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A Note on Inverses of Non-decreasing Lévy Processes

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ABSTRACT
We show that, apart from deterministic processes, compound Poisson processes with exponential jumps are the only (shifted) non-decreasing Lévy processes whose inverses are also (shifted) non-decreasing Lévy processes.

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1 Introduction

It is known (see Bertoin (1996), p. 72, or Sato (1999), pp. 137-8) that a Lévy process $X = \{X_t : t \geq 0\}$ is non-decreasing (i.e., its realizations are a.s. non-decreasing functions) if and only if the Laplace transform of X_t is of the form

$$L_{X_t}(u) = e^{-t\Psi(u)}, \tag{1.1}$$

where $\Psi(u) = \gamma - \int_0^\infty (e^{-ux} - 1)\nu(dx)$, $\gamma \geq 0$, and ν is a measure on $(0, \infty)$ satisfying $\int_0^1 x\nu(dx) < \infty$. A non-decreasing Lévy process is also called a subordinator.

Example A
A non-decreasing compound Poisson process X is defined in terms of a Poisson process $N = \{N_t : t \geq 0\}$ with rate λ and of an independent, non-negative i.i.d. sequence $\xi_1, \xi_2, \ldots$ by the formula $X_t = \sum_{i=1}^{N_t} \xi_i$, $t \geq 0$. Clearly, X has independent increments and $L_{X_t}(u) = \exp\{-\lambda t(1 - L_\xi(u))\}$, where L_ξ is the Laplace transform of the ξ_i's. □

Let us define a process X^{-1} whose realizations work as inverses of those of X by putting

$$X_x^{-1} = \inf\{t : X_t > x\}, \qquad x \geq 0.$$

It is well known (e.g. Revuz and Yor (1991), p. 7) that X^{-1} is non-decreasing and right-continuous if X is, and that $X_{x-}^{-1} = \inf\{t : X_t \geq x\}$. Moreover, $X_t \leq x$ if and only if $X_{x-}^{-1} > t$, so

$$P(X_{x-}^{-1} > t) = P(X_t \leq x), \qquad x, t \geq 0. \tag{1.2}$$

This shows that if X^{-1} is continuous in probability then $P(X_x^{-1} > t) = P(X_t \le x)$ for almost all t.

If we think of X as describing the growth in time of some physical quantity, then an event such as $\{X_x^{-1} > t\}$ corresponds to a statement about the time it takes for X to exceed the threshold x. This suggests that the inverse process X^{-1} may occur in some practical problems. Since one usually hopes to work with processes which are in some sense amenable or have a nice structure, it is natural to ask, for example, whether X^{-1} can itself be a (non-decreasing) Lévy process. The following example suggests a slightly different form of this question:

Example B
Consider a compound Poisson process with $\xi_1, \xi_2, \ldots$ exponentially distributed with mean $1/\mu$. Then

$$L_{X_t}(u) = \exp\left\{-t\frac{\lambda u}{\mu + u}\right\}, \qquad u > -\mu, \tag{1.3}$$

which corresponds to (1.1) with $\gamma = 0$ and $\nu(dx) = \lambda\mu e^{-\mu x}dx$.

Let $\psi_1, \psi_2, \ldots$ be the lengths of the successive constancy intervals of N, and denote by M the Poisson process determined by $\xi_1, \xi_2, \ldots$ We can see that

$$X^{-1}(x) = \psi_1 \;\Leftrightarrow\; 0 \le x < \xi_1 \;\Leftrightarrow\; M_x = 0,$$

$$X^{-1}(x) = \psi_1 + \psi_2 \;\Leftrightarrow\; \xi_1 \le x < \xi_1 + \xi_2 \;\Leftrightarrow\; M_x = 1,$$

and so on, so that $X^{-1}(x) = \psi_1 + \sum_{i=1}^{M_x} \psi_{i+1}$, $x \ge 0$. It follows that $\tilde{X}^{-1} := X^{-1} - \psi_1$ is a Lévy process with

$$L_{\tilde{X}_t^{-1}}(u) = \exp\left\{-t\frac{\mu u}{\lambda + u}\right\}, \qquad u > -\lambda,$$

which is the same as (1.3) with λ and μ interchanged. In other words, if X is a compound Poisson process with exponential jumps then X^{-1} is a compound Poisson process with exponential jumps plus an additional independent exponential jump.

It can be checked that the inverse of X^{-1} is X itself. Thus we have the converse statement that if X^{-1} is a compound Poisson process with exponential jumps plus an additional independent exponential jump then its inverse X is a compound Poisson process with exponential jumps. $\square$

Let us say that Y is a *shifted* Lévy process if Y is the sum of a Lévy process and an independent, non-negative random variable, and that Y is *deterministic* if, with probability 1, $Y_t \propto t$ for all $t \ge 0$. We next show that, apart from deterministic processes, the only non-decreasing, shifted Lévy processes whose inverses are also shifted Lévy processes are the compound Poisson processes with exponential jumps and their inverses, which are compound Poisson processes with exponential jumps shifted by an additional independent exponential jump.

In view of the representation (1.1) this result is almost trivial, but it may be of some curiosity value. We do not know whether such models have been, or can be, successfully used in statistical modelling, but the result suggests an obvious model checking procedure.

2 Proof of the result

We know that if Y and Y^{-1} are non-decreasing, shifted Lévy processes then

$$L_{Y_t}(u) = \varphi(u)e^{-t\Psi(u)} \qquad \text{and} \qquad L_{Y_t^{-1}}(u) = \tilde{\varphi}(u)e^{-t\tilde{\Psi}(u)},$$

where φ and $\tilde{\varphi}$ are Laplace transforms of some random variables, and

$$\Psi(u) = \gamma - \int_0^\infty (e^{-ux} - 1)\nu(dx), \qquad \tilde{\Psi}(u) = \tilde{\gamma} - \int_0^\infty (e^{-ux} - 1)\tilde{\nu}(dx),$$

with $\int_0^1 x\nu(dx), \int_0^1 x\tilde{\nu}(dx) < \infty$, $\gamma, \tilde{\gamma} \geq 0$.

Put

$$\hat{L}_Y(u,v) := \int_0^\infty e^{-vt} L_{Y_t}(u)dt = \frac{\varphi(u)}{\Psi(u) + v},$$

$$\hat{L}_{Y^{-1}}(u,v) := \int_0^\infty e^{-vt} L_{Y_t^{-1}}(u)dt = \frac{\tilde{\varphi}(u)}{\tilde{\Psi}(u) + v}.$$

Noting $L_{Y_t^{-1}}(u) = u\int_0^\infty e^{-us}P(Y_t^{-1} \leq s)ds - P(Y_t^{-1} = 0)$, recalling that $P(Y_t^{-1} \leq s) = P(Y_s > t)$ for almost all s from the discussion around (1.2), using the definition of $\hat{L}_Y(u,v)$, and writing $p(v) = \int_0^\infty e^{-vt}P(Y_t^{-1} = 0)dt$, we see that

$$\begin{aligned}
\hat{L}_{Y^{-1}}(u,v) &= \int_0^\infty e^{-vt} u \int_0^\infty e^{-us} P(Y_t^{-1} \leq s)\, ds\, dt - p(v) \\
&= \int_0^\infty e^{-vt} u \int_0^\infty e^{-us} P(Y_s > t)\, ds\, dt - p(v) \\
&= \int_0^\infty u e^{-us} \int_0^\infty e^{-vt} P(Y_s > t)\, dt\, ds - p(v) \\
&= \int_0^\infty u e^{-us} \left(\frac{1}{v} - \frac{1}{v} L_{Y_s}(v) \right) ds - p(v) \\
&= \frac{1}{v} - \frac{u}{v}\hat{L}_Y(v,u) - p(v),
\end{aligned}$$

which gives

$$\frac{v\tilde{\varphi}(u)}{\tilde{\Psi}(u) + v} = 1 - \frac{u\varphi(v)}{\Psi(v) + u} - p(v),$$

for all u and v for which the Laplace transforms involved are defined. But with $v = 0$ this reads $0 = 1 - u/(\gamma + u) - p(0)$, implying $p(0) = 0$ and hence $p \equiv 0$. Thus we have the simpler equation

$$\frac{v\tilde{\varphi}(u)}{\tilde{\Psi}(u) + v} = 1 - \frac{u\varphi(v)}{\Psi(v) + u}. \tag{2.1}$$

It can be checked that (2.1) is satisfied by $\tilde{\varphi}(u) = \lambda/(\lambda + u)$, $\tilde{\Psi}(u) = \mu u/(\lambda + u)$, $\varphi(u) = 1$, $\Psi(u) = \lambda u/(\mu + u)$, which correspond to the shifted compound Poisson processes with exponential jumps of Example B.

To solve (2.1), assume first that $\varphi \equiv 1$, i.e., that φ is the Laplace transform of a random variable concentrated at 0. Then (2.1) becomes

$$\frac{v\tilde{\varphi}(u)}{\tilde{\Psi}(u)+v} = \frac{\Psi(v)}{\Psi(v)+u}. \tag{2.2}$$

Letting $v \to \infty$ in this equation we get $\tilde{\varphi}(u) = \lambda/(\lambda+u)$, where $\lambda := \tilde{\gamma} + \tilde{\nu}(0,\infty)$. If $\lambda = \infty = \tilde{\nu}(0,\infty)$, so that $\tilde{\varphi} \equiv 1$, (2.2) reduces to $\tilde{\Psi}(u)/v = u/\Psi(v)$, which implies $\Psi(v) = v/\tilde{\Psi}(1)$, $\tilde{\Psi}(u) = u/\Psi(1)$ and shows that $Y_t \propto t$, $t \geq 0$, with probability 1, i.e., that Y is deterministic.

If $\lambda < \infty$, (2.2) is equivalent to

$$\frac{\lambda v}{\lambda+u}\frac{1}{\tilde{\Psi}(u)+v} = \frac{\Psi(v)}{\Psi(v)+u}. \tag{2.3}$$

Taking $u = 1$ here and solving for $\Psi(v)$, we find

$$\Psi(v) = \frac{\lambda v}{\mu+v}, \qquad v > -\mu,$$

where $\mu := (\lambda+1)\tilde{\Psi}(1)$; thus Y must be a compound Poisson process with exponential jumps.

Now suppose $\varphi \not\equiv 1$. Letting $v \to +\infty$ in (2.1) we find $\tilde{\varphi} \equiv 1$ and

$$\frac{v}{\tilde{\Psi}(u)+v} = 1 - \frac{u\varphi(v)}{\Psi(v)+u}, \qquad \text{or} \qquad \frac{u\varphi(v)}{\Psi(v)+u} = \frac{\tilde{\Psi}(u)}{\tilde{\Psi}(u)+v}.$$

But this last equation is the same as (2.2) with u and Ψ interchanged respectively with v and $\tilde{\Psi}$ and $\tilde{\varphi}$ replaced by φ; therefore, the same arguments as before show that Y^{-1} is either a deterministic process or a compound Poisson process with exponential jumps, or, equivalently, that Y is either a deterministic process or the inverse of a compound Poisson process with exponential jumps.

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