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Compactification of G-spaces Revisited

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ABSTRACT

We discuss compactifications of G–spaces from a new point of view that completely differs from our earlier approaches. From a topologist's point of view, this new approach is more natural than the previous ones. In addition, it enables a unified discussion of the compactification and the linearization problem for G–spaces (which we shall discuss in a subsequent report). The central idea is to get a sufficiently large 'natural' family of elementary compact G–spaces which can play the same role as the interval $[0\,;1]$ in ordinary topology.

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1. Introduction

Before stating the main problem we establish some notation and terminology and give the necessary definitions. For any topological space X and any point $x \in X$ the set of all neighbourhoods of x will be denoted by $\mathcal{N}_x$. The symbol G will always denote an arbitrary topological Hausdorff group. An *action* of G on a space X is a mapping $\pi : G \times X \to X$ satisfying the following conditions: $\pi(e,x) = x$ and $\pi(s, \pi(t,x)) = \pi(st,x)$ for all $x \in X$ and $t,s \in G$ (here e denotes the unit element of G). A *G–space* $\mathfrak{X} := \langle X, \pi \rangle$ consists of a Hausdorff space X with a *continuous* action π of G on X (that is, the action $\pi : G \times X \to X$ is jointly continuous). In that case we call X the *phase space* or *underlying topological space* of $\mathfrak{X}$; usually we employ the same letter for a G–space and its underlying topological space, using different type faces (thus, $\mathfrak{Y}$ has underlying space Y, $\mathfrak{Z}$ has underlying space Z, etc.). If **(P)** is topological property that is applicable to topological spaces then we speak of a **(P)**–G–space $\mathfrak{X}$, or we say that the G–space $\mathfrak{X}$ has **(P)**, whenever its underlying space has **(P)**; for example, a compact G–space is a G–space with a compact (and always Hausdorff) phase space. If $\mathfrak{X} = \langle X, \pi \rangle$ is a G–space then the mapping π is certainly separately continuous; thus, for every $t \in G$ the mapping $\pi^t : x \mapsto \pi(t,x) : X \to X$ is a homeomorphism; similarly, for every $x \in X$ the mapping $\pi_x : t \mapsto \pi(t,x) : G \to X$ is continuous.

If $\langle X, \pi \rangle$ is a G–space then a subset Y of X is said to be *invariant* whenever $\pi[G \times Y] \subseteq Y$, or, equivalently, $\pi^t[Y] = Y$ for every $t \in G$. In that case $\pi|_{G\times Y} : G \times Y \to Y$ is a continuous action of G on Y. For simplicity, in such a case we shall always write π instead of $\pi|_{G\times Y}$; thus, an invariant subset Y of $\langle X, \pi \rangle$ gives rise to the G–space $\langle Y, \pi \rangle$.

Usually there is no need for a notational distinction between various actions and we shall often just write tx for $\pi(t,x)$. If $\mathfrak{X}$ and $\mathfrak{Y}$ are G–spaces then a mapping $\varphi : X \to Y$ is called *equivariant* whenever $\varphi(tx) = t\varphi(x)$ for all $(t,x) \in G \times X$. A continuous equivariant mapping $\varphi : X \to Y$ will also be called a *morphism of G–spaces*; notation: $\varphi : \mathfrak{X} \to \mathfrak{Y}$. Properties for mappings will also be used for morphisms of G–spaces; thus, $\varphi : \mathfrak{X} \to \mathfrak{Y}$ is said to be open, an embedding, etc. whenever the 'underlying' continuous mapping $\varphi : X \to Y$ is open, or an embedding, etc.

We need one more definition: let A be a set and for each $a \in A$, let $\mathfrak{X}_a = \langle X_a, \pi_a \rangle$ be a G–space. Then the *product* $\prod_{a\in A} \mathfrak{X}_a$ is the G–space $\mathfrak{X} := \langle X, \pi \rangle$ with

$$X := \prod_{a\in A} X_a \quad \text{and} \quad \pi(t,x) := (\pi_a(t,x_a))_{a\in A}$$

for $t \in G$ and $x = (x_a)_{a\in A} \in X$. Clearly, every projection $\mathfrak{X} \to \mathfrak{X}_a$ is equivariant. If $\mathfrak{Z}$ is a G–space and for every $a \in A$ we have a morphism of G–spaces $\varphi_a : \mathfrak{Z} \to \mathfrak{X}_a$, then the induced continuous mapping $\varphi : z \mapsto (\varphi_a(z))_{a\in A} : Z \to X$ is equivariant, hence a morphism of G–spaces from $\mathfrak{Z}$ to $\mathfrak{X}$. From ordinary topology we know that φ is an (equivariant) embedding iff the morphisms $\{\varphi_a\}_{a\in A}$ separate points and closed subsets of Z, that is: for every closed subset B of Z and every point $z_0 \in Z \setminus B$ there exists an $a \in A$ such that $\varphi_a(z_0) \notin \overline{\varphi_a[B]}$ (closure in X_a).

Back in the late seventies, we were involved in a program investigating to which extend the category of G–spaces has properties similar to the category of 'ordinary' topological spaces (which can be seen as G–spaces for the trivial group $G = \{e\}$). See e. g. [1], [6], [9], [5] and [2]. In [7] and [8] we considered the analog of the well-known result from ordinary topology that a topological space can be topologically embedded in a compact Hausdorff space iff it is a Tychonov space. This is the *compactification problem* for G–spaces: for which topological groups G is it true that every Tychonov G–space $\mathfrak{X}$ can equivariantly be embedded in a compact G–space? In [8] it was shown that this is true if G is locally compact. The methods used to prove this involved uniform structures. In [7] a later proof (though appearing earlier) was published, using function algebras.

In the present paper we reconsider this problem, in a slightly more general form: given an arbitrary (not necessarily locally compact) topological group G, characterize the G–spaces that admit an equivariant embedding into a compact G–space. Of course, these G–spaces are necessarily Tychonov, but this is not a sufficient condition, as an example by Megrelishvili shows (see [4]). The present approach looks very much like the simple method used to prove that every Tychonov space can be embedded in a compact space: consider all continuous maps from that space to the 'elementary' compact interval $[0;1]$; these separate points and closed subsets, hence can be used to embed the space into a product of copies of $[0;1]$. In the same vain, for any Tychonov G–space $\mathfrak{X}$ we shall look for the class of all morphisms from $\mathfrak{X}$ to 'elementary' compact G–spaces, and we shall try to single out a subset $\{\varphi_a : \mathfrak{X} \to \mathfrak{Y}_a\}_{a\in A}$ of this class such that the family $\{\varphi_a\}_{a\in A}$ separates the points and the closed subsets of X. Then

$$\varphi : x \mapsto (\varphi_a(x))_{a\in A} : \mathfrak{X} \to \prod_{a\in A} \mathfrak{Y}_a$$

is an equivariant topological embedding of $\mathfrak{X}$ into a compact G–space.

So the whole problem boils down to finding a sufficiently large set of compact G–spaces. At first sight this seems not so easy: if G is some 'wild' topological group it will probably have no non-trivial continuous actions at all on simple compact spaces like the unit interval . The crucial step here is to realize that the concept of 'elementary' G–spaces may depend on the group G.

The basic question is: where do we find sufficiently many (compact) spaces on which G acts? A natural candidate is the space $C(G)$ of all continuous real-valued functions. The group G acts on this space by means of the action ρ, which is defined by

$$\rho(t,f)(s) := f(st) \quad \text{for all} \quad (t,f) \in G \times C(G) \quad \text{and} \quad s \in G\,.$$

Note that $\rho^e = \mathrm{id}_{C(G)}$ and that $\rho^s \circ \rho^t = \rho^{st}$ for all $s,t \in G$, so that, indeed, ρ is an action of G on $C(G)$ (action by *right* translation). The subsets of $C(G)$ that are invariant under this action will be called *ρ-invariant* or *right invariant*. It will turn out that we will get sufficiently many compact G–spaces when we consider the pointwise bounded, non-empty equicontinuous ρ-invariant subsets of $C(G)$ endowed with the topology of pointwise convergence. The strength of our methods lies in their simplicity; in fact, anybody who has understood the proof of the well-known Ascoli Theorem should be able to follow the arguments in this paper.

2. Elementary compact G–spaces

The set off all right-uniformly {left-uniformly} continuous functions on G will be denoted by $RUC(G)$ {by $LUC(G)$, respectively}; the set of all bounded members of $C(G)$ and $RUC(G)$ will be denoted by $C^*(G)$ and $RUC^*(G)$, respectively. We shall use the subscript p when these sets are considered with the topology of pointwise convergence, as follows: $C_p(G)$, etc. Closures in this topology are denoted by the operator cl_p. (We shall use similar notation for spaces of continuous functions on *any* topological space X.) It is important to note that, in general, the action ρ is not continuous on $C_p(G)$ and $RUC_p(G)$. Indeed, without further conditions on G we may only assume that for $Z := C_p(G)$ or $RUC_p(G)$ the mapping $\rho\colon G \times Z \to Z$ is separately continuous. Fortunately, there are many subsets Z of $C_p(G)$ such that ρ is continuous on $G \times Z$:

(2.1) ***Lemma.*** *Let Z be a ρ-invariant equicontinuous subset of $C_p(G)$. Then $\rho\colon G \times Z \to Z$ is jointly continuous. Moreover, $Z \subseteq RUC(G)$.*

Proof: It is sufficient to show that for every fixed $s_0 \in G$ the mapping $(t, f) \mapsto \rho^t f(s_0)\colon G \times Z \to Z$ is continuous at the (arbitrarily chosen) point $(t_0, f_0) \in G \times Z$. To this end, let $(t, f) \in G \times Z$ and consider the inequality

$$|\rho^t f(s_0) - \rho^{t_0} f_0(s_0)| \le |f(s_0 t) - f(s_0 t_0)| + |f(s_0 t_0) - f_0(s_0 t_0)|\,. \qquad (1)$$

Let $\varepsilon > 0$. The condition on f that the second term of the right-hand side of this inequality is smaller than $\varepsilon/2$ defines a neighbourhood U of f_0 in Z. In view of the equicontinuity of Z at the point $s_0 t_0$ there is a neighbourhood V of t_0 in G such that the first term of the right-hand side of (1) is smaller than $\varepsilon/2$ for $t \in V$. Hence the left-hand side of (1) is smaller than ε for all $(t, f) \in V \times U$. This completes the proof of the continuity of ρ on $G \times Z$.

Finally, if $f \in Z$ then $\rho^t f \in Z$ for all $t \in G$, hence the family $\{\rho^t f\}_{t\in G}$ is equicontinuous, being a subset of the equicontinuous set Z. Writing down what this means at the point $e \in G$ shows that for every $\varepsilon > 0$ there exists $W \in \mathcal{N}_e$ such that $|f(st) - f(t)| < \varepsilon$ for all $s \in W$ and all $t \in G$. This means precisely that $f \in RUC(G)$. □

(2.2) ***Remarks.*** **1.** An alternative presentation of the same proof would be: as Z is equicontinuous, the evaluation mapping $(s, f) \mapsto f(s)\colon G \times Z \to Z$ is jointly continuous (see e. g. [3], 7.15, essentially the same proof as the one presented above). This is easily seen to imply joint continuity of ρ on $G \times Z$. In fact, we are talking about equicontinuous subsets of $C(G)$. On such sets the topologies of pointwise convergence and the compact-open topology (which is the same as the topology of uniform convergence on compact subsets of G) coincide. Taking this into account, the above lemma and its corollary below are reformulations of well-known facts.

2. Actually, the above proof shows that ρ is already continuous on $G \times Z$ if every point of Z has a neighbourhood (in the topology of pointwise convergence, of course) that is equicontinuous on G . Conversely, if ρ is continuous on $G \times Z$ then the inequality

$$|f(s_0 t) - f(s_0)| \le |\rho^t f(s_0) - \rho^e f_0(s_0)| + |f_0(s_0) - f(s_0)|$$

for $f, f_0 \in Z$ and $s_0, t \in G$, shows that every point of Z has a neighbourhood that is equicontinuous on G.

3. The second part of (2.1) implies that it may not limit generality too much if we restrict our attention to (equicontinuous) subsets of $RUC(G)$.

In what follows, $\mathrm{co}\,(Y)$ will denote the convex hull of the subset Y of a vector space. Thus, $\mathrm{co}\,(Y)$ is the set of all finite sums of the form $\sum_i \alpha_i y_i$ with $y_i \in Y$ for all i, each $\alpha_i \ge 0$ and $\sum_i \alpha_i = 1$.

(2.3) *Corollary.* **1.** *Let Y be a pointwise bounded ρ-invariant equicontinuous subset of $C^*(G)$, let $Z := \mathrm{cl}_p(Y)$ and let $K := \mathrm{cl}_p\,\mathrm{co}\,(Y)$. Then Z and K are compact ρ-invariant equicontinuous subsets of $RUC^*_p(G)$, so $\mathfrak{Z} := \langle Z, \rho\rangle$ and $\mathfrak{K} := \langle K, \rho\rangle$ are compact G–spaces.*

2. *For $f \in RUC^*(G)$, let $Z_f := \mathrm{cl}_p\{\rho^t f \colon t \in G\}$ and $K_f := \mathrm{cl}_p(\mathrm{co}\,\{\rho^t f \colon t \in G\})$. Then Z_f and K_f are compact ρ-invariant equicontinuous subsets of $RUC^*_p(G)$, and $\mathfrak{Z}_f := \langle Z_f, \rho\rangle$ and $\mathfrak{K}_f := \langle K_f, \rho\rangle$ are compact G–spaces.*

Proof: 1. We start with the observation that Z equals the closure of Y in $\mathbb{R}^G$ with its product topology: as is well-known, this closure in $\mathbb{R}^G$ is an equicontinuous set (this is part of most proofs of the Ascoli Theorem; see e. g. [3], 7.14), hence it is included in $C(G)$ and therefore equals Z. Next, note that Z is pointwise bounded on G. The easy proof is as follows: if $s \in G$ then continuity of the evaluation mapping $\delta_s : f \mapsto f(s) : C_p(G) \to \mathbb{R}$ implies that

$$Z(s) \;=\; \delta_s\big[\,\overline{Y}\,\big] \;\subseteq\; \overline{\delta_s[Y]} \;=\; \overline{Y(s)}\,,$$

which is a bounded set in $\mathbb{R}$. As Z is the clusure of Y in $\mathbb{R}^G$, this implies that Z is compact: it is a closed subset of the compact product $\prod_{s\in G}\overline{Y(s)}$. Finally, because every $\rho^s : C_p(G) \to C_p(G)$ is continuous, ρ-invariance of Y implies ρ-invariance of $\mathrm{cl}_p Y = Z$. Combine these observations with Lemma (2.1), and the proof of the first part of this Corollary for Z is complete.

For K similar conclusions hold, because (like Z) it is the pointwise closure of an equicontinuous, pointwise bounded ρ-invariant set, namely, $\mathrm{co}\,(Y)$. We first show that the set $\mathrm{co}\,(Y)$ is pointwise bounded. Indeed, because Y is pointwise bounded there exists for every $s \in G$ a real number $M_s > 0$ such that $Y(s) \subseteq [\,-M_s\,; M_s\,]$. Now consider an element f of $\mathrm{co}\,(Y)$, $f = \sum_i \alpha_i f_i$, a finite sum with $f_i \in Y$ for all i, $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. Then for every $s \in G$ we have $|f(s)| \leq \sum_i \alpha_i M_s = M_s$ This shows that $\mathrm{co}\,(Y)(s) \subseteq [\,-M_s\,; M_s\,]$. Hence $\mathrm{co}\,(Y)$ is pointwise bounded. Moreover, since Y is equicontinuous, an easy argument (very similar to the above) shows that $\mathrm{co}\,(Y)$ is equicontinuous as well. Finally, each ρ^s is linear so it preserves convex combinations of elements of Y. This implies that the set $\mathrm{co}\,(Y)$ is ρ-invariant.

2. Let $f \in RUC^*(G)$. Then it is straightforward to show that $O_f := \{\rho^t f\}_{t\in G}$ is an equicontinuous set; it is also obvious that it is a ρ-invariant set. Moreover, O_f is pointwise bounded: for every $s \in G$ we have $O_f(s) = f[sG] = f[G]$ which is a bounded set as f is a bounded function. So the second part of the Corollary is a special case of the first part. □

(2.4) *Remarks.* **1.** The above result can also be obtained as an easy consequence of the Ascoli Theorem. To keep our results as clear as possible we instead choose to present the relevant parts of the proof of that theorem.

2. There is a converse to Ascoli's theorem, saying that compactness (in the compact-open) topology implies equicontinuity, *provided* the evaluation mapping is jointly continuous, which is e. g. the case on locally compact spaces. Though G is not supposed to be locally compact we do have a jointly continuous evaluation on the relevant sets. *Claim*: if Z is a compact ρ-invariant subset of $C_p(G)$ then Z is equicontinuous. The proof is easy: in (2.2)2 we have seen that joint continuity of ρ on $G\times Z$ implies that each point of Z has an equicontinuous neighbourhood; compactness of Z implies that Z can be covered by finitely many of such neighbourhoods, which implies that Z is equicontinuous. (Alternative proof: use that continuity of ρ on $G\times Z$ implies that the evaluation mapping $(s,f) \mapsto f(s) : G\times Z \to Z$ is jointly continuous.)

3. Categorical arguments show that in the case that G is locally compact the G–space $\langle C_c(G), \rho\rangle$ plays the same role in the category of all G–spaces and equivariant continuous maps as $\mathbb{R}$ plays in the category of ordinary topological spaces and continuous mappings (see [6] and [9]). Here $C_c(G)$ is the space $C(G)$ endowed with the compact-open topology. The compact orbit closures in this G–space coincide with the sets Z_f with $f \in RUC^*(G)$, so these sets are, in a sense, the 'smallest' compact invariant subsets of $RUC^*(G)$. It are these sets which play the role of the closed bounded intervals in $\mathbb{R}$ (or, if convexity is an issue, consider the sets K_f; but note that always $Z_f \subseteq K_f$). If G is not locally compact we loose the G–space $\langle C_c(G), \rho\rangle$ — the mapping $\rho\colon G \times C_c(G) \to C_c(G)$ need not be continuous — but we retain the compact G–spaces $\langle Z_f, \rho\rangle$ (or $\langle K_f, \rho\rangle$) and, more generally, the compact G–spaces $\langle Z, \rho\rangle$ with Z a closed pointwise bounded equicontinuous ρ-invariant subset of $C_p(G)$ (which, by (2.1), areous included in $RUC(G)$).

3. Compactification of G–spaces

(3.1) ***Lemma.*** *Let $\langle X, \pi\rangle$ be a G–space and let $f \in C(X)$. The following conditions are equivalent:*
(i) The set $\{f \circ \pi_x\}_{x\in X}$ is equicontinuous at e;
(ii) The set $\{f \circ \pi_x\}_{x\in X}$ is right uniformly equicontinuous on G.
Moreover, if $f \in C^*(X)$ then this set is pointwise bounded on G.

Proof: Elementary (see [7]). (Note that 'right uniformly equicontinuous' means: uniformly equicontinuous (or: equi-uniformly continuous) when G is considered with its right uniform structure). $\square$

(3.2) ***Remark.*** Condition (i) above means that for every $\varepsilon > 0$ there exists $U \in \mathcal{N}_e$ such that for all $t \in U$ we have $|f(tx) - f(x)| < \varepsilon$ for all $x \in X$. In the particular case that $X = G$ and π is the multiplication mapping in G this condition is nothing but the condition that f is right uniformly continuous on G.

Let $\mathfrak{C}(\mathfrak{X})$ be the set of all $f \in C^*(X)$ satisfying the conditions (i) and (ii) above. (Here one should see the symbol $\mathfrak{C}$ as the initial letter of 'compact' rather than of 'continuous': see (3.4) and (3.6)2 below.)

(3.3) ***Proposition.*** *Let $\mathfrak{Z} = \langle Z, \sigma\rangle$ be a compact G–space; then we have $\mathfrak{C}(\mathfrak{Z}) = C(Z)$. In addition, if $\varphi\colon \mathfrak{X} \to \mathfrak{Z}$ is a morphism of G–spaces then $f \circ \varphi \in \mathfrak{C}(\mathfrak{X})$ for every $f \in C(Z)$.*

Proof: For the first statement it is sufficient to show that $C(Z) \subseteq \mathfrak{C}(\mathfrak{Z})$. Let $f \in C(Z)$. Then f is bounded. In addition, the mapping $f \circ \sigma\colon (t, z) \mapsto f(tz)\colon G \times Z \to \mathbb{R}$ is jointly continuous, hence an elementary compactness argument shows that the set of all mappings $f \circ \sigma_z : G \to \mathbb{R}$ with $z \in Z$ is equicontinuous. This shows that $C(Z) \subseteq \mathfrak{C}(\mathfrak{Z})$.

Now consider the G–space $\mathfrak{X} = \langle X, \pi\rangle$ and a morhism of G–spaces $\varphi\colon \mathfrak{X} \to \mathfrak{Z}$, and let $f \in C(Z)$. Then for every $x \in X$ we have $(f \circ \varphi) \circ \pi_x = f \circ \sigma_{\varphi(x)}$, hence $\{(f \circ \varphi) \circ \pi_x\}_{x\in X} \subseteq \{f \circ \sigma_z\}_{z\in Z}$. As the latter set is equicontinuous and pointwise bounded, also the former set is equicontinuous and pointwise bounded. $\square$

(3.4) Let $\mathfrak{X} = \langle X, \pi\rangle$ be a G–space and let $f \in \mathfrak{C}(\mathfrak{X})$. Then it is obvious from (2.3)1 and the definition of $\mathfrak{C}(\mathfrak{X})$ that $\widetilde{X_f} := \mathrm{cl}_p\{f \circ \pi_x\}_{x\in X}$ and $\widetilde{K_f} := \mathrm{cl}_p\,\mathrm{co}\,(\{f \circ \pi_x\}_{x\in X})$ are closed, pointwise bounded, ρ-invariant equicontinuous subsets of $C_p(X)$. So we have the compact G–spaces $\widetilde{\mathfrak{X}_f} := \langle \widetilde{X_f}, \rho\rangle$ and $\widetilde{\mathfrak{K}_f} := \langle \widetilde{K_f}, \rho\rangle$, the latter being a G–space on a compact convex subset of a locally convex topological vector space with an action by means of *affine* mappings.

(3.5) ***Proposition.*** *Let $\mathfrak{X} := \langle X, \pi\rangle$ be an arbitrary G–space and let $f \in \mathfrak{C}(\mathfrak{X})$. Then the mapping $\Phi_f : x \mapsto f \circ \pi_x : X \to \widetilde{X_f}$ is a morphism of G–spaces, $\Phi_f : \mathfrak{X} \to \widetilde{\mathfrak{X}_f}$. Moreover, there exists $\tilde f \in C(\widetilde{X_f})$ such that $f = \tilde f \circ \Phi_f$.*

Proof: That Φ_f is continuous follows immediately from the easy observation that, for every $s \in G$, the mapping $x \mapsto (\Phi_f x)(s) = f(sx)\colon X \to \mathbb{R}$ is continuous. A straightforward verification shows that the mapping Φ_f is equivariant. Finally, it is easily verified that the rule $\tilde f(g) := g(e)$ for $g \in \widetilde{X_f}$ defines a continuos mapping $\tilde f : \widetilde{X_f} \to \mathbb{R}$ which satisfies the required relationship $f = \tilde f \circ \Phi_f$. $\square$

(3.6) ***Remarks.*** **1.** In this proposition we can, of course, replace $\mathfrak{X}_f$ by $\mathfrak{K}_f$. In that case $\tilde{f}$ may be assumed to be a continuous *affine* function.
2. The previous propositions show that the members of $\mathcal{C}(\mathfrak{X})$ are precisely the continuous functions on X that 'come from' equivariant compactifications of $\mathfrak{X}$. Differently from the procedure in [7] we shall not use $\mathcal{C}(\mathfrak{X})$ as an algebra of functions defining the 'largest' G–compactification of $\mathfrak{X}$. Rather, we shall use the members of $\mathcal{C}(\mathfrak{X})$ as the 'topological counterparts' of the morphisms from $\mathfrak{X}$ to the 'elementary' compact G–spaces (the latter are indexed by the former). In particular, they can be used to see if the morphisms of G–spaces from $\mathfrak{X}$ to compact G–spaces separate points and closed subsets of X:

(3.7) ***Lemma.*** *Let $\mathfrak{X}$ be a G–space. Then $\mathcal{C}(\mathfrak{X})$ separates points and closed subsets of X iff the morphisms of G–spaces from $\mathfrak{X}$ to compact G–spaces separate points and closed subsets of X.*

Proof: Let A be a closed, non-empty subset of X and let $x_0 \in X \setminus A$.
"Only if": Let $f \in \mathcal{C}(\mathfrak{X})$ be such that $f(x_0) \notin \overline{f[A]}$ (closure in $\mathbb{R}$). This can be written as

$$(f \circ \pi_{x_0})(e) \notin \overline{\{(f \circ \pi_x)(e)\}_{x \in A}}\,.$$

As the mapping $\delta_e : g \mapsto g(e) : X_f \to \mathbb{R}$ is continuous, this implies that $f \circ \pi_{x_0} \notin \overline{\{f \circ \pi_x\}_{x \in A}}$ (closure in X_f). Thus, using the notation of (3.5), we have $\Phi_f(x_0) \notin \overline{\Phi_f[A]}$, where Φ_f is a morphism from $\mathfrak{X}$ to a compact G–space.

"If": Consider a morphism of G–spaces $\varphi : \mathfrak{X} \to \mathfrak{Z}$ with Z a compact Hausdorff space and assume that $\varphi(x_0) \notin \overline{\varphi[A]}$ (closure in Z). There exists $g \in C(Z)$ such that $g(x_0) = 0$ and $g[A] = \{1\}$. Now let $f := g \circ \varphi$. Then by (3.3) we have $f \in \mathcal{C}(\mathfrak{X})$, and it is clear that $f(x_0) \notin \overline{f[A]}$. □

(3.8) ***Theorem.*** (See e. g. [8].) *A G–space $\mathfrak{X}$ admits an equivariant embedding in a compact G–space iff $\mathcal{C}(\mathfrak{X})$ separates points and closed subsets of X.*

Proof: Clear from (3.7) and the fact that $\mathfrak{X}$ admits an equivariant embedding in a compact G–space iff there exists a set of morphisms from $\mathfrak{X}$ to compact G–spaces which separate points and closed subsets of X. □

(3.9) There is an amusing reformulation of this Theorem. To place it in its proper context we remind the reader to the following known facts. If $\mathfrak{X} = \langle X, \pi \rangle$ is an arbitrary G–space then the mapping $\overline{\pi} : t \mapsto \pi^t$ from G into the group $H(X)$ of all homeomorphisms of X onto itself is a group homomorphism. If we give $\overline{\pi}[G]$ the quotient topology induced by $\overline{\pi}$ then the evaluation mapping $(h, x) \mapsto h(x) : \overline{\pi}[G] \times X \to X$ is easily seen to be jointly continuous (this is because π is jointly continuous and the quotient map $\overline{\pi} : G \to \overline{\pi}[G]$ is open).

Consider the case that X is a dense invariant subset of the compact G–space $\mathfrak{Z} = \langle Z, \sigma \rangle$ and let $\pi := \sigma|_{G \times X}$. By the preceding paragraph (but with $\mathfrak{Z}$ instead of $\mathfrak{X}$), the evaluation mapping $\overline{\sigma}[G] \times Z \to Z$ is jointly continuous when $\overline{\sigma}[G]$ is given the quotient topology induced by $\overline{\sigma}$. A well-known result (see, for example, [3], 7.5) implies that on $\overline{\sigma}[G]$ this quotient topology is finer than the compact-open topology — which is the topology of uniform convergence because Z is compact.

Now identify the groups $\overline{\pi}[G]$ and $\overline{\sigma}[G]$ with each other by identifying $\pi^t = \sigma^t|_X$ with σ^t $(t \in G)$. Then on this group the quotient topologies induced by $\overline{\pi}$ and $\overline{\sigma}$ are the same, and, because X is dense in Z, the topologies of uniform convergence on X and on Z coincide as well (of course, we give X the relative uniformity of Z). It follows that also on $\overline{\pi}[G]$ the former topology is finer than the latter. Stated otherwise: the mapping $\overline{\pi} : G \to H_u(X)$ is continuous; here $H_u(X)$ is the group of all homeomorphisms of X endowed with the topology of uniform convergence.

This proves half of the following:

(3.10) *Corollary.* *The following conditions for a G–space $\mathfrak{X} = \langle X, \pi\rangle$ are mutually equivalent:*
(i) *$\mathfrak{X}$ admits an equivariant embedding into a compact G–space;*
(ii) *There exists a uniformity on X (compatible with its topology) such that, if we give the group $H(X)$ of all homeomorphisms of X into itself the topology of uniform convergence, then the mapping $t \mapsto \pi^t : G \to H(X)$ is continuous.*

Proof: $(i) \Longrightarrow (ii)$: In the remarks above we have shown that (i) implies (ii); we give an alternative proof. Give X the uniform structure generated by $\mathcal{C}(\mathfrak{X})$ (i. e., the weakest uniformity making all members of $\mathcal{C}(\mathfrak{X})$ uniformly continuous). By Theorem (3.8), condition (i) means that $\mathcal{C}(\mathfrak{X})$ separates points and closed subsets of X; it follows that this uniformity is compatible with the topology of X. We show that the mapping $t \mapsto \pi^t : G \to H(X)$ is continuous when $H(X)$ is given the topology of uniform convergence with respect to this uniformity. Let α be a member of this uniformity: there are finitely many elements $f_i \in \mathcal{C}(\mathfrak{X})$, $i = 1, \dots, n$, such that for every pair of points $x, y \in X$ the conditions $|f_i(x) - f_i(y)| < \varepsilon$ for $i \in \{1, \dots, n\}$ imply that $(x, y) \in \alpha$. Now let $s \in G$. By the definition of $\mathcal{C}(\mathfrak{X})$, for every $i \in \{1, 2, \dots, n\}$ there exists $U_i \in \mathcal{N}_s$ such that $|f_i(tx) - f_i(sx)| < \varepsilon$ for all $t \in U_i$ and $x \in X$. Let U be the intersection of the neighbourhoods U_i of s for $i = 1, 2, \dots, n$. Then $U \in \mathcal{N}_s$ and for all $t \in U$ we have $(tx, sx) \in \alpha$ for all $x \in X$. Thus, condition (ii) holds.
$(ii) \Longrightarrow (i)$: Let $\mathcal{U}(\mathfrak{X})$ denote the set of all bounded real-valued functions on X that are uniformly continuous with respect to the uniformity in question. It is a straightforward exercise to show that condition (ii) implies that $\mathcal{U}(X) \subseteq \mathcal{C}(\mathfrak{X})$. As the set $\mathcal{U}(X)$ separates point and closed subsets of X, the set $\mathcal{C}(\mathfrak{X})$ does so as well. In view of (3.8) it follows that condition (i) is fulfilled. □

4. A sufficient condition

In this final section we present a construction that, for any G–space $\mathfrak{X} = \langle X, \pi\rangle$, transforms a certain type of real-valued continuous functions on X into members of $\mathcal{C}(\mathfrak{X})$. This transformation is such that if we start with a family of functions that separates points and closed subsets of the space X then we end up with a subset of $\mathcal{C}(\mathfrak{X})$ separating points and closed sets as well. This result generalizes the old result (see [7], [8]) that if G is locally compact then every Tychonov G–space has an equivariant compactification. Another consequence will be that every G–space that can equivariantly be embedded in a locally convex linear G–space has an equivariant compactification, thus providing a new and simple proof of one of the results in [5]; see [10].

A real valued function f on X is called *locally equicontinuous* (with respect to the action π) whenever $f \in C^*(X)$ and, in addition, there exists a neighbourhood U of e in G (depending on f) such that the family $\{f \circ \pi^t\}_{t\in U}$ is equicontinuous on X. Hence a non-negative continuous function f on X is locally equicontinuous iff the following condition holds: there exists $U \in \mathcal{N}_e$ such that

$$\forall x \in X\ \forall \varepsilon > 0\ \exists W_x \in \mathcal{N}_x \text{ such that } |f(tx) - f(ty)| < \varepsilon \text{ for all } y \in W_x \text{ and } t \in U\,. \qquad (*)$$

The set of all bounded non-negative locally equicontinuous functions on X will be denoted $\mathcal{LE}^+(\mathfrak{X})$. Note that if G is locally compact then each f of $C(X)$ satisfies condition $(*)$ with the same U. Indeed, if U is a compact neighbourhood of e in G then joint continuity of the mapping $f \circ \pi$ implies equicontinuity of the family $\{f \circ \pi^t\}_{t\in U}$ on X.

(4.1) *The construction.* Let $f \in \mathcal{LE}^+(\mathfrak{X})$, and fix $U \in \mathcal{N}_e$ such that $(*)$ holds. Since G is a Tychonov space, the left uniformly continuous real-valued functions on G separate points and closed subsets of G, hence there exists $\varphi \in LUC(G)$ such that $0 \le \varphi(t) \le \|f\| + 2$ for all $t \in G$, $\varphi(e) = 0$ and $\varphi(t) = \|f\| + 2$ for all $t \in G \setminus U$; here $\|f\| := \sup\{|f(x)| : x \in X\}$. Observe that there is quite some freedom here; if necessary, we can replace U by any smaller neighbourhood of e, replacing at the same time φ by a function on G that fulfills the above conditions with respect to this smaller neighbourhood. Using any such φ we define a new function $\tilde{f}_\varphi : X \to \mathbb{R}$ by

$$\tilde{f}_\varphi(x) := \inf_{t\in G}\{\varphi(t) + f(tx)\} \quad \text{for} \quad x \in X\,.$$

Our aim is to show that $\tilde{f}_\varphi \in \mathcal{C}(\mathfrak{X})$ and that, if we apply this construction to a subset of $\mathcal{LE}^+(\mathfrak{X})$ separating points and closed subsets of X then we obtain a subset of $\mathcal{C}(\mathfrak{X})$ separating points and closed subsets of X as well.

(4.2) *Lemma.* *Let $f \in \mathcal{LE}^+(\mathfrak{X})$ and φ be as in (4.1). Then $\tilde{f}_\varphi \in \mathcal{C}(\mathfrak{X})$.*

Proof: First note that for every $x \in X$ we have

$$0 \leq \tilde{f}_\varphi(x) \leq \varphi(e) + f(x) = f(x) \leq \|f\| \,. \tag{1}$$

Hence $\tilde{f}_\varphi$ is bounded. In order to prove that $\tilde{f}_\varphi$ is continuous we first introduce the set

$$A_\varphi := \{t \in G \colon \varphi(t) < \|f\| + 1\} \,.$$

It is clear that $A_\varphi \subseteq U$. Moreover, for all $t \in G \setminus A_\varphi$ we have, by inequality (1) and our definitions,

$$\varphi(t) + f(tx) \geq \|f\| + 1 + f(tx) \geq \|f\| + 1 \geq \tilde{f}_\varphi(x) + 1$$

for every $x \in X$. This implies that

$$\tilde{f}_\varphi(x) = \inf_{t \in A_\varphi} \{\varphi(t) + f(tx)\} \quad \text{for} \quad \text{all} \quad x \in X \,. \tag{2}$$

Now let $\varepsilon > 0$ and consider a point $x \in X$. By condition $(*)$ there exists $W \in \mathcal{N}_x$ such that for all $y \in W$ and all $t \in U$ we have $|f(tx) - f(ty)| < \varepsilon$. Moreover, it follows from equation (2) that there exists $t_1 \in A_\varphi \subseteq U$ such that $\varphi(t_1) + f(t_1 x) < \tilde{f}_\varphi(x) + \varepsilon$. Fix any $y \in W$. Because $t_1 \in U$ we have, by the choice of W, $f(t_1 y) < f(t_1 x) + \varepsilon$, hence

$$\tilde{f}_\varphi(y) \leq \varphi(t_1) + f(t_1 y) \leq \varphi(t_1) + f(t_1 x) + \varepsilon \leq \tilde{f}_\varphi(x) + 2\varepsilon \,.$$

Similarly, $\tilde{f}_\varphi(x) < \tilde{f}_\varphi(y) + 2\varepsilon$. This completes the proof that $\tilde{f}_\varphi$ is continuous at x.

Finally, we show that $\tilde{f}_\varphi \in \mathcal{C}(\mathfrak{X})$. To this end, consider a point $(t, x) \in G \times X$. Then

$$\begin{aligned} \tilde{f}_\varphi(tx) &= \inf_{s \in G} \{\varphi(s) + f(stx)\} \\ &= \inf_{u \in G} \{\varphi(ut^{-1}) - \varphi(u) + \varphi(u) + f(ux)\} \\ &\geq \inf_{u \in G} \{\varphi(ut^{-1}) - \varphi(u)\} + \tilde{f}_\varphi(x) \,. \end{aligned}$$

But $\varphi \in LUC(G)$, so for given $\varepsilon > 0$ there exists $V \in \mathcal{N}_e$ such that $|\varphi(ut^{-1} - \varphi(u)| < \varepsilon$ for all $u \in G$ and $t \in V^{-1}$. It follows that $\tilde{f}_\varphi(tx) > \tilde{f}_\varphi(x) - \varepsilon$ for all $x \in X$ and all $t \in V^{-1}$. Now replace in this inequality x by tx with $t \in V$; because then $t^{-1} \in V^{-1}$ we get $\tilde{f}_\varphi(x) = \tilde{f}_\varphi(t^{-1}tx) > \tilde{f}_\varphi(tx) - \varepsilon$. Combining these results we see that $|\tilde{f}_\varphi(tx) - \tilde{f}_\varphi(x)| < \varepsilon$ for all $x \in X$ and $t \in V \cap V^{-1}$. This shows that $\tilde{f}_\varphi \in \mathcal{C}(\mathfrak{X})$. $\square$

(4.3) *Lemma.* *Let $g \in \mathcal{LE}^+(\mathfrak{X})$, let K be a closed set in X and let $x_0 \in X \backslash K$ such that $g(x_0) \notin \overline{g[K]}$. Then there exists a uniformly continuous function $\psi \colon \mathbb{R} \to [0\,;1]$ and an element $\varphi \in LUC(G)$ such that for $f := \psi \circ g \in \mathcal{LE}^+(\mathfrak{X})$ we have $\tilde{f}_\varphi(x_0) \notin \overline{\tilde{f}_\varphi[K]}$.*

Proof: There exists a uniformly continuous mapping $\psi\colon \mathbb{R} \to [0;1]$ such that $\psi(g(x_0)) = 1$ and $\psi[\overline{g[K]}] = \{0\}$. Then it is easily seen that $f := \psi \circ g \in \mathcal{LE}^+(\mathfrak{X})$; moreover, $f(x_0) = 1$ and $f[K] = \{0\}$. As $f(x_0) = 1$, there exists $V \in \mathcal{N}_e$ such that $f(tx_0) > 1/2$ for all $t \in V$. Now observe that if $U \in \mathcal{N}_e$ satisfies condition $(*)$ of (4.1) then we may replace here U by the smaller neighbourhood $U \cap V$ of e. So we may assume from the outset that U has, in addition to the properties mentioned in (4.1), the property that $f(tx_0) > 1/2$ for all $t \in U$.

With this particular U select φ according to (4.1) and consider $\tilde{f}_\varphi \in \mathcal{C}(\mathfrak{X})$ (see (4.2)). Let A_φ be defined as in the proof of (4.2). By equation (2) above we infer that

$$\tilde{f}_\varphi(x_0) = \inf_{t \in A_\varphi} \{\varphi(t) + f(tx_0)\}\,, \tag{3}$$

and because $A_\varphi \subseteq U$ it follows that the right-hand side of (3) is at least 1/2. So $\tilde{f}_\varphi(x_0) \geq 1/2$. In addition, by equation (1) in (4.2) it follows that for every point $x \in K$ we have $0 \leq \tilde{f}_\varphi(x) \leq f(x) = 0$. Hence $\tilde{f}_\varphi[K] = \{0\}$. □

(4.4) ***Corollary.*** *If $\mathcal{LE}^+(\mathfrak{X})$ separates points and closed subsets of X then so does $\mathcal{C}(\mathfrak{X})$.*

Proof: Clear from (4.3). □

(4.5) ***Remark.*** Close inspection of the above proof shows that if a subset of $\mathcal{LE}^+(\mathfrak{X})$ separates points and closed subsets of X, then a subset of $\mathcal{C}(\mathfrak{X})$ *of the same cardinality* separates points and closed sets of X. This is useful if one wants to make estimations of the weight of a possible equivariant compactification of $\mathfrak{X}$. See [7], [5].

(4.6) ***Corollary.*** *If $\mathfrak{X}$ is a Tychonov G–space such that a neighbourhood of e acts equicontinuously on X with respect to a certain uniformity compatible with the topology of X, then $\mathfrak{X}$ can be equivariantly embedded in a compact G–space.*

Proof: Clearly, for any uniformly continuous, bounded, non-negative function $f : X \to \mathbb{R}$ we have in this situation $f \in \mathcal{LE}^+(\mathfrak{X})$. As those uniformly contious functions separate points and closed subsets of X, the result follows from (4.4) and (3.8). □

(4.7) ***Remarks.*** **1.** This Corollary was presented by the present author at at the conference 'Topological Dynamics' in Oberwolfach in 1980; see Tagungsbericht 25/1980. It appears also in [5].
2. If G is locally compact then in every G-space $\mathfrak{X}$ a (compact) neighbourhood of e acts equicontinuously on X. In that case every Tychonov G–space has an equivariant compactification. This was the main result in [7, 8].

References

[1] Baayen, P. C. & J. de Groot, Linearization of locally compact transformation groups in Hilbert space, *Math. Systems Theory* **6** (1972), 363–379.

[2] Ball, R. N. & Hagler, J. N., The Gleason cover of a flow, in: E. Coplakova & K. P. Hart (eds.), *Papers on general topology and applications* (Amsterdam, 1994), New York Acad. Sci., New York, 1996, pp. 13–33.

[3] Kelley, J. L., *General Topology,* Van Nostrant Reinhold Co., New York, 1955.

[4] Megrelishvili, M. G., A Tychonov G-space not admitting a compact Hausdorff G-extension or G-linearization, *Russian Math. Surveys* **43:2** (1988), 177–178.

[5] Megrelishvili, M. G., Free topological G-groups, *New Zealand J. Math.* **25** (1996), 59–72.

[6] Vries, J. de, *Topological transformation groups 1: a categorical approach,* Mathematical Centre Tracts 65, Mathematisch Centrum, Amsterdam, 1975.

[7] Vries, J. de, Equivariant embeddings of G-spaces, in: J. Novak (ed.), *General Topology and its Relations to Modern Analysis and Algebra IV,* (Proc. 4th Prague Topol. Symp. 1976), Part B, Prague, 1977, pp. 485–493.

[8] Vries, J. de, On the existence of G-compactifications, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys.* **26** (1978), 275–280.

[9] Vries, J. de, Linearization of actions of locally compact groups, *Trudy. Mat. Inst. Steklov* **154** (1983), 57-70.

[10] Vries, J. de, Linearization of G-spaces revisited, *in preparation.*