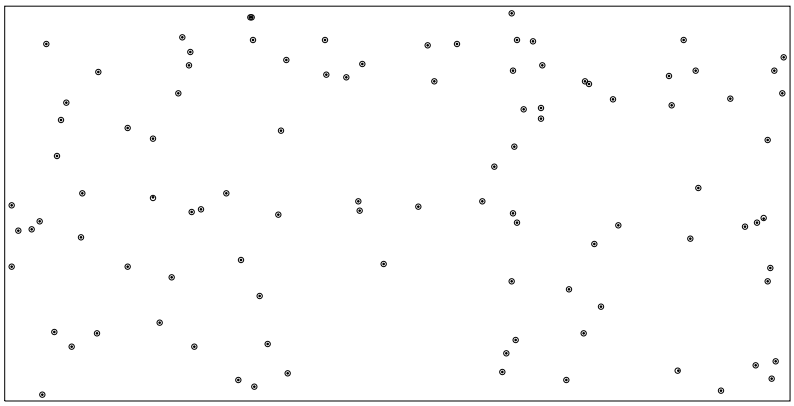

Figure 50: Problem *kroA100*.

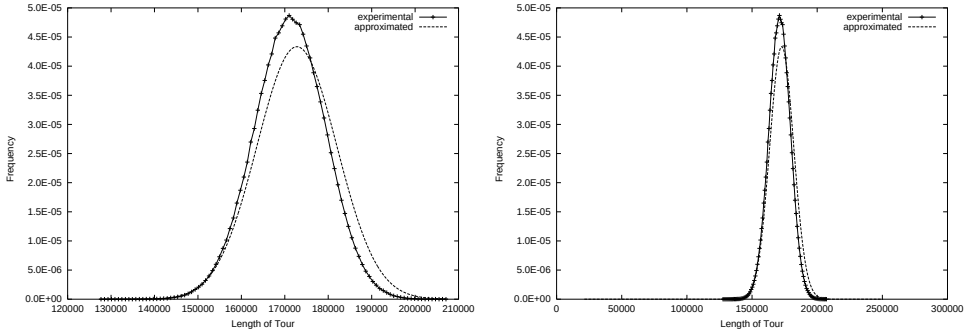

Figure 51: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.25 kroA150

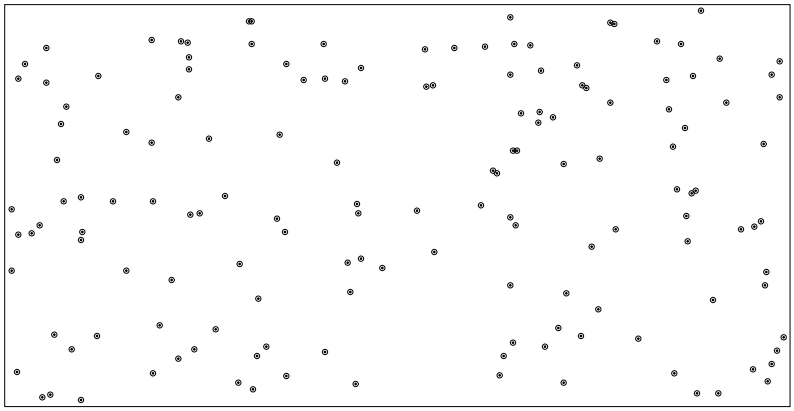

Figure 52: Problem *kroA150*.

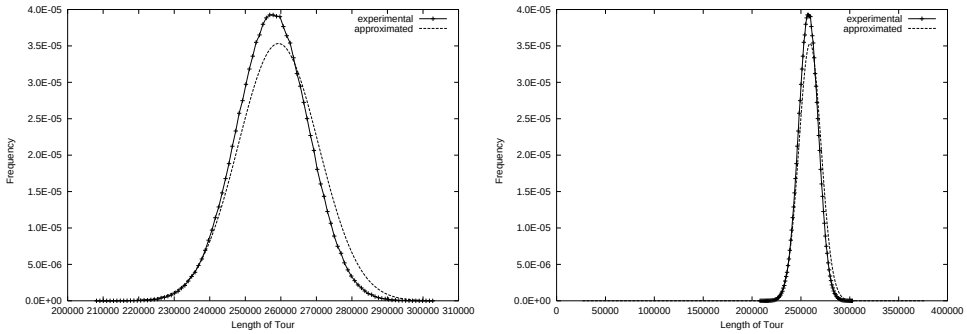

Figure 53: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.26 kroA200

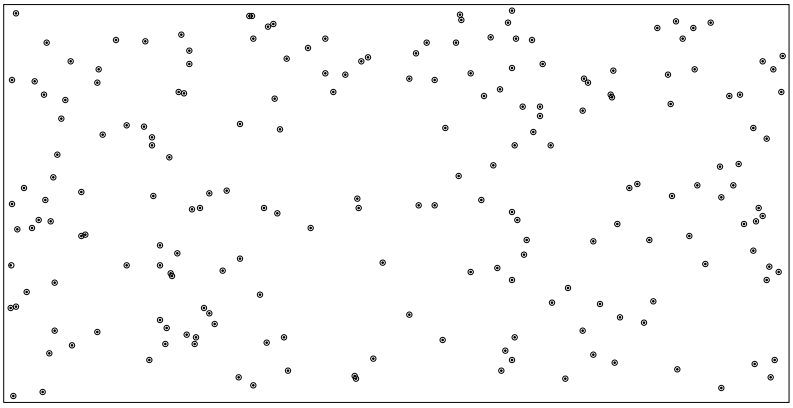

Figure 54: Problem *kroA200*.

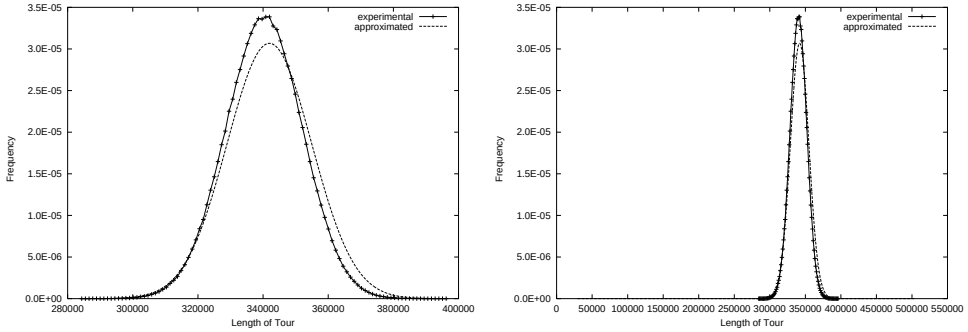

Figure 55: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.27 kroB100

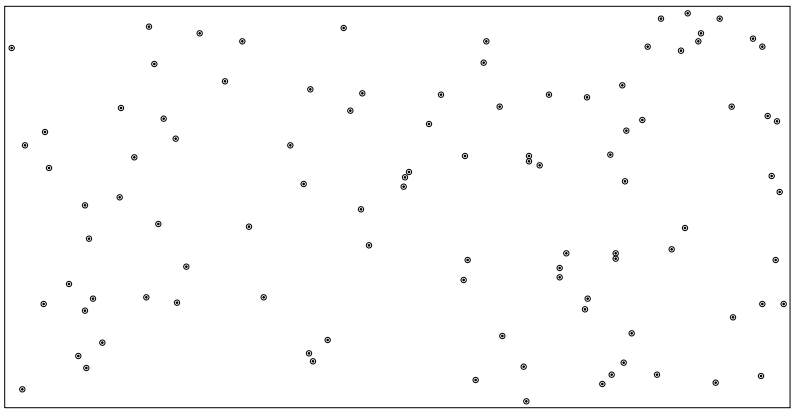

Figure 56: Problem *kroB100*.

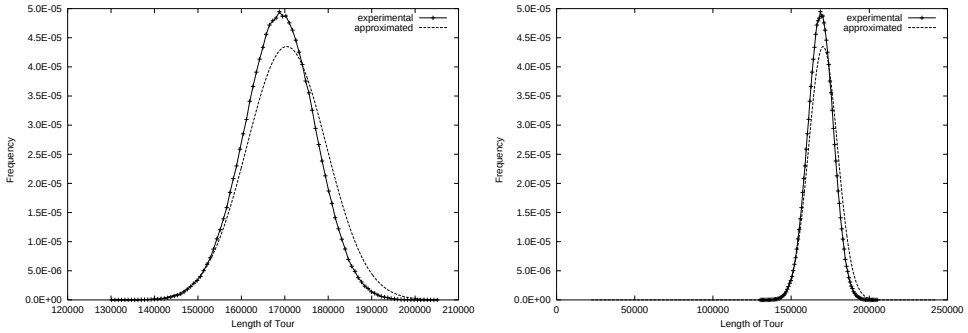

Figure 57: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.28 kroB150

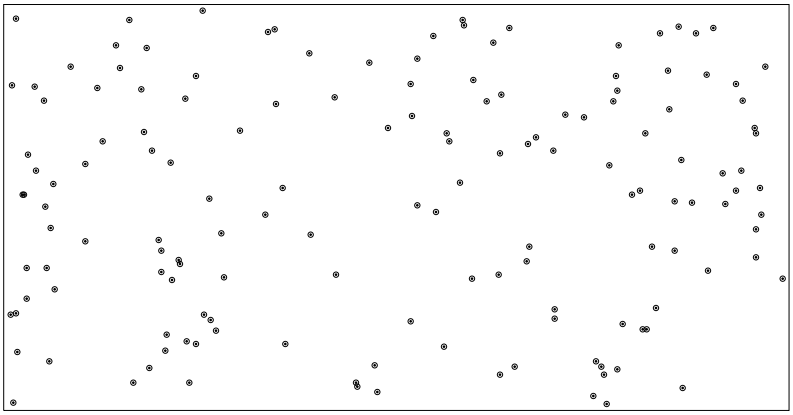

Figure 58: Problem *kroB150*.

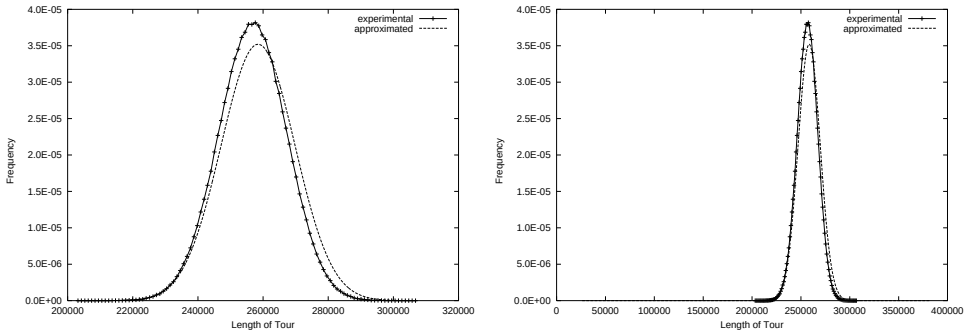

Figure 59: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.29 kroB200

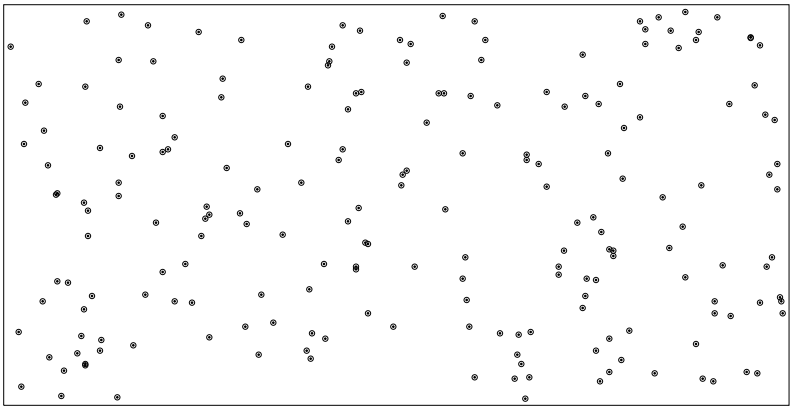

Figure 60: Problem *kroB200*.

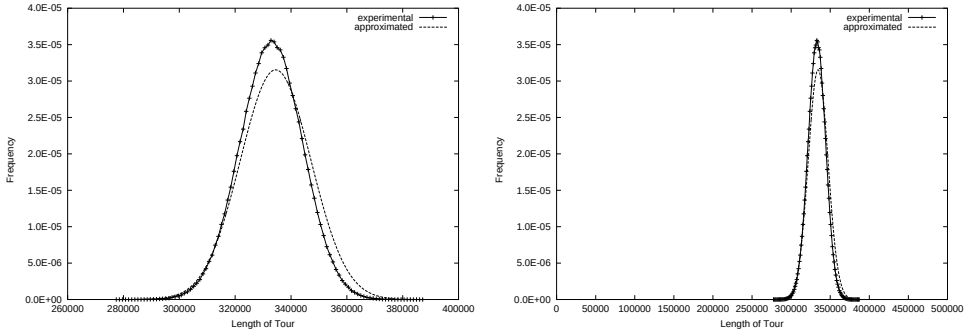

Figure 61: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.30 kroC100

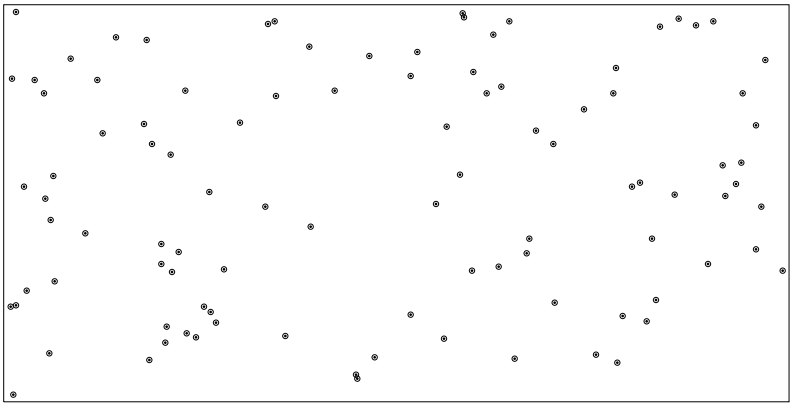

Figure 62: Problem *kroC100*.

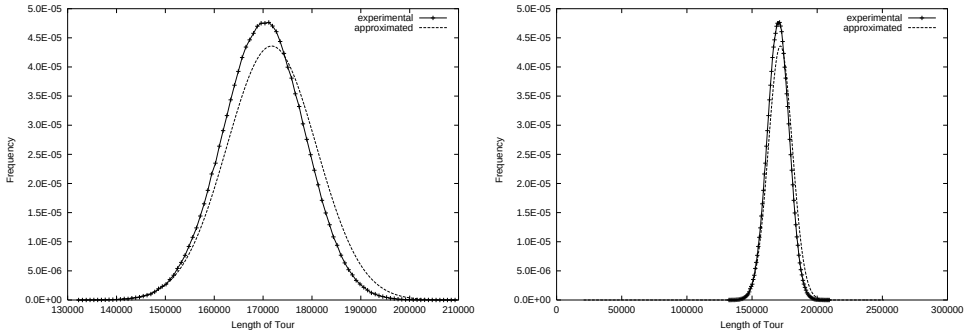

Figure 63: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.31 kroD100

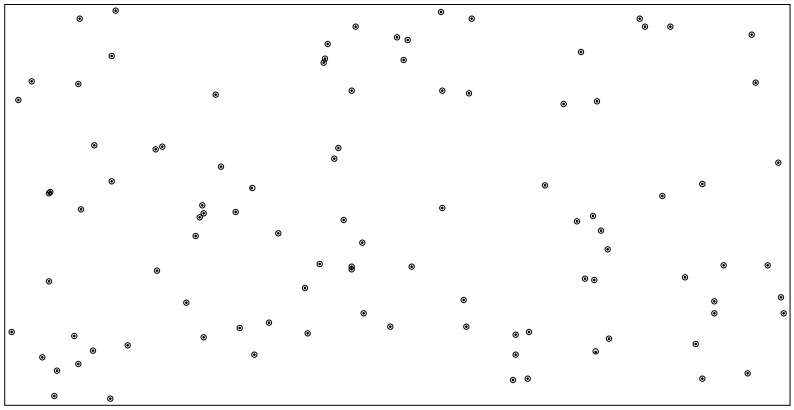

Figure 64: Problem *kroD100*.

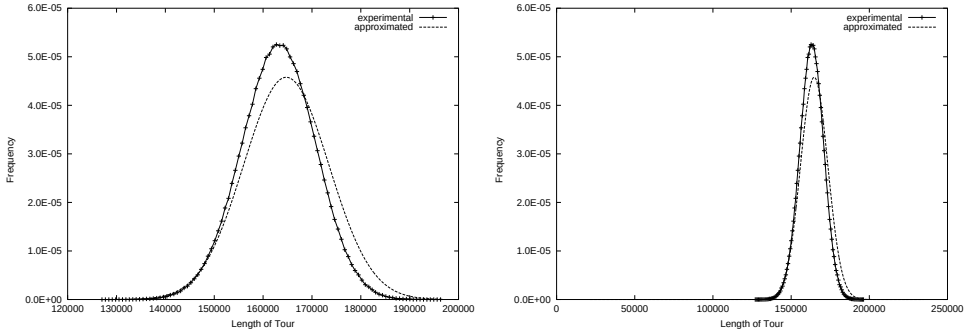

Figure 65: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.32 kroE100

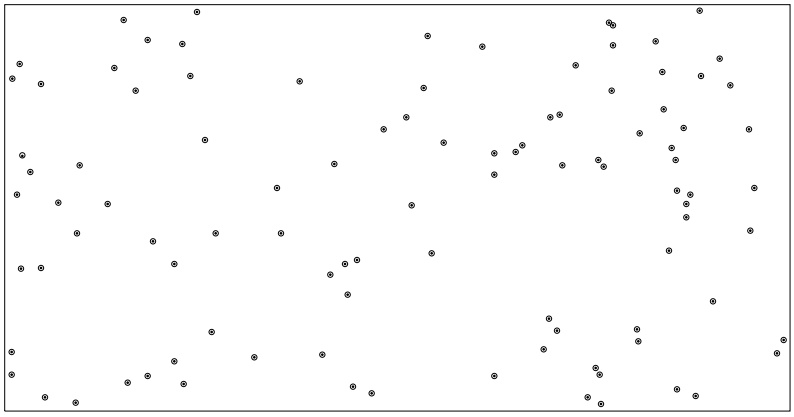

Figure 66: Problem *kroE100*.

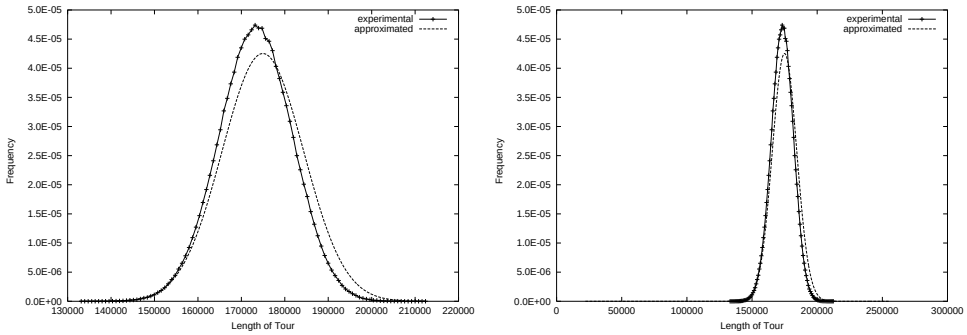

Figure 67: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.33 lin105

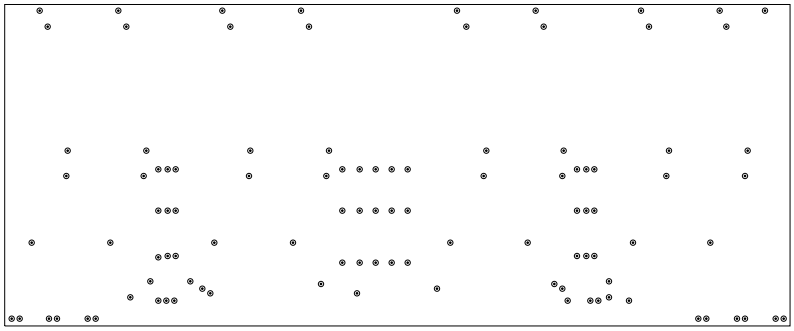

Figure 68: Problem *lin105*.

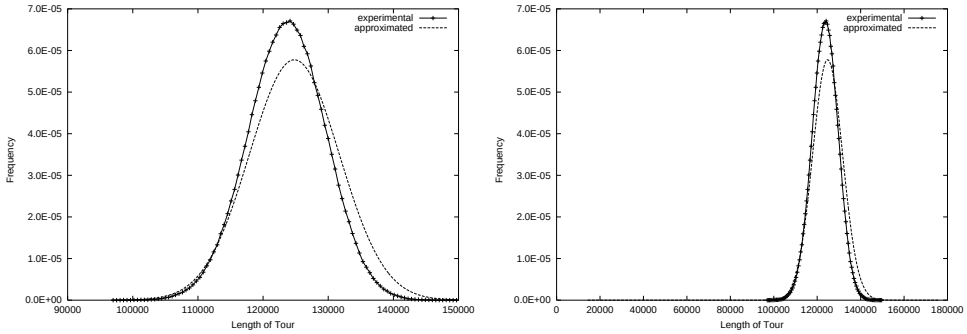

Figure 69: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.34 lin318

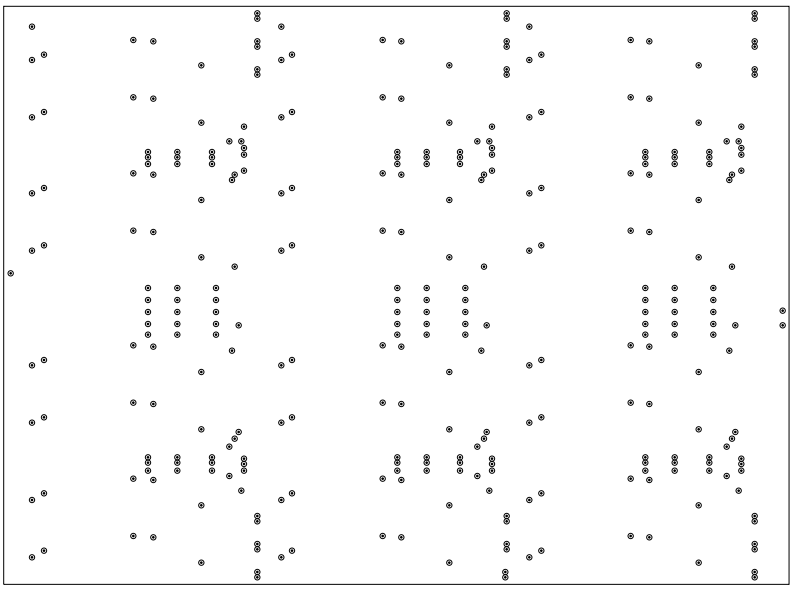

Figure 70: Problem *lin318*.

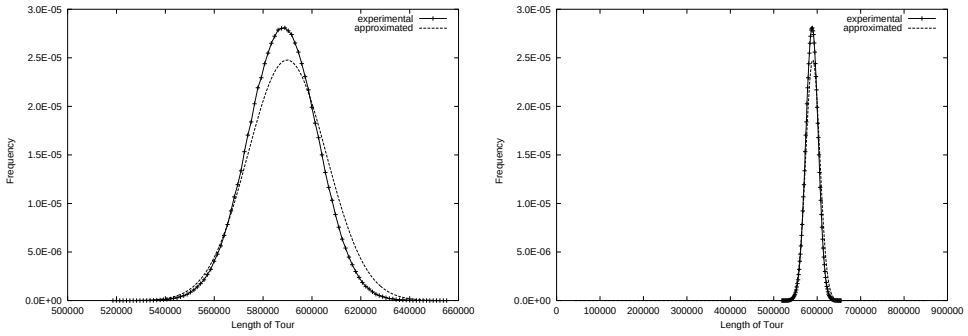

Figure 71: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.35 nrw1379

Figure 72: Problem *nrw1379*.

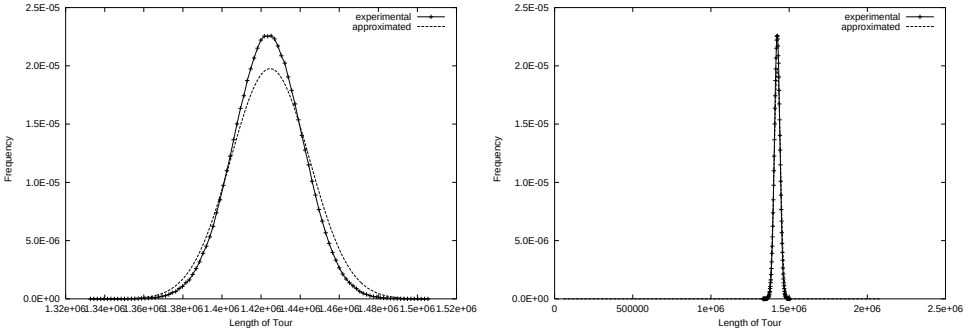

Figure 73: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.36 p654

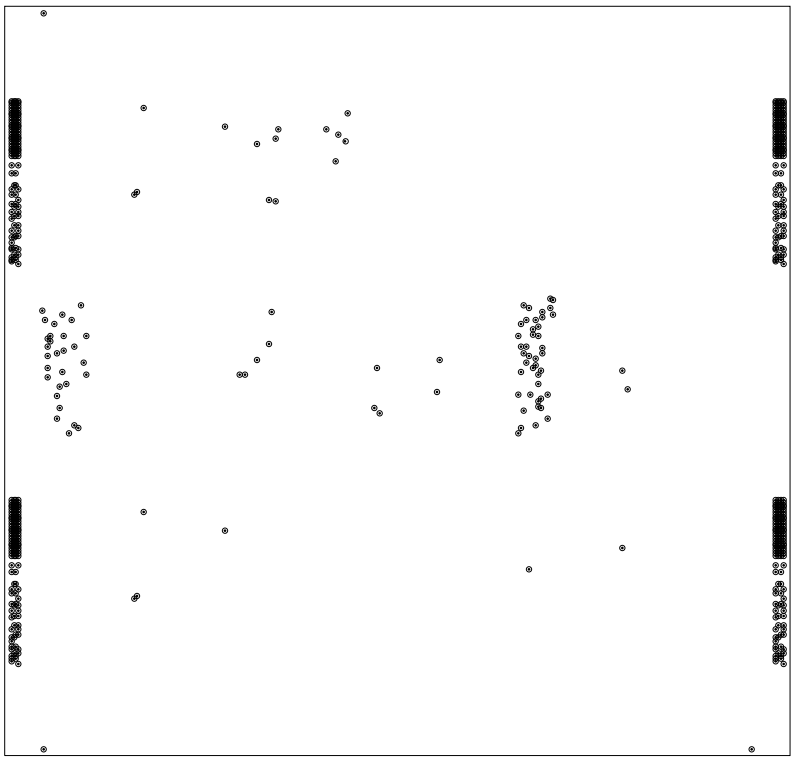

Figure 74: Problem *p654*.

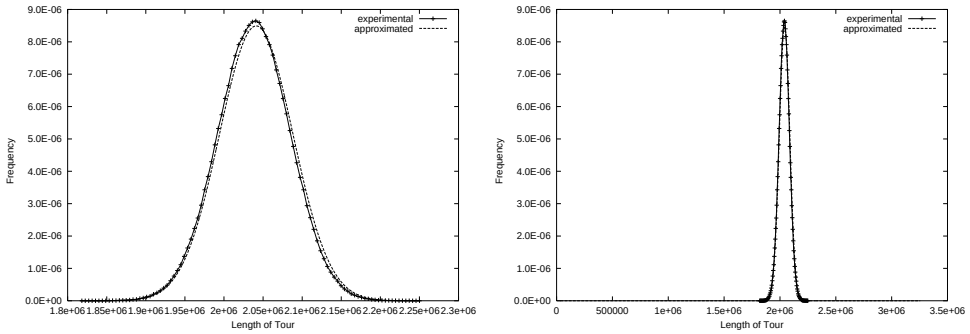

Figure 75: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.37 pcb1173

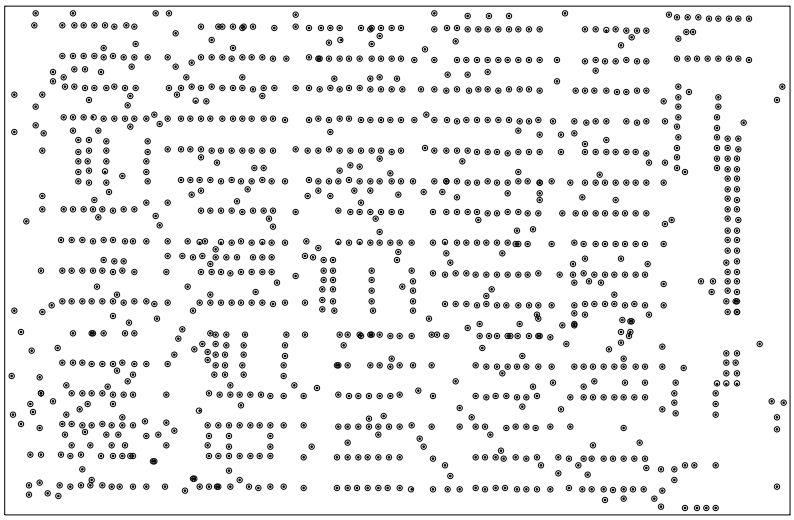

Figure 76: Problem *pcb1173*.

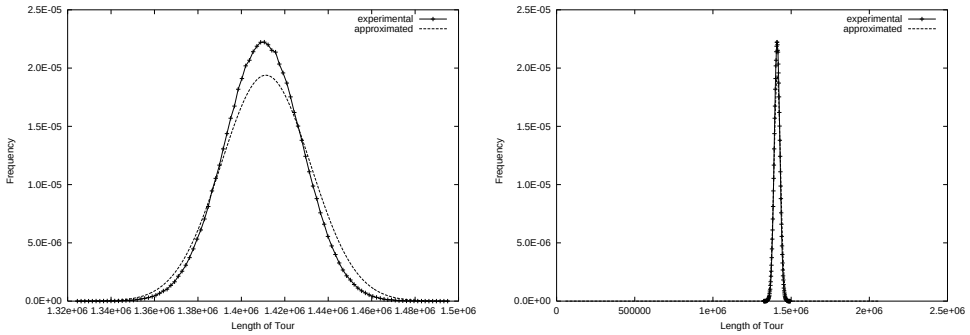

Figure 77: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.38 pcb3038

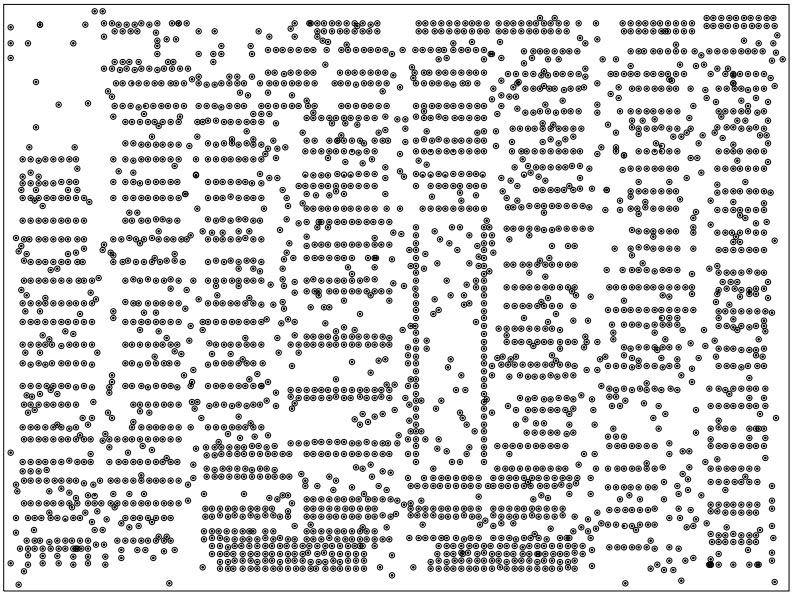

Figure 78: Problem *pcb3038*.

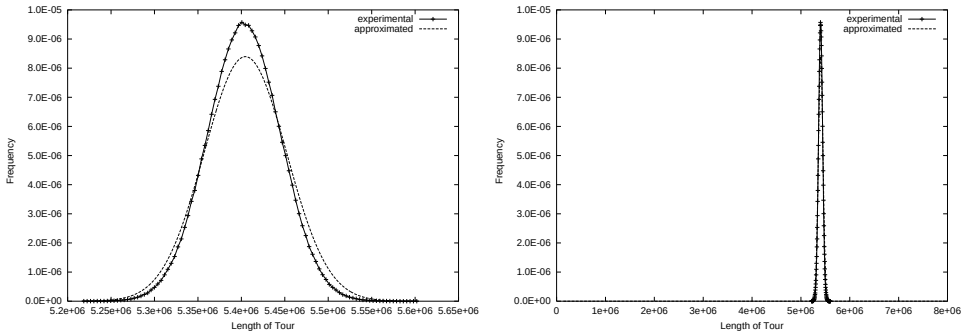

Figure 79: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.39 pcb442

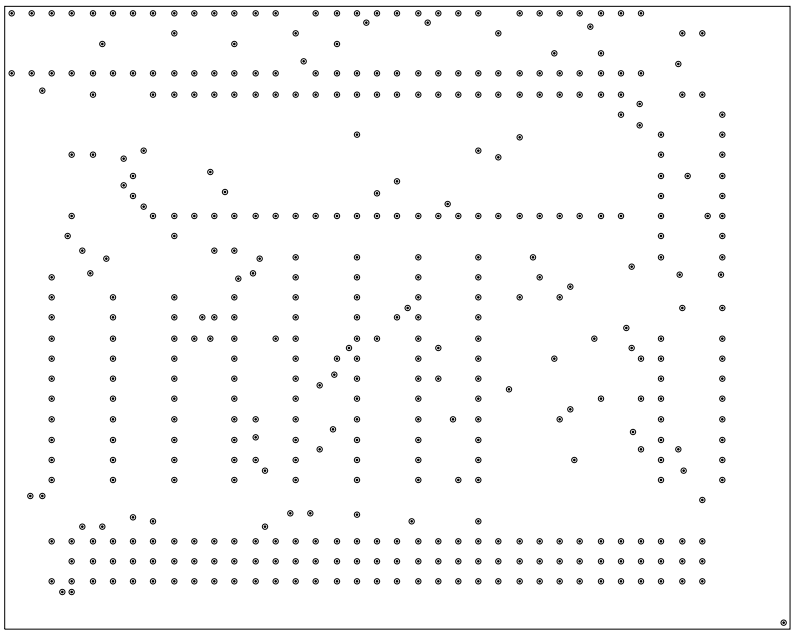

Figure 80: Problem *pcb442*.

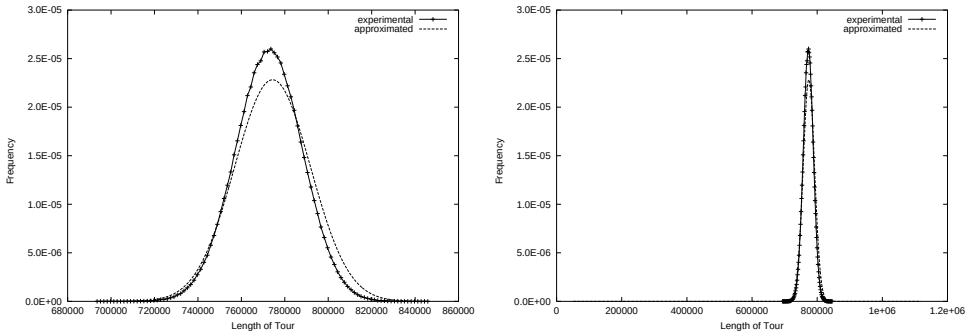

Figure 81: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.40 pr1002

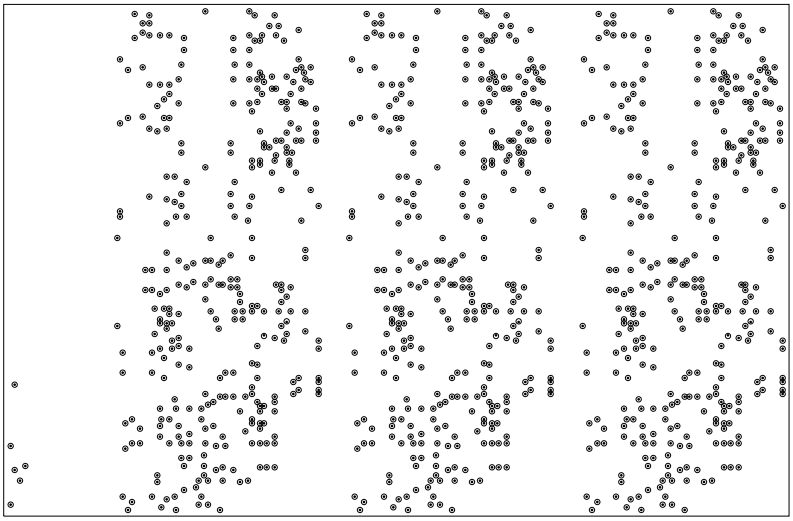

Figure 82: Problem *pr1002*.

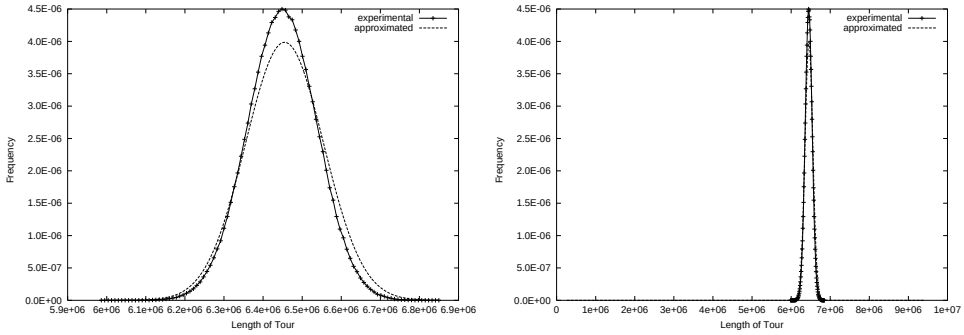

Figure 83: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.41 pr107

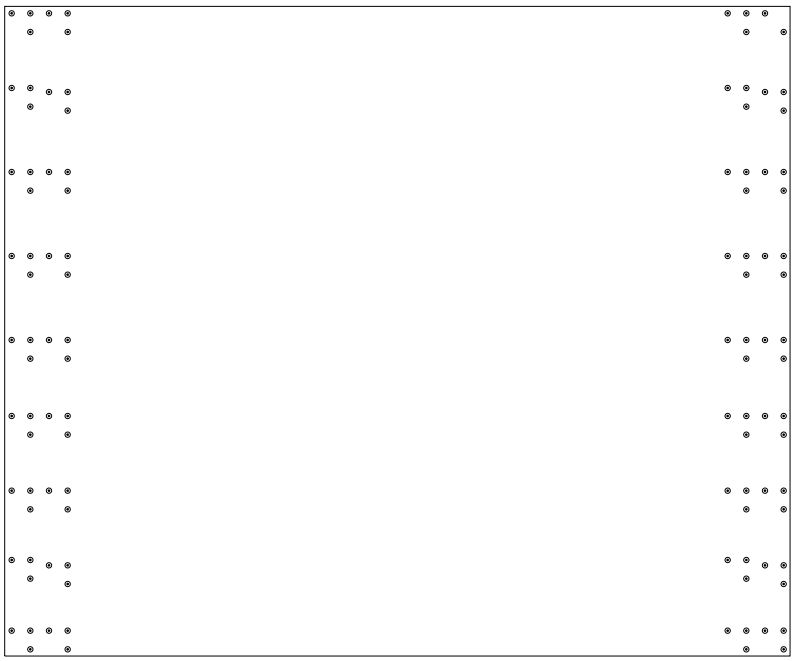

Figure 84: Problem *pr107*.

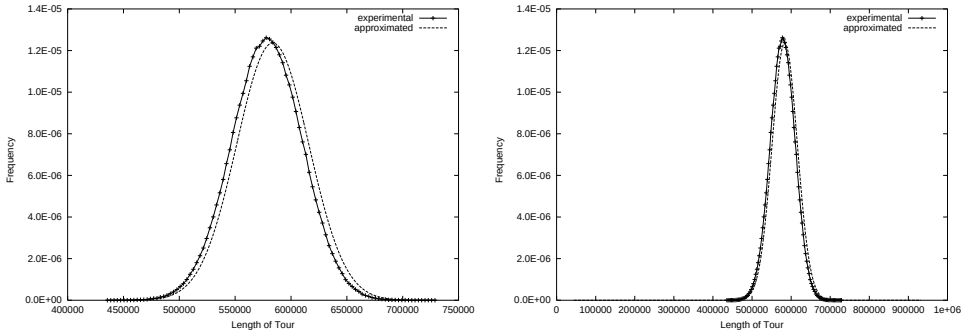

Figure 85: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.42 pr124

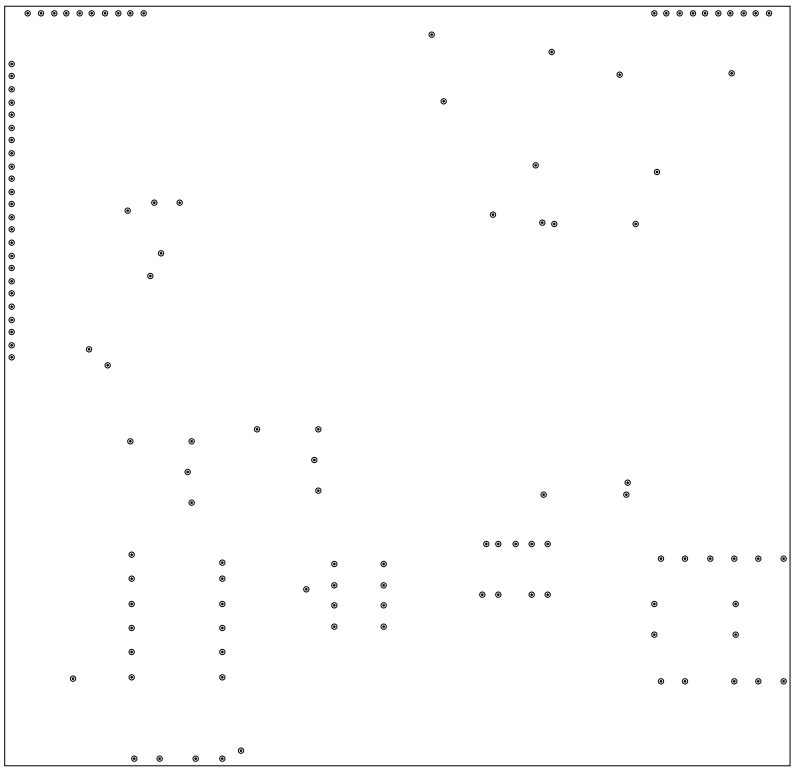

Figure 86: Problem *pr124*.

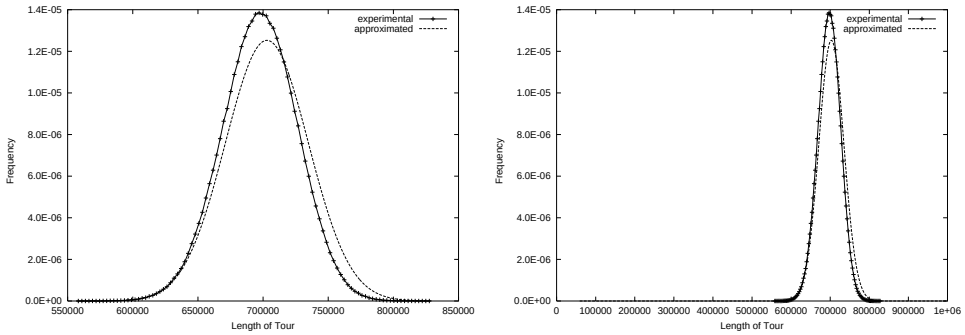

Figure 87: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.43 pr136

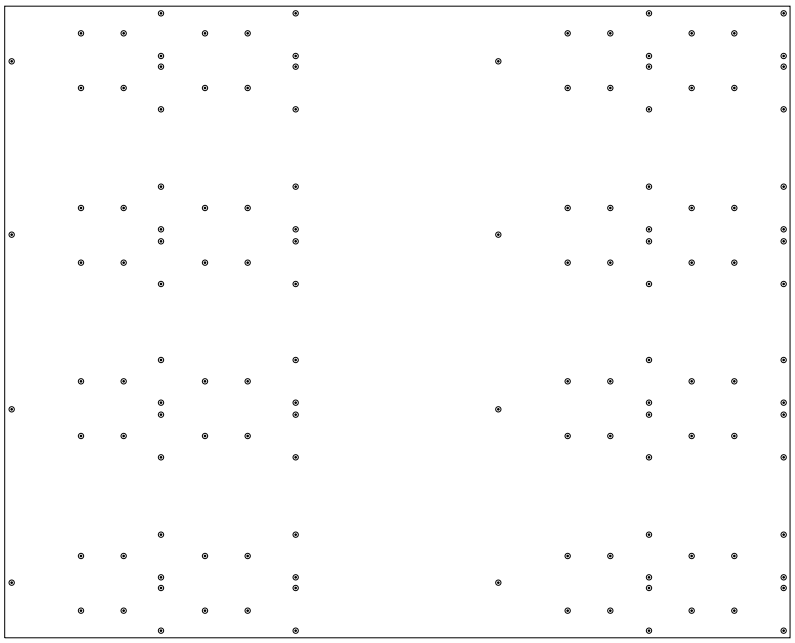

Figure 88: Problem *pr136*.

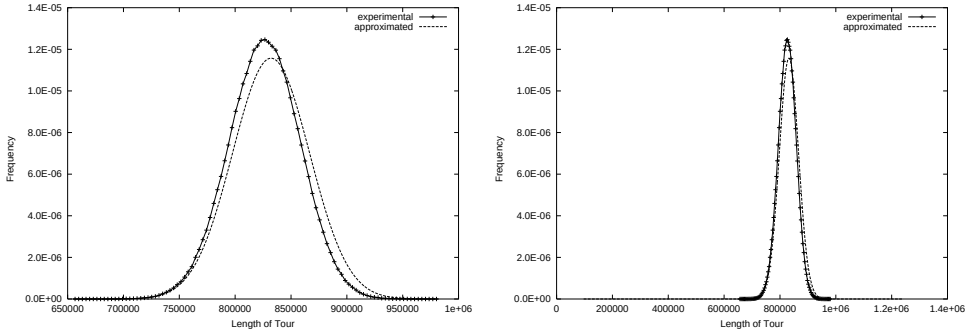

Figure 89: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.44 pr144

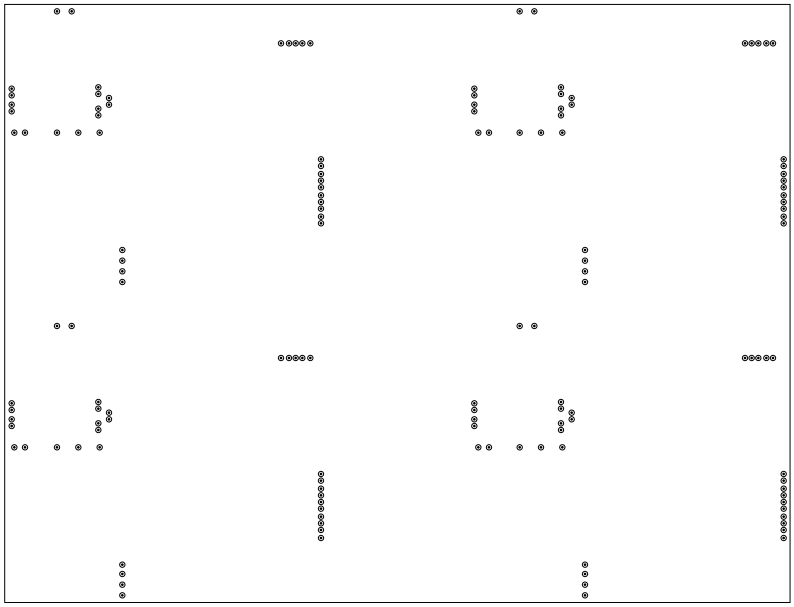

Figure 90: Problem *pr144*.

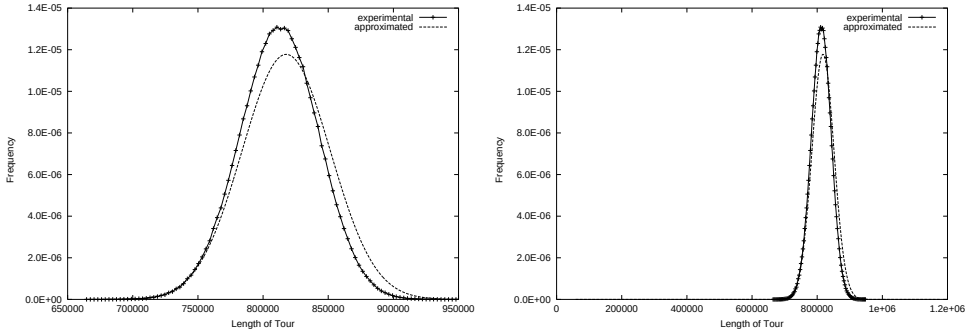

Figure 91: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.45 pr152

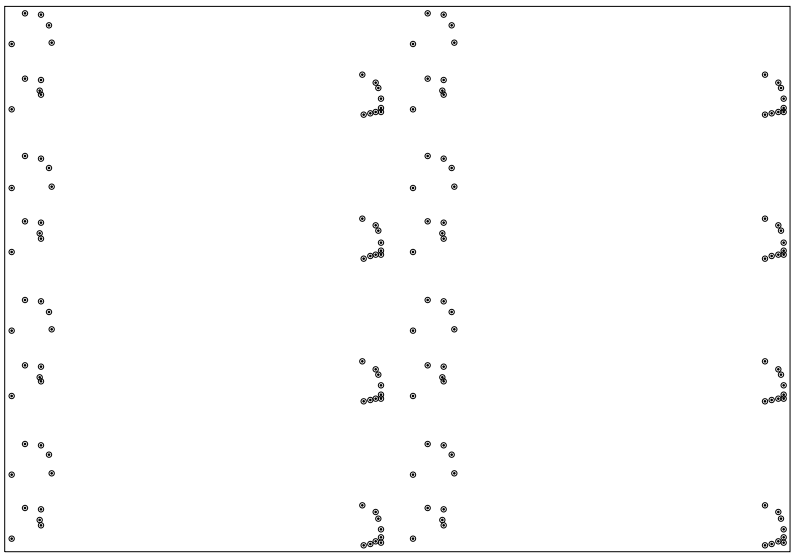

Figure 92: Problem *pr152*.

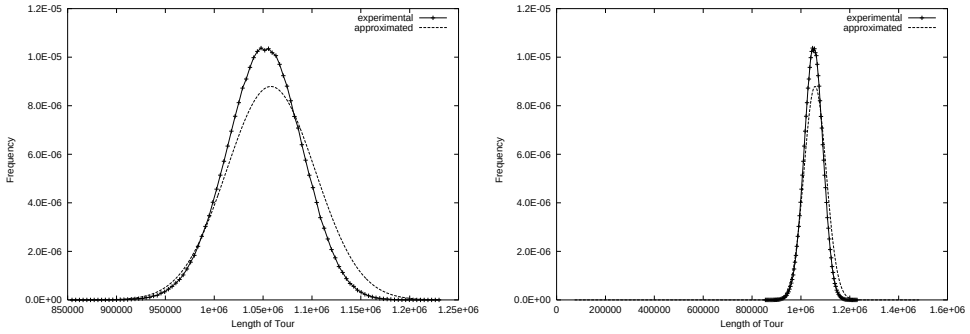

Figure 93: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.46 pr226

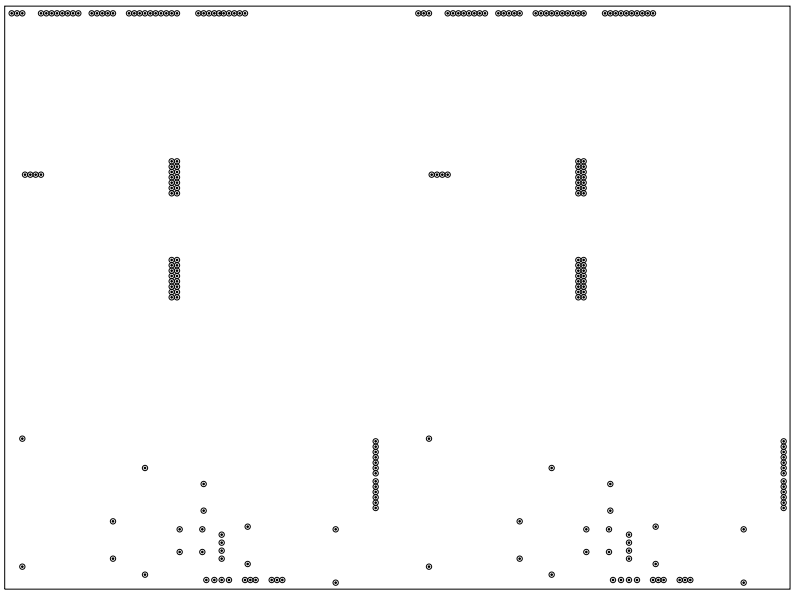

Figure 94: Problem *pr226*.

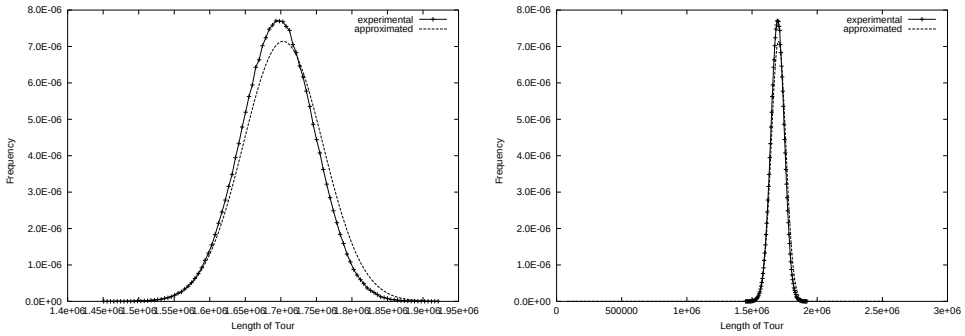

Figure 95: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.47 pr2392

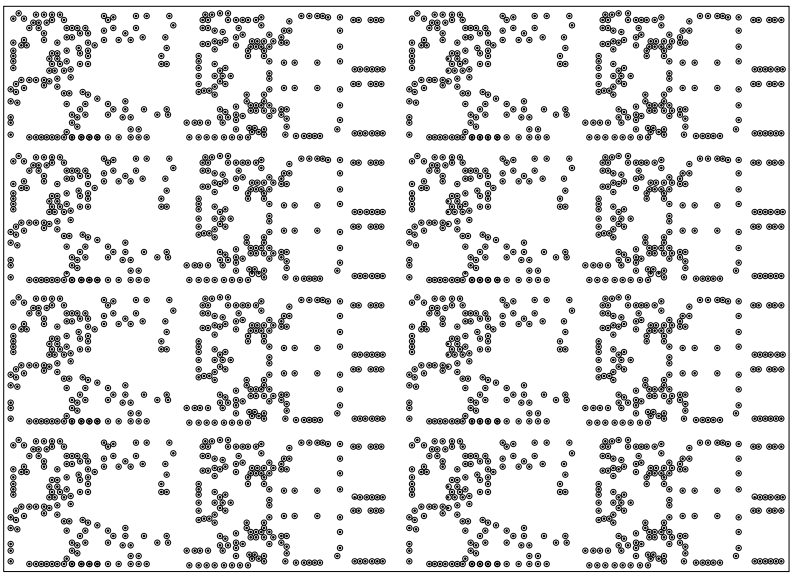

Figure 96: Problem *pr2392*.

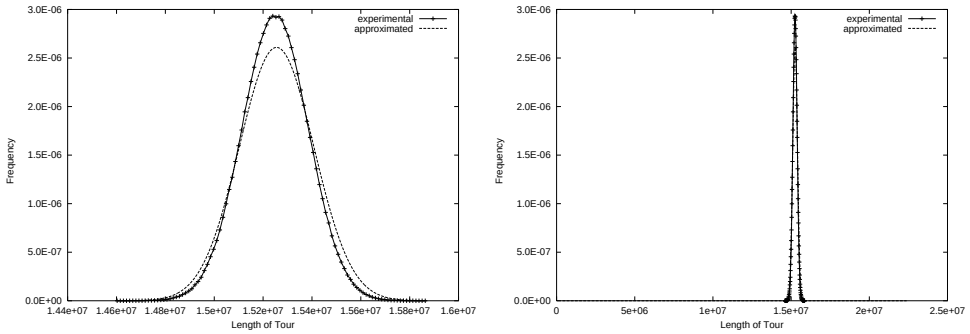

Figure 97: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.48 pr264

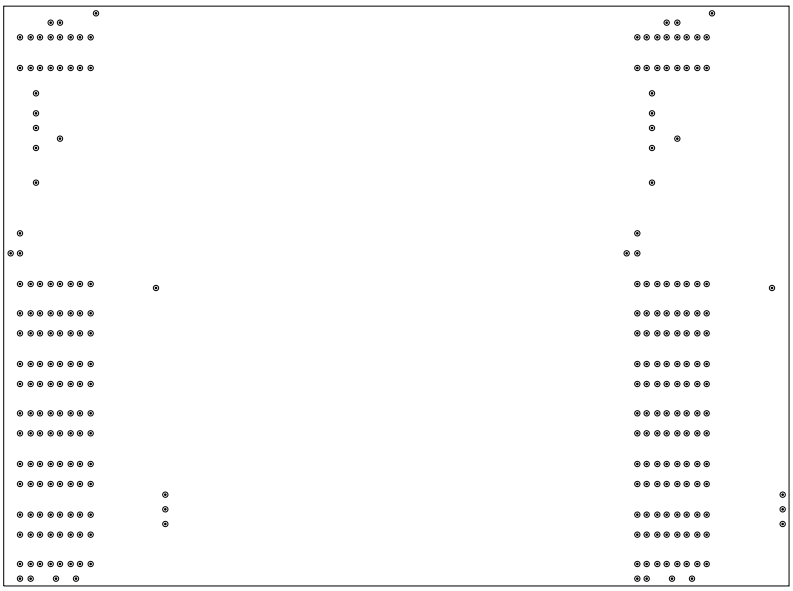

Figure 98: Problem *pr264*.

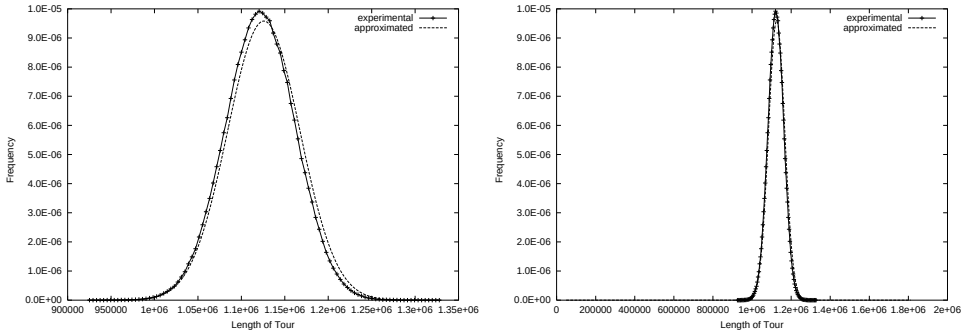

Figure 99: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.49 pr299

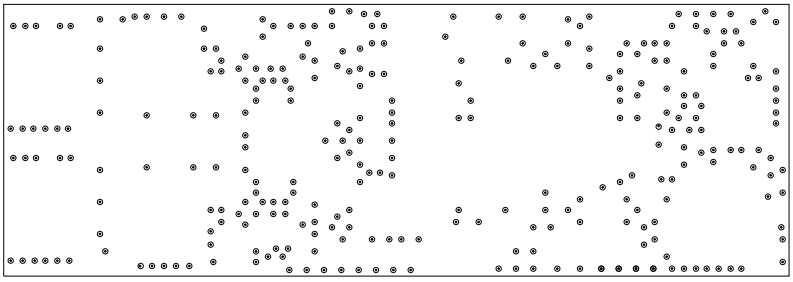

Figure 100: Problem *pr299*.

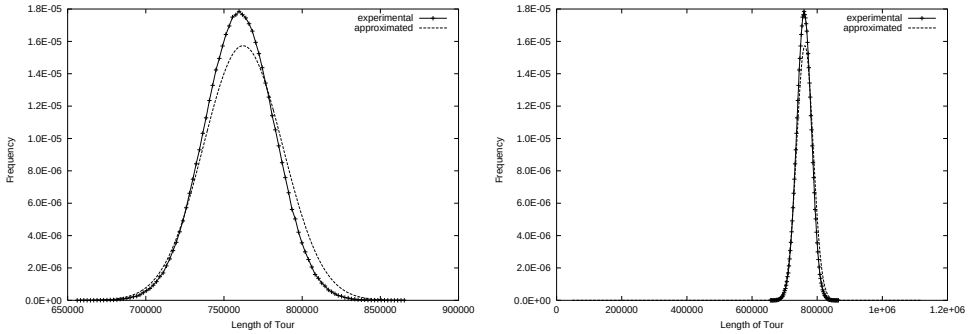

Figure 101: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.50 pr439

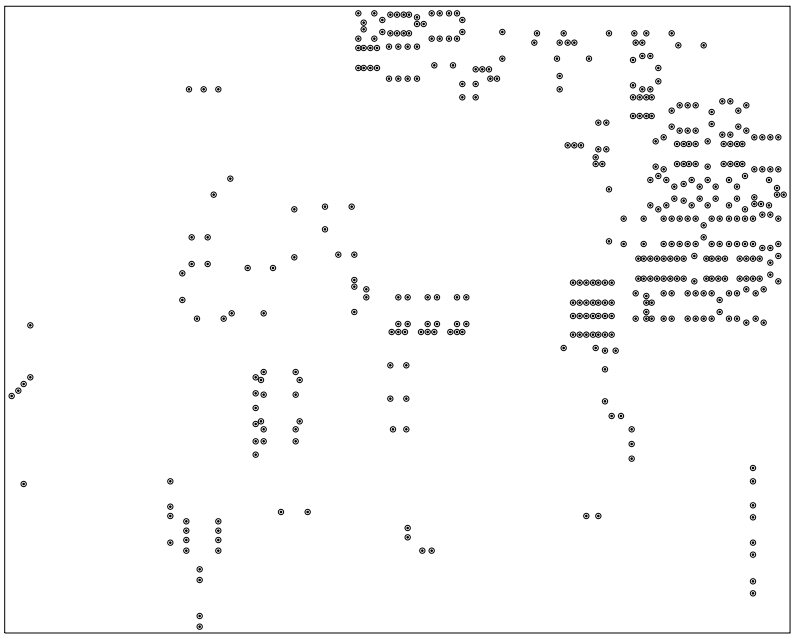

Figure 102: Problem *pr439*.

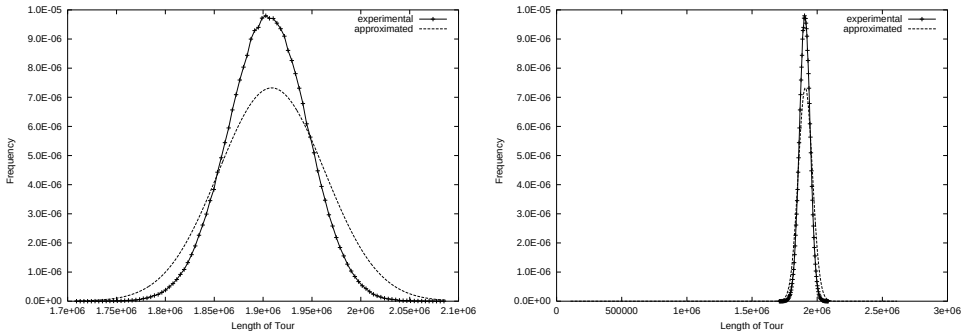

Figure 103: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.51 pr76

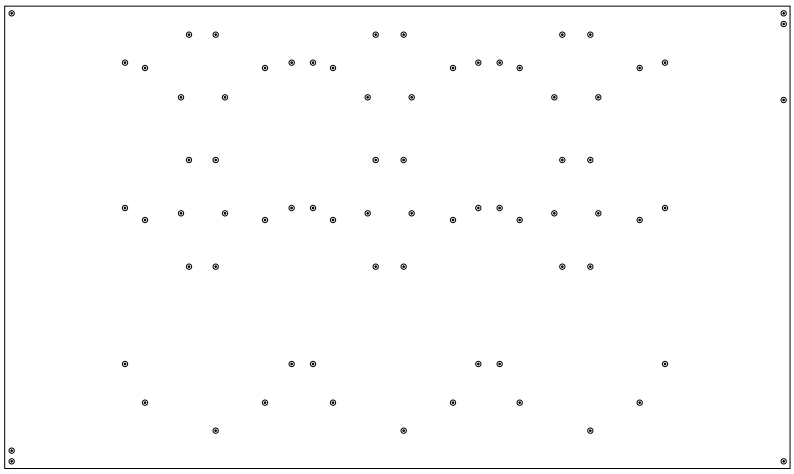

Figure 104: Problem *pr76*.

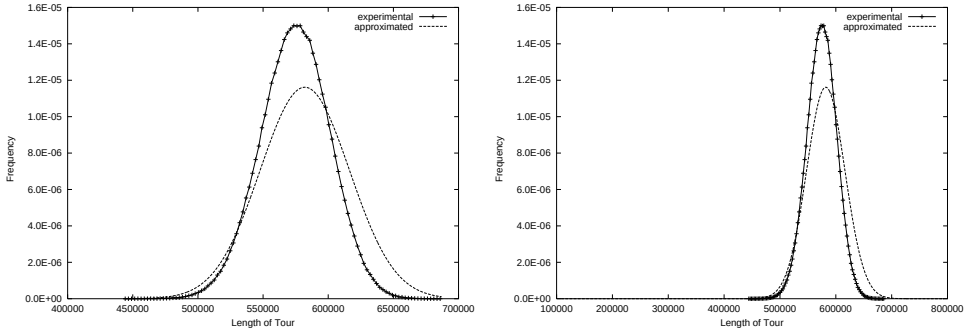

Figure 105: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.52 rat195

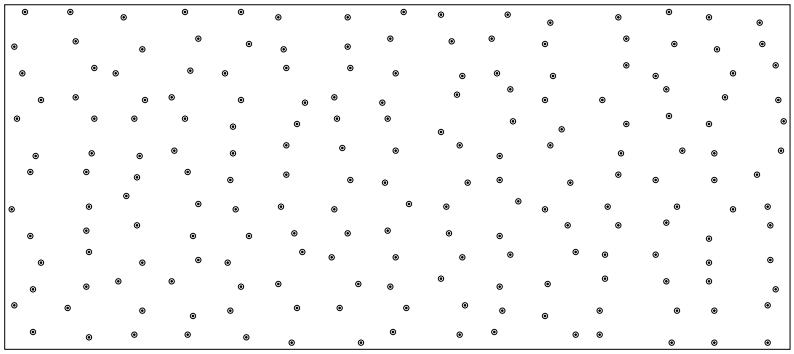

Figure 106: Problem *rat195*.

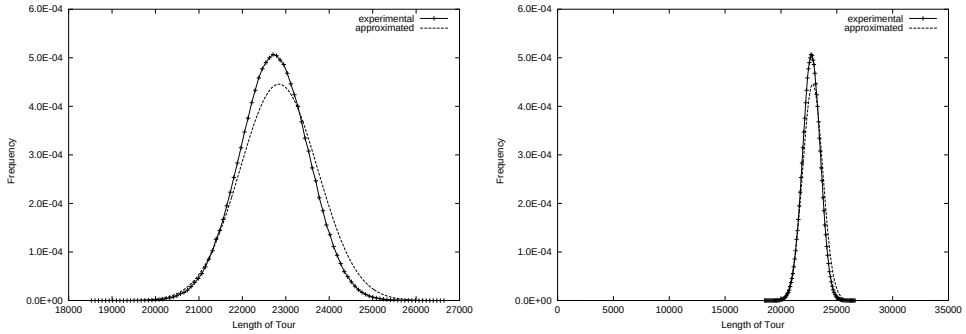

Figure 107: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.53 rat575

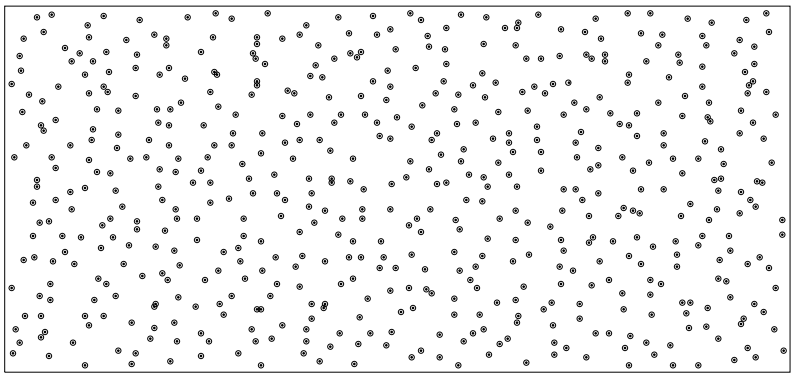

Figure 108: Problem *rat575*.

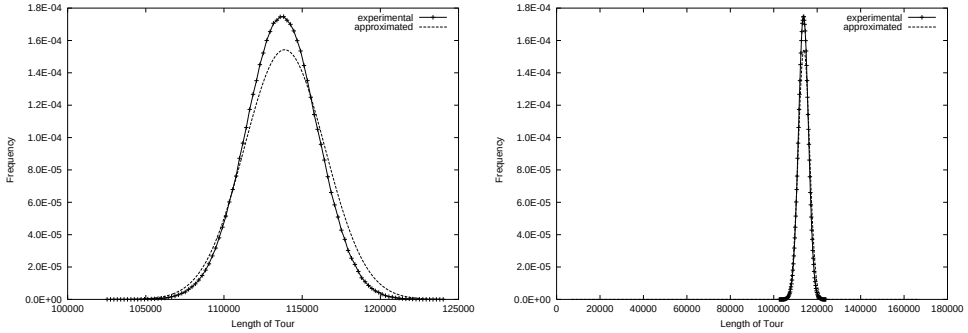

Figure 109: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.54 rat783

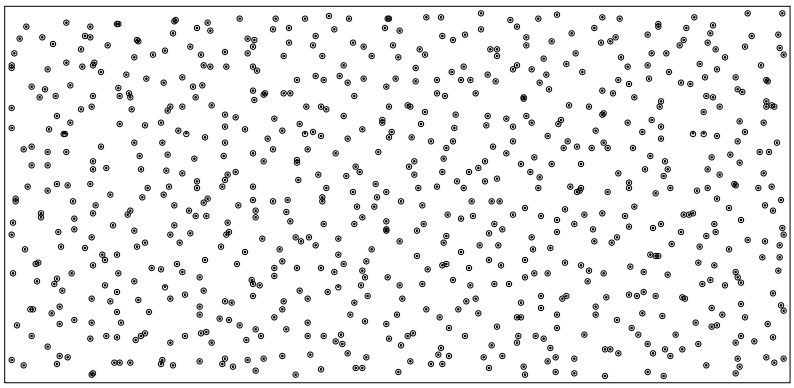

Figure 110: Problem *rat783*.

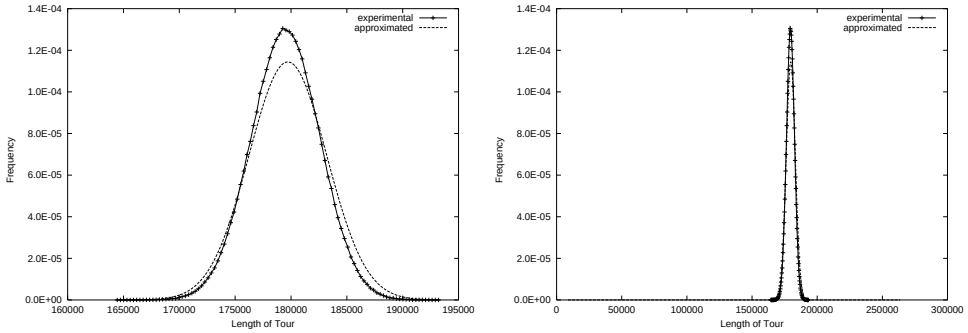

Figure 111: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.55 rat99

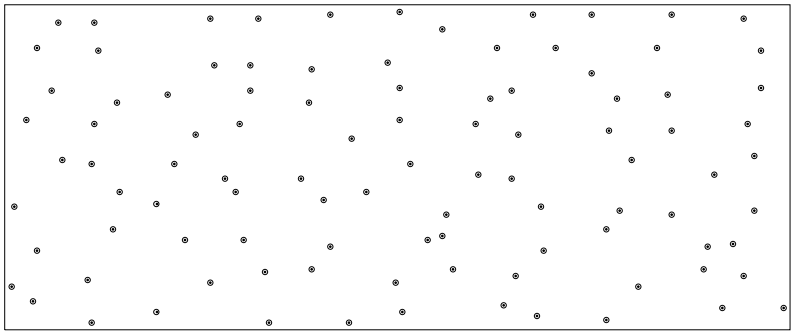

Figure 112: Problem *rat99*.

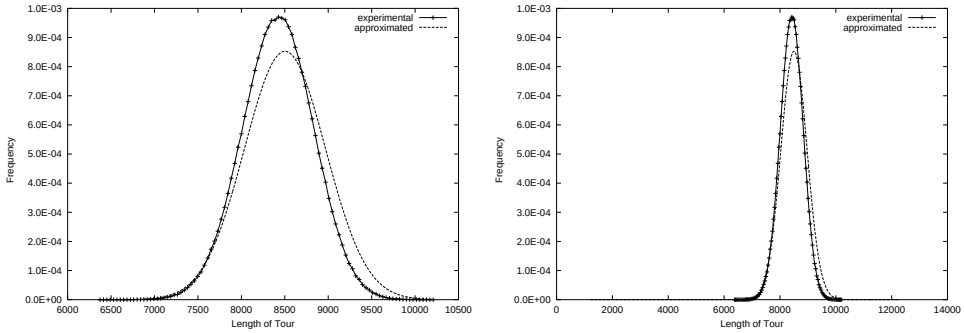

Figure 113: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.56 rd100

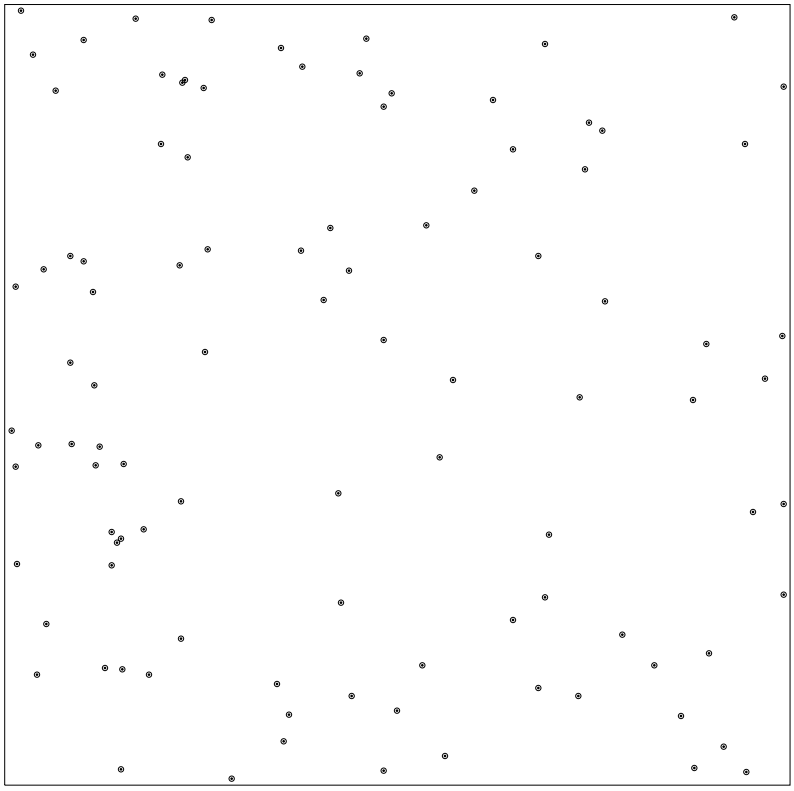

Figure 114: Problem *rd100*.

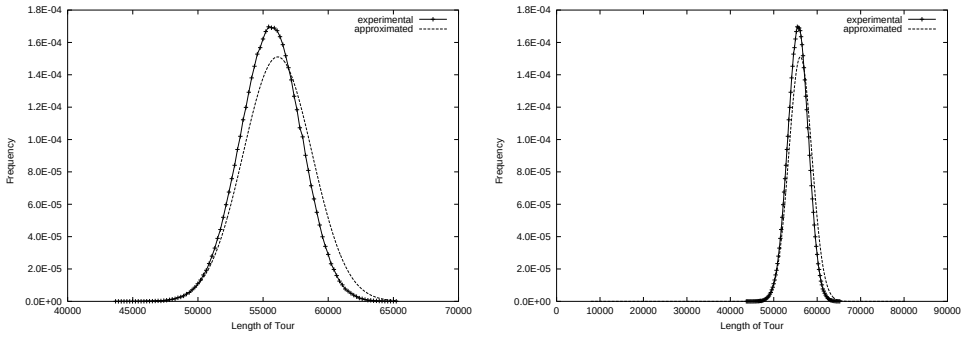

Figure 115: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.57 rd400

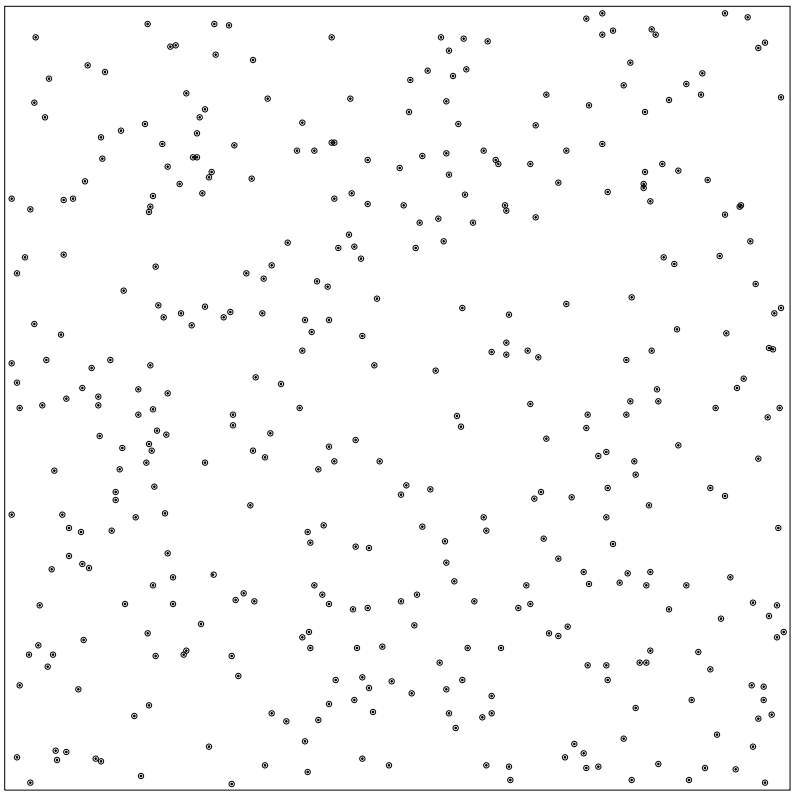

Figure 116: Problem *rd400*.

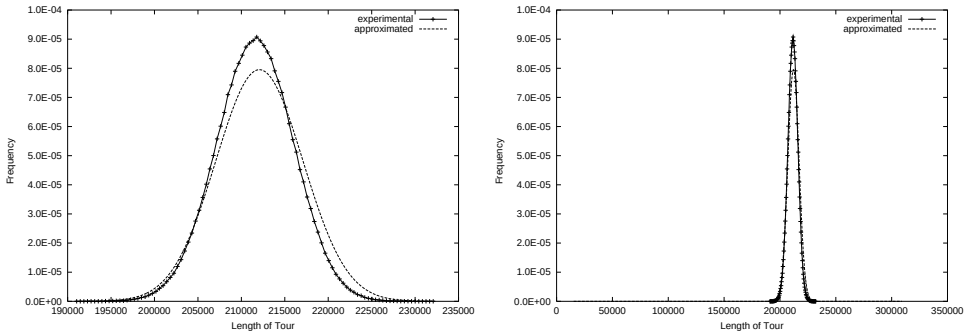

Figure 117: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.58 rl11849

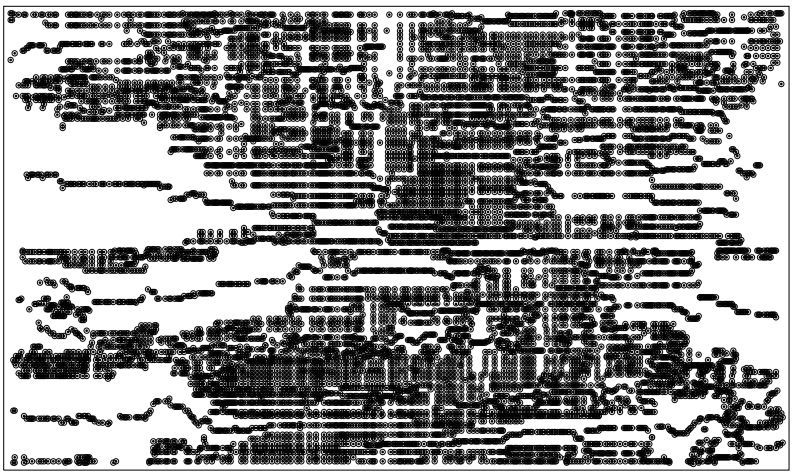

Figure 118: Problem *rl11849*.

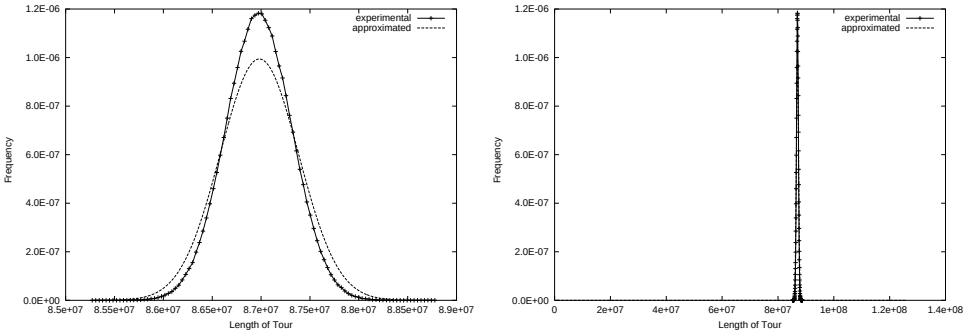

Figure 119: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.59 rl1304

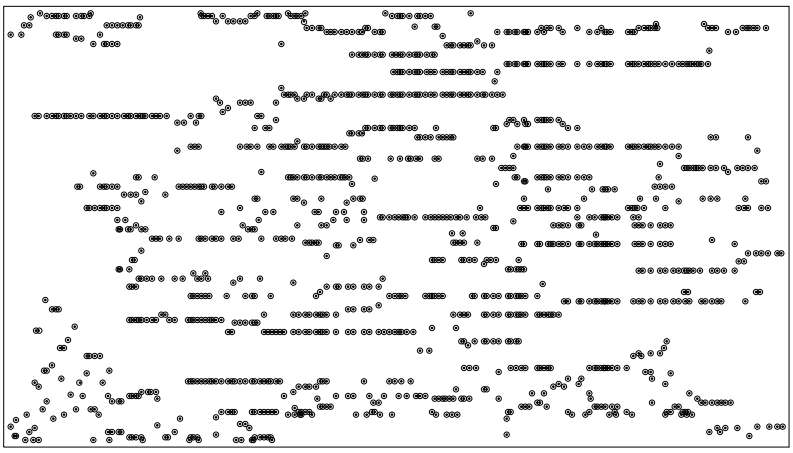

Figure 120: Problem *rl1304*.

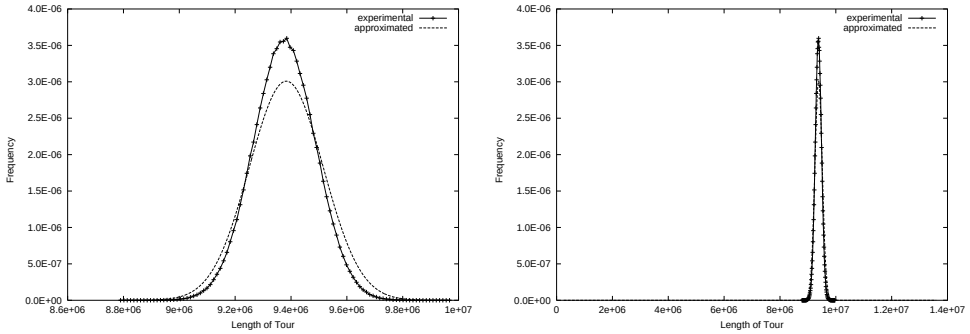

Figure 121: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.60 rl1323

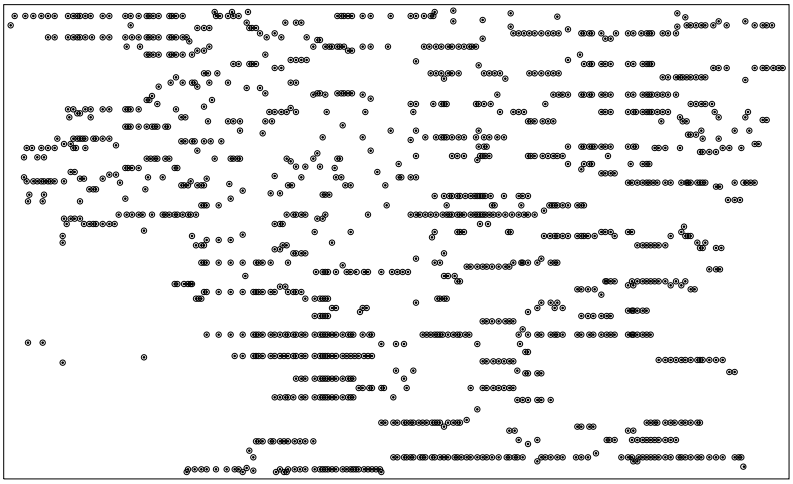

Figure 122: Problem *rl1323*.

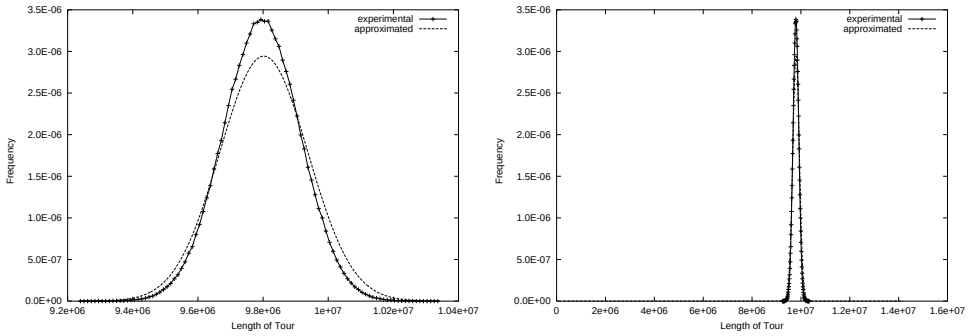

Figure 123: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.61 rl1889

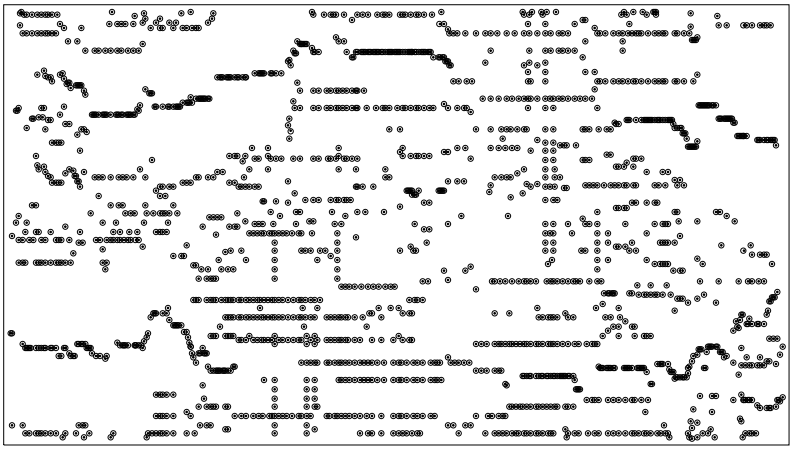

Figure 124: Problem *rl1889*.

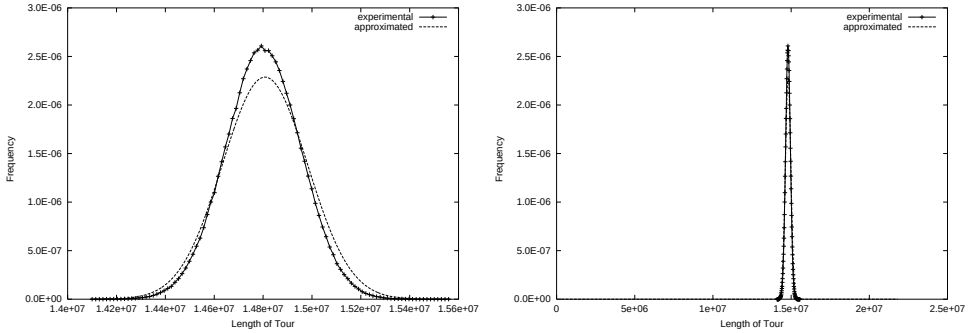

Figure 125: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.62 rl5915

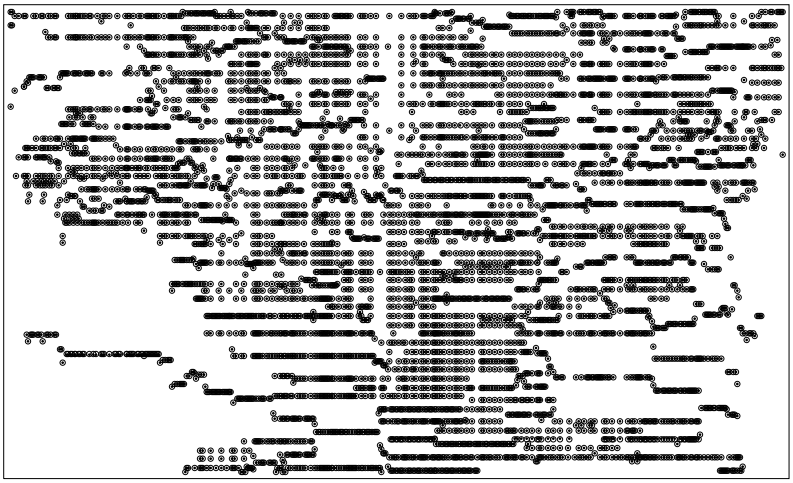

Figure 126: Problem *rl5915*.

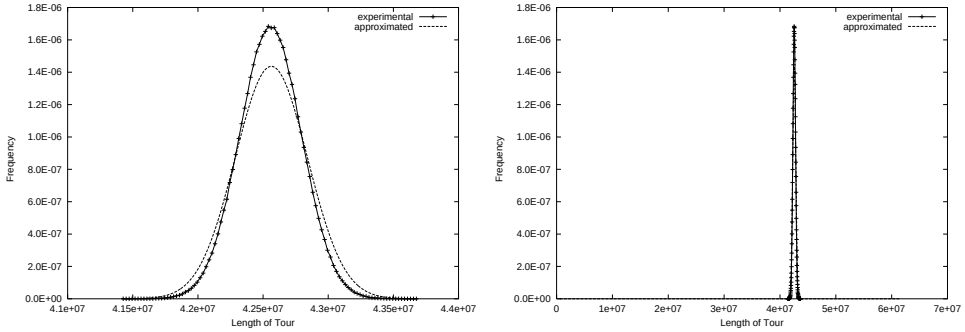

Figure 127: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.63 rl5934

Figure 128: Problem *rl5934*.

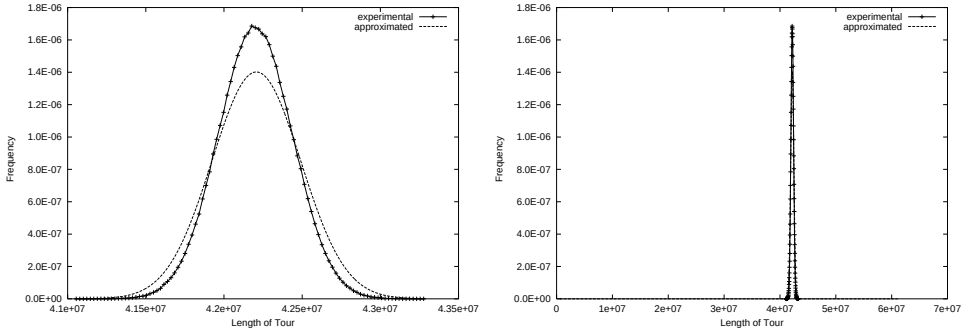

Figure 129: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.64 st70

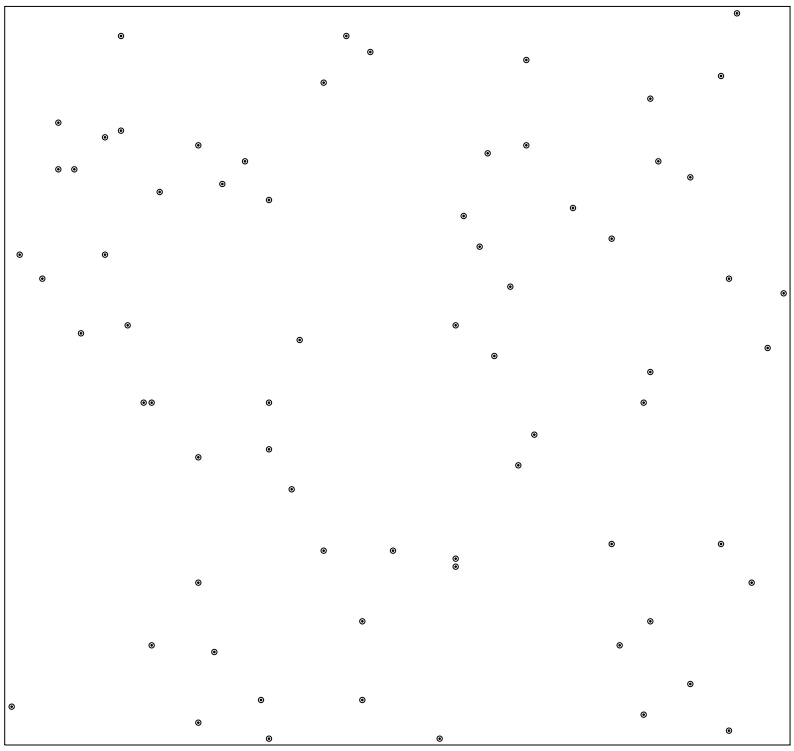

Figure 130: Problem *st70*.

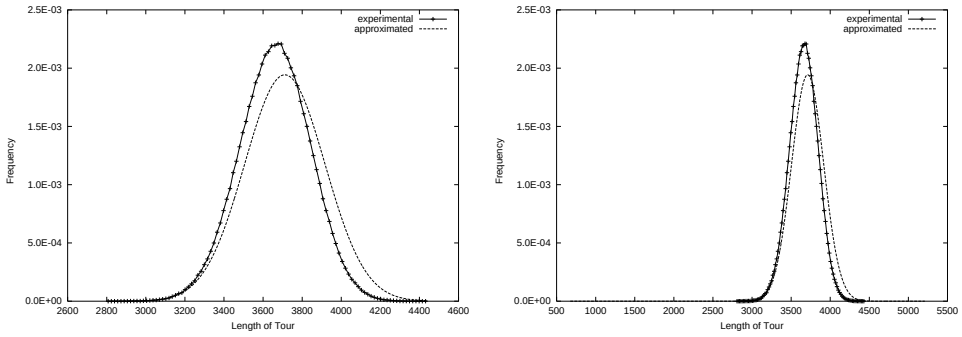

Figure 131: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.65 ts225

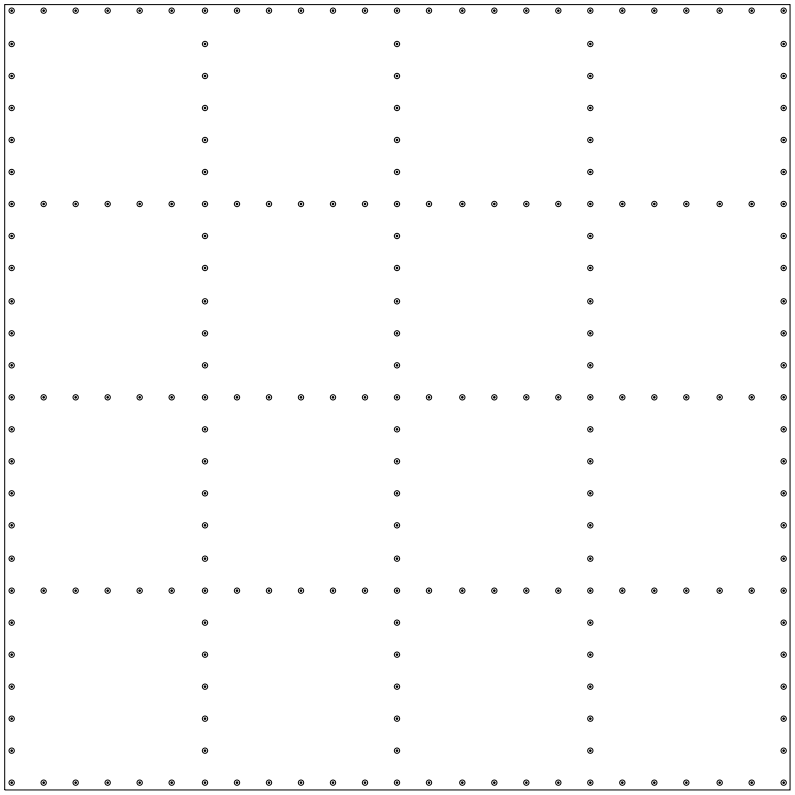

Figure 132: Problem *ts225*.

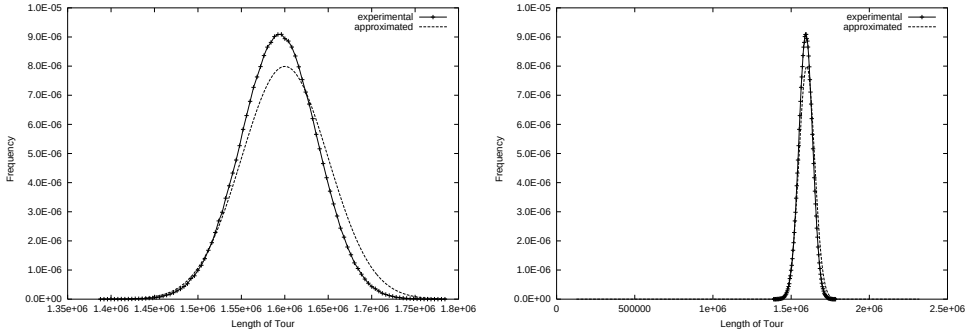

Figure 133: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.66 tsp225

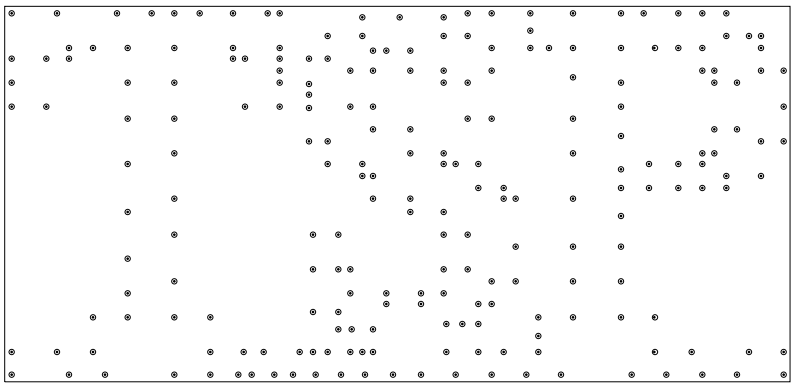

Figure 134: Problem *tsp225*.

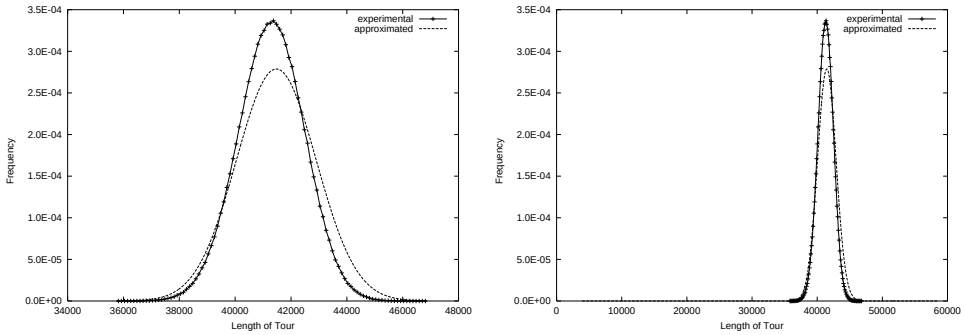

Figure 135: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.67 u1060

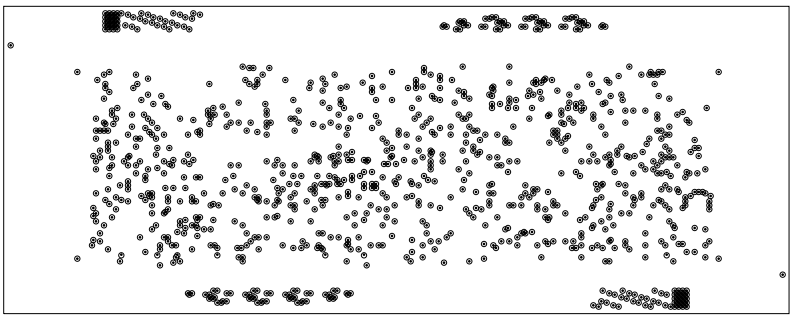

Figure 136: Problem *u1060*.

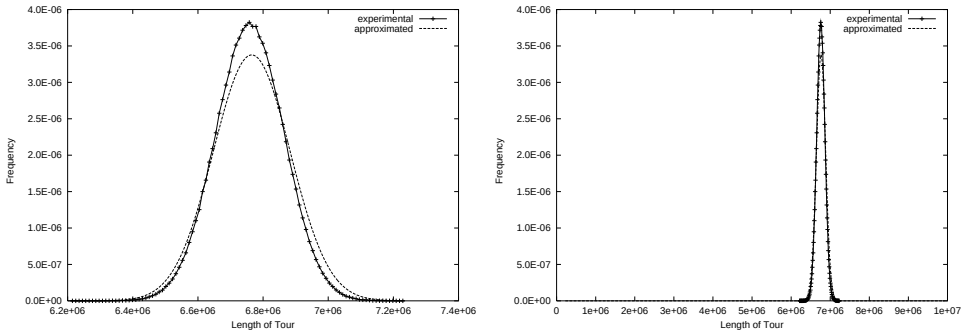

Figure 137: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.68 u1432

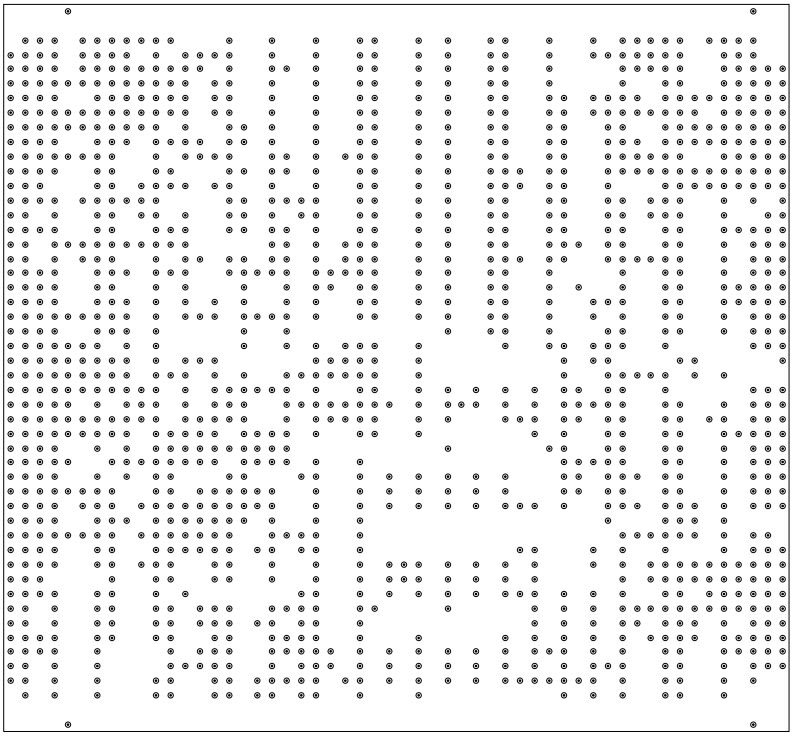

Figure 138: Problem *u1432*.

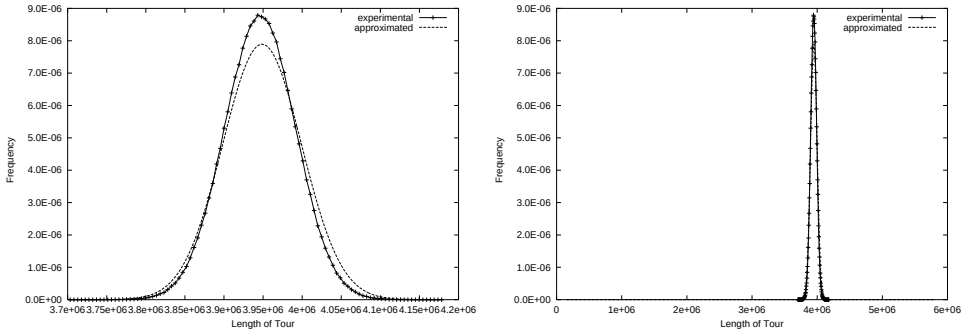

Figure 139: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.69 u159

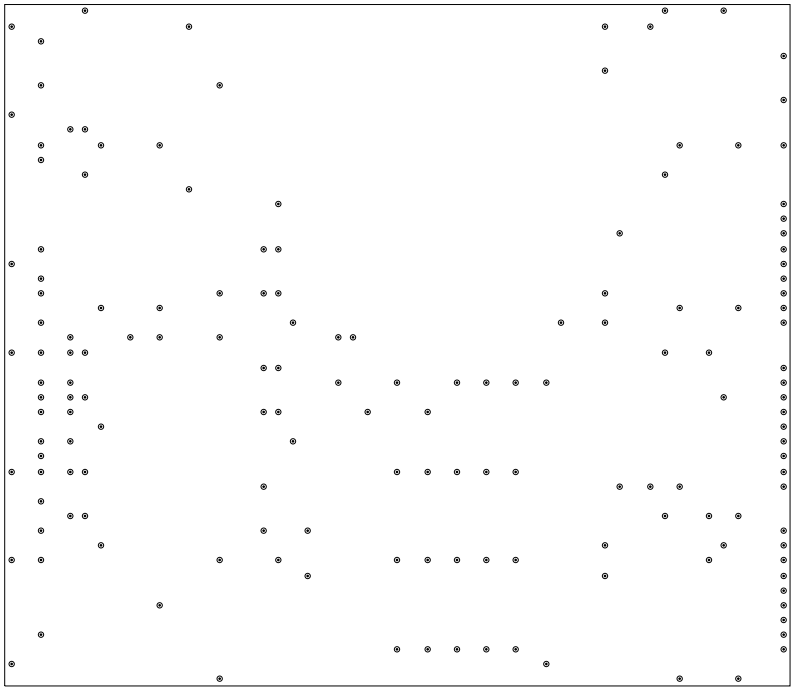

Figure 140: Problem *u159*.

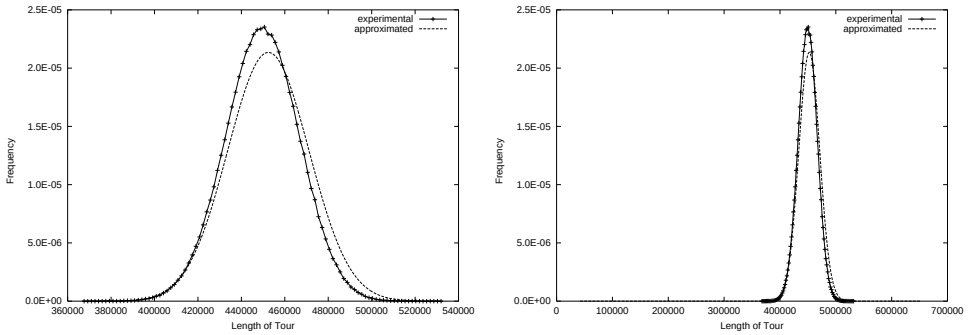

Figure 141: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.70 u1817

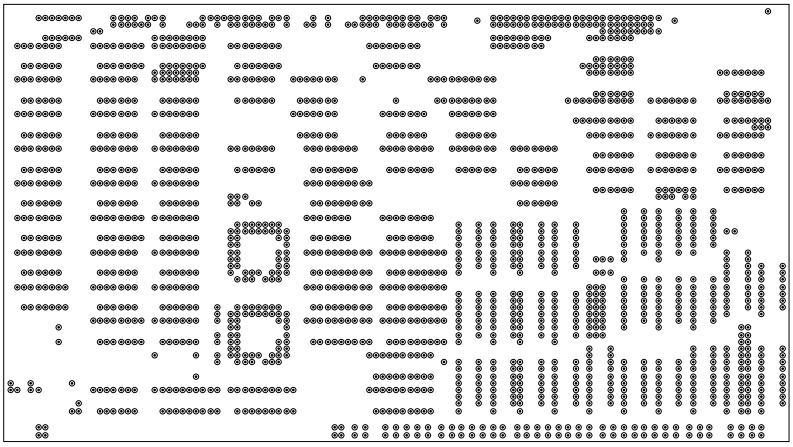

Figure 142: Problem *u1817*.

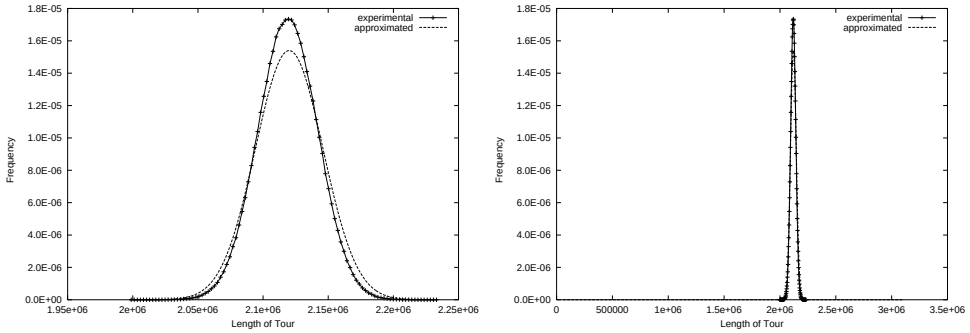

Figure 143: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.71 u2152

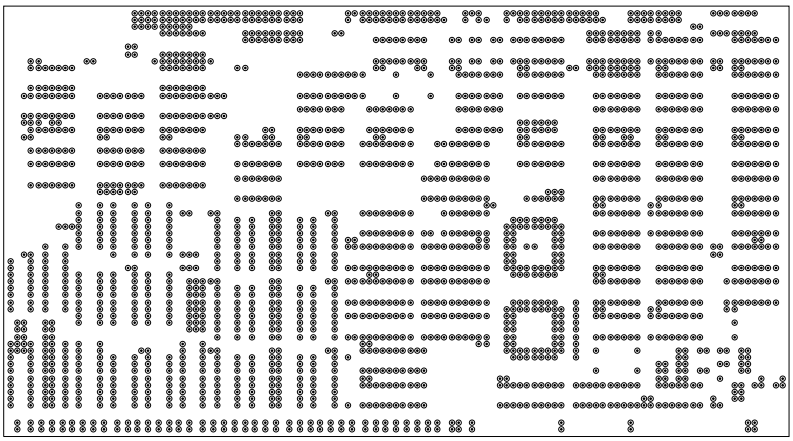

Figure 144: Problem *u2152*.

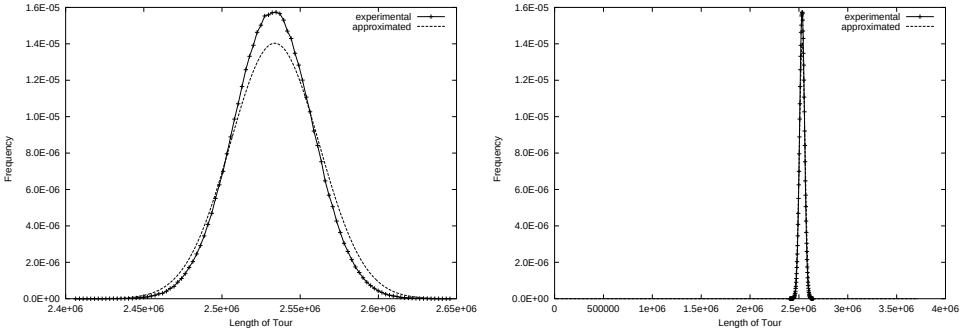

Figure 145: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.72 u2319

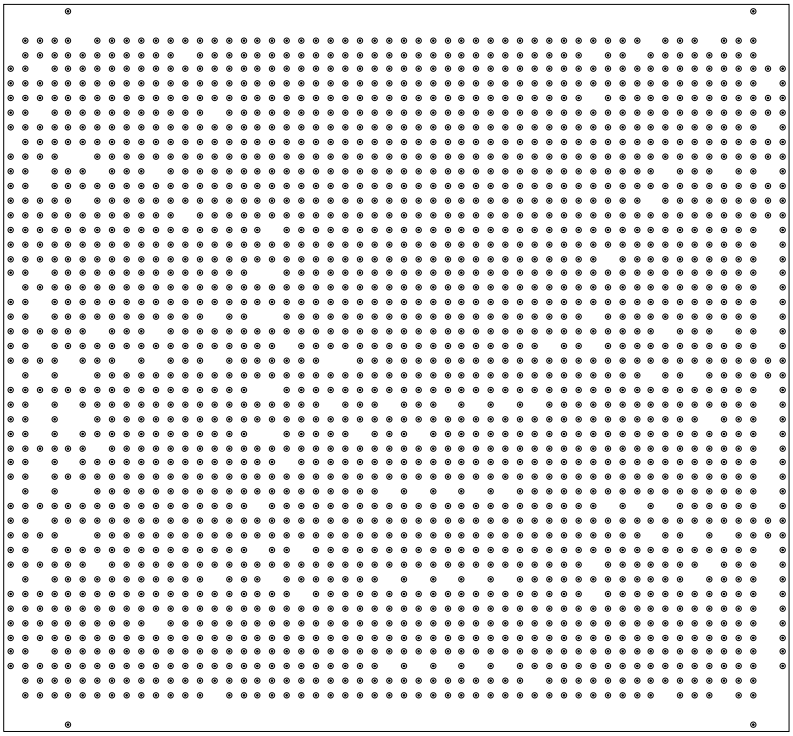

Figure 146: Problem *u2319*.

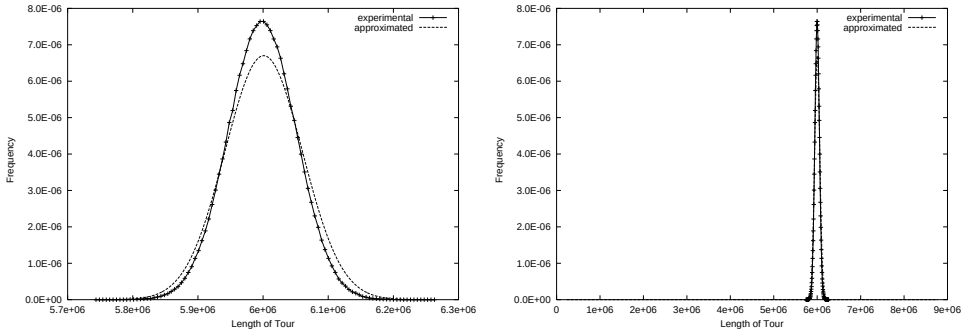

Figure 147: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.73 u574

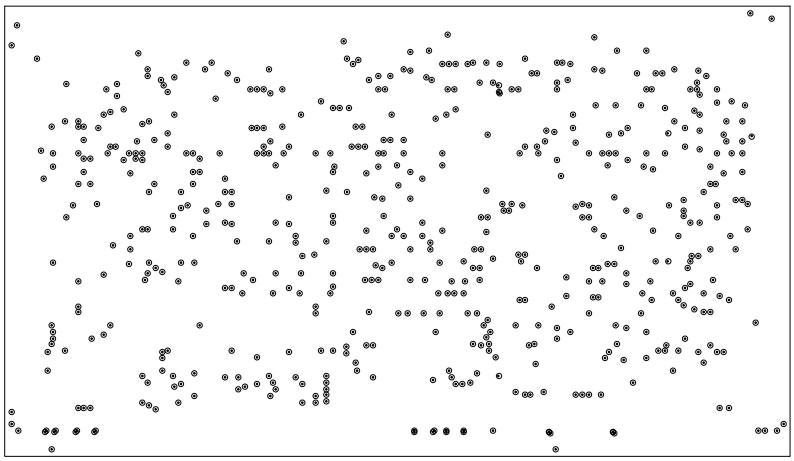

Figure 148: Problem *u574*.

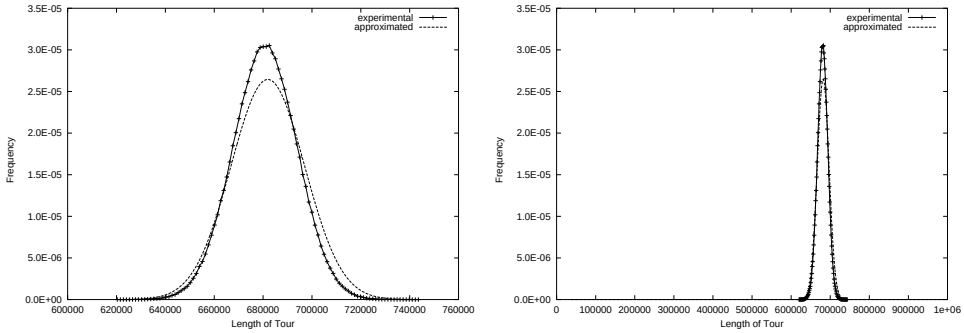

Figure 149: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.74 u724

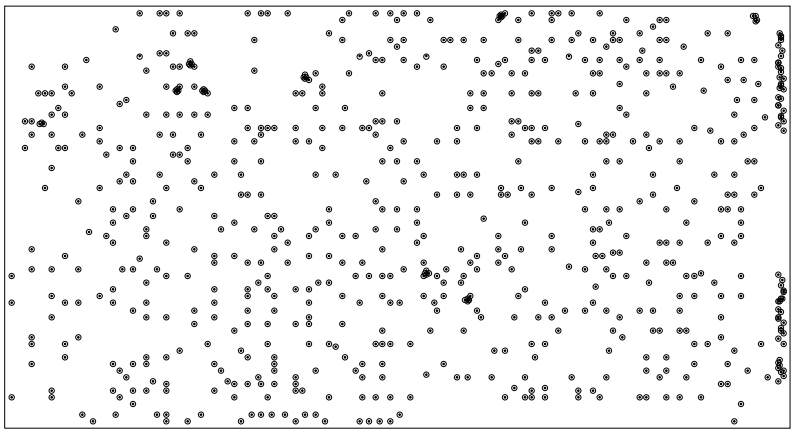

Figure 150: Problem *u724*.

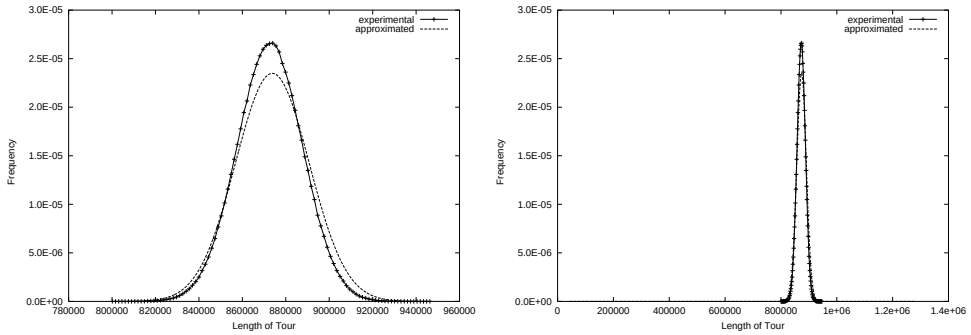

Figure 151: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.75 usa13509

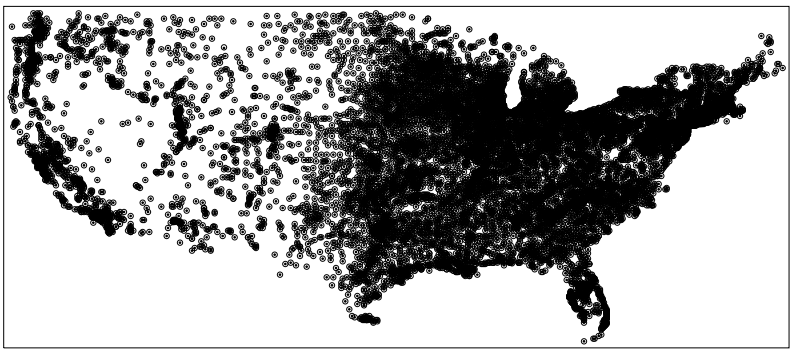

Figure 152: Problem *usa13509*.

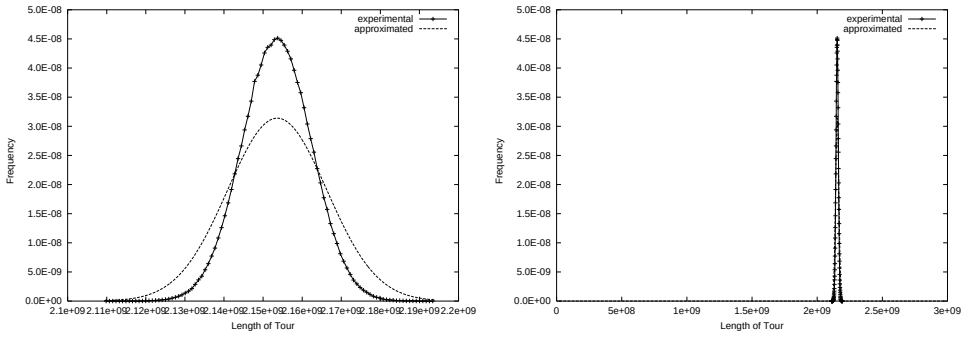

Figure 153: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.76 vm1084

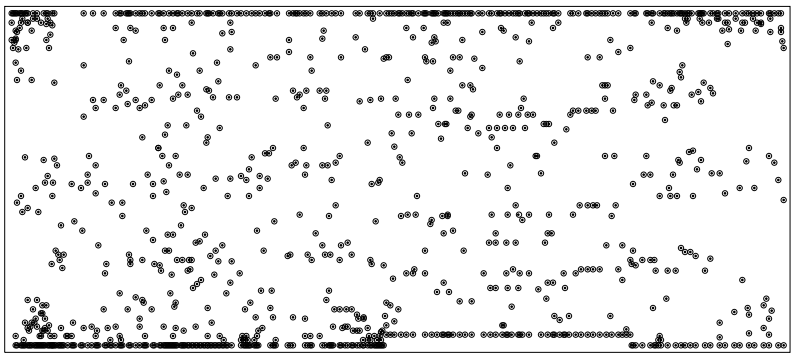

Figure 154: Problem *vm1084*.

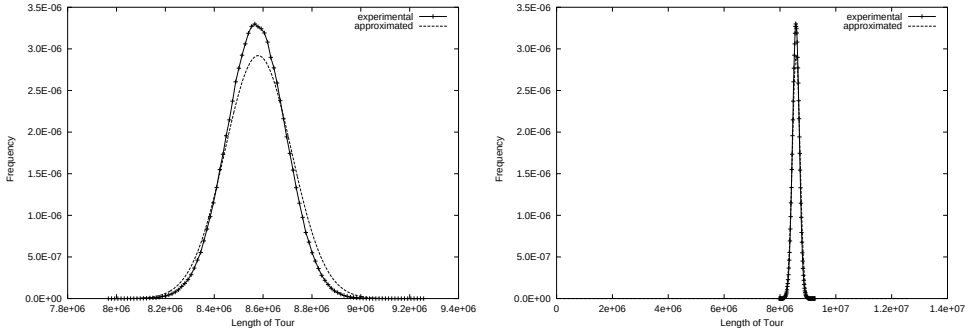

Figure 155: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).

A.77 vm1748

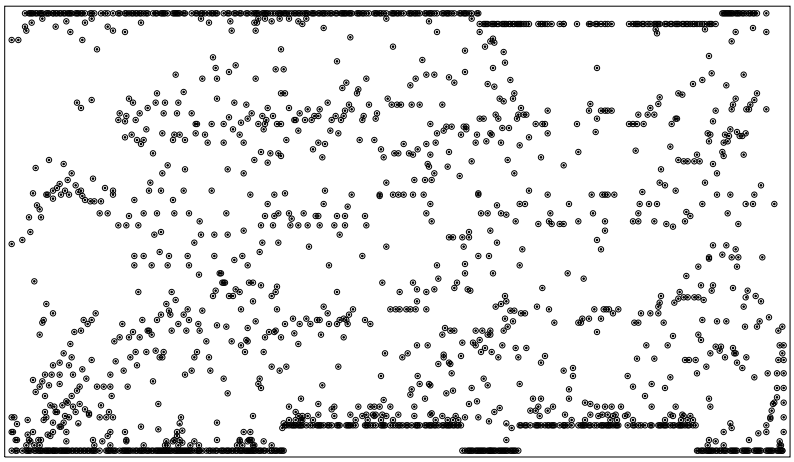

Figure 156: Problem *vm1748*.

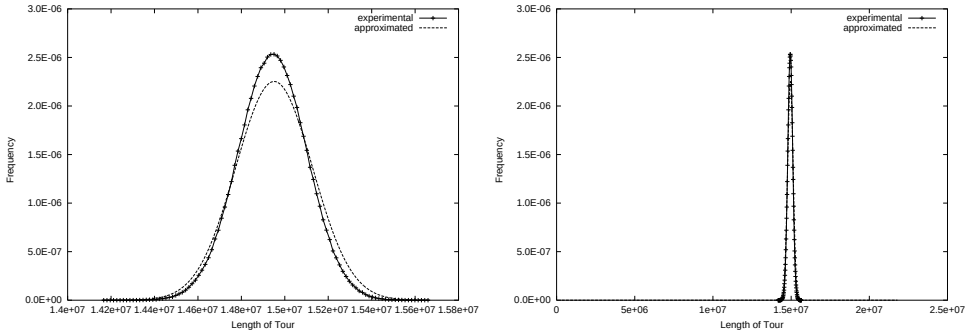

Figure 157: Comparison of cost distributions found analytically and by sampling. (Left, sampled distribution; Right, sampled distribution with adjusted range).