ON THE ISING PROBLEM

C. HOEDE

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PROEFSCHRIFT

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CONTENTS

CHAPTER I

INTRODUCTION

§1. INTRODUCTION

In 1925 ISING [1] published an article on the calculation of the sum of states or partition function for a one-dimensional system. This system served as a model for a ferromagnetic needle. Its constituent elements were thought of as equidistant elementary magnets that could be either parallel or antiparallel. Any two parallel neighbouring elements of the chain of magnets were supposed to have an interaction energy J_1, whereas for any two antiparallel neighbouring elements the interaction energy was to be J_2 with $J_1 < J_2$, as the model was to be an idealization of a ferromagnetic system*. To keep things simple it was further assumed that only nearest neighbours were interacting. The problem of calculating the partition function for this model was posed to ISING by LENZ and is now known as the ISING problem for the one-dimensional ISING model.

The formalism of statistical mechanics being developed, for the description of systems in equilibrium practically only one problem remains, i.e. the calculation of the partition function, which is a mathematical problem. Nowadays investigations in connection with the ISING problem are mainly aiming at carrying the calculation of partition functions for generalized models further. The accent lies on the quantitative aspects, rather than on the qualitative behaviour of the models. It was ISING's aim, however, to calculate the partition function of a nontrivial model, with special attention to the question whether this model would give rise to a singularity in the specific heat curve as is known for a ferromagnet. This curve, approximately with the shape of the Greek letter lambda, shows a singularity at the CURIE temperature T_C, also known as CURIE point or λ-point. At this temperature the spontaneous magnetization disappears. This phenomenon is, of course, very strongly related to the singular behaviour of the specific heat curve and it was in fact this phenomenon that had ISING's main interest.

ISING's approach was essentially of a combinatorial nature and resulted in not giving the spontaneous magnetization or the singularity in the specific

* In case $J_1 > J_2$ the model idealizes an antiferromagnetic system.

heat curve he was probably hoping for. He argumented that an extension of the model to one in two or three dimensions would not cause any of these phenomena either. Although his argument was invalid, the study of generalized ISING models did not seem very promising as HEISENBERG [2] demonstrated in 1928 that in ferromagnetic materials the interaction was to be described by quantum mechanics. Even though the possibility of only two interaction states is "quantumlike" and atoms mainly interact with their direct neighbours, ISING's model was a somewhat poor one. On the other hand the alternative HEISENBERG model proved to be much more difficult to handle.

Meanwhile the interest in ferromagnetism was not waning and a comparatively long period was marked by approximate methods for generalized models. A first review on the subject was published by NIX and SHOCKLEY [3] in 1938, dealing with important contributions like that of BRAGG and WILLIAMS, BETHE, PEIERLS, etc. In 1941 three articles were published, by KRAMERS and WANNIER [4], MONTROLL [5] and LASSETTRE and HOWE [6], all three introducing matrices in the formulation of the partition function for ISING models. In the same year VAN DER WAERDEN [7] published an article, in which he used an essentially combinatorial approach to the calculation of the partition function for a two-dimensional lattice, with the accent on the investigation of the convergence radius of a series development for certain physical quantities. The direct impulse for this revival of interest can be seen in the article of PEIERLS [8] in 1936, showing the possibility of spontaneous magnetization for a two-dimensional model*.

In ref. [4] an eventual transition point was located by an investigation of the matrices in this "matrix method". This method gives a very elegant solution for the original one-dimensional problem and on generalization led to the solution of the ISING problem for a two-dimensional rectangular lattice by ONSAGER [9] in 1944. An important modification of his approach was given by KAUFMAN [10] in 1949. The matrix method seemed very promising as the method produced a "closed" expression for the free energy per element of the two-dimensional model, resulting in a specific heat curve of the λ-type. At least qualitatively the ISING model gave interesting results. Moreover YANG [11] was able to deduce a formula for the spontaneous magnetization in 1952, giving rise to a magnetization curve in qualitative accordance with experiment.

However, the year 1952 also marked a decline in the interest in the matrix method. Firstly the method did not seem to allow a solution for the generalization to the three-dimensional problem or for the two-dimensional problem with an external magnetic field present. Secondly KAC and WARD [12] were able to devise a combinatorial method for the two-dimensional problem by picking up the trend of thought of VAN DER WAERDEN's paper, although this was done by inspection of the known ONSAGER-KAUFMAN solution. The method appeared to be able to give the same results as the matrix method. The time was there to write a second review on the subject, which was done by NEWELL and MONTROLL [13] in 1953. The combinatorial method had the great advantage that several terms could be obtained of series developments for relevant physical quantities like the free energy or the susceptibility for those models that evaded an approach by the matrix method.

* The interrelation between the disappearing of this magnetization and the λ-point inferred the existence of a convergence radius, i.e. a singularity.

From these series much information can be obtained on the critical behaviour of e.g. the three-dimensional model. Numerous articles on series developments have been published in the last decade. In addition to a third review by DOMB [14] on cooperative phenomena in 1960, a review by FISHER [15] in 1963 should be mentioned in this connection.

Another line of development in the combinatorial method has been more stubbornly aiming at attaining exact solutions. By this is meant the method using Pfaffians which was recently developed. In this connection the reader is referred to the monograph by GREEN and HURST [16] of 1964 on order-disorder phenomena. As was the case with the method of KAC and WARD, the essentially combinatorial problem is reduced to the calculation of the determinant of a matrix. Although the ISING problem gained importance by this line of development because of its relationship to other combinatorial problems hardly any new results were achieved but for the deepened insight in the nature of the problem. In 1964 the situation is such, that there are two rather complicated but equally powerful methods that each failed so far to solve e.g. the three-dimensional problem or the two-dimensional problem with external magnetic field, the three-dimensional problem with external magnetic field being rather utopical. In addition to this, problems that generally include other than nearest neighbour interactions have not yet been solved.

Now one thing that can be done is refining the existing methods, if necessary by introducing techniques developed in other fields. Here the articles of SCHULTZ, MATTIS and LIEB [17] (1964), THOMPSON [18] (1965), HURST [19] (1966) and GIBBERD and HURST [20] (1967) should be mentioned specially. Another thing that can be done is to try and get a better insight in the links and correspondences between both the matrix- and the combinatorial approach. This was done in 1965 and 1966 by HURST [21], [22] . Here it may be said that it is the aim to find out what are the corresponding difficulties and what method should be looked upon as the most promising assuming that after all a solution to open problems will appear to be possible by applying a method directly connected with the familiar methods.

The author's contribution to the theory can be seen as belonging to this second trend of development. Not because two rather unwieldy mathematical apparatuses are probed to fit but because, next to an investigation of the limitations of the KAUFMAN method, it is tried to develop a method by generalizing the matrix method for the one-dimensional problem in another way than has been done up to now. This leads a.o. to an interesting relationship between the two rather different approaches mentioned before. It is hoped that eventually a method will be obtained being possible the method meant by ONSAGER as he wrote: *It seems rather likely that this result (the expression for the free energy per element of the two-dimensional model) could be derived from direct algebraic and topological considerations without recourse to the operator method used in the present work. Such a development might well amount to a great improvement of the theory.*

§2. PHILOSOPHY OF THE INVESTIGATION

In comparison with the original method of ISING for the one-dimensional problem, the matrix method solution for this problem is very elegant. The

partition function is written as the spur of a certain matrix of order 2, which can be calculated readily, being the sum of the eigenvalues of the matrix. The two-dimensional problem leads in various ways to the calculation of the spur of a matrix of order 2^{ℓ}, when the rectangular ISING lattice (the most direct generalization of the linear lattice) contains ℓ latticepoints per row in one of both directions. The eigenvalues are to be determined again. Because of the order of the matrix this is quite a problem, that can only be solved quite accidentally. When an external magnetic field is present, the problem is essentially the same, i.e. the calculation of the spur of a matrix of order 2^{ℓ}, but difficulties appear that are seemingly unsurmountable. The non-solved three-dimensional problem can also be formulated as the calculation of the spur of a matrix. When the lattice, which is now cubic, consists of rectangular layers of m rows of ℓ latticepoints the order of the matrix is $2^{\ell m}$ for various ways of formulation.

We now start by making a trivial remark. For the solution of ISING problems, using the matrix method, two things are necessary:

a) It must be possible to write the partition function as the spur of a matrix;
b) It must be possible to determine the spur or, equivalently, the eigenvalues of the matrix.

As the second point necessitates the first point, this remark is trivial indeed. In literature a) is dealt with in a few cases only* and in most cases attention is paid to a) as well as to b). The operator method used produces the "transfer matrix" whose eigenvectors can be interpreted probabilistically and whose eigenvalues are to give the partition function. In several other cases the attention is almost entirely centred on b)** which is, of course, the more interesting part of the whole problem.

The mathematical apparatus of ONSAGER and KAUFMAN to satisfy b) is rather complex. When attempts were made to generalize the method in order to solve the three-dimensional problem or the two-dimensional problem with external magnetic field, the reason for failure was looked for in the structure of the mathematical apparatus. More in detail: the LIE algebras that can be generated by the operators respectively matrices occurring in the formulation for the various problems were investigated and the number of structure constants turned out to be much larger for those problems that could not be solved. We quote NEWELL and MONTROLL: *Attempts to apply this procedure to the three-dimensional problem or even the two-dimensional problem with a magnetic field are seriously hindered at an early stage because the operators of interest generate a much larger LIE algebra, so large in fact that it would seem to be of little value.* *** Now this could be the deeper reason for failure but it is unsatisfactory that it was not indicated how far one could come and what should be solved to come any further, as was clearly indicated in the case of the Pfaffian method in ref. [16].

It has been remarked that the order of the matrices involved in the one-, two- and three-dimensional problem was the same for various ways of formulation. However, the structure of the matrices varies with the way the lattices are built up, when using the operator method. We shall only distinguish three ways of building up the lattice, whereas throughout this investigation we shall mainly consider three lattices i.e. the "linear", the "quadratic" and the "cubic" lattice.

* See for example ref. [23]

** See for example ref. [10] and ref. [18]

*** From the quotation it can be seen that the three-dimensional problem is considered to be more difficult than the two-dimensional problem with field. We shall come back to this in chapter V.

One can build up a lattice by adding to the units that are present already:

1°) a single element
2°) a string of elements
3°) a layer of elements

KRAMERS and WANNIER built up the quadratic $\ell \times m$-lattice by adding a single element at the time. ONSAGER builds up the lattice by adding a string of ℓ elements at the time. Both ways of using the operator method are leading to the calculation of the spur of a (transfer) matrix of order 2^{ℓ}. The matrices involved are quite different. However, NEWELL [24] was able to apply KAUFMAN's method to the problem in the formulation of KRAMERS and WANNIER in 1950. The cubic $\ell \times m \times p$-lattice can be built up by adding single lattice points, as was shown by DOMB [23] in 1949, strings of e.g. ℓ lattice points or layers of $\ell.m$ lattice points. We shall revert to this in chapter IV. It should be remarked that the second possibility has hardly drawn any attention up to now. This small gap in the theory can be filled up easily by a formulation using a helix construction with strings à la KRAMERS and WANNIER. The author supposes that, in spite of NEWELL's success, possibly because of the failure to satisfy b) for the third possibility the first two possibilities are considered to be dead tracks.

Summarizing we see that for the lattices indicated before:

i) To satisfy a), for application of the transfer matrix method, the lattice can be considered to be built up with different units, leading to:
ii) Matrices whose order is 2 for the linear lattice, 2^{ℓ} for the quadratic and $2^{\ell.m}$ for the cubic lattice, irrespective of the way the lattice is built up, but whose structure does depend on the way of building up the lattice.

We now make a second remark, apparently hardly less trivial. For writing the partition function as the spur of a matrix it is not necessary to have recourse to the operator method and the associated building-up principle. Indeed, the partition function for a system that can be in M states with corresponding BOLTZMANN factors $f_1, f_2, \ldots, f_M$ can always be written as the spur of a diagonal matrix of order M, the numbering of the rows and columns in correspondence with a numbering of the states, having the BOLTZMANN factors f_i ($i = 1, \ldots, M$) as diagonal elements. However, the crucial point is that the partition functions for the ISING models in question can be written in a nontrivial way as the spur of a matrix, whose order is not determined by the dimension of the lattice, as is the case when the building-up principle is used.

We now drop the idea of building up the lattice but stress the fact that the lattice can be thought of as being constituted by units of e.g. the three types mentioned before, viz. element, string or layer. Corresponding to the way we "mentally dissect" the lattice, the orders of the matrices involved turn out to be different for the same lattice. These orders are:

2 for considering single lattice points as the constituents
2^{ℓ} for considering strings of ℓ lattice points as the constituents
$2^{\ell.m}$ for considering layers of $\ell.m$ lattice points as the constituents

We might add: 2^N for considering the whole ISING lattice of N elements as the sole constituent of the lattice, which is the trivial possibility mentioned before.

Now a clear statement can be made what the philosophy of the investigation is: *As the operator method for satisfying a) is not necessary,*

and in consequence of which the order of the matrices involved is not necessarily determined by the dimension of the lattice, the other ways to formulate the partition functions may now be investigated. As we consider only three types of units for the lattices there are three nontrivial ways left; to formulate the partition function of the two-dimensional problem in terms of matrices of order 2 and to formulate the partition function of the three-dimensional lattice in terms of matrices of order 2 or 2^{ℓ}.

§3. CLASSIFICATION OF PROBLEMS

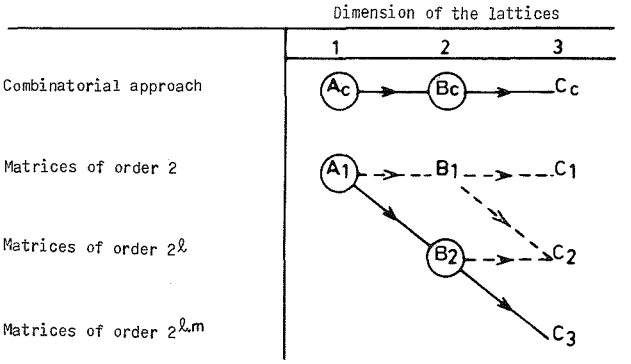
There is a host of ISING problems. Let us indicate what kind of generalizations might be considered and what kind we shall consider actually, starting from the original one-dimensional ISING model. Only nearest neighbours were considered to interact with an interaction energy that could assume only two values for any pair of neighbours. We shall consider the generalization by including interactions between next-nearest neighbours, but we shall maintain that there are only two values for the interaction energy for a pair of elements of a certain type*. We might drop this latter restriction and consider the case that there are more values possible for the interaction energy of a pair of elements or admit the possibility that the value of the interaction energy is different for various pairs of elements of the same type. The regularity of crystals in nature justifies that we shall only consider the case that the interaction energies of pairs of elements of a certain type is the same for every pair. More values for the interaction energy, no doubt a realistic generalization, could be considered but this would influence the order of the matrices leading too far away for the present.

In all these cases we may speak of an ISING model. In two and three dimensions there is the extra possibility of varying the structure of the lattice. It has already been said that we shall consider linear, quadratic and cubic lattices only. The question arises how severe this restriction is. In the two-dimensional case next to the quadratic lattice other lattices, like the triangular lattice, the honeycomb lattice or the KAGOME lattice, have been studied. However, no essentially different features were found in these cases, see e.g. ref. [14] . The quadratic lattice is therefore representative of the two-dimensional lattices and it seems quite possible that the cubic lattice will be so of other three-dimensional lattices like the body-centred cubic and the face-centred cubic lattices next to the simple cubic lattice. Further there may be the presence of an external magnetic field and we shall consider this as well as the possibility of impurities in the lattice, i.e. the occurrence of elements that do not interact or interact "improperly" with other lattice points. As the values of the interaction energies may still vary with the one, two or three directions that can be distinguished in the lattice, or eventually may be partly or entirely antiferromagnetic, we shall have to specify the interactions for every model.

If we call the original one-dimensional problem A, we might indicate the inclusion of a magnetic field by A^M , the inclusion of next-nearest neighbour interactions by ${}_{n.n.}A$ and the occurrence of impurities by ${}^{imp.}A$. We shall indicate the two-dimensional problem by B and the three-dimensional pro-

* The "type" could e.g. be determined in the one-dimensional case by the distance between the elements, measured in links of the chain.

blem by C. Next to the nature of the problem there is the nature of the approach, as has been discussed in the former paragraph. We shall use the indices 1, 2 and 3 to denote that the formulation is in terms of matrices of order 2, 2^{ℓ} and $2^{\ell.m}$ respectively*. The various ways of using the building-up principle will be indicated by numbers 1, 2 or 3 between parentheses, depending on the unit, element, row or layer respectively, used in the building-up procedure. This will be done in chapter IV only. Finally we shall use the index c in case a combinatorial approach is made. We can now give the following scheme:



The lines drawn indicate realized generalizations as far as the formulation is concerned. The dotted lines indicate non-realized generalizations. The circles indicate problems solved.

* See § 2 for the order of the matrices

CHAPTER II

A NEW FORMULATION OF THE PROBLEM FOR THE QUADRATIC LATTICE

§ 1. THE PROBLEM FOR THE LINEAR CHAIN

The starting point for the matrix method is problem A_1. From here we can generalize to problem B_1 or to problem B_2. Only problem B_1 is a new problem and the generalization $A_1 \rightarrow B_1$ is what this chapter is about. We shall give the solution for problem A_1 as an example.

The linear ISING lattice consists of, say, N elements, situated on a line. Only nearest neighbours interact. In order to facilitate the calculation we let the first element interact with the last, so that essentially we consider a simple closed chain. Two interacting elements will be said to form a "bond". It is intuitively clear and can be proved rigorously that the inclusion of the extra bond has a vanishing influence on the significant quantities to be calculated as N becomes infinite. However, we shall simply consider also in future problems the ISING model to include this bond and similar bonds. We shall now specify the interaction energy for a pair of neighbouring elements, i.e. for any pair, in accordance with the restrictions we have made. With each element i is connected a variable σ_i, called the "spin" variable, that can assume the values +1 and -1.

The interaction energy $E(\sigma_i, \sigma_{i+1})$* is defined as follows:

$$\begin{aligned} E(+1,+1) &= E(-1,-1) = -J \\ E(+1,-1) &= E(-1,+1) = +J \\ J &> 0^{**}. \end{aligned} \qquad \text{(II, 1)}$$

When k is the BOLTZMANN constant and T the absolute temperature, the partition function Z_N is the sum of the "BOLTZMANN factors" $\exp\left[-E_n/kT\right]$, where E_n is the total interaction energy of the n-th of the 2^N possible states.

* N.B. $\sigma_{N+i} = \sigma_i$ $(i = 1, 2, \ldots, N)$.

** The model is "ferromagnetic", as the spin variables tend to have the same values, i.e. the elements, thought of as elementary magnets, tend to stand in line. Thus they try to constitute order, the basic concept underlying ferromagnetism to the extent that the occurrence of spontaneous magnetization is synonymous with the occurrence of (long range) order.

Thus $$Z_N = \sum_{n=1}^{2^N} \exp\left[-E_n/kT\right] \qquad \text{(II, 2)}$$

or, as the total interaction energy is $\sum_{i=1}^{N} E(\sigma_i, \sigma_{i+1})$,

$$Z_N = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \exp\left[-\sum_{i=1}^{N} E(\sigma_i, \sigma_{i+1})/kT\right] . \qquad \text{(II, 3)}$$

We now think of the lattice as dissected in lattice points and in all of the 2^N terms we consider the factor $\exp\left[-E(\sigma_i,\sigma_{i+1})/kT\right]$, stemming from the interaction between the i-th and the i+1-th element, and perform the following trick. We consider a 2×2-matrix V, the rows numbered by the spin states +1 and -1 of the i-th element and the columns by the spin states of the i+1-th element, i.e.

$$V = \begin{array}{c} \sigma_i \backslash \sigma_{i+1} \\ \\ +1 \\ -1 \end{array} \begin{array}{c} \begin{array}{cc} +1 & \quad -1 \end{array} \\ \begin{pmatrix} e^{J/kT} & e^{-J/kT} \\ e^{-J/kT} & e^{J/kT} \end{pmatrix} \end{array}, \qquad \text{(II, 4)}$$

and indicate the factor $\exp\left[-E(\sigma_i, \sigma_{i+1})/kT\right]$ as $V_{\sigma_i;\sigma_{i+1}}$. This makes it possible to write

$$Z_N = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} V_{\sigma_1;\sigma_2} V_{\sigma_2;\sigma_3} \cdots V_{\sigma_{N-1};\sigma_N} V_{\sigma_N;\sigma_1} . \qquad \text{(II, 5)}$$

Now $$\sum_{\sigma_i = \pm 1} V_{\sigma_{i-1};\sigma_i} V_{\sigma_i;\sigma_{i+1}} = (V^2)_{\sigma_{i-1};\sigma_{i+1}}, \qquad \text{(II, 6)}$$

and summation over all but the variable σ_1 leads, as the spur is invariant under similarity transformations, to

$$Z_N = \sum_{\sigma_1 = \pm 1} (V^N)_{\sigma_1;\sigma_1} = \mathrm{Sp}\, V^N = \lambda_1^N + \lambda_2^N , \qquad \text{(II, 7)}$$

where λ_1 and λ_2 are the eigenvalues of V. As these are 2 ch J/kT and 2 sh J/kT we obtain

$$Z_N = (2 \text{ ch } J/kT)^N + (2 \text{ sh } J/kT)^N . \qquad \text{(II, 8)}$$

We have thus arrived at the solution in an extremely simple way. Note that

the numbering of the indices σ_i, that occur as the second index and the first index of two neighbouring matrices, was done by counting off the elements of the lattice, thus acting as what we shall call a "counting chain".

The connection with the thermodynamics of the system is established via the free energy

$$F_N = -kT \log Z_N . \qquad (II, 9)$$

One is mainly interested in the quantity

$$f = \lim_{N\to\infty} f_N = \lim_{N\to\infty} \frac{1}{N} \log Z_N , \qquad (II, 10)$$

which is the free energy per element divided by -kT for an infinite lattice. Introducing

$$H = J/kT , \qquad (II, 11)$$

we finally obtain for the problem A :

$$\begin{aligned} f &= \lim_{N\to\infty} \frac{1}{N} \log (2\ \mathrm{ch}H)^N \left\{1 + (\mathrm{th}H)^N\right\} = \\ &= \log 2\ \mathrm{ch}H + \lim_{N\to\infty} \frac{1}{N} \log \left\{1 + (\mathrm{th}H)^N\right\} = \\ &= \log 2\ \mathrm{ch}H . \end{aligned} \qquad (II, 12)$$

It is easy to see that the specific heat per element shows no singularity.

§2. CHOICE OF THE TWO DIMENSIONAL LATTICE. COUNTING CHAIN. A LEMMA

We now try to give a formulation of the problem B_1. Thinking of the lattice as dissected in single lattice points, in order to write the partition function as the spur of a matrix, the factors in the terms of the partition function are considered as being elements of square matrices of order 2, as was done in problem A_1. Again the spin variables σ are to take care of the numbering of rows and columns of these matrices. A lattice point shall be characterized by the pair of indices (i, j), indicating that it is the element in the i-th row and the j-th column of the lattice.

The spin variable of the element (i, j) is written as $\sigma_{i,j}$. The lattice consists of N elements arranged in m strings (columns) of ℓ elements. The interaction between neighbouring elements in different strings will be characterized, as in (II, 1), by J and for neighbouring elements on a string this will be done by J'. As in (II, 11) we introduce

$$H = J/kT \qquad \text{and} \qquad H' = J'/kT \qquad (II, 13)$$

planning a formulation in terms of the matrices

$$V_1 = \begin{pmatrix} e^{H} & e^{-H} \\ e^{-H} & e^{H} \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} e^{H'} & e^{-H'} \\ e^{-H'} & e^{H'} \end{pmatrix} , \qquad (II, 14)$$

the analoga of the matrix V in (II, 4).

Taking into account that only nearest neighbours interact, the factors that appear in the terms of the partition function can be written as either

$$(V_1)_{\sigma_{i,j};\sigma_{i,j+1}} \qquad \text{or} \qquad (V_2)_{\sigma_{i,j};\sigma_{i+1,j}}.$$

When we want to do something similar to what we have done in problem A_1 the following is necessary. The indices $\sigma_{i,j}$ have to be second index and first index of neighbouring matrices in order to carry out a summation as in (II, 6). The sequence of the matrices is to be given by a counting chain of elements, corresponding with a walk along the bonds in such a way that every bond is passed once and ending at the same lattice point the walk started from, in order to end the summing over indices with the calculation of a spur. We consider the model to include bonds between the first and the last element of a string as well as bonds between corresponding elements of the first and the last string. The lattice can then be considered as being wrapped on a torus. There are 2N bonds between the N elements of the lattice and every element has to act in the counting chain twice. It is readily verified that, whatever the walk along the bonds, every element is to give its name twice to a pair of indices that are second and first index of consecutive matrix elements. Thus every index $\sigma_{i,j}$ occurs 4 times instead of twice, as in problem A_1.

The lattice is chosen as follows. Each of the m strings contains m-1 elements, i.e. we chose $\ell = m-1$. This restriction is irrelevant as we shall finally be interested in a lattice that is infinite in both directions*. The

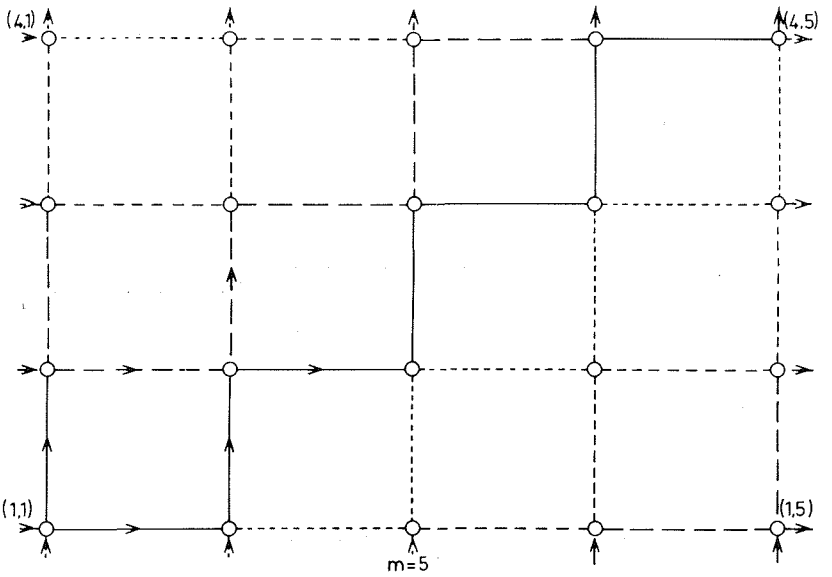


figure 1

A counting chain of bonds respectively lattice points for m = 5. On passing the elements (2,1), (3,1) and (4,1) for the first time the dotting of the bonds has been changed

* Other choices are possible and will be discussed in chapter III. If we want the model to exhibit a second order phase transition, allowing the lattice to become infinite in both directions is necessary. See e.g. ref. [14], p. 170.

counting chain is indicated in fig. 1 (see page 11) for a lattice consisting of 5 strings of 4 elements. The counting chain starts and ends at element (1, 1), making consecutively "length steps" and "breadth steps". Length step plus breadth step form what is called an "elementary counting pattern".

The occurrence of twice a pair of indices $\sigma_{i,j}$ makes it impossible to sum over this index immediately. We need the following lemma.

Lemma I: If A, B, C and D are arbitrary matrices of order 2 and I is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then for arbitrary indices a, b, c and d

$$\sum_{j=1}^{2} A_{aj}B_{jb}C_{cj}D_{jd} = \tfrac{1}{2}\{(AB)_{ab}(CD)_{cd} + (AIB)_{ab}(CID)_{cd}\}. \qquad \text{(II, 15a)}$$

Proof:

$$(AB)_{ab} = A_{a1}B_{1b} + A_{a2}B_{2b} \quad (CD)_{cd} = C_{c1}D_{1d} + C_{c2}D_{2d}$$

$$(AIB)_{ab} = A_{a1}B_{1b} - A_{a2}B_{2b} \quad (CID)_{cd} = C_{c1}D_{1d} - C_{c2}D_{2d}$$

and the proof follows simply from writing out the right hand side of (II, 15a). We shall further need

Corollary I :

$$\sum_{j=1}^{2} A_{aj}B_{jj}C_{jc} = \tfrac{1}{2}\{(ABC)_{ac} + (AIBIC)_{ac}\}, \qquad \text{(II, 15b)}$$

following also directly from the fact that B + IBI is diagonal, and

Corollary II:

$$\sum_{j=1}^{2} A_{jj}B_{jj} = \tfrac{1}{2}\{\mathrm{Sp}\ AB + \mathrm{Sp}\ AIBI\}. \qquad \text{(II, 15c)}$$

§3. THE PARTITION FUNCTION FOR THE QUADRATIC LATTICE

The partition function Z_N for the two-dimensional model that we have specified is

$$Z_N = \sum_{\{\sigma_{i,j}\}} \exp\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m}(H\sigma_{i,j}\sigma_{i,j+1} + H'\sigma_{i,j}\sigma_{i+1,j})\right], \qquad \text{(II, 16)}$$

where $\{\sigma_{i,j}\}$ below the summation sign indicates that we have to sum over all 2^N combinations of the $\sigma_{i,j}$, that can be +1 or -1. We have indicated how the bonds are counted off and we can write

$$Z_N = \sum_{\{\sigma_{i,j}\}} (V_1)_{\sigma_{1,1};\sigma_{1,2}} (V_2)_{\sigma_{1,2};\sigma_{2,2}} (V_1)_{\sigma_{2,2};\sigma_{2,3}} \cdots$$

$$(V_2)_{\sigma_{m-1,m};\sigma_{1,m}} (V_1)_{\sigma_{1,m};\sigma_{1,1}} (V_2)_{\sigma_{1,1};\sigma_{2,1}} \cdots$$

$$(V_1)_{\sigma_{m-1,m};\sigma_{m-1,1}} (V_2)_{\sigma_{m-1,1};\sigma_{1,1}} . \qquad \text{(II, 17)}$$

The choice of the element (1, 1) as starting point (and ending point) is of course irrelevant, as the lattice is translation invariant in both directions. We now sum over all indices $\sigma_{i,j}$ except $\sigma_{1,1}$, using the lemma (II, 15a) or its corollary (II, 15b). To see clearly what happens let us start by summing over an index $\sigma_{k,l}$. At first sight two cases can be distinguished. In the first case we reach the element for the first time along the counting chain, on starting from element (1, 1), by a length step and application of the lemma separates Z_N into two partial sums:

$$Z_N = \tfrac{1}{2} \sum_{\{\sigma_{i,j}\ ;\ i,j \neq k,l\}} (V_1)_{\sigma_{1,1};\sigma_{1,2}} \cdots (V_1 V_2)_{\sigma_{k,l-1};\sigma_{k+1,l}} \cdots$$

$$\cdots (V_2 V_1)_{\sigma_{k-1,l};\sigma_{k,l+1}} \cdots (V_2)_{\sigma_{m-1,1};\sigma_{1,1}} +$$

$$+ \tfrac{1}{2} \sum_{\{\sigma_{i,j}\ ;\ i,j \neq k,l\}} (V_1)_{\sigma_{1,1};\sigma_{1,2}} \cdots (V_1 I V_2)_{\sigma_{k,l-1};\sigma_{k+1,l}} \cdots$$

$$\cdots (V_2 I V_1)_{\sigma_{k-1,l};\sigma_{k,l+1}} \cdots (V_2)_{\sigma_{m-1,1};\sigma_{1,1}} . \qquad \text{(II, 18a)}$$

In the second case we reach the element by a breadth step. This leads to:

$$Z_N = \tfrac{1}{2} \sum_{\{\sigma_{i,j}\ ;\ i,j \neq k,l\}} (V_1)_{\sigma_{1,1};\sigma_{1,2}} \cdots (V_2 V_1)_{\sigma_{k-1,l};\sigma_{k,l+1}} \cdots$$

$$\cdots (V_1 V_2)_{\sigma_{k,l-1};\sigma_{k+1,l}} \cdots (V_2)_{\sigma_{m-1,};\sigma_{1,1}} +$$

$$+ \tfrac{1}{2} \sum_{\{\sigma_{i,j}\ ;\ i,j \neq k,l\}} (V_1)_{\sigma_{1,1};\sigma_{1,2}} \cdots (V_2 I V_1)_{\sigma_{k-1,l};\sigma_{k,l+1}} \cdots$$

$$\cdots (V_1 I V_2)_{\sigma_{k,l-1};\sigma_{k+1,l}} \cdots (V_2)_{\sigma_{m-1,1};\sigma_{1,1}} . \qquad \text{(II, 18b)}$$

The summation over a second index splits each of the partial sums again in two partial sums and an extra factor $\frac{1}{2}$ appears in front of the summation signs. After summing over all indices except $\sigma_{1,1}$ the partition function consists of 2^{N-1} partial sums of the form

$$2^{-N+1} \sum_{\{\sigma_{1,1} = \pm 1\}} (A)_{\sigma_{1,1};\sigma_{1,1}} (B)_{\sigma_{1,1};\sigma_{1,1}} . \qquad \text{(II, 19)}$$

Let us discuss the matrices A and B. It is clear that these matrices are products of a number of matrices V_1 and V_2 that alternate, as they correspond to alternating length steps and breadth steps of the counting chain

casu quo bonds between neighbouring elements of different strings or of the same string. Between these matrices V_1 and V_2 a certain number of matrices I occur, in a certain way. Irrespective of the factors I in every partial sum, there are m matrices V_1 and m-1 matrices V_2 in A and consequently N-m matrices V_1 and N-m+1 matrices V_2 in B. This follows from inspection of the counting chain. The distance between element (1, 1) occurring for the first time and the same element occurring for the second time in the counting chain is 2m-1 when measured in "links" of the chain. This is also the case for other elements when the distance is to be the shortest distance along the closed counting chain. For this reason (II, 18a) and (II, 18b) are essentially the same. In (II, 18a) the distance between the two factors I is 2N-2m+1 links, when following the counting chain, but 2m-1 links, as in (II, 18b) when the shortest* distance is to be considered, thus "sandwiching" m factors V_1 and m-1 factors V_2.

Knowing this, we may now apply (II, 15c) to (II, 19) which leads to a partition function consisting of 2^N terms of the form

$$2^{-N}\ \mathrm{Sp}\ (V_1 V_2 \ldots V_1 V_2)\,, \tag{II, 20}$$

where in every term the matrix product contains a combination of pairs of factors I on a distance 2m-1. It is easily seen that the summation process is such that every combination of the 2^N combinations of pairs of factors I occurs just once. Thus we have succeeded in reducing the calculation of Z_N to the calculation of spurs of matrices of order 2. However, the price for getting an analogous situation to the one expressed in (II, 7) is that there are now 2^N spurs whose calculation might be difficult and that have to be summed subsequently.

The calculation of the spurs turns out not to be the crucial point. We start by considering some commutators and anticommutators. The matrices V_1 and V_2 commute, i.e.

$$[V_1, V_2]_- = V_1 V_2 - V_2 V_1 = 0\,. \tag{II, 21}$$

This feature, due to the fact that the elements on the diagonal and on the counter diagonal are equal, is not shared by the combinations V_1 and I or V_2 and I. However, in these cases we consider the anticommutator

$$[V, I]_+ = VI + IV = 2e^H I \tag{II, 22}$$

with the consequence that

$$VI = I(2e^H E - V) = IV'\,, \tag{II, 23}$$

where E is the unit matrix and

$$V' = \begin{pmatrix} e^{H} & -e^{-H} \\ -e^{-H} & e^{H} \end{pmatrix}. \tag{II, 24}$$

Similar results hold for V_1 and V_2. After commutation with the matrix I these matrices are "infected", i.e. the elements on the counter diagonal

* There is, of course, the trivial restriction $m \geq 5$.

get a minus sign. Also in V_1' and V_2', however, the elements are equal on both diagonals and thus

$$[V_1, V_1']_- = [V_1, V_2']_- = [V_1', V_2]_- = [V_1', V_2']_- = 0 . \qquad \text{(II, 25)}$$

Now consider a pair of matrices I. By commutation with the 2m-1 matrices that separate them they can be brought together and then "annihilate" each other as $I^2 = E$. The 2m-1 matrices V_1 and V_2 are infected. Other matrices I simply commute with these particular matrices I. Carrying out this annihilation process, in order to get rid of the matrices I, leaves us with the calculation of spurs of commuting matrices V_1, V_2, V_1' and V_2'. Suppose in a special case the numbers of these matrices are n_1, n_2, n_1', and n_2' respectively. We then have to calculate expressions of the form

$$2^{-N} \mathrm{Sp}\ V_1^{n_1} V_2^{n_2} V_1'^{n_1'} V_2'^{n_2'} . \qquad \text{(II, 26)}$$

The matrix V_1 has normed eigenvectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, that are simultaneously eigenvectors of V_2, V_1' and V_2'. A similarity transformation with

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{(II, 27)}$$

leads to

$$2^{-N} \mathrm{Sp}\ S^{-1} V_1^{n_1} V_2^{n_2} V_1'^{n_1'} V_2'^{n_2'} S =$$

$$= 2^{-N} \mathrm{Sp} \begin{pmatrix} 2\mathrm{ch}H & 0 \\ 0 & 2\mathrm{sh}H \end{pmatrix}^{n_1} \begin{pmatrix} 2\mathrm{ch}H' & 0 \\ 0 & 2\mathrm{sh}H' \end{pmatrix}^{n_2} \begin{pmatrix} 2\mathrm{sh}H & 0 \\ 0 & 2\mathrm{ch}H \end{pmatrix}^{n_1'} \begin{pmatrix} 2\mathrm{sh}H' & 0 \\ 0 & 2\mathrm{ch}H' \end{pmatrix}^{n_2'} =$$

$$= \frac{2^{n_1+n_2+n_1'+n_2'}}{2^N} \left\{ (\mathrm{ch}H)^{n_1} (\mathrm{ch}H')^{n_2} (\mathrm{sh}H)^{n_1'} (\mathrm{sh}H')^{n_2'} + (\mathrm{sh}H)^{n_1} (\mathrm{sh}H')^{n_2} (\mathrm{ch}H)^{n_1'} (\mathrm{ch}H')^{n_2'} \right\} =$$

$$= (2\ \mathrm{ch}H\ \mathrm{ch}H')^N \left\{ (\mathrm{th}H)^{n_1'} (\mathrm{th}H')^{n_2'} + (\mathrm{th}H)^{N-n_1'} (\mathrm{th}H')^{N-n_2'} \right\}, \qquad \text{(II, 28)}$$

where use has been made of the fact that $n_1 + n_1' = n_2 + n_2' = N$.

Introducing

$$w = \mathrm{th}H \qquad \text{and} \qquad w' = \mathrm{th}H' , \qquad \text{(II, 29)}$$

the partition function can be written as

$$Z_N = (2\ \mathrm{ch}H\ \mathrm{ch}H')^N \sum \left\{ w^{n_1'} w'^{n_2'} + w^{N-n_1'} w'^{N-n_2'} \right\} =$$

$$= (2\ \mathrm{ch}H\ \mathrm{ch}H')^N\ L_N(w, w') , \qquad \text{(II, 30)}$$

where the summation is over all possible combinations of n_1' and n_2', that are the numbers of infected matrices V_1 and V_2 respectively. The relevant quantity f now is

$$f = \log 2 \; \mathrm{ch}H \; \mathrm{ch}H' + \lim_{N\to\infty} \frac{1}{N} \log L_N(w,w') \quad , \qquad \text{(II, 31)}$$

where of course $N\to\infty$ along with $m\to\infty$. This formula should be compared with (II, 12), that, when putting $1+w^N = L_N(w)$, can be written in the form

$$f = \log 2 \; \mathrm{ch}H + \lim_{N\to\infty} \frac{1}{N} \log L_N(w) \quad . \qquad \text{(II, 12)}$$

After calculation of the spurs the crucial point appears to be the summation of these 2^N spurs. $L_N(w,w')$ is a polynomial in w and w'. The coefficient of an expression like the one between the braces in (II, 30) is the number of ways the combination of n_1' infections of matrices V_1 and n_2' infections of matrices V_2 can be established by annihilation of pairs of matrices I. Thus we have to solve the combinatorial problem of finding a function $L_N(w,w')$, that is a generating function, for the combinations just mentioned, in the sense of combinatorial analysis. Let us consider the combinatorial aspect in a little bit more detailed way. Essentially we have the following situation. A closed chain consists of 2^N alternating elements a and b. The links between these elements can be infected and disinfected in groups of 2m-1 links, that form parts of the chain in such a way that, going around clockwise say (see fig. 2), the first element involved is an element a and the last one an element b. Schematically the infection by one group is indicated in figure 2.

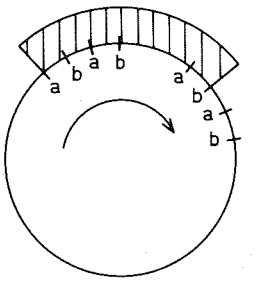


figure 2

Infection of a group of 2m-1 links on a chain of 2N links

The infection of a second group may infect non-infected links or disinfect infected links, as the annihilation of a corresponding second pair of matrices I may cause a second minus sign on the counter diagonal in certain matrices that were infected in the process of annihilation of the first pair of matrices I. Each combination of groups will be said to give rise to an "infection pattern" on the counting chain. When all combinations of groups, with the restriction mentioned, can occur we have to find out the number $A(n_1', n_2')$ of ways to obtain n_1' infected links a→b and n_2' infected links b→a in order to obtain the series development

$$L_N(w,w') = \sum A(n_1', n_2') \left\{ w^{n_1' w'^{n_2'}} + w^{N-n_1' w'^{N-n_2'}} \right\} \quad . \qquad \text{(II, 32)}$$

At first sight this is rather disappointing. There is a much simpler way to arrive at a formulation as a combinatorial problem also starting of course from the expression (II, 16) but manipulating with it in another way. A few things have been achieved, however. *Firstly, we have arrived at a combinatorial problem, in a quite natural way, starting from the matrix method for problem* A_7. *Secondly, the combinatorial problem is one on the counting chain, i.e. on a linear arrangement and, what is more important, the generating principle for the formation of the infection patterns comes forward.*

We want to remark the following. It is quite important for the simplicity of our formulation to choose the counting chain in the way we have done. Let us illustrate this. The elementary counting pattern consisted of one of each of the occurring types of bonds, namely a length bond and a breadth bond successively. For using on repetition this counting pattern, the lattice had to be chosen in the special way indicated. As a result all elements on the chain that correspond to it are on a distance of 2m-1 links when the lattice is chosen as a $m \times (m-1)$ lattice. Suppose we have a $m \times n$ lattice, then a rather regular counting chain is the following. We start with element (1, 1) and count off the bonds in the first row up to the bond between (1, m) and (1, 1), then count the bond between (1, 1) and (2, 1) and count off the bonds of the second row up to the bond between (2, m) and (2, 1), then count the bond between (2, 1) and (3, 1), etc. After counting off all bonds in the various rows we arrive at (n, 1) and after counting the length bond between (n, 1) and (n, 2) we count off the bonds of the second column up to the one between (1, 2) and (n, 2), then count the bond between (n, 2) and (n, 3), etc. After counting off all bonds in the various columns we arrive at (n, m) for the second time and by counting off the bond between (n, m) and (n, 1) and between (n, 1) and (1, 1) we close the counting chain. Now, with this counting chain let us apply the lemma needed for the various summations and let us see what kind of pairs of factors I occur on this counting chain of matrices V_1 and V_2 that, by the way, do not alternate any more now. The elements in the first column, except element (n, 1), have a corresponding pair of elements on a distance of m links. The elements of the n-th row, except element (n-1), have corresponding elements on a distance of n links. Element (1, 2) e.g. has corresponding elements on a distance of m.n-m+2n-2 links, element (2, 2) has corresponding elements on a distance of m.n-2m+2n-4 links etc. A chaos of types of groups of infections comes out instead of precisely one type of group in the case of the counting chain we indicated. Thus, in a way, the counting chain we have given is an optimal choice.

§4. THE CONNECTION WITH THE COMBINATORIAL METHOD

We shall now outline how the usual formulation as a combinatorial problem can be set in correspondence with the combinatorial problem derived in the former paragraph.

The usual way of arriving at a combinatorial problem is by writing the partition function (II, 16) as

$$Z_N = \sum_{\{\sigma_{i,j}\}} \text{n.n.} \prod_{\substack{i=1,\cdots m-1 \\ j=1,\cdots m}} \exp\left[H\sigma_{i,j}\,\sigma_{i,j+1} \right] \cdot \exp\left[H'\sigma_{i,j}\,\sigma_{i+1,j} \right] =$$

$$= \sum_{\{\sigma_{i,j}\}} \prod_{\substack{i=1,\ldots,m-1 \\ j=1,\ldots,m}}^{n.n.} (chH+\sigma_{i,j}\,\sigma_{i,j+1}\,shH).(chH'+\sigma_{i,j}\,\sigma_{i+1,j}\,shH') =$$

$$= (chH\ chH')^N \sum_{\{\sigma_{i,j}\}} \prod_{\substack{i=1,\ldots,m-1 \\ j=1,\ldots,m}}^{n.n.} (1+\sigma_{i,j}\,\sigma_{i,j+1}\,thH)(1+\sigma_{i,j}\,\sigma_{i+1,j}\,thH')$$

$$= (chH\ chH')^N \sum_{\{\sigma_{i,j}\}} \prod_{\substack{i=1,\ldots m-1 \\ j=1,\ldots m}}^{n.n.} (1+\sigma_{i,j}\ \sigma_{i,j+1}\,w)(1+\sigma_{i,j}\ \sigma_{i+1,j}\ w'). \qquad (II,33)$$

The product sign $\prod^{n.n.}$ indicates that for every bond between nearest neighbours a factor is present. Writing out the product produces 2^{2N} terms that contain certain numbers of factors $\sigma_{i,j}$.

When a spin variable occurs 0, 2 or 4 times the summation over this variable gives a factor 2. When it occurs 1 or 3 times the result is 0*. Thus summing over all variables $\sigma_{i,j}$ either gives a factor 2^N or a factor 0. Only those terms survive that contain variables that occur an even number of times. Thus we are again able to write

$$Z_N = (2\ chH\ chH')^N L_N(w,w'). \qquad (II,30)$$

The polynomial $L_N(w,w')$ is of course the same as in (II,32), which brings clearly out the feature that only 2^{N+1} terms survive after summation in (II,33). In order to see in what way it is the generating function of a combinatorial problem we write the polynomial as

$$L_N(w,w') = \sum B(n_1',n_2')\ w^{n_1'} w'^{n_2'}. \qquad (II,34)$$

As every element (i,j) takes part in the formation of 0, 2 or 4 bonds, depending on the number of times the corresponding variable $\sigma_{i,j}$ occurs in a specific term of the partition function, with every term a closed "polygon" that is not necessarily simply-connected corresponds on the lattice. The coefficients $B(n_1',n_2')$ give the numbers of closed polygons, consisting of n_1' bonds in one direction and n_2' bonds in the other, that can be formed on the lattice. The combinatorial problem is one of counting closed polygons instead of counting infection patterns. A comparison of (II,32) and (II,34) shows that $A(n_1',n_2')$ and $B(n_1',n_2')$ do not have entirely the same function. $A(n_1',n_2')$ plays the role of $B(n_1',n_2')$ as well as of $B(N-n_1',N-n_2')$. Here we should note that this approach is more general with respect to the fact that it is not necessary to choose the lattice in a special way.

It is rather instructive to see how the combination of groups of infected links on the counting chain corresponds to the building up of closed polygons on the lattice. To an infected link we set in correspondence a bond of a graph. We shall consider some simple graphs. The first term in (II,32) we consider is the one corresponding to the case that there are no infections

* If g denotes the other factors in a term:

$$\sum_{\sigma_{i,j}=\pm1} g\,\sigma_{i,j} = \sum_{\sigma_{i,j}=\pm1} g\,\sigma_{i,j}^3 = 0 \quad \text{and} \quad \sum_{\sigma_{i,j}=\pm1} g\,\sigma_{i,j}^0 = \sum_{\sigma_{i,j}=\pm1} g\,\sigma_{i,j}^2 = \sum_{\sigma_{i,j}=\pm1} g\,\sigma_{i,j}^4 = 2g.$$

on the counting chain. This occurs only once corresponding to the one possibility that there is no graph. Thus $A(0,0) = B(0,0) = B(N,N) = 1$. Note that there are two graphs corresponding to a certain infection pattern as in (II, 32) terms are occurring in pairs. These are the graph corresponding to the infected links and the "complementary" graph consisting of bonds that correspond to non-infected links. We shall restrict ourselves now to the graphs corresponding to the infected links.

Let one group of infections be present. The situation shown in figure 3 is possible, where with a and b those elements are indicated where a length or a breadth step on the lattice begins in the corresponding lattice point. Numbers are inserted to facilitate comparison.

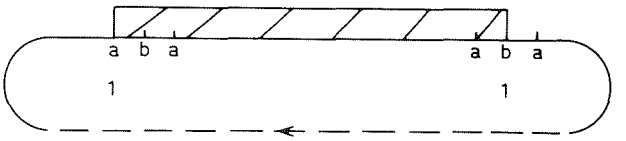


figure 3

2m-1 infections

On the lattice this corresponds to the graph indicated in figure 4, i.e. to a graph "looping" the torus.

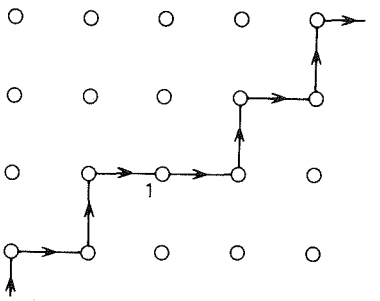


figure 4

Not possible is the situation shown in figure 5.

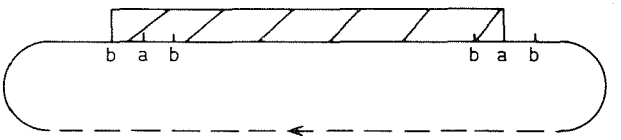


figure 5

2m-1 infections

Combination of two groups of infections causes at least 4 infections as is shown in figure 6.

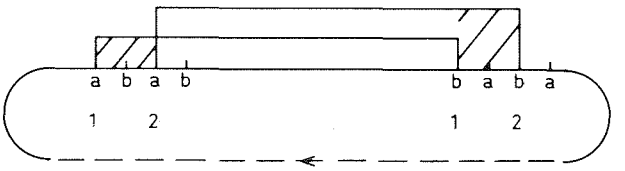


figure 6

2m-3 disinfections

The corresponding graph on the lattice is indicated in figure 7.

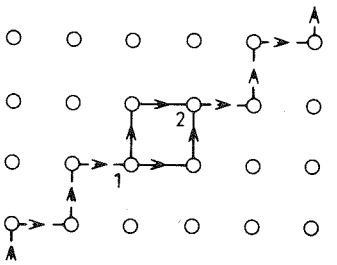


figure 7

In figure 7 the thick-drawn bonds correspond to infected links and the dotted bonds to disinfected links (respectively matrices). This graph does not loop the torus but inclusion, in any way, of a third group of infections causes a graph that does. It is of course possible that with two groups of infections no disinfections occur and the corresponding graph then consists of two loops of the type of figure 4. Non-looping graphs with a small number of bonds can be obtained in correspondence with the combination of two groups of infections e. g. in the way shown in figure 8.

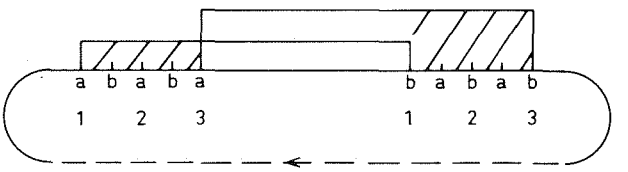


figure 8

2m-5 disinfections

The corresponding graph is indicated in figure 9 on page 21. In figure 6 the infections can be considered to form a barbell. In figure 8 there is a barbell too, but with 4 infections at each end. Now in the spur that corresponds to this particular configuration we can insert the matrix E in the form I^2 in the places in the counting chain of matrices that correspond to the element

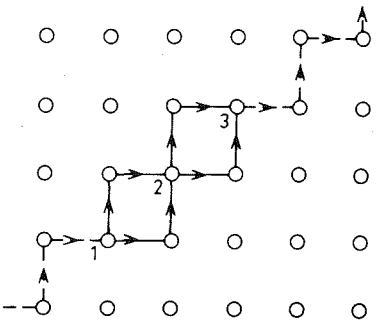


figure 9

indicated by 2 in the figures 8 and 9. Doing this means imposing two groups on the chain both spanning the links between the elements indicated by 2. Thus the barbell in figure 8 can be seen as consisting of two barbells of the type of figure 6. Infections may also correspond to a graph consisting of say 2 squares that have a side in common that is disinfected, thus giving rise to a graph consisting of 6 bonds. This can be done by considering 4 groups of infections forming two barbells of the type of figure 6 in the way shown in figure 10.

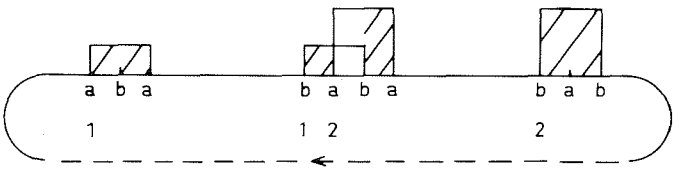


figure 10

The corresponding graph is indicated in figure 11.

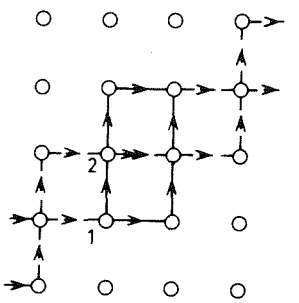
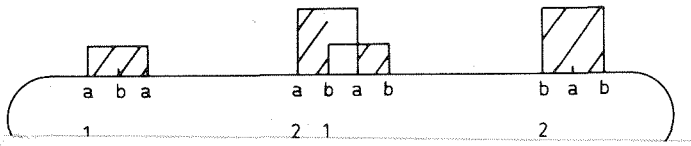


figure 11

The bond corresponding to the disinfected link belonging to both barbells is indicated by two arrow points

are counted off m breadth bonds and m + 1 depth bonds. Thus we may choose p = m + 1, but do not come back to the element in the first layer we started

Another way is shown in figure 12,



from because we counted off m breadth bonds whereas the breadth is only m - 1 elements. More precisely: when α, β and γ are the indices of the lattice point (α, β, γ), indicating the location of the element ($1 \leq \alpha \leq m$, $1 \leq \beta \leq m-1$, $1 \leq \gamma \leq m+1$), and $(\alpha, \beta, 1)$ is the starting point in the first layer of our counting procedure, then the next element in the first layer that will be counted off is element $(\alpha, \beta + 1, 1)$. After passing element $(\alpha + 1, \beta + 1, 1)$ and $(\alpha + 1, \beta + 2, 1)$ the counting chain goes to the second layer and from the $m+1$-th layer arrives in the first layer at element $(\alpha + 1, \beta + 3, 1)$ etc. In the first layer the following pattern appears (similar patterns appear in the other layers).

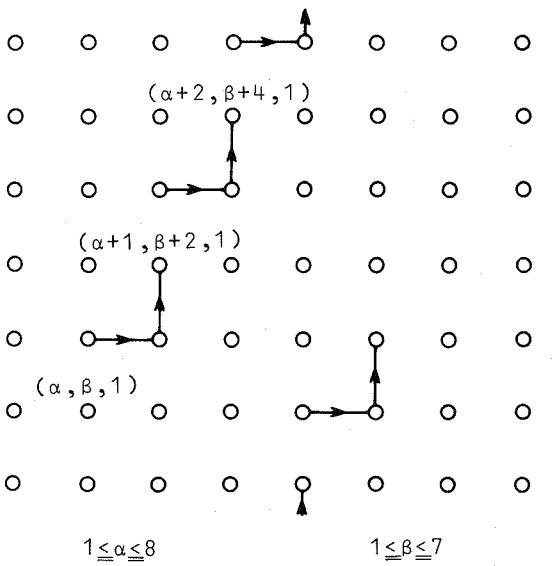


figure 14

The counting pattern is not identical with the one in problem B_1 but the question is whether with this counting off of bonds in the first (and any other) layer we can pass all bonds just once and finally come back to element $(\alpha, \beta\ \ 1)$ by counting off the bond between $(\alpha, \beta - 1, 1)$ and $(\alpha, \beta, 1)$.

We notice that the counting chain goes to the next layer at elements with second index β, $\beta + 2$, $\beta + 4$, etc. Thus when $\beta_{max} = m - 1$ is even, only breadth bonds will be counted having as "second" element an element with an even number as second index. In this way not all bonds can be counted off (only half of them). However, when $\beta_{max} = m - 1$ is odd, for e.g. even β for the starting point $(\alpha, \beta, 1)$ (see figure 14) the element where the counting chain leaves the first layer after using the boundary conditions in the breadth has an odd second index, which becomes even again after repeated use of the (breadth) boundary condition etc. It is readily verified that in this way all bonds in the first layer are counted off just once and automatically this is the case with all other bonds as each element of the first layer acts once as a starting point for a tour into the depth*.

* As there are $m(m-1)$ elements in the first layer and on every tour $3m+1$ bonds are passed, next to the $2m(m-1)$ bonds in the first layer, the total number of bonds counted is $2m(m-1)+m(m-1)(3m+1)=3m(m-1)(m+1)$

Thus we have found a closed counting chain of bonds respectively elements, that is completely analogous to the counting chain for the problem B_1, by imposing on the lattice the restrictions that it consists of m x (m-1) x (m+1) elements with m even. These restrictions are irrelevant when we are interested in infinite systems i.e. let $m \to \infty$.

Our next problem is to cope with the summations over indices that will appear in the formulation of the partition function in a way analogous to the way they appeared in problem B_1. There the problem was that a certain index appeared twice as second and first index of subsequent matrices of order 2. Here an index will appear three times in such a way and we shall need the following lemma.

Lemma II: Let A, B, C, D, E and F be arbitrary matrices of order 2 and let I be the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then for arbitrary indices a, b, ... , f

$$\sum_{j=1}^{2} A_{aj} B_{jb} C_{cj} D_{jd} E_{ej} F_{jf} = \frac{1}{4} \Big\{ (AB)_{ab} (CD)_{cd} (EF)_{ef} +$$

$$(AIB)_{ab} (CID)_{cd} (EF)_{ef} + (AIB)_{ab} (CD)_{cd} (EIF)_{ef} + (AB)_{ab} (CID)_{cd} (EIF)_{ef} \Big\}. \quad \text{(III, 1)}$$

Proof: By writing out, analogous to the proof of the corresponding lemma in paper I.

$$\frac{1}{4} \Big\{ (A_{a1}B_{1b} + A_{a2}B_{2b})(C_{c1}D_{1d} + C_{c2}D_{2d})(E_{e1}F_{1f} + E_{e2}F_{2f}) +$$
$$(A_{a1}B_{1b} - A_{a2}B_{2b})(C_{c1}D_{1d} - C_{c2}D_{2d})(E_{e1}F_{1f} + E_{e2}F_{2f}) +$$
$$(A_{a1}B_{1b} - A_{a2}B_{2b})(C_{c1}D_{1d} + C_{c2}D_{2d})(E_{e1}F_{1f} - E_{e2}F_{2f}) +$$
$$(A_{a1}B_{1b} + A_{a2}B_{2b})(C_{c1}D_{1d} - C_{c2}D_{2d})(E_{e1}F_{1f} - E_{e2}F_{2f}) \Big\} =$$
$$= A_{a1}B_{1b}C_{c1}D_{1d}E_{e1}F_{1f} + A_{a2}B_{2b}C_{c2}D_{2d}E_{e2}F_{2f}.$$

The following corollaries come into play as well:

Corollary I :

$$\sum_{j=1}^{2} A_{aj} (BC)_{jj} D_{jd} E_{ej} F_{jf} = \frac{1}{4} \Big\{ (ABCD)_{ad} (EF)_{ef} +$$
$$+ (AIBCID)_{ad} (EF)_{ef} + (AIBCD)_{ad} (EIF)_{ef} + (ABCID)_{ad} (EIF)_{ef} \Big\}. \quad \text{(III, 2a)}$$

Corollary II : An analogous expression for $\sum_{j=1}^{2} A_{aj} B_{jb} C_{cj} (DE)_{jj} F_{jf}$.

Corollary III :

$$\sum_{j=1}^{2} A_{aj} (BC)_{jj} (DE)_{jj} F_{jf} = \frac{1}{4} \Big\{ (ABCDEF)_{af} +$$
$$+ (AIBCIDEF)_{af} + (AIBCDEIF)_{af} + (ABCIDEIF)_{af} \Big\}, \quad \text{(III, 2b)}$$

which can also be verified by noting that (BC + IBCI) as well as (DE + IDEI) are diagonal and that the right hand side reads

$$\left\{ A.(BC + IBCI).(DE + IDEI).F \right\}_{af} .$$

§2. THE PARTITION FUNCTION FOR THE SIMPLE CUBIC LATTICE

We now start the calculation of the partition function, along the same lines as in problem B_1, for the lattice discussed in the former paragraph.

Each element (i, j, k) possesses a spin variable $\sigma_{i,j,k}$ that again can assume the values +1 and -1. The interaction energy $E(\sigma_{i,j,k}, \sigma_{p,q,r})$ * is defined as follows. Analogous to (II, 1) we have

$$\begin{aligned} &E(+1,+1) = E(-1,-1) = \text{resp. } -J,-J',-J'' \\ &E(+1,-1) = E(-1,+1) = \text{resp. } +J,+J',+J'' \\ &J,J' \text{ and } J'' > 0 , \end{aligned} \tag{III, 3}$$

where J, J' and J'' stand for the values of the interaction energies of nearest neighbours in the three directions, length, breadth and depth in this order. When now

$$H = J/kT \ , \quad H' = J'/kT \ , \quad H'' = J''/kT \ , \tag{III, 4}$$

the partition function can be written as

$$Z_N = \sum_{\{\sigma_{i,j,k}\}} \exp\left[\sum_{i=1}^{m} \sum_{j=1}^{m-1} \sum_{k=1}^{m+1} (H\sigma_{i,j,k}\sigma_{i+1,j,k} + H'\sigma_{i,j,k}\sigma_{i,j+1,k} + H''\sigma_{i,j,k}\sigma_{i,j,k+1}) \right] \tag{III, 5}$$

where $\{\sigma_{i,j,k}\}$ below the summation sign indicates that we have to sum over all 2^N combinations of the $\sigma_{i,j,k}$. We define

$$V_1 = \begin{pmatrix} e^{H} & e^{-H} \\ e^{-H} & e^{H} \end{pmatrix} , V_2 = \begin{pmatrix} e^{H'} & e^{-H'} \\ e^{-H'} & e^{H'} \end{pmatrix} , V_3 = \begin{pmatrix} e^{H''} & e^{-H''} \\ e^{-H''} & e^{H''} \end{pmatrix} \tag{III, 6}$$

similar to the matrices introduced in chapter II. The spin variables act as the indices of these matrices. Using the counting chain we have indicated, starting with a length bond from element (1, 1, 1), we can write

$$\begin{aligned} Z_N = \sum_{\{\sigma_{i,j,k}\}} & (V_1)_{\sigma_{1,1,1};\sigma_{2,1,1}} (V_2)_{\sigma_{2,1;1};\sigma_{2,2,1}} (V_3)_{\sigma_{2,2,1};\sigma_{2,2,2}} \ldots (V_1)_{\sigma_{m,1,1};\sigma_{1,1,1}} \\ & (V_2)_{\sigma_{1,1,1},\sigma_{1,2,1}} \ldots (V_2)_{\sigma_{1,m-1,1};\sigma_{1,1,1}} (V_3)_{\sigma_{1,1,1};\sigma_{1,1,2}} \\ & \ldots (V_3)_{\sigma_{1,1\,m+1};\sigma_{1,1,1}} . \end{aligned} \tag{III, 7}$$

* Note the cyclic boundary conditions, thus $\sigma_{a,b,m+2} = \sigma_{a,b,1}$ etc.

We remark that the choice of element (1, 1, 1) as the starting point is again irrelevant, as the lattice is translation invariant in all three directions.

We now make use of the lemma and its corollaries and sum over the values of the spin variables which is the same as a summation over matrix indices. These indices occur 6 times because each element takes part in the formations of 6 bonds and with each bond corresponds one of the factors

$$(V_1)_{\sigma_{i,j,k}\,;\sigma_{i+1,j,k}} \,, (V_2)_{\sigma_{i,j,k}\,;\sigma_{i,j+1,k}} \,, (V_3)_{\sigma_{i,j,k},\,;\sigma_{i,j,k+1}} \,. \qquad \text{(III, 8)}$$

Moreover a certain index $\sigma_{i,j,k}$ occurs 3 times as second index and first index of consecutive matrices, that stand in line according to the sequence of the bonds they correspond with on the counting chain. Thus the lemma and its corollaries can be applied and, analogous to the partition function of problem B_1, Z_N splits into 4^N terms of the form

$$4^{-N}\ \mathrm{Sp}\ V_1 V_2 V_3 V_1 \ldots V_3 V_1 V_2 V_3 \,. \qquad \text{(III, 9)}$$

Between the matrices V_1, V_2 and V_3 all kinds of combinations of pairs of factors I occur.

For each element three types of pairs of factors I occur, corresponding with the three types of pairs that occur in the expression on the right hand side of lemma II. The location of these pairs of factors I on the counting chain, seen as consisting of matrices instead of links, will be discussed in paragraph 6. The important fact is that the matrices I again occur in pairs. We note that V_1, V_2 and V_3 are commuting matrices. As was shown in chapter II,

$$V_i\, I = V_i'\, I \quad (i = 1,2,3), \qquad \text{(III, 10)}$$

where V_i' is equal to V_i but for two minus signs on the counterdiagonal. These matrices V_i' were said to be infected. After annihilation of the pairs of matrices I in each term there will be a certain number of infected matrices, say n_1' matrices V_1', n_2' matrices V_2' and n_3' matrices V_3'. These matrices commute with each other and with the non-infected matrices. When n_1, n_2 and n_3 are the numbers of these non-infected matrices respectively, for which $n_1 + n_1' = n_2 + n_2' = n_3 + n_3' = N$, a term of the partition function is of the form

$$4^{-N}\ \mathrm{Sp}\ V_1^{n_1} V_2^{n_2} V_3^{n_3} V_1^{n_1'} V_2^{n_2'} V_3^{n_3'} \,. \qquad \text{(III, 11)}$$

The calculation of this spur is as simple as the calculation of the corresponding spurs in (II, 26). Using the same similarity transformation and with

$$w = \mathrm{th}H, \quad w' = \mathrm{th}H', \quad w'' = \mathrm{th}H'', \qquad \text{(III, 12)}$$

we obtain in a strictly analogous way terms of the form

$$2^N (\mathrm{ch}H)^N (\mathrm{ch}H')^N (\mathrm{ch}H'')^N \left\{ w^{n_1'} w'^{n_2'} w''^{n_3'} + w^{N-n_1'} w'^{N-n_2'} w''^{N-n_3'} \right\} \,. \qquad \text{(III, 13)}$$

Thus the partition function can be written as

$$Z_N = (2\ chH\ chH'\ chH'')^N \sum (w^{n_1'} w'^{n_2'} w''^{n_3'} + w^{N-n_1'} w'^{N-n_2'} w''^{N-n_3'}) \equiv$$

$$\equiv (2\ chH\ chH'\ chH'')^N\ L_N(w, w', w''). \qquad \text{(III, 14)}$$

$L_N(w, w', w'')$ is a sum of pairs of terms over all 4^N possible combinations of n_1', n_2' and n_3' for the number of infected matrices V_1, V_2 and V_3 respectively. The relevant quantity f is

$$f = \lim_{N \to \infty} \frac{1}{N} \log Z_N = \log 2\ chH\ chH'\ chH'' +$$

$$+ \lim_{N \to \infty} \frac{1}{N} \log L_N(w,w',w''). \qquad \text{(III, 15)}$$

This expression should be compared with the formulae (II, 12) and (II, 31) for the problems A_1 and B_1 derived in chapter II.

The combinations of infections that can occur are determined by the combinations of pairs of matrices I, that on annihilation create groups of infections in the array of matrix factors or, equivalently, create groups of "infected" links on the counting chain. We shall consider these groups more detailed in paragraph 5.

Starting from a formulation analogous to the matrix method formulation for problem A_1 we have thus arrived at the problem to determine

$$L_N(w,w',w'') = \sum A(n_1', n_2', n_3') \left\{ w^{n_1'} w'^{n_2'} w''^{n_3'} + w^{N-n_1'} w'^{N-n_2'} w''^{N-n_3'} \right\} \qquad \text{(III, 16)}$$

as the generating function for a combinatorial problem, again on a one-dimensional arrangement, namely the counting chain.

§3. THE CONNECTION WITH THE COMBINATORIAL METHOD

In the former paragraph some effort has been made to formulate the calculation of the partition function as a combinatorial problem, the problem we denoted as C_1 in the scheme of paragraph 3 in chapter I. This combinatorial problem is different from the usual combinatorial problem C_c as this is a combinatorial problem on a three-dimensional arrangement, that can be arrived at in a much simpler way. Completely analogous to the way the formulation of problem B_c was derived in chapter II the partition function for the same lattice can be written as

$$Z_N = (2\ chH\ chH'\ chH'')^N \sum B(n_1', n_2', n_3',)\ w^{n_1'} w'^{n_2'} w''^{n_3'}, \qquad \text{(III. 17)}$$

where $B(n_1', n_2', n_3')$ is the number of closed polygons on the lattice consisting of n_1', n_2' and n_3' bonds in the three directions. Evidently again $A(n_1', n_2', n_3')$ plays the same role as $B(n_1', n_2', n_3')$ and as $B(N-n_1', N-n_2', N-n_3')$.

In chapter II it was shown how the two problems B_1 and B_c can be set in correspondence when letting an infected link correspond to a bond of a polygon

on the lattice once and letting a non-infected link correspond to a bond a second time as the terms occur in pairs in the formulation B_1. It is interesting to see how combinations of groups of infections correspond to closed polygons in the case of the three-dimensional lattice. We shall consider some of the simplest polygons. The one-one correspondence of infection patterns with closed polygons is again clear from the fact that both expressions (III, 16) and (III, 17) are exact.

We have seen that there are three types of pairs of matrices I. Each type sandwiches a number of matrices V_1, V_2 and V_3. We shall see in paragraph 5 that, when we speak of a l-step, b-step and d-step on the counting chain when the link corresponds to V_1, V_2 and V_3 respectively, each group of infections, that remain after annihilation of the pair of matrices I in question, begins and ends with the same type of step. Let the number of infections be N_1, N_2 and N_3 for the groups that start and end with a l-step, b-step and d-step respectively. These numbers indicate the smaller of the two numbers of infections that can be caused. When the matrices I are annihilated by commutation "the other way around" the complementary part of the links on the counting chain is infected in number $N_1^c = 3N - N_1$, $N_2^c = 3N - N_3$ and $N_3^c = 3N - N_3$. Schematically the situation can be indicated by drawing the counting chain as a circle. The three places corresponding to the occurrence of a certain element (i,j,k) along the chain are indicated and divide the chain into the three groups of links that can be infected in the process of annihilation of the matrices I that come in on summing over $\sigma_{i,j,k}$.

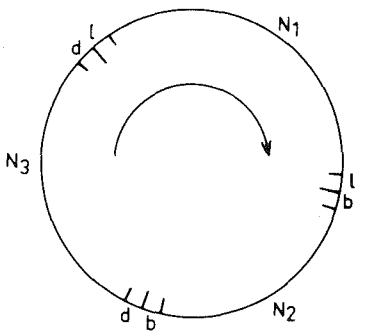


figure 15

The direction of the counting off of links is indicated by the arrow

Note that the numbers of infections are independent of the choice of the element in question.

In problem B_1 infection patterns corresponding to simple figures could be made by combination of two groups of infections, only one type of pairs of matrices I occurring in the formulation. We now try to combine two groups of the same type, say two groups of N_i (i=1, 2, 3) infections. One readily sees that 6 is the minimum number of infections, $N_i - 3$ links being disinfected. Again we indicate the elements where a l-step, b-step or d-step begins by respectively the letters a, b and c. The pattern

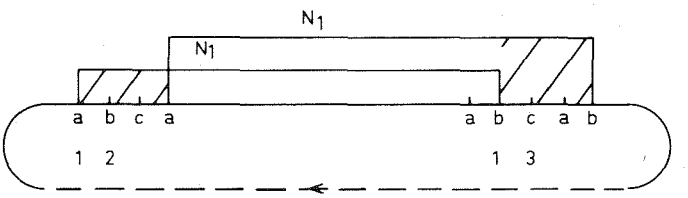


figure 16

corresponds on the lattice to the polygon

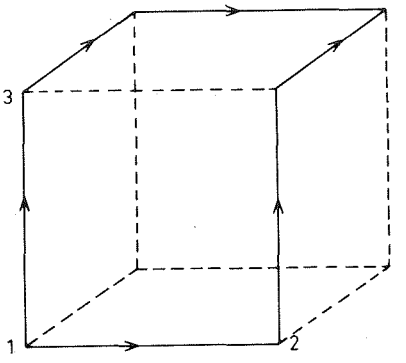


figure 17

The pattern

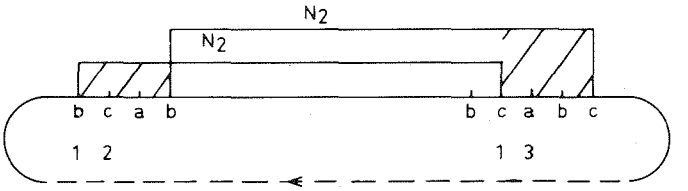


figure 18

corresponds to

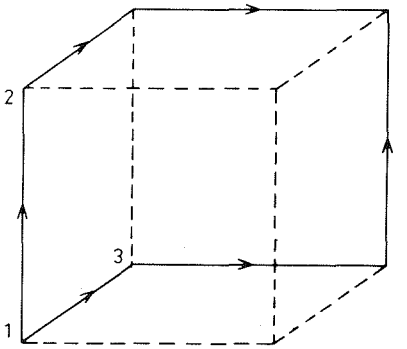


figure 19

The pattern

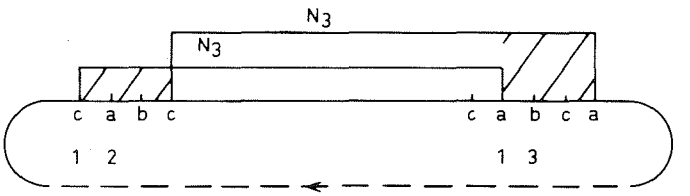


figure 20

corresponds to

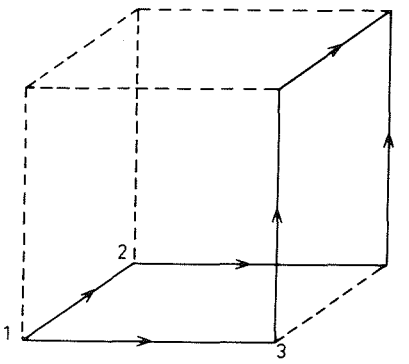


figure 21

The figures 16, 18 and 20 should be compared with figure 6 in chapter II. There 4 infections formed a barbell of 2 times 2 infections. Here 6 infections form a barbell of 2 times 3 infections. Now it is known from formulation C_c that patterns with 4 infections occur and one readily sees that this cannot be effected by combination of barbells of the type of figure 16, 18 or 20. Whereas in problem B_1 the smaller numbers of infections can all be formed by combination of barbells that are formed by two groups of infections, in problem C_1 the combination of different types of groups is to be made in order to generate a pattern corresponding to e. g. a simple square of 4 bonds on the lattice.

In formula (III, 16) one sees that the way of annihilating pairs of matrices I is irrelevant as the terms occur in pairs. Thus we might as well consider combinations of groups of infections of number N_1^c, N_2^c, and N_3^c with groups of number N_1, N_2 and N_3, noting that $N_1^c = N_2 + N_3$, $N_2^c = N_1 + N_3$ and $N_3^c = N_1 + N_2$. Actually the pattern

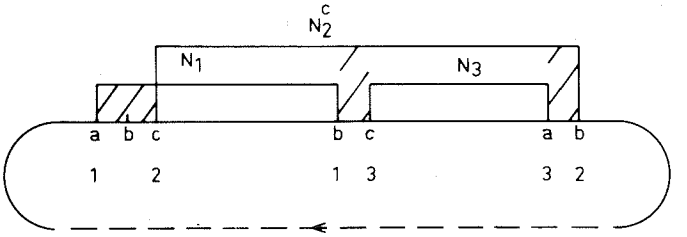


figure 22

corresponds on the lattice to

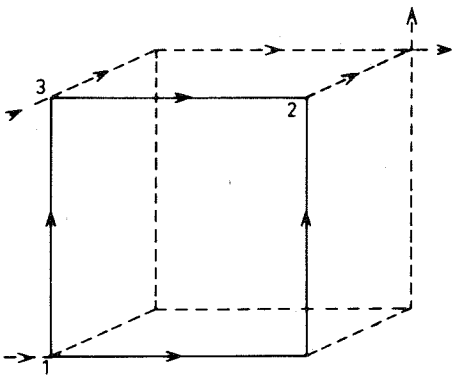


figure 23

Dotted bonds with arrow points indicate disinfected bonds

The pattern

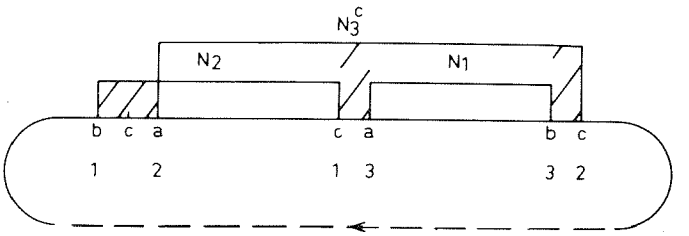


figure 24

corresponds on the lattice to

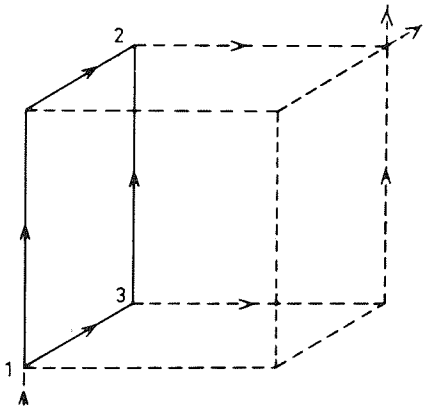


figure 25

The pattern

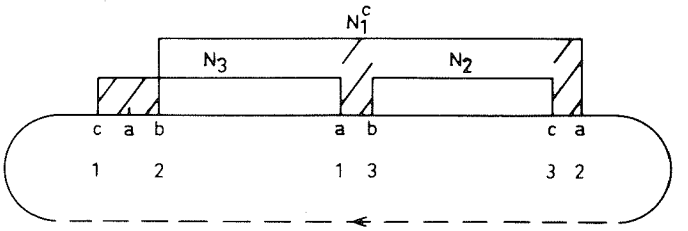


figure 26

corresponds on the lattice to

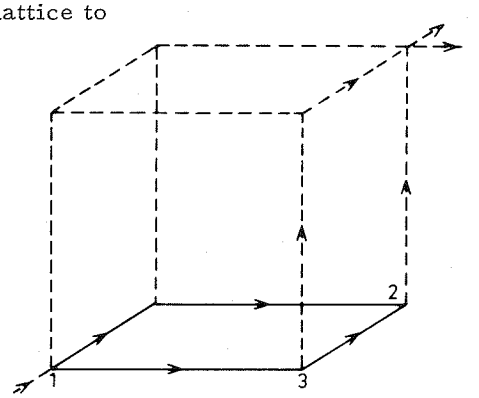


figure 27

So far our discussion of the correspondence of the problems C_1 and C_c.

§4. THE QUADRATIC LATTICE WITH NEXT NEAREST NEIGHBOUR INTERACTIONS

Next to problem C, interesting non-solved ISING problems are a. o. B^M and $_{n.n.}B$. We shall discuss the latter problem as it follows quite naturally on the problems considered so far. The formulation $_{n.n.}B_1$ can be derived along the same lines as the problem B_1 and C_1. As the technique is now known we shall give the main steps.

a) The lattice is of the form

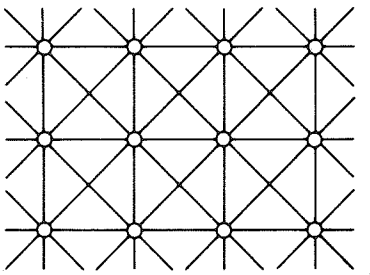


figure 28

with cyclic boundary conditions.

b) We choose the lattice to consist of m x (6m+2) elements with m odd. A counting chain can then be indicated using on repetition the following elementary counting pattern.

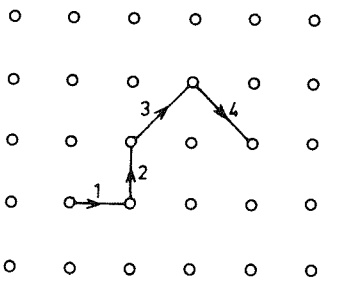


figure 29

The numbers indicate the various types of bonds

We shall discuss this choice further on.

c) As each element takes part in the forming of 8 bonds the coordination number q is 8 for this problem, whereas q = 4 for problem B and q = 6 for problem C. This means that each element is passed 4 times along the counting chain. Thus a certain index $\sigma_{i,j}$ is occuring 4 times as second and first index of consecutive matrices V_1, V_2, V_3 and V_4 thus necessitating the following lemma.

Lemma III: When A, B, C, D, E, F, G and H are arbitrary matrices of order 2 and $I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then for arbitrary indices a, b, ..., h

$$\sum_{j=1}^{2} A_{aj}B_{jb}C_{cj}D_{jd}E_{ej}F_{jf}G_{gj}H_{jh} = \frac{1}{8}\Big\{(AB)_{ab}(CD)_{cd}(EF)_{ef}(GH)_{gh} +$$
$$+ (AIB)_{ab}(CID)_{cd}(EF)_{ef}(GH)_{gh} + (AIB)_{ab}(CD)_{cd}(EIF)_{ef}(GH)_{gh} +$$
$$+ (AIB)_{ab}(CD)_{cd}(EF)_{ef}(GIH)_{gh} + (AB)_{ab}(CID)_{cd}(EIF)_{ef}(GH)_{gh} +$$
$$+ (AB)_{ab}(CID)_{cd}(EF)_{ef}(GIH)_{gh} + (AB)_{ab}(CD)_{cd}(EIF)_{ef}(GIH)_{gh} +$$
$$+ (AIB)_{ab}(CID)_{cd}(EIF)_{ef}(GIH)_{gh}\Big\} \qquad \text{(III, 18)}$$

with corollaries analogous to those in the case of the corresponding lemma for the problems B_1 and C_1.

d) Summing over the indices splits the partition function into 8^N terms of the form

$$8^{-N}\ \mathrm{Sp}\ V_1V_2V_3V_4V_1 \ldots V_4V_1V_2V_3V_4 \qquad \text{(III, 19)}$$

where between the matrices V_i (i = 1, 2, 3, 4) matrices I occur in 6 types of pairs that, on annihilation, cause infections of the matrices V_i corresponding to infection patterns on the counting chain that can be afflicted

by 6 types of groups of different numbers. Again a combinatorial problem on a one-dimensional arrangement is arrived at.

e) Calculation of the spurs leads to the partition function

$$Z_N = (2\ \mathrm{chH}\ \mathrm{chH'}\ \mathrm{chH''}\ \mathrm{chH'''})^N\ L_N(w,w',w'',w''') , \qquad \text{(III, 20)}$$

where

$$L_N(w, w', w'', w''') = \sum A(n_1', n_2', n_3', n_4') \cdot \left\{ w^{n_1'} w'^{n_2'} w''^{n_3'} w'''^{n_4'} + w^{N-n_1'} w'^{N-n_2'} w''^{N-n_3'} w'''^{N-n_4'} \right\} , \qquad \text{(III, 21)}$$

with obvious meaning of the symbols involved. The relevant quantity is

$$f = \log 2\ \mathrm{chH}\ \mathrm{chH'}\ \mathrm{chH''}\ \mathrm{chH'''} + \lim_{N\to\infty} \frac{1}{N} \log L_N(w,w',w'',w'''), \qquad \text{(III, 22)}$$

to be compared with formulae (II, 12), (II, 31) and (III, 15).

§5. COMPARISON OF THE COUNTING CHAINS FOR THE VARIOUS PROBLEMS

We shall now investigate somewhat more detailed the various counting chains that we used. For a two-dimensional lattice of $m \times (m-1)$ elements, starting from an arbitrary element (i, j) with a length step, $2m-1$ bonds are counted off before one reaches element (i, j) for the second time. The pair of matrices I that can occur because of summation over $\sigma_{i,j}$ thus causes, on annihilation, a group of $2m-1$ infections of the matrices V_1 and V_2 involved or, equivalently, of the links on the counting chain. These links belong alternately to two different types. The complementary part of the chain consists of $2m(m-1) - (2m-1) = 2m^2 - 4m + 1$ links.

For problem C_1 we have seen that there are three types of pairs of matrices I due to the summation over the spin variable of an arbitrary element that is passed twice when starting the counting procedure in the element in question. For the three-dimensional lattice consisting of $m \times (m-1) \times (m+1)$ elements with m even let us start at element $(\alpha, \beta, 1)$ with a depth step. We reach $(\alpha, \beta, 1)$ again for the first time after $\frac{1}{2}(m-2)m(3m+3)+3m+1$ steps via a depth step from $(\alpha, \beta, m+1)$.

EXPLICATION: To get back in the α-th row of the first layer one needs m length steps and thus $2m = 2(m-1)+2$ breadth steps in the pattern of the first layer, thus arriving at $(\alpha, \beta+2, 1)$. This proces is repeated and we arrive at $(\alpha, \beta+4, 1)$. Each time $m(3m+3)$ steps are made. Doing this $\frac{1}{2}(m-2)$ times we arrive at $(\alpha, \beta-1, 1)$ and after $3m+1$ steps we finally arrive at $(\alpha, \beta, 1)$ via $(\alpha, \beta, m+1)$.

We arrive at $(\alpha, \beta, 1)$ the second time after $(m-1)(3m+3)+1$ steps making a length step from $(\alpha-1, \beta, 1)$.

EXPLICATION: To arrive at the $(\alpha-1)$-th row $m-1$ length steps and thus $2(m-1)$ breadth steps have to be made in the pattern of the first layer and a length step from $(\alpha-1, \beta, 1)$ brings us back to $(\alpha, \beta, 1)$ for the second time.

To get back to our starting point $(\alpha, \beta, 1)$ via a breadth step from $(\alpha, \beta-1, 1)$, thereby closing the counting chain, $\frac{1}{2}(m-2)m(3m+3)+1$ steps are necessary.

EXPLICATION: 1 step is necessary to arrive at $(\alpha, \beta+1, 1)$ and then $\frac{1}{2}(m-2m)m(3m+3)$ to arrive at $(\alpha, \beta, 1)$ (see the first explication).

The latter result could have been obtained by subtracting the first two numbers of steps from the total number of bonds. However, now we easily check that $\frac{3}{2}(m-2)m(m+1)+3m+1+3(m-1)(m+1)+1+\frac{3}{2}(m-2)m(m+1)+1 = 3m(m-1)(m+1)$. We have thus shown, as was anticipated in paragraph 3, that each group of infections starts and ends with the same type of step and we have determined the numbers of infections in the three different groups.

Finding the dimensions of the lattice and investigating the counting chain for problem C_1 was rather solving a puzzle than applying a general method. This is also the case with the lattice and the counting chain for problem ${}_{nn}B_1$. For the moment, however, this is enough. The main aim of this chapter is to show that the formulation of problem B_1 is not restricted to that problem and as a matter of fact is quite basic. The choice made for the elementary counting pattern shown in figure 29 is quite arbitrary. Now suppose that with this choice, for a lattice consisting of $m \times (6m+2)$ elements with m odd, one starts at element $(1,1)$ with a step of type 3, then $(1,1)$ is reached for the first time after $8m+2$ steps. Then one continues with a step of type 1 and arrives at $(1,1)$ after $12m^2+8m+1$ steps, goes on with a step of type 2 and comes back at $(1,1)$ after $12m^2-16m-6$ steps. Finally, starting with a step of type 4, one ends the walk along the bonds after $8m+3$ steps. Indeed $(8m+2)+(12m^2+8m+1)+(12m^2-16m-6)+(8m+3)=24m^2+8m=4m(6m+2)$. The number of links in each of the 4 parts the counting chain is divided into by the 4-fold occurrence of an element along the chain in their dependence on m can be established by inspection as was done for problem C_1. One can easily verify the result e.g. for the three lattices 3×20, 5×32 and 7×44.

The information about the distinct form of the lattice and the bonds that come into play has transformed into information about the numbers of links that are contained in the parts the counting chain is divided into by the elements of the chain corresponding to a certain element of the lattice and into information about the types of links that occur. Denoting the numbers of links by N_1, N_2, etc. we may now summarize:

B_1:

$$N_1 = 2m - 1$$
$$N_2 = 2m^2 - 4m + 1$$

Total number of links
$2m \times (m-1)$ (2 types of links)

(III, 23a)

C_1:

$$N_1 = \tfrac{3}{2}m^3 - \tfrac{3}{2}m^2 + 1$$
$$N_2 = 3m^2 - 2$$
$$N_3 = \tfrac{3}{2}m^3 - \tfrac{3}{2}m^2 - 3m + 1$$

$3(m-1) \times m \times (m+1)$
m even (3 types of links)

(III, 23b)

$$
{}_{n.n.}B_1:\quad
\left.\begin{aligned}
N_1 &= 8m + 2\\
N_2 &= 12m^2 + 8m + 1\\
N_3 &= 12m^2 - 16m - 6\\
N_4 &= 8m + 3
\end{aligned}\right\}
\quad 4m \times (6m + 2)\ \text{m odd (4 types of links)}
\qquad \text{(III, 23c)}
$$

§6. A GENERALIZED LEMMA. A HIERARCHY OF PROBLEMS

The two important features of the formulation of the problems that we have considered are the counting chain and the summation lemma. The natural question to be asked at this moment is whether the procedure can be generalized. The really difficult point is the counting chain. We have found ad hoc solutions for two important outstanding problems. It is an interesting mathematical problem on its own to see to what extent the idea of counting chain, consisting of successive elementary counting patterns, can be generalized. The following can be said in this connection. The counting chains that we considered are special cases of "Eulerian circuits", a concept in graph theory stemming from the famous Königsberger bridge problem. From the theorem that proves the impossibility to solve this particular problem, see ref. [25], it can be seen that lattices like the triangular lattice, the body-centred cubic lattice and the face-centred cubic lattice permit a Eulerian circuit as long as toroidal boundary conditions are assumed. However, the question is whether these lattices can be chosen in such a way that this circuit consists of successive elementary counting patterns that contain each type of bond exactly once*. So we are looking for very special Eulerian circuits. Only these give rise to a formulation where exactly one, two, three, etc. types of groups of infections occur in the formulation. A generalized lemma, however, can be given easily. The lemmas I, II and III are special cases n=2, 3, and 4 of the following lemma.

Lemma IV: Are $A_1, A_2, \ldots A_{2n}$ arbitrary 2 x 2 matrices and is $I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then for arbitrary indices $a_1, a_2, \ldots, a_{2n}$

$$
\sum_{j=1}^{2} A_{1\,a_1,j}\, A_{2\,j,a_2}\, A_{3\,a_3,j}\, A_{4\,j,a_4} \ldots A_{2n-1\,a_{2n-1},j}\, A_{2n\,j,a_{2n}} \qquad \text{(III, 24)}
$$

$$
= 2^{-n} \sum{}' (A_1, A_2)_{a_1,a_2} (A_3, A_4)_{a_3,a_4} \ldots (A_{2n-1}, A_{2n})_{a_{2n-1},a_{2n}} ,
$$

* During the preparation of this thesis GÖBEL deduced necessary and sufficient conditions for the dimensions of a lattice in order to permit a counting chain consisting of consecutive elementary counting patterns of a certain type. The group of lattices ${}_{n.n.}B$ first found by the author is rather awkward. The lattices m x (3m + 1) with m natural for example do just as well.
When all bonds between N elements come into play and N = 2n + 1 is odd there exists such a circuit involving n types of bonds in the elementary counting pattern. This was proved by JETTEN and the author. It is an important result for our story because now ISING models that include all possible bonds might be considered along the lines developed in this thesis.

where $\sum'$ indicates the sum of 2^{n-1} terms consisting of factors that are matrix elements of $A_i\, A_{i+1}$ or of $A_i\, I\, A_{i+1}$ with i odd. There are n factors. All combinations of matrices I are present with the restriction that in each term the total number of matrices I is even.

Proof: The partitions of the factors I over the n available places are given by the terms with an even number of factors I in $(E+I)(E+I)\ldots(E+I)=(E+I)^n$, where E is e.g. the unit matrix. Each factor can or cannot give a factor I for the corresponding place in the array of places where a matrix I can occur. We shall speak of the "even" part of $(E+I)^n$ and denote the other part as the "odd" part. When n = 2 we have for the odd part

$$\tfrac{1}{2}\left\{(A_1 I A_2)_{a_1,a_2}\,(A_3 A_4)_{a_3,a_4} + (A_1 A_2)_{a_1,a_2}\,(A_3 I A_4)_{a_3,a_4}\right\} = \tag{III, 25}$$

$$= A_{1\,a_1,1}\,A_{2\,1,a_2}\,A_{3\,a_3,1}\,A_{4\,1,a_4} - A_{1\,a_1,2}\,A_{2\,2,a_2}\,A_{3\,a_3,2}\,A_{4\,2,a_4}\,.$$

For the even part we know that the plus sign occurs on the right hand side of (III, 25). In view of these two results we shall use induction.

The even part of $(E+I)^{n-1}$, with a factor 2^{-n+2} in front of the summation sign, will by hypothesis give A+B for n-1 pairs of indices where A denotes the term on the left hand side with j = 1 wherever j occurs as an index and B the term with j = 2 everywhere. The odd part of $(E+I)^{n-1}$ will by hypothesis give A-B. Now consider the even part of $(E+I)^n$ and separate the terms on the right hand side of (III, 24) into two groups. The first group has no matrix I in the n-th place and the matrices I are then partitioned over the n-1 other places according to the even part of $(E+I)^{n-1}$. The second group has a matrix I in the n-th place and the other matrices I are partitioned according to the odd part of $(E+I)^{n-1}$. Now

$$(A_{2n-1}A_{2n})_{a_{2n-1},a_{2n}} = (A_{2n-1})_{a_{2n-1},1}\,(A_{2n})_{1,a_{2n}} + (A_{2n-1})_{a_{2n-1},2}\,(A_{2n})_{2,a_{2n}} \overset{p.d.}{=} x+y$$

and

$$(A_{2n-1}A_{2n})_{a_{2n-1},a_{2n}} = (A_{2n-1})_{a_{2n-1},1}\,(A_{2n})_{1,a_{2n}} - (A_{2n-1})_{a_{2n-1},2}\,(A_{2n})_{2,a_{2n}} = x-y\,.$$

The right hand side of (III, 24) is now, by virtue of the induction hypotheses, equal to

$$\tfrac{1}{2}\left[(A+B)(x+y) + (A-B)(x-y)\right] = A\,x + B\,y \equiv P+Q\,,$$

P is a term with j = 1 everywhere and Q a term with j = 2 everywhere.

Q.E.D.

If we have a counting chain for a certain problem with coordination number $q = 2n$, each element is passed n times leading to a n-fold occurrence of the corresponding spin variable as second and first index of consecutive matrices. The summation over the spin variables can then be carried out using this lemma and its obvious corollaries. After summation the partition function of the problem in question will consist of $(2^{n-1})^N$ spurs, where N is the total number of elements. In each spur all possible combinations out of $\binom{n}{2}$ types of pairs of matrices I will occur between the matrices that occur analogously to the matrices V_i encountered so far.

Some simplification can be made due to the fact that it is irrelevant for the calculation of a certain spur in what way the pairs of matrices I are annihilated, as has already been discussed in chapter II. Let us consider for instance the counting chain for problem C_1 (figure 15). One of the three groups of infections gives the same result as the combined infections of the two other groups as $N_1^c = N_2 + N_3$ etc. Thus it suffices to consider only the infections that can be caused by all combinations of two groups of infections that we now allow to be placed against each other on the counting chain.* Likewise in problem ${}_{n.n.}B_1$ a reduction to three types of groups of infections is possible and it is easily seen, e.g. from figure 30, that for the general case with coordination number 2n, n-1 groups need to be taken into consideration only.

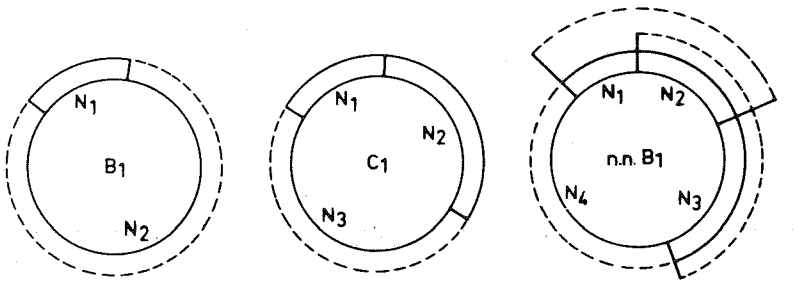


figure 30

Problem A_1 with an obvious counting chain and 0 groups of infections and the problems B_1 (1 group), C_1 (2 groups), ${}_{n.n.}B_1$ (3 groups) are problems on the first four levels of a hierarchy of problems.

* This was not possible before as a matrix I can only participate in the infection of one group. Actually the situation described corresponds to the interjection of the unit matrix in the form I^2 on the third place of the three places on the counting chain that correspond to a certain element of the lattice, two of these places being occupied by matrices I.

§7. DISCUSSION OF OTHER LATTICES. THE PROBLEM FOR THE LINEAR CHAIN WITH NEXT NEAREST NEIGHBOUR INTERACTIONS

Up to now we have only considered special forms of the quadratic and the cubic lattice that we considered as being representative for other two- and three-dimensional lattices. Although we do not want to study other lattices some remarks must be made here.

As for the finite lattices of different size the following should be remarked. Different ratios of the numbers of elements in the various directions can be established while maintaining the counting chain, consisting of successive elementary counting patterns, for the lattices that we have considered. In problem B_1 for example the length of the lattice may be chosen to be $2m-1$, when measured in elements, instead of m. However, we are mainly interested in infinite lattices and therefore the specific shape of the finite lattice we start with is irrelevant as far as those alternatives are concerned.

Now consider lattices that differ in structure from the quadratic or the cubic lattice. From the example of the honeycomb lattice it is clear that a counting chain as in problem B cannot always be found. In this case the coordination number $q = 3$ and a condition necessary for finding a counting chain that counts each bond once, is that q is even. It is sufficient, however, to have a counting chain for the quadratic lattice. The quadratic UTIYAMA lattice gives the solution for the triangular and the honeycomb lattice on variation of the values of the interaction energies. A transformation of the latter lattice can be applied to obtain the properties of still another two-dimensional lattice, the KAGOME lattice. The reader is referred to DOMB's review [14] for more details in this connection.

The main point we want to discuss is the following. Other two-dimensional lattices have turned out to give essentially the same results as the quadratic lattice that is therefore representative of two-dimensional lattices. Now when it would be possible to obtain an appropriate counting chain for the triangular lattice with $q = 6$ this would lead to a problem on the same level (third level) as problem C_1 in the hierarchy that we have outlined. Thus this problem would be a problem on another level than the problem B_1 that is known to give similar results. Now either problem C_1 would give results of the same nature as problem B_1 too or the level is not significant. The latter appears to be the case. The solution for the problem C_1 and the solution for the problem for the triangular lattice may therefore have a completely different nature.

Likewise other three-dimensional lattices that are often considered, like the body-centred lattice with $q=8$ or the face-centred cubic lattice with $q=12$, would lead to problems on the fourth level and the sixth level and might yet essentially give the same results as problem C_1. We did not indicate how for these lattices the counting chain and the lattice are to be chosen. This is a flaw in our reasoning as we compare the well-formulated problem C_1 with hypothetical formulations of the other problems. Nevertheless, with this restriction, our points are made once we can give an example of problems on the same level with solutions of a different nature.

Such problems are the problems B_1 and ${}_{n.n.}A_1$.* We shall consider the formulation of the latter problem that is also of some interest of its own.

* In problem ${}_{n.n.}A$ there is no singularity for the specific heat. We shall revert to this problem in chapter IV.

The significant features are:

i) the coordination number, that is 4 as each element has two nearest neighbours and two next-nearest neighbours

ii) the counting chain that will consist alternately of bonds between nearest neighbours and between next-nearest neighbours

iii) the number N of elements that is chosen to be N = 3m + 2 (m natural).

It is clear from i) that lemma (III, 24) is to be used with n = 2, which means that we have a problem on the same level as problem B_1 . In ii) the choice of counting off alternately bonds of the two types is in accordance with that for the other problems. The important feature is iii) that can be seen to be correct in the following way. Let us make a distinction between short steps and long steps for bonds between nearest neighbours and next-nearest neighbours respectively.

Consider N elements that are places as corners of a (regular) N-sided polygon. Starting from element 1 with a short step we reach element 2, making a long step we reach element 4, then 5, 7, 8, 10, 11 etc. are passed reaching element N by a short step after 2m+1 steps. Passing element 1 on reaching element 2 by a long step, we reach element 1 by a short step after "going around" once more which takes another 2m+1 steps. As N is a three-fold plus 2 it is clear that we need only one tour around to come back to element 1, again starting and ending with a long step. See figure 31 for an example.

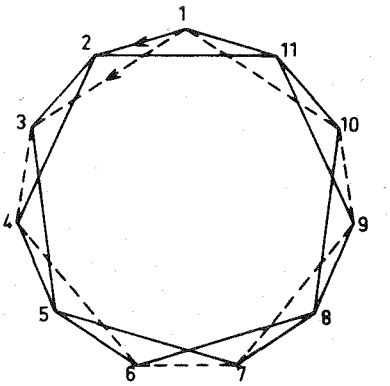


figure 31

This counting chain consists of 2N links and the groups of infections that can be afflicted on the counting chain consist of 2m+1 links on a total of 6m+4 links. This should be compared with problem B_1 where there are groups of 2m-1 links on a total of 2m(m-1) links. In problem $_{n.n.}A_1$ the counting chain is divided into two parts that in number of links are both of the same order of magnitude, as m becomes large, as the total number of links. In problem B_1 only one part is of the same order as the total number of links. As, next to the specification of the type of the links on the counting chain, the information about the original constellation of the lattice is contained in the numbers of links in these parts the difference just mentioned must be responsible for the different nature of the results for these two problems on the same (second) level of the hierarchy.

CHAPTER IV

ON THE MATRIX METHOD

§ 1. MATRIX FORMULATIONS BY MEANS OF THE BUILDING-UP PRINCIPLE

The original considerations in deducing the matrix formulations of the various problems are most clearly demonstrated by the method of KRAMERS and WANNIER. The main steps are the following.

i) A connection is laid between the probabilities for the constellations of the lattice before and after the adding of a new unit that, as said, can be a single element, a row of elements or a layer of elements.

ii) A summation over the spin variables of non-relevant elements simplifies this connection to a system of equations that can be read as a matrix eigenvalue problem when the states before and after the adding of the new unit can be considered to be identical on physical grounds.

iii) The partition function is expressed in the eigenvalues of the matrix in question.

The matrix operates on the probabilities of elements or groups of elements of the lattice before the adding of the new unit that are transformed in probabilities for that new unit. Therefore one can speak of a "transfer" matrix. We shall now shortly discuss the derivation of the formulations for the problems A, B and C.

A The linear lattice with a single element as unit in the building-up procedure. When n-1 elements are present the probability $P(\sigma_1, \ldots, \sigma_{n-1})$ for a certain constellation of the spin variables σ_i is proportional to

$$\exp\left[H(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \ldots + \sigma_{n-2}\sigma_{n-1})\right], \qquad (IV,1)$$

where $H = J/kT$. An analogous expression is valid for $P(\sigma_1, \ldots, \sigma_n)$ after adding the n-th element. Summation over the variables σ_1 up to σ_{n-2} gives that the probability $P(\sigma_{n-1})$ is proportional to

$$\sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \cdots \sum_{\sigma_{n-2}=\pm 1} \exp\left[H(\sigma_1\sigma_2 + \ldots + \sigma_{n-2}\sigma_{n-1})\right]. \qquad (IV,2)$$

Doing the same with $P(\sigma_1, \ldots, \sigma_n)$ leads to $P(\sigma_{n-1}, \sigma_n)$ i. e. the probability for the four combinations of σ_{n-1} and σ_n. With a proportionality factor we can write

$$\exp[H\sigma_{n-1}\sigma_n]\ P(\sigma_{n-1}) = \lambda P(\sigma_{n-1}, \sigma_n) \qquad \text{(IV, 3)}$$

and after summation over σ_{n-1} we read

$$\sum_{\sigma_n = \pm 1} \exp[H\sigma_{n-1}\sigma_n]\ P(\sigma_{n-1}) = \lambda P(\sigma_n). \qquad \text{(IV, 4)}$$

The assumption that for long chains both P's describe the same situation and that thus $P(\sigma_{n-1})$ is the same function of σ_{n-1} as $P(\sigma_n)$ of σ_n, makes that (IV, 9) can be read as an eigenvalue problem for the matrix

$$V_{\sigma_{n-1};\sigma_n} = \exp[H\sigma_{n-1}\sigma_n]\ . \qquad \text{(IV, 5)}$$

For a lattice consisting of ℓ elements with cyclic boundary conditions one finds that

$$Z_\ell = \mathrm{Sp}\ V^\ell \simeq \lambda_{max}^\ell\ , \qquad \text{(IV, 6)}$$

where λ_{max} is the largest eigenvalue of V. See e.g. ref. [4] and [5]. This of course is the same result as in our treatment of problem A_1 given in chapter II. The matrix V is of order 2 and has only positive elements.

B [2] The quadratic lattice consisting of ℓ rows of m elements with a row as unit for the building-up procedure. With $\sigma_{i,j}$ instead of σ_i and with ν_i for the constellation of the i-th row (2^m possibilities) we obtain with cyclic boundary conditions analogously

$$\exp[H(\sigma_{n-1,1}\sigma_{n,1} + \sigma_{n-1,2}\,\sigma_{n,2} + \ldots + \sigma_{n-1,m}\sigma_{n,m}) + H'(\sigma_{n-1,1}\,\sigma_{n-1,2} + \ldots + \sigma_{n-1,2}\,\sigma_{n-1,3} + \sigma_{n-1,m}\,\sigma_{n-1,1})]\ P(\nu_{n-1}) = \rho P(\nu_{n-1}, \nu_n), \qquad \text{(IV, 7)}$$

where $H' = J'/kT$, to be compared with formula (IV, 3). After summation over the variables of the (n-1)-th row this leads to

$$\sum_{\nu_{n-1}} \exp[H(\sigma_{n-1,1}\sigma_{n,1} + \ldots + \sigma_{n-1,m}\,\sigma_{n,m}) + H'(\sigma_{n-1,1}\,\sigma_{n-1,2} + \ldots + \sigma_{n-1,m}\sigma_{n-1,1})]\ P(\nu_{n-1}) = \rho P(\nu_n). \qquad \text{(IV, 8)}$$

We assume that this equation can be read as an eigenvalue problem, from now on this will not be mentioned specially, for the matrix

$$V_{\nu_{n-1};\nu_n} = \exp[H(\sigma_{n-1,1}\,\sigma_{n,1} + \ldots + \sigma_{n-1,m}\sigma_{n,m}) + H'(\sigma_{n-1,1}\sigma_{n-1,2} + \ldots + \sigma_{n-1,m}\sigma_{n-1,1})]\ . \qquad \text{(IV, 9)}$$

The partition function is expressed in this matrix and its eigenvalues according to

$$Z_{\ell x m} = \mathrm{Sp}\ V^{\ell} \simeq \rho_{max}^{\ell} . \qquad (IV,10)$$

See e.g. ref. [4]. The matrix V is of the order 2^m and has only positive elements.

B [1] The quadratic lattice consisting of ℓ rows of m elements with a single element as unit in the building-up procedure. The lattice is built up as a helix so the boundary conditions are somewhat different from those in B[2]. The helix is closed. Now a connection is laid between the probabilities for m subsequent elements of the helix and for these elements plus the added unit. This leads to

$$\exp[H\sigma_m\sigma_o + H'\sigma_m\sigma_{m-1}]\ P(\sigma_o, \ldots, \sigma_{m-1}) = \lambda P(\sigma_o, \ldots, \sigma_m), \qquad (IV,11)$$

to be compared with the formulae (IV,3) and (IV,7). From (IV,11) follows

$$\sum_{\sigma_o = \pm 1} \exp[H\sigma_m\sigma_o + H'\sigma_m\sigma_{m-1}]\ P(\sigma_o, \ldots, \sigma_{m-1}) = \lambda P(\sigma_1, \ldots, \sigma_m). \qquad (IV,12)$$

So the formulation is in terms of the matrix

$$V_{\nu_1;\nu_o} = \exp[H\sigma_m\sigma_o + H'\sigma_m\sigma_{m-1}], \qquad (IV,13)$$

where now ν_1 indicates the 2^m possibilities for σ_1 up to σ_m and ν_o the 2^m possibilities for σ_o up to σ_{m-1}. The partition function is now

$$Z_{\ell x m} = \mathrm{Sp}\ V^{\ell.m} \simeq \lambda_{max}^{\ell.m} . \qquad (IV,14)$$

See e.g. ref. [4]. The order of the matrix V is 2^m, just as in B[2], but only two positive elements appear in each row and each column. The other elements are 0.

By a handy numbering of the states ν_1 and ν_o using the binary code the matrix can be written in a "duodiagonal" form. Using a symmetry argument, see § 3, the problem can be reduced to an eigenvalue problem for a matrix of the order 2^{m-1} that has a beautiful "V-form". With

$$\alpha = \exp[H + H'], \quad \alpha' = \exp[-H + H'],$$

$$\beta = \exp[-H - H'], \quad \beta' = \exp[H - H'],$$

this matrix reads (m=4)

$$M(H,H') = \begin{pmatrix} \alpha & \alpha' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & \alpha' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \alpha' \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta' \\ 0 & 0 & 0 & 0 & \beta & \beta' & 0 & 0 \\ 0 & 0 & \beta & \beta' & 0 & 0 & 0 & 0 \\ \beta & \beta' & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \qquad \text{(IV, 15)}$$

The form for higher values of m is obvious.

C [3] The simple cubic lattice consisting of ℓ layers of m×p elements with a layer as unit in the building-up procedure. With $\sigma_{i,j,k}$ for the spin variables and τ_i for the constellation of the i-th layer ($2^{m.p}$ possibilities) we obtain with cyclic boundary conditions (hypertorus)

$$\exp\left[\sum_{j=1}^{m} \sum_{k=1}^{p} \left(H\sigma_{n-1,j,k}\,\sigma_{n,j,k} + H'\sigma_{n,j,k}\sigma_{n,j+1,k} + H''\sigma_{n,j,k}\,\sigma_{n,j,k+1} \right) \right] P(\tau_{n-1}) = \qquad \text{(IV, 16)}$$

$$= \omega P(\tau_{n-1}, \tau_n),$$

to be compared with (IV, 3), (IV, 7) and (IV, 11). Of course H'' = J''/kT. After summation over the spin variables of the (n-1)-th layer this leads to

$$\sum_{\tau_{n-1}} \exp\left[\sum_{j=1}^{m} \sum_{k=1}^{p} \left(H\sigma_{n-1,j,k}\sigma_{n,j,k} + H'\sigma_{n,j,k}\,\sigma_{n,j+1,k} + H''\sigma_{n,j,k}\,\sigma_{n,j,k+1} \right) \right] P(\tau_{n-1}) = \qquad \text{(IV, 17)}$$

$$= \omega P(\tau_n) .$$

We obtain an eigenvalue problem for the matrix

$$V_{\tau_{n-1};\tau_n} = \exp\left[\sum_{j=1}^{m} \sum_{k=1}^{p} \left(H\sigma_{n-1,j,k}\,\sigma_{n,j,k} + H'\sigma_{n,j,k}\,\sigma_{n,j+1,k} + H''\sigma_{n,j,k}\,\sigma_{n,j,k+1} \right) \right] . \qquad \text{(IV, 18)}$$

The partition function becomes

$$Z_{\ell \times m \times p} = \mathrm{Sp}\, V^{\ell} \simeq \omega_{max}^{\ell} . \qquad \text{(IV, 19)}$$

See e.g. ref. [5]. The matrix is of the order $2^{m.p}$ and has positive elements only.

C [2] The same lattice as in C[3] but now with a row of m elements as a unit that is added in the same way as the unit in B[1] along a "helix" of rows. The p-th row of a layer is now interacting with the first row of the next layer. The helix is closed again. When ν_i indicates the constellation in a row (2^m possibilities) we find in a way analogous to the treatment of B[1] for p subsequent rows of the helix

$$\exp[H \sum_{o,p} + H' \sum_{p} + H'' \sum_{p-1,p}] P(\nu_o, \ldots, \nu_{p-1}) = \rho P(\nu_o, \ldots, \nu_p), \quad (IV,20)$$

to be compared with (IV,3), (IV,7), (IV,11) and (IV,16). $\sum_{o,p}$, $\sum_{p}$ and $\sum_{p-1,p}$ indicate summations over m σ-variables namely

$$\sum_{j=1}^{m} \sigma_{j,o}\sigma_{j,p} \quad \sum_{j=1}^{m} \sigma_{j,p}\sigma_{j+1,p} \text{ and } \sum_{j=1}^{m} \sigma_{j,p-1}\sigma_{j,p} \quad \text{respectively.}$$

Note that $\sigma_{j,p}$ indicates the spin variable of the j-th element of the p-th row in the array of rows along the helix. From (IV,20) follows

$$\sum_{\nu_o} \exp[H \sum_{o,p} + H' \sum_{p} + H'' \sum_{p-1,p}] P(\nu_o, \ldots, \nu_{p-1}) = \rho P(\nu_1, \ldots, \nu_p). \quad (IV,21)$$

Thus the formulation is in terms of the matrix

$$V_{\tau_1;\tau_o} = \exp[H \sum_{o,p} + H' \sum_{p} + H'' \sum_{p-1,p}], \quad (IV,22)$$

where τ_1 indicates the $2^{m.p}$ possibilities for the combinations of ν_1 up to ν_p and τ_o indicates the $2^{m.p}$ possibilities for the combinations of ν_o up to ν_{p-1}. The partition function becomes

$$Z_{\ell x m x p} = Sp\ V^{\ell.p} \simeq \rho_{max}^{\ell.p}. \quad (IV,23)$$

See the next paragraph. The matrix is of the order $2^{m.p}$, just as in C[3], but now in each row and each column only 2^m elements appear. The other elements are 0.

C [1] A third possibility for the lattice considered in C[3] and C[2] is a building-up procedure with a single element as a unit. The elements of a row are added subsequently letting the m-th element of a row interact with the first element of the next row. The rows are ordered as in C[2]. The m-th element of the p-th row in the n-th layer is now interacting with and followed by the first element of the first row in the (n+1)-th layer. So the boundary conditions are again a little different from those in C[3] and C[2]. The helix of the complete lattice is again closed and for a "pitch" of m×p elements we find

$$\exp[H\sigma_{m.p}\sigma_o + H'\sigma_{m.p}\sigma_{m.p-1} + H''\sigma_{m.p}\sigma_{m.(p-1)}] P(\sigma_o, \ldots, \sigma_{m.p-1}) = \lambda P(\sigma_o, \ldots, \sigma_{m.p}), \quad (IV,24)$$

to be compared with the formulae (IV,3), (IV,7), (IV,11), (IV,16) and (IV,20). From (IV,24) follows

$$\sum_{\sigma_o} \exp\left[H\sigma_{m.p}\,\sigma_o + H'\sigma_{m.p}\,\sigma_{m.p-1} + H''\sigma_{m.p}\,\sigma_{m.(p-1)}\right] P(\sigma_o, \ldots, \sigma_{m.p-1}) =$$

$$= \lambda P(\sigma_1, \ldots, \sigma_{m.p}). \qquad \text{(IV,25)}$$

The formulation is in terms of the matrix

$$V_{\tau_1;\tau_o} = \exp\left[H\sigma_{m.p}\,\sigma_o + H'\sigma_{m.p}\,\sigma_{m.p-1} + H''\sigma_{m.p}\,\sigma_{m.(p-1)}\right], \qquad \text{(IV,26)}$$

where τ_1 and τ_o indicate the $2^{m.p}$ possibilities for the constellation of the spin variables σ_1 up to $\sigma_{m.p}$ and σ_o up to $\sigma_{m.p-1}$ respectively. The partition function becomes

$$Z_{\ell x m x p} = \mathrm{Sp}\; V^{\ell.m.p} \simeq \lambda_{max}^{\ell.m.p}. \qquad \text{(IV,27)}$$

See e.g. the article written by OGUCHI [28] in 1950 for a detailed treatment. Also DOMB's article [14] of 1949 is mentioned in this connection. The matrix is again of the order $2^{m.p}$ but contains, as in B[1], only two positive elements in each row and each column. All other elements are 0.

Again the problem can be reduced to an eigenvalue problem for a matrix with a "V-form" of the order $2^{m.p-1}$ as has already been remarked by KRAMERS and WANNIER. λ_{max} in A, B[1] and C[1] can be considered as the partition function per element, ρ_{max} in B[2] and C[2] as the partition function per row of m elements and ω_{max} in C [3] as the partition function per layer of mxp elements.

§ 2 . A TREATMENT OF THE ONE-DIMENSIONAL MODEL WITH NEXT-NEAREST NEIGHBOUR INTERACTIONS BY THE METHOD OF KRAMERS AND WANNIER

In the former paragraph no reference was given for the formulation C[2]. NEWELL and MONTROLL leave the formulations C[i], i=1,2,3, to the reader as an exercise in their review article of 1953. Another exercise is the treatment of problem $_{n.n.}$A, probably solved by many. The author's solution is given here as an illustration of the deduction and treatment of the formulations in the former paragraph.

As a unit we choose a single element, as was done in A, B[1] and C[1]. Let the interaction energy between nearest neighbours be J and between next-nearest neighbours be j for equal values of the spin variables then, with H = J/kT and h = j/kT, we have for the constellation of the spin variables σ_1 up to σ_{n-1} that

$$P(\sigma_1, \ldots, \sigma_{n-1}) = \exp\left[H(\sigma_1\sigma_2 + \ldots + \sigma_{n-2}\sigma_{n-1}) + \right. \qquad \text{(IV,28)}$$

$$\left. + h(\sigma_1\sigma_3 + \sigma_2\sigma_4 + \ldots + \sigma_{n-3}\sigma_{n-1})\right].$$

Summation over n-3 spin variables leads to

$$\sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_{n-3}=\pm 1} P(\sigma_1, \ldots, \sigma_{n-1}) = P(\sigma_{n-2}, \sigma_{n-1}). \qquad \text{(IV,29)}$$

For a chain of n elements we find analogously

$$\sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_{n-3}=\pm 1} P(\sigma_1, \ldots, \sigma_n) = P(\sigma_{n-2}, \sigma_{n-1}, \sigma_n). \qquad \text{(IV,30)}$$

From (IV,28), (IV,29) and (IV,30) follows with a proportionally factor λ that

$$\exp[H\sigma_{n-1}\sigma_n + h\sigma_{n-2}\sigma_n]\, P(\sigma_{n-2}, \sigma_{n-1}) = \lambda\, P(\sigma_{n-2}, \sigma_{n-1}, \sigma_n). \qquad \text{(IV,31)}$$

After summation over σ_{n-2} we obtain

$$\sum_{\sigma_{n-2}=\pm 1} \exp[H\sigma_{n-1}\sigma_n + h\sigma_{n-2}\sigma_n]\, P(\sigma_{n-2}, \sigma_{n-1}) = \lambda\, P(\sigma_{n-1}, \sigma_n). \qquad \text{(IV,32a)}$$

Assuming that the lattice is so long that $P(\sigma_{n-2}, \sigma_{n-1})$ and $P(\sigma_{n-1}, \sigma_n)$ are the same functions of their arguments, i.e. that the state of the last two elements in the chain of elements is independent of the length of that chain, which seems reasonable for long chains, (IV,32a) can be read as an eigenvalue problem for the matrix

$$V_{(\sigma_n, \sigma_{n-1});(\sigma_{n-1}, \sigma_{n-2})} = \exp\,[H\sigma_{n-1}\sigma_n + h\sigma_{n-2}\sigma_n]. \qquad \text{(IV,33)}$$

The matrix is of the order 4 and has two positive elements on each row and each column. The other elements are zero thus the explicit form is

(σ_n, σ_{n-1}) \ $(\sigma_{n-1}, \sigma_{n-2})$	(+,+)	(+,−)	(−,+)	(−,−)
(+,+)	e^{H+h}	0	e^{H-h}	0
(+,−)	e^{-H-h}	0	e^{-H+h}	0
(−,+)	0	e^{-H+h}	0	e^{-H-h}
(−,−)	0	e^{H-h}	0	e^{H+h} .

(IV,34)

The most important problem is now to express the partition function in terms of this matrix or in terms of its eigenvalues.

We let the matrix operator in (IV,32a) operate on $P(\sigma_{n-2}, \sigma_{n-1})$ another time to find

$$\sum_{\sigma_{n-2}=\pm 1} \sum_{\sigma_{n-1}=\pm 1} \exp[\, H(\sigma_{n-1}\sigma_n + \sigma_n\sigma_{n+1}) + h(\sigma_{n-2}\sigma_n + \sigma_{n-1}\sigma_{n+1})]\, P(\sigma_{n-2},\sigma_{n-1}) = \lambda^2 P(\sigma_n,\sigma_{n+1}). \quad \text{(IV,35a)}$$

The vectors P are right eigenvectors of V. As V is not symmetrical we also consider the left eigenvectors Q of V, for which

$$\sum_{\sigma_n=\pm 1} Q(\sigma_{n-1},\sigma_n)\, \exp[H\sigma_{n-1}\sigma_n + h\sigma_{n-2}\sigma_n] = \lambda\, Q(\sigma_{n-2},\sigma_{n-1}) \quad \text{(IV,32b)}$$

and

$$\sum_{\sigma_n=\pm 1} \sum_{\sigma_{n+1}=\pm 1} Q(\sigma_n,\sigma_{n+1})\, \exp[H(\sigma_{n-1}\sigma_n + \sigma_n\sigma_{n+1}) + h(\sigma_{n-2}\sigma_n + \sigma_{n-1}\sigma_{n+1})] = \lambda^2 Q(\sigma_{n-2},\sigma_{n-1}) . \quad \text{(IV,35b)}$$

The further treatment is based on two facts*. Firstly right and left eigenvectors can be normed according to

$$\sum_{\sigma_{n-1}=\pm 1} \sum_{\sigma_n=\pm 1} Q_i(\sigma_{n-1},\sigma_n)\, P_k(\sigma_{n-1},\sigma_n) = \delta_{ik} , \quad \text{(IV,36)}$$

where δ_{ik} is the KRONECKER delta symbol. Secondly we can write V as the dyadic product of right and left eigenvectors. Thus for V^2 with eigenvalues λ_i^2 we have

$$\exp[H(\sigma_{n-1}\sigma_n + \sigma_n\sigma_{n+1}) + h(\sigma_{n-2}\sigma_n + \sigma_{n-1}\sigma_{n+1})] = \sum_{i=1}^{4} \lambda_i^2 P_i(\sigma_n,\sigma_{n+1}) Q_i(\sigma_{n-2},\sigma_{n-1}) . \quad \text{(IV,37)}$$

Analogously we have

$$\exp[H(\sigma_{n+1}\sigma_{n+2} + \sigma_{n+2}\sigma_{n+3}) + h(\sigma_n\sigma_{n+2} + \sigma_{n+1}\sigma_{n+3})] = \sum_{i=1}^{4} \lambda_i^2 P_i(\sigma_{n+2},\sigma_{n+3}) Q_i(\sigma_n,\sigma_{n+1}) . \quad \text{(IV,38)}$$

Multiplying left-hand sides and right-hand sides of the equations (IV,37) and (IV,38) with each other leads after summation over σ_n and σ_{n+1} and making use of (IV,36) to

$$\sum_{\sigma_n=\pm 1} \sum_{\sigma_{n+1}=\pm 1} \exp[H(\sigma_{n-1}\sigma_n + \ldots + \sigma_{n+2}\sigma_{n+3}) + h(\sigma_{n-2}\sigma_n + \ldots + \sigma_{n+1}\sigma_{n+3})] = \sum_{i=1}^{4} \lambda_i^4 P_i(\sigma_{n+2},\sigma_{n+3})\, Q_i(\sigma_{n-2},\sigma_{n-1}) . \quad \text{(IV,39)}$$

* See e.g. ref. [29] p. 155 and p. 183.

The right-hand side of (IV,39) is again a dyadic product. The left-hand side begins to show some resemblance to the partition function.

Repeated application of this procedure leads when choosing n=3 for a lattice of N elements (N even) to

$$\sum_{\sigma_3=\pm 1} \cdots \sum_{\sigma_{N-2}=\pm 1} \exp\left[H(\sigma_2\sigma_3 + \ldots + \sigma_{N-1}\sigma_N) + h(\sigma_1\sigma_3 + \sigma_{N-2}\sigma_N)\right] = \tag{IV,40}$$

$$= \sum_{i=1}^{4} \lambda_i^{N-2} P_i(\sigma_{N-1},\sigma_N) Q_i(\sigma_1,\sigma_2) \ .$$

With the usual cyclic boundary conditions the bonds between the elements N and 1 and between 1 and 2 can be added, as well as those between N-1 and 1 and between N and 2. Therefore we multiply (IV,40) another time with (IV,37) for n=1 (n-1 = N, n-2 = N-1) and a summation over σ_{N-1} and σ_N gives on the left-hand side the partition function but for a summation over σ_1 and σ_2, whereas the right-hand side becomes

$$\sum_{i=1}^{4} \lambda_i^N P_i(\sigma_1,\sigma_2) Q_i(\sigma_1,\sigma_2) \ .$$

Thus after summation over σ_1 and σ_2 we finally obtain

$$Z_N = \sum_{i=1}^{4} \lambda_i^N \ . \tag{IV,41}$$

The characteristic equation for V is

$$\lambda^4 - 2\lambda^3 e^{H+h} + \lambda^2 e^{2h}(e^{2H}-e^{-2H}) + 2\lambda e^{-H+h}(e^{2h}-e^{-2h}) - (e^{2h}-e^{-2h})^2 = \tag{IV,42}$$

$$= \left[\lambda^2 - e^h(e^H+e^{-H})\lambda + 2\ \mathrm{sh}2h\right] \cdot \left[\lambda^2 - e^h(e^H-e^{-H})\lambda - 2\ \mathrm{sh}2h\right] = 0.$$

Thus

$$\begin{aligned} \lambda_{1,2} &= e^h\left[\mathrm{ch}H \pm \sqrt{\mathrm{sh}^2H + e^{-4h}}\right] \\ \lambda_{3,4} &= e^h\left[\mathrm{sh}H \pm \sqrt{\mathrm{ch}^2H - e^{-4h}}\right] \ . \end{aligned} \tag{IV,43}$$

For the relevant quantity $f = \lim_{N\to\infty} N^{-1} \log Z_N$ we find the logarithm of the largest eigenvalue. Thus

$$f = h + \log\left[\mathrm{ch}H + \sqrt{\mathrm{sh}^2H + e^{-4h}}\right] \ . \tag{IV,44}$$

For h=0 we rederive the solution for problem A

$$f = \log 2\,\mathrm{ch}H \ . \tag{II,12}$$

This problem has already been solved as far back as 1942 by MONTROLL [26], who builds up the lattice by adding two elements as a unit and finds the partition function as the spur of the N/2-th power of a matrix with a largest eigenvalue that is the square of ours. So far our discussion of KRAMERS and WANNIER's original way to formulate the problems. One remark should be made yet. Usually in matrix formulations symmetric matrices are considered to ensure diagonalization and to facilitate some calculations. However, this is not necessary or needed in our considerations and we have taken the opportunity to present the problems in such a way that the building-up principle comes forward most clearly.

§ 3. ON THE TRANSITION POINT

The most spectacular result in the article of KRAMERS and WANNIER was the conjecture about the transition point while assuming that such a point would turn out to be present for the quadratic lattice. Their method is based on certain transformation properties of the "V-matrix" indicated in § 1 in the treatment of the formulation B[1]. As also in the formulation C[1] a reduction of the problem to study of a "V-matrix" is possible the question can be posed why a similar conjecture cannot be made for the simple cubic lattice. To answer this question let us reproduce the argument of KRAMERS and WANNIER and try to apply it in the formulation C[1].

The matrix M given in the formula (IV, 13) is of the order 2^m. The numbering of the rows and columns is done by the numbers obtained when writing a zero for a plus one value and a one for a minus one value in the ordered constellation of the spin variables σ_m up to σ_1 and reading the resulting array of zeros and ones in the binary code. In this way a so-called duo-diagonal matrix comes out of that for m=3 has the following form

$$V = \begin{pmatrix} \alpha & \alpha' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta' & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta' & \beta \\ \beta & \beta' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & \beta' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha' & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha' & \alpha \end{pmatrix}, \qquad \text{(IV, 45)}$$

where

$$\alpha = \exp\,[H + H'], \quad \alpha' = \exp\,[-H + H'],$$

$$\beta = \exp\,[-H - H'], \quad \beta' = \exp\,[H - H'].$$

Following the treatment of OGUCHI, we read the last $4(=2^{m-1})$ rows and columns in reverse order to find the form

$$V = \begin{pmatrix} \alpha & \alpha' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta' \\ 0 & 0 & 0 & 0 & \beta & \beta' & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & \alpha' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \alpha' \\ 0 & 0 & \beta & \beta' & 0 & 0 & 0 & 0 \\ \beta & \beta' & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \qquad \text{(IV,46)}$$

A similarity transformation by means of the unitary matrix

$$S = \tfrac{1}{2}\sqrt{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{(IV,47)}$$

transforms V into $\tilde{V} = S\,V\,S$ of the form

$$\tilde{V} = \begin{pmatrix} \alpha & \alpha' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha' & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & \beta' & 0 & 0 & 0 & 0 \\ \beta & \beta' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & \alpha' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \alpha' \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta & -\beta' \\ 0 & 0 & 0 & 0 & -\beta & -\beta' & 0 & 0 \end{pmatrix} . \qquad \text{(IV,48)}$$

The investigation is now reduced to the eigenvalue problem for two matrices of the order $4(=2^{m-1})$, that both have the "V-form".

Suppose V in (IV,47) is considered a transformation with respect to the basisvectors

$$A_i \begin{pmatrix} + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ +, & -, & +, & -, & -, & +, & -, & + \end{pmatrix},$$

where $i=1,\ldots,8(2^m)$ and we have used the combinations of plus- and minus-signs as indices of comparents of the vectors, then $\tilde{V}$ denotes the same transformation with respect to the basisvectors

$$\tfrac{1}{2}\sqrt{2}\left(A_i\begin{pmatrix}+\\+\\+\end{pmatrix} + A_i\begin{pmatrix}-\\-\\-\end{pmatrix}, A_i\begin{pmatrix}+\\+\\-\end{pmatrix} + A_i\begin{pmatrix}-\\-\\+\end{pmatrix}, A_i\begin{pmatrix}+\\-\\+\end{pmatrix} + A_i\begin{pmatrix}-\\+\\-\end{pmatrix}, A_i\begin{pmatrix}+\\-\\-\end{pmatrix} + A_i\begin{pmatrix}-\\+\\+\end{pmatrix},\right.$$

$$\left., A_i\begin{pmatrix}+\\+\\+\end{pmatrix} - A_i\begin{pmatrix}-\\-\\-\end{pmatrix}, A_i\begin{pmatrix}+\\+\\-\end{pmatrix} - A_i\begin{pmatrix}-\\-\\+\end{pmatrix}, A_i\begin{pmatrix}+\\-\\+\end{pmatrix} - A_i\begin{pmatrix}-\\+\\-\end{pmatrix}, A_i\begin{pmatrix}+\\-\\-\end{pmatrix} + A_i\begin{pmatrix}-\\+\\+\end{pmatrix}\right).$$

Thus the "V-matrix" in the upper left part of $\tilde{V}$ is working in the subspace of vectors that are symmetric ("even") with respect to "inversion of all spins", the "V-matrix" in the lower right part of $\tilde{V}$ is working in the subspace of vectors that are antisymmetric ("odd") with respect to that operation. Now the eigenvector corresponding to the largest eigenvalue of V has only non-negative components as V has solely non-negative elements*. The corresponding eigenvector of $\tilde{V}$ has positive components in the space of the "even" vectors. Thus for finding the largest eigenvalue of V we can restrict ourselves to the first "V-matrix". This brings us to the matrix M already mentioned in § 1.

Now consider the unitary matrix

$$D = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}. \qquad \text{(IV,49)}$$

* See e.g. ref. [30] chapter XIII.

This matrix is of the order 4, corresponding with m=3. The similar matrix D' of the order 8 is obtained by writing each row of D twice and supplementing by the same row one time and the same row multiplied by -1 the other time and this alternately. An extra factor $\frac{1}{2}\sqrt{2}$ is involved to maintain unitarity. For m=4 we thus find

$$D' = \tfrac{1}{4}\sqrt{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix} \qquad \text{(IV,50)}$$

Matrices for higher values of m are constructed in the same way.
The important feature now is that

$$D\, M(H,H')D = (\mathrm{sh2H}\ \mathrm{sh2H'})^{\frac{1}{2}}\, M^{T}(H^{*},H^{*\prime}) \ , \qquad \text{(IV,51)}$$

where M^{T} is the transposed matrix of M and

$$e^{2H^{*}} = \mathrm{cothH'} \text{ and } e^{2H^{*\prime}} = \mathrm{cothH} \qquad \text{(IV,52a)}$$

or

$$\mathrm{sh2H}\ \mathrm{sh2H}^{*\prime} = 1 \text{ and } \mathrm{sh2H'}\ \mathrm{sh2H}^{*} = 1 \qquad \text{(IV,52b)}$$

or

$$\frac{\mathrm{sh2H}}{\mathrm{ch}^{2}2H} = \frac{\mathrm{sh2H}^{*\prime}}{\mathrm{ch}^{2}2H^{*\prime}} \text{ and } \frac{\mathrm{sh2H'}}{\mathrm{ch}^{2}2H'} = \frac{\mathrm{sh2H}^{*}}{\mathrm{ch}^{2}2H^{*}} \ . \qquad \text{(IV,52c)}$$

By the substitution

$$B(H,H') = (\mathrm{ch2H}\ \mathrm{ch2H'})^{-\frac{1}{2}}\, M(H,H') \ . \qquad \text{(IV,53)}$$

the matrix relation (IV,51) is changed into

$$D\, B(H,H')D = B^{T}(H^{*},H^{*\prime}) \ . \qquad \text{(IV,54)}$$

Because of $D = D^{T}$ and $D^{2} = E$, where E is the unit matrix, this relation is reversible. As transformation with D and transposition do not change the eigenvalues λ of M we obtain

$$x(H,H') = \frac{\lambda(H,H')}{(\mathrm{ch2H\ ch2H'})^{\frac{1}{2}}} = \frac{\lambda(H^*,H^{*\prime})}{(\mathrm{ch2H^*\ ch2H^{*\prime}})^{\frac{1}{2}}} = x(H^*,H^{*\prime}) \ , \qquad \text{(IV,55)}$$

the analogon of formula (30) in ref. [4].

The importance of this formula is the fact that any singularity for the values (H,H') has a counterpart in the values (H*,H*'). Multiplication of the formulae (IV,52) leads to

$$\mathrm{sh2H\ sh2H'\ sh2H^*\ sh2H^{*\prime}} = 1 \ . \qquad \text{(IV,56)}$$

The significance of the formulae (IV,52) is that twice two temperatures are coupled. As one of them rises from 0 to ∞ the other drops from ∞ to 0. This because H = J/kT. A small T gives a large H, a coth H $\approx$ 1 and thus a H*' $\approx$ 0 or a large value of T*', in H*' = J'/kT*'. Likewise a low temperature T' in H' and a high temperature T* in H* = J/kT* are coupled. Note that there is a cross coupling, H with H*' and H' with H*. The exceptional case is that in which H and H' determine H*' and H* in such a way that H = H* and H' = H*' in which case it follows from (IV,56) that

$$\mathrm{sh2H\ sh2H'} = 1 \ . \qquad \text{(IV,57)}$$

This formula determines a transition point if such a point exists. Along this way KRAMERS and WANNIER conjectured the CURIE temperature for H=H'. Their conjecture was confirmed three years later by ONSAGER.

The matrix that corresponds to the matrix M in the case of the simple cubic lattice can be deduced from the formula (IV,26) and reads for m=2, p=2

$$M(H,H',H'') = \begin{pmatrix}
\alpha & \alpha' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & \alpha' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta & \beta' & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta' \\
0 & 0 & 0 & 0 & 0 & 0 & \gamma & \gamma' \\
0 & 0 & 0 & 0 & \gamma & \gamma' & 0 & 0 \\
0 & 0 & \delta & \delta' & 0 & 0 & 0 & 0 \\
\delta & \delta' & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} , \qquad \text{(IV,58)}$$

where

$$\alpha = \exp\,[+H + H' + H''], \quad \alpha' = \exp\,[-H + H' + H'']$$
$$\beta = \exp\,[+H + H' - H''], \quad \beta' = \exp\,[-H + H' - H'']$$
$$\gamma = \exp\,[-H - H' + H''], \quad \gamma' = \exp\,[+H - H' + H'']$$
$$\delta = \exp\,[-H - H' - H''], \quad \delta' = \exp\,[+H - H' - H''] \ .$$

For $H = H' = H''$ this matrix is

$$M(H) = \begin{pmatrix} e^{3H} & e^{H} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3H} & e^{H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{H} & e^{-H} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{H} & e^{-H} \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-H} & e^{H} \\ 0 & 0 & 0 & 0 & e^{-H} & e^{H} & 0 & 0 \\ 0 & 0 & e^{-3H} & e^{-H} & 0 & 0 & 0 & 0 \\ e^{-3H} & e^{-H} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad \text{(IV,59)}$$

An attempt to make a conjecture about a transition point in an analogous way now fails because the matrix $D'\,M(H)\,D'$, with the matrix D' in formula (IV,50), has no "V-form". Attempts to find an alternative of D' for this matrix $M(H)$ have failed.

So we see that it has been an ad hoc find for the quadratic lattice to lay a connection between the partition function, represented by the eigenvalues of M, for some value of the temperature and the partition function for a complementary value of the temperature by means of the matrices D. The situation for the combinatorial method is rather analogous.

We introduce the so-called dual lattice consisting of elements one in each elementary rectangle of the quadratic lattice. In the case of the triangular lattice the dual lattice is the honeycomb lattice and vice versa. As the dual lattice of the quadratic lattice is again quadratic this lattice is called self-dual. Now suppose the totally ordered state is the state with the value +1 for all spin variables. If N elements are contained in the lattice with this state the BOLTZMANN factor $\exp[N(H+H')]$ corresponds. For a certain distribution of elements with the value -1 for the spin variables let there be $n_1'+n_2'$ connections between elements of opposite values of the spin variables, n_1' of one type and n_2' of the other type, then for this state the corresponding BOLTZMANN factor is $\exp[(N-2n_1')H+(N-2n_2')H']$. Connecting the elements of the dual lattice on both sides of such a connection for every distribution of elements with value -1 for the spin variable on the lattice a pattern is made on the dual lattice. With the exception of patterns as e.g. a single "loop" on the torus that cannot act as a boundary between elements with different values for the spin variables, exactly those patterns occur that by means of their number determine the coefficients in the series development of the partition function in terms of hyperbolic tangents of H and H'.

As the numbers of bonds n_1' and n_2' of a pattern on the dual lattice is equal to the number of connections, resp. n_2' and n_1', between elements with opposite values for the spin variables we have for the partition function

$$Z_N(H,H') = e^{N(H+H')} \sum B(n_1',n_2') \exp[-2n_1'H - 2n_2'H'] = \qquad \text{(IV,60)}$$

$$= \frac{e^{N(H+H')}}{(2\ \mathrm{ch}H\ \mathrm{ch}H^{*\prime})^N} (2\ \mathrm{ch}H^*\ \mathrm{ch}H^{*\prime})^N \sum B(n_1',n_2')\ (\mathrm{th}H^*)^{n_2'} (\mathrm{th}H^{*\prime})^{n_1'},$$

where the formulae (IV,52) again give the various relations between the variables. The summation on the right-hand side is practically over all closed polygons on the dual lattice and with this restriction we write

$$Z_N(H,H') = \frac{e^{N(H+H')}}{(2\ chH^*\ chH^{*\prime})^N} Z_N(H^{*\prime},H^*) = (sh2H\ sh2H')^{\frac{1}{2}N} Z_N(H^{*\prime},H^*). \quad (IV,61)$$

Again we have found a relation between the partition function, now in its integral form, for some value of the temperature and the same expression for a corresponding other temperature with the same formulae to describe this correspondence as in the method of KRAMERS and WANNIER and therefore with the same result. We see that for $H^* = H'$ and $H^{*\prime} = H$ no discontinuity occurs in Z_N because of formula (IV,57). The above argument can be found in the article of NEWELL and MONTROLL. The reason that this way does not lead to a conjecture for the transition point of a simple cubic lattice is situated in the fact that although this lattice is selfdual there is no natural one-one correspondence between connections in the dual lattice, the elements are now each in an elementary cube of the lattice, and connections between elements with opposite values for the spin variables on the lattice. To such a connection on the lattice we can set in correspondence a rectangle of the dual lattice and vice versa, but this does not lead to a result like (IV,61).

In the chapters II and III an alternative formulation of both classical formulations was given and one may ask whether it is possible to say something about the CURIE temperature from this formulation. Let us recall that both for the problem B and the problem C a so-called (closed) counting chain was indicated, consisting on repetition of elementary counting patterns that consisted of a length bond and a breadth bond in the first case and of a length bond, a breadth bond and a depth bond in the second case.

Firstly the "topological" argument of NEWELL and MONTROLL can be translated into the language of the infection-disinfection problem B_1 by letting correspond each "first" element of a, say, West-East bond of the counting chain to the element of the dual lattice on the North side and each first element of a South-North bond to the element of the dual lattice on the East side. In this way there is again a correspondence between bonds on the lattice and bonds on the dual lattice, as before, and the counting chain reproduces itself on the dual lattice. When an element (-1) is different from the others (+1) four bonds of the counting chain are involved. To these four bonds, that form a barbell of 2x2 bonds, correspond four bonds of the dual lattice that form a barbell of 2x2 bonds on the counting chain of the dual lattice of the type that could be used, as can be seen in chapter II, to generate the graphs that do not loop the torus. Now it is exactly the graphs that loop the torus that have as a subset the graphs that were omitted in the argument of NEWELL and MONTROLL. Omitting all graphs that loop the torus we can reproduce their argument in view of the one-one correspondence of barbells.

An attempt to generalize this argument first asks for a one-one correspondence between bonds of the lattice with bonds of the dual lattice for the problem C_1. This is possible by letting, for any pair of consecutive bonds of the counting chain of the lattice, correspond the bond of the dual lattice between the elements on both sides of the plane that pair of bonds lies in with e.g. the "first" bond of that pair. Again the counting chain is reproduced on the dual lattice. Now when one element (-1) is different from the others (+1) this involves six bonds and with these no closed pattern corresponds on the

dual lattice. Still this is not so bad because these non-closed patterns respectively the corresponding infection patterns on the counting chain of the dual lattice nicely follow up the infection-disinfection game played by the groups of six bonds occurring when one element (-1) differs from the others (+1). So the patterns of connections between elements with opposite values for the spin variables have their counter parts in patterns generated by elementary patterns of six bonds on the counting chain of the dual lattice via the infection-disinfection principle.

All patterns that thus arise on the dual lattice have numbers that are in accordance with that for the patterns of connections between elements with opposite values for the spin variables on the lattice. Now when these patterns would be practically the same in number as the patterns that occur in the full partition function, the argument could be generalized. However, it is easy to indicate a pattern on the dual lattice that cannot correspond to a certain pattern of connections between elements with opposite values for the spin variables on the lattice. This pattern is representative of a large group of patterns and is the simple square that, as was also discussed in chapter III, plays quite another role in problem C_1 than in problem B_1. There is no possibility to have a distribution of elements with "wrong" sign on the lattice such that the corresponding pattern on the dual lattice is a square. So a low temperature development cannot be translated into a high temperature development for the dual lattice which makes us fail in spite of the fact that in this case a bond-bond correspondence has been established.

A second way could be a pure algebraic transformation of the partition function into an expression in which, as in (IV, 61), the partition function occurs as function of another variable. Such a transformation, without taking into account the way in which the patterns are generated, does not seem possible.

§ 4. ON THE PROBLEM FOR THE QUADRATIC LATTICE

In the former paragraph an important result of KRAMERS and WANNIER was discussed together with the analoga in other formulations. These authors as well as MONTROLL brought forward the idea to write the partition function as the spur of some power of a matrix and the conjecture about the transition point was an extra in comparison with MONTROLL's article. Although the building-up principle, as we have seen in § 1, has the great advantage that various formulations can be deduced and is of great use for getting some insight into the structure of the matrices concerned, there is the following objection. In point ii) of § 1 it is assumed that we have to deal with very long chains (large lattices) to be able to read the formulae (IV, 4), (IV, 8) etc. as eigenvalue problems. However, in the deduction of the formulae (IV, 6), (IV, 10) etc. the above assumption is also made for small chains, as can be seen in § 2 in the treatment of problem ${}_{n.n.}A_1$. Where KRAMERS and WANNIER use closed chains, i.e. always assume cyclic boundary conditions, it comes down to the fact that we can use their argument when we admit that we have always considered the lattice in its integral form, that is as being large, and that we have only used the building-up principle for the advantages named above.

In the treatment of MONTROLL, that does not consider cyclic boundary conditions, the dissection into units and the ordering of these units is only considered to have insight into the structure of the matrices and to be able to interpret the summation over spin variables as the summation involved in matrix multiplication. This is also the approach that ONSAGER and KAUFMAN have followed, be it that the treatment of ONSAGER still leans a little on the building-up principle. By standing extremely on the side of MONTROLL it became possible for the author to find the alternative formulations B_1, C_1, etc. The assumption of cyclic boundary conditions appeared to be a condition sine qua non.

The now following survey of some points in the treatment of the problem B_2 with KAUFMAN's method is mainly given to back up of the considerations given in the next paragraphs.

a. We consider the formula

$$V_{\nu_{n-1};\nu_n} = \exp\left[H(\sigma_{n-1,1}\,\sigma_{n,1} + \ldots + \sigma_{n-1,m}\,\sigma_{n,m}) + H'(\sigma_{n-1,1}\sigma_{n-1,2} + \ldots \right. \qquad \text{(IV,9)}$$
$$\left. \ldots + \sigma_{n-1,m}\,\sigma_{n-1,1})\right] .$$

It is recognized that this expression corresponds with the interactions in the (n-1)-th row and the interactions between the (n-1)-th and n-th row for a certain constellation of those rows. V_{ν_{n-1},ν_n} can be considered as the product of two factors that are matrix elements, viz.

$$(V_1)_{\nu_{n-1};\nu_{n-1}} = \exp\left[H'(\sigma_{n-1,1}\,\sigma_{n-1,2} + \ldots + \sigma_{n-1,m}\,\sigma_{n-1,1})\right] \qquad \text{(IV,62)}$$

and

$$(V_2)_{\nu_{n-1};\nu_n} = \exp\left[H(\sigma_{n-1,1}\,\sigma_{n,1} + \ldots + \sigma_{n-1,m}\sigma_{n,m})\right], \qquad \text{(IV,63)}$$

where the 2^m values of the indices ν take care of the numbering of rows and columns.

As a summation over the values $\sigma_{n,1}$ up to $\sigma_{n,m}$ corresponds with a summation over the 2^m values of ν_n the partition function can be written as

$$Z_{\ell x m} = \sum_{\{\nu_i\}} (V_1)_{\nu_1;\nu_1} (V_2)_{\nu_1;\nu_2} (V_1)_{\nu_2;\nu_2} \ldots (V_1)_{\nu_\ell;\nu_\ell} (V_2)_{\nu_\ell;\nu_1} . \qquad \text{(IV,64)}$$

As V_1 is a diagonal matrix the summation over the values ν_i can be carried out because of

$$\sum_{\nu_i} (V_2)_{\nu_{i-1},\nu_i} (V_1)_{\nu_i,\nu_i} (V_2)_{\nu_i,\nu_{i+1}} = (V_2 V_1 V_2)_{\nu_{i-1},\nu_{i+1}} . \qquad \text{(IV,65)}$$

This formula should be compared with formula (II,6) in chapter II. Carrying out all summations we obtain

$$Z_{\ell x m} = \mathrm{Sp}(V_1 V_2)^\ell . \qquad \text{(IV,66)}$$

For coming to this formulation we have mentally dissected the lattice into rows of m elements.

b. One can look at the matrices in two ways. The first way is practically that of ONSAGER. We consider the matrices as representations of operators $\mathbf{V}_1$ and $\mathbf{V}_2$ that work on a set of orthonormal basisvectors $\psi_j(\sigma_1,\ldots,\sigma_m)$ with 2^m components, where $1 \leq j \leq 2^m$ and the spin variables are related to the elements of a row. Introducing the operators $\mathbf{I}_i$ and $\mathbf{C}_i$ defined by

$$\begin{aligned} \mathbf{I}_i\ \psi_j(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_m) &= \sigma_i\ \psi_j(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_m) \\ \mathbf{C}_i\ \psi_j(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_m) &= \psi_j(\sigma_1,\ldots,-\sigma_i,\ldots,\sigma_m) \end{aligned} \qquad \text{(IV,67)}$$

we have

$$\mathbf{V}_1 = \exp\left[H' \sum_{i=1} \mathbf{I}_i\ \mathbf{I}_{i+1}\right] \qquad \text{(IV,68)}$$

and

$$\mathbf{V}_2 = \prod_{i=1}^{m} (e^{H} + e^{-H}\mathbf{C}_i) = (2\mathrm{sh}2H)^{\frac{1}{2}m} \exp\left[H^* \sum_{i=1}^{m} \mathbf{C}_i\right], \qquad \text{(IV,69)}$$

where again $\mathrm{th}H^* = e^{-2H}$.

ONSAGER saw a component of ψ_j as a general term of the partition function (ref. [9], page 121). The variables σ_i are related to the elements of the last row, the summation over the variables of the foregoing rows has already been carried out. So the operators describe the effect of the adding of a new unit (row). This view is the same as that of SCHULTZ, MATTIS and LIEB [17] in their alternative derivation, by means of the concept of reduced density operator, of the transfer matrix.

The operators $\mathbf{I}_1$ up to $\mathbf{I}_m$ and $\mathbf{C}_1$ up to $\mathbf{C}_m$ can be considered as generators of a group, that forms a complete basis for the algebra of linear operators working in the space of the functions $\psi_j(\sigma_1,\ldots,\sigma_m)$. The algebras based on the subgroups, that have a simple structure, are quaternion algebras. Now the point that we are interested in is the fact that ONSAGER showed that the only irreducible representation of the algebra has the form of a KRONECKER product of irreducible representations of the named quaternion algebras, which are two by two matrices. From this KAUFMAN leant the special form of the matrix representation of the operators $\mathbf{I}_i$ and $\mathbf{C}_i$ as

$$\begin{aligned} I_i &\equiv E \times E \times \ldots \times E \times I \times E \times \ldots \times E \\ C_i &\equiv E \times E \times \ldots \times E \times C \times E \times \ldots \times E\ , \end{aligned} \qquad \text{(IV,70)}$$

where I and C stand on the i-th place of the m places and

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad \text{(IV,71)}$$

Adding $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ with $Y = E \times \ldots \times Y \times \ldots \times E$ we recognize the PAULI matrices. Along this way we reach the starting point of KAUFMAN viz.

$$Z_N = Z_{\ell x m} = (2sh2H)^{\frac{1}{2}m} Sp(V_1 V_2')^{\ell} \qquad (IV,72)$$

with

$$V_1 = \exp\left[H' \sum_{i=1}^{m} I_i I_{i+1}\right] \qquad (IV,73)$$

and

$$V_2' = \exp\left[H^* \sum_{i=1}^{m} C_i\right] . \qquad (IV,74)$$

Once we know or expect that the matrices can be written in the form of KRONECKER products, a second way of looking at the matrices is the following. Just as in problem A_1 the matrix

$$V = \begin{pmatrix} e^{H} & e^{-H} \\ e^{-H} & e^{H} \end{pmatrix} \qquad (II,4)$$

correctly represented the factor contribution to a term of the partition function of a certain bond between element j and element j+1 when σ_j and σ_{j+1} number the rows and the columns respectively, here the matrix

$$V_2 = V \times V \times \ldots \times V \qquad (m \text{ factors}) \qquad (IV,75)$$

correctly represents the factor contribution of the combined effect of the bonds between the pairs of elements (j,n) and (j+1,n), where $1 \leq n \leq m$, of neighbouring rows. Now the spin variables of the i-th pair $\sigma_{j,i}$ and $\sigma_{j+1,i}$ number the rows respectively columns of the matrix V on the i-th place of the KRONECKER product. Due to the fact that

$$\begin{aligned} V = e^{H}E + e^{-H}C = e^{H}(E + e^{-2H}C) = e^{H}(E + thH^*C) = \\ = (2sh2H)^{\frac{1}{2}} \exp\left[H^*C\right] , \end{aligned} \qquad (IV,76)$$

where use has been made of the fact that $C^2 = E$, we find again that

$$V_2 = (2sh2H)^{\frac{1}{2}m} \prod_{i=1}^{m} \exp\left[H^*C_i\right] = (2sh2H)^{\frac{1}{2}m} \exp\left[H^* \sum_{i=1}^{m} C_i\right] . \qquad (IV,77)$$

That the interactions in a row are correctly represented by the matrix

$$V_1 = \exp\left[H' \sum_{i=1}^{m} I_i I_{i+1}\right] \qquad (IV,73)$$

can be verified by inspection. This method can be found in the book of HUANG [31]. In this presentation no reference is made to the building-up principle, as was indirectly the case in KAUFMAN's presentation. We now come to the quintessence of KAUFMAN's method.

c. We introduce the 2m matrices

$$\begin{aligned}
\Gamma_1 &= I_1 & \Gamma_2 &= Y_1 \\
\Gamma_3 &= C_1 I_2 & \Gamma_4 &= C_1 Y_2 \\
\Gamma_5 &= C_1 C_2 I_3 & \Gamma_6 &= C_1 C_2 Y_3 \\
&\vdots & &\vdots \\
\Gamma_{2m-1} &= C_1 \cdots C_{m-1} I_m & \Gamma_{2m} &= C_1 \cdots C_{m-1} Y_m
\end{aligned} \qquad \text{(IV,78)}$$

matrix representations of quantities wherefore, with E' as the 2^m dimensional unit matrix,

$$\left[\Gamma_\alpha, \Gamma_\beta\right]_+ \equiv \Gamma_\alpha \Gamma_\beta + \Gamma_\alpha \Gamma_\beta = 2\, E'\, \delta_{\alpha\beta} \,. \qquad \text{(IV,79)}$$

In terms of these matrices we can deduce by the aid of the well-known properties of the PAULI matrices

$$C^2 = Y^2 = I^2 = E^2$$

$$[CY]_+ = [Y,I]_+ = [I,C]_+ = 0 \qquad \text{(IV,80)}$$

$$CY = iI,\ YI = iC,\ IC = iY \,,$$

that with $1 \leq \alpha \leq m$

$$\begin{aligned}
\Gamma_{2\alpha+1} \Gamma_{2\alpha} &= i I_\alpha I_{\alpha+1} \,, \quad \Gamma_{2\alpha} \Gamma_{2\alpha-1} = i C_\alpha \\
\Gamma_1 \Gamma_{2m} &= -i I_1 I_m (C_1 \ldots C_m) = -i I_1 I_m U \,,
\end{aligned} \qquad \text{(IV,81)}$$

which contains a definition of U. Using (IV,81) we have

$$W = V_1 V_2' = \exp\left[iH'U\, \Gamma_1 \Gamma_{2m}\right] \exp\left[-iH' \sum_{\alpha=1}^{m-1} \Gamma_{2\alpha+1} \Gamma_{2\alpha}\right] \cdot$$

$$\cdot \exp\left[-iH^* \sum_{\alpha=1}^{m} \Gamma_{2\alpha} \Gamma_{2\alpha-1}\right] \,. \qquad \text{(IV,82)}$$

We see that the matrix W is written as the product of matrices of approximately the same form. Only the first factor contains the matrix U next to a product of two Γ's in the exponent. Because of

$$\exp\left[iH'U\ \Gamma_1\Gamma_{2m}\right] = \tfrac{1}{2}(E'+U)\exp\left[iH'\ \Gamma_1\Gamma_{2m}\right]+\tfrac{1}{2}(E'-U)\ \exp\left[-iH'\Gamma_1\Gamma_{2m}\right] \quad \text{(IV,83)}$$

we obtain

$$W = \tfrac{1}{2}(E'+U)W^{+} + \tfrac{1}{2}(E'-U)W^{-} , \quad \text{(IV,84)}$$

where now in W^{+} and W^{-} all factors are of the same type namely $\exp[\text{constant}.\Gamma_\mu\Gamma_{\mu+1}]$. Thus in the exponents only forms occur that are quadratic in the Γ-matrices, that have consecutive indices.

Now the method is based on the fact thas, as we shall see in a moment, the matrices W^{+} and W^{-} are (spinor)-representations of rotations in the 2m-dimensional space of the Γ-matrices. Because the relation between the eigenvalues of the rotations (2m-dimensional matrices), that are compositions of m commuting rotations in two-dimensional planes, and the eigenvalues of the representations (2^m-dimensional matrices) is known, the problem is reduced to determining the eigenvalues of the matrices of the order 2m that effectuate the transformation in the space of the Γ-matrices. These matrices have a simple cyclic structure and this makes it possible to find the eigenvalues quite easily. After finding out which eigenvalues of W^{+} and W^{-} are relevant (both give 2^m eigenvalues) one can write down the relevant quantity f in the famous integral form of ONSAGER.

The most important feature for our further considerations is the character of the exponent in the factors of W^{+} and W^{-}. We therefore discuss this in a somewhat more detailed way. Let ω be a 2m-dimensional matrix that effectuates a linear orthogonal transformation in the space of the Γ-matrices, viz.

$$\Gamma^{*}_{\mu} = \sum_{\nu=1}^{2m} \omega_{\mu\nu}\ \Gamma_{\nu} \quad , \quad \text{(IV,85)}$$

with

$$\sum_{\nu=1}^{2m} \omega_{\mu\nu}\ \omega_{\mu\lambda} = \delta_{\nu\lambda} . \quad \text{(IV,86)}$$

Consider in particular the rotation

$$\begin{aligned} \Gamma^{*}_{\lambda} &= \Gamma_{\lambda} && (\lambda \neq \mu,\ \lambda \neq \nu) \\ \Gamma^{*}_{\mu} &= \Gamma_{\mu}\cos\gamma - \Gamma_{\nu}\sin\gamma && (\mu \neq \nu) \\ \Gamma^{*}_{\nu} &= \Gamma_{\mu}\sin\gamma + \Gamma_{\nu}\cos\gamma && (\mu \neq \nu) \end{aligned} \quad \text{(IV,87)}$$

$S(\omega) = \exp[\frac{\gamma}{2}\ \Gamma_\mu\Gamma_\mu]$ is a (spinor)-representation of this transformation, for we have from series development that

$$S(\omega) = E' \cos\tfrac{1}{2}\gamma + \Gamma_\mu \Gamma_\nu \sin\tfrac{1}{2}\gamma \qquad \text{(IV,88)}$$

and therefore

$$S(\omega)\,\Gamma_\mu\,S^{-1}(\omega) = (E'\cos\tfrac{1}{2}\gamma + \Gamma_\mu\Gamma_\nu\sin\tfrac{1}{2}\gamma)\cdot$$
$$\cdot\,\Gamma_\mu\,(E'\cos\tfrac{1}{2}\gamma - \Gamma_\mu\Gamma_\nu\sin\tfrac{1}{2}\gamma) = \Gamma_\mu\cos\gamma - \Gamma_\nu\sin\gamma = \Gamma_\mu^* \qquad \text{(IV,89)}$$

respectively

$$S(\omega)\,\Gamma_\nu\,S^{-1}(\omega) = \Gamma_\mu\sin\gamma + \Gamma_\nu\cos\gamma = \Gamma_\nu^* \qquad \text{(IV,90)}$$

Now as soon as in an anlogous formulation of some problem there appears a factor like $\exp[\text{constant}.\Gamma_\mu\Gamma_{\mu+1}\ldots\Gamma_{\mu+n}]$, thus factors with exponents that are no longer quadratic in the Γ-matrices, our analysis fails because such factors are no longer representations of transformations that are rotations in the space of the Γ's, as is easily verified*. This means that we have here a criterium (a necessary condition) for the solvability of a problem along the KAUFMAN method.

For completeness let us note here the alternative for problem B_2 treated by NEWELL [24] in 1950, starting from the formulae (IV,13) and (IV,14). The matrix

$$V_{\nu_1;\nu_0} = \exp\left[H\sigma_m\sigma_0 + H'\sigma_m\sigma_{m-1}\right] \qquad \text{(IV,13)}$$

again can be seen as a representation of an operator **V** working on vectors $\psi_j(\sigma_1,\ldots,\sigma_m)$ in which the spin variables are now referring to m succesive elements of the helix. Defining the operator **R** by

$$\mathbf{R}\,\psi_j(\sigma_1,\sigma_2,\ldots,\sigma_{m-1},\sigma_m) = \psi_j(\sigma_2,\sigma_3,\ldots,\sigma_m,\sigma_1) \quad , \qquad \text{(IV,91)}$$

R could be called a shifting operator, we have, see ref. [24],

$$\mathbf{V} = \exp\left[H'\,\mathbf{I}_{m-1}\mathbf{I}_m\right]\,(e^{H} + e^{-H}\mathbf{C}_m)\mathbf{R} =$$
$$= (2\mathrm{sh}2H)^{\frac{1}{2}m}\exp\left[H'\,\mathbf{I}_{m-1}\mathbf{I}_m\right]\exp\left[H^*\mathbf{C}_m\right]\mathbf{R}\,, \qquad \text{(IV,92)}$$

a form already indicated by ONSAGER.

This operator has the same structure as $\mathbf{V}_1\mathbf{V}_2$, but for **R**, and an analogous treatment is possible. A splitting up of **V** appears necessary analogous to that of **W** in formula (IV,84), again by means of the operator, case quo matrix, $\mathbf{U} = \mathbf{C}_1.\mathbf{C}_2.\ldots.\mathbf{C}_m$. This operator **U** inverts all spin variables in the functions $\psi(\sigma_1,\ldots,\sigma_m)$ and as C has the eigenvalues +1 and -1, U has

* Write down the analoga of the formulae (IV,88) and (IV,89). Such a factor does represent a transformation but this is a.o. one (no rotation) in the $(\Gamma_\mu,\Gamma_{\mu+1}\Gamma_{\mu+2}\cdots\Gamma_{\mu+n})$-plane in a space of Γ's and products of Γ's.

2^{m-1} eigenvalues +1 and 2^{m-1} eigenvalues -1 in the space of "even" and of "odd" functions respectively. The splitting up of V is completely analogous to the reduction of the same matrix given in §3.

We finally remark that it is hard to find a formulation like this one without use of the building-up principle.

§5. ON THE PROBLEMS FOR THE SIMPLE CUBIC LATTICE AND THE QUADRATIC LATTICE WITH NEXT-NEAREST NEIGHBOUR INTERACTIONS

This chapter has the character of a survey on the matric method. In the former paragraphs mostly well-known facts were compilated. Only a few items cannot be found in literature. However, the author is not aware of an account of most of the following material.

We first intend to give a formulation of the problems C_3 and ${}_{n.n.}B_2$ in terms of KRONECKER products, either by inspection or by using operators, strictly analogous to the formulations $B_2[2]$ and $B_2[1]$ of the former paragraph. We start with formula (IV, 18) for formulation $C_3[3]$ viz.

$$V_{\tau_{n-1};\tau_n} = \exp\left[\sum_{j=1}^{m}\sum_{k=1}^{p}\left(H\sigma_{n-1,j,k}\,\sigma_{n,j,k} + H'\sigma_{n,j,k}\,\sigma_{n,j+1,k} + H''\sigma_{n,j,k}\,\sigma_{n,j,k+1}\right)\right]. \quad \text{(IV, 18)}$$

The interaction between the (n-1)-th and n-th layers is represented by

$$\exp\left[\sum_{j=1}^{m}\sum_{k=1}^{p} H\sigma_{n-1,j,k}\sigma_{n,j,k}\right]$$

and these factors can be seen as matrix elements of the matrix V_2 that is a KRONECKER product of m.p matrices V, specified in formula (II, 4), and whose rows and columns are numbered by the $2^{m.p}$ combinations of the values of the spin variables in the (n-1)-th and the n-th layer respectively. The main difficulty is to find the analogon of the matrix V_1 that represents the interactions in a layer. We have to prescribe an ordering of the elements in the layer and this can be done quite arbitrarily as the form of the matrix V_2 is independent of this ordering. We therefore have anyhow

$$V_2 = (2\mathrm{sh}2H)^{\frac{1}{2}m.p.} \exp\left[H^* \sum_{i=1}^{m.p} C_i\right] \quad \text{(IV, 93)}$$

analogous to formula (IV, 77).

Now for bringing the diagonal matrix

$$(V_1)_{\tau_n;\tau_n} = \exp\left[\sum_{j=1}^{m}\sum_{k=1}^{p}\left(H'\sigma_{n,j,k}\sigma_{n,j+1,k} + H''\sigma_{n,j,k}\sigma_{n,j,k+1}\right)\right] \quad \text{(IV, 94)}$$

in a form analogous to formula (IV, 73) we are tempted to choose p = m+1

and an ordering of the elements like in problem B_1. As in the matrix V_1 of problem B_2 the matrix $I_i I_{i+1}$ corresponded to the form $\sigma_i\sigma_{i+1}$ in the exponent of the factor representing the contribution of the bond between the i-th and the (i+1)-th element in a row, now matrices like $I_i I_{i+1}$ will correspond to forms $\sigma_i\sigma_{i+1}$ but with a numbering of the elements along the counting chain. The first 2m+1 bonds in our specific counting-off procedure thus give rise to similar forms in the exponents as in the matrix V_1 of problem B_2. However, the second time we pass the starting point (or any point), we cannot let this element take care of the numbering of a matrix factor of the KRONECKER product as this would make the order of the matrices $2^{2m(m+1)}$ instead of $2^{m(m+1)}$ as must be because of the number of states in which the layer can be*. Thus the bond between the elements (1,m+1) and (1,1) gives rise to a matrix $I_1 I_{2m}$ of the structure

$$I \times E \times \ldots \times E \times I \times E \times \ldots \times E$$

with matrices I on the first and on the 2m-th places of the m(m+1) places. The bond between the element (1,1) and the "new" element (2,1) along the counting chain will give rise to a matrix $I_1 I_{2m+1}$ of the structure

$$I \times E \times \ldots \times E \times I \times E \times \ldots \times E$$

with matrices I on the first and on the 2m+1-th place . The bond between (2,1) and (2,2) gives $I_2 I_{2m+1}$ etc. The products of matrices I_i that come out are not as regular as one would want. This is a fortiori the case when we do not choose p = m+1**.

We therefore maintain the choice p = m+1 but let the place of a factor in the KRONECKER product be determined alternately by an element along the counting chain. Thus the array of elements is chosen to be (1,1), (2,2),(3,3), ..., (1,m+1), (2,1), (3,2), etc., when starting from (1,1), or any other point as the resulting formulation will be the same. In this way the form $\sigma_{n,i,j}\,\sigma_{n,i+1,j}$ will give rise to a product of two matrices I_i that is of the form

$$E \times \ldots \times E \times I \times E \times \ldots \times E \times I \times E \times \ldots \times E ,$$

where the factors I are "separated" by m factors E as can easily be verified by sketching this "counting along the diagonals" in a lattice of say 4x5 points. With this ordering of the elements one finds, with the convention $I_{m(m+1)+i} \equiv I_i$, that

$$V_1 = \exp\left[H' \sum_{\alpha=1}^{m(m+1)} I_\alpha I_{\alpha+m+1} + H'' \sum_{\alpha=1}^{m(m+1)} I_\alpha I_{\alpha+m}\right] . \qquad (IV,95)$$

Other orderings give a less regular expression for V_1. No ordering will give only forms where the indices of the two matrices I in a product are consecutive.

When we try to make an analysis like the one in problem B_2 we notice that

$$I_\alpha I_{\alpha+m} = I_\alpha I_{\alpha+1} I_{\alpha+1} I_{\alpha+2} \cdots I_{\alpha+m-1} I_{\alpha+m-1} I_{\alpha+m} , \qquad (IV,96)$$

* See further chapter IV, §6 in this connection.
** See chapter II, §3 in this connection.

as $I_{\alpha+1}^2 = E'$. A product $I_\alpha I_{\alpha+1}$ is transformed into $-i\Gamma_{2\alpha+1}\Gamma_{2\alpha}$, as is known from (IV, 81), and thus using (IV, 79) we have

$$I_\alpha I_{\alpha+m} = (i)^m \Gamma_{2\alpha}\Gamma_{2\alpha+1}\Gamma_{2\alpha+2}\cdots\Gamma_{2\alpha+2m-1}\Gamma_{2\alpha+2m} \qquad (IV, 97)$$

and the KAUFMAN analysis fails in view of the discussion in the former paragraph.

Starting from the formulation C[3] of §1 we have thus formulated the problem C_3 in terms of matrices that were given in the form of KRONECKER products. We have based ourselves on the second way of deducing the formulation B_2 in the former paragraph namely that by inspection. For the two alternative formulations, starting from the formulations C[2] and C[1] direct inspection is not so simple. Just as in the formulation given by NEWELL for the two-dimensional problem starting from formulation B[1], it is simpler to write the matrices in terms of KRONECKER products along the operator formalism. Let us begin with the somewhat simpler case starting from the formulation C[1].

We consider the matrix given in formula (IV, 26) for a lattice consisting of ℓ layers of m x p elements

$$V_{\tau_1;\tau_o} = \exp\left[H\sigma_{m.p}\sigma_o + H'\sigma_{m.p}\sigma_{m.p-1} + H''\sigma_{m.p}\sigma_{m.(p-1)}\right], \qquad (IV, 26)$$

where τ_1 and τ_o indicate the $2^{m.p}$ possible constellations of the spin variables $\sigma_{m.p}$ up to σ_1 and $\sigma_{m.p-1}$ up to σ_o respectively. We may also consider the matrix

$$V_{\sigma'_{m.p},\ldots,\sigma'_1;\sigma_{m.p},\ldots,\sigma_1} = \exp\left[H\sigma'_{m.p}\sigma_1 + H'\sigma'_{m.p}\sigma_{m.p} + H''\sigma'_{m.p}\sigma_{m.(p-1)+1}\right] \cdot \prod_{j=1}^{m.p-1}\delta_{\sigma'_j,\sigma_{j+1}}, \qquad (IV, 98)$$

where $\delta_{\alpha,\beta}$ is again the KRONECKER delta symbol. In terms of this matrix we have

$$Z_{\ell x m x p} = \prod_{i=1}^{\ell.m.p} V_{\sigma_{m.p+i},\ldots,\sigma_{1+i};\sigma_{m.p+i-1},\ldots,\sigma_i} = Sp\, V^{\ell.m.p}. \qquad (IV, 99)$$

Following NEWELL we shall show that this matrix can be seen as a representation of the operator $\mathbf{V}$ that, as can be guessed from (IV, 92), has the form

$$\mathbf{V} = \exp\left(H'\mathbf{I}_{m.p}\mathbf{I}_{m.p-1}\right).\exp\left(H''\mathbf{I}_{m.p}\mathbf{I}_{m.(p-1)}\right).\left(e^H + e^{-H}\mathbf{C}_{m.p}\right).\mathbf{R}. \qquad (IV, 100)$$

This operator works in the space of orthonormal basis vector functions $\psi(\sigma_1,\ldots,\sigma_{m.p})$. The operators $\mathbf{I}_j$, $\mathbf{C}_j$ and $\mathbf{R}$ have been specified in the former paragraph.

We see that

$$\mathbf{V}\psi(\sigma_1,\sigma_2,\ldots,\sigma_{m.(p-1)+1},\ldots,\sigma_{m.p}) = \exp\,(H'\mathbf{I}_{m.p}\mathbf{I}_{m.p-1})\cdot$$

$$\cdot\ \exp(H''\ \mathbf{I}_{m.p}\mathbf{I}_{m.(p-1)})\ \cdot\ \Big\{e^{H}\ \psi(\sigma_2,\sigma_3,\ldots,\sigma_{m.p},\sigma_1)\ +$$

$$+\ e^{-H}\ \psi(\sigma_2,\sigma_3,\ldots,\sigma_{m.p},-\sigma_1)\Big\} = \exp\,(H'\sigma_1\sigma_{m.p})\exp(H''\sigma_1\sigma_{m.(p-1)+1})\ \cdot \qquad \text{(IV,101)}$$

$$\cdot\ e^{H}\ \psi(\sigma_2,\ldots,+\sigma_1)+\exp(-H'\sigma_1\sigma_{m.p})\ \exp(-H''\sigma_1\sigma_{m.(p-1)+1})e^{-H}\ \psi(\sigma_2,\ldots,-\sigma_1).$$

For the matrix element $V_{\sigma'_{m.p},\ldots,\sigma_1';\sigma_{m.p},\ldots,\sigma_1}$ therefore find

$$\Big\{\exp\,(H'\sigma_1\sigma_{m.p})\ \exp\,(H''\sigma_1\sigma_{m.(p-1)+1})e^{H}\delta_{\sigma'_{m.p},\sigma_1}\ +$$

$$+\ \exp\,(-H'\sigma_1\sigma_{m.p})\ \exp\,(-H''\sigma_1\sigma_{m.(p-1)+1})e^{-H}\delta_{\sigma'_{m.p},-\sigma_1}\Big\}\cdot\ \prod_{j=1}^{m.p-1}\delta_{\sigma'_j,\sigma_{j+1}}\ , \qquad \text{(IV,102)}$$

which is the right-hand side of (IV,98). The relevance of the form (IV,100) for the operator $\mathbf{V}$ is now again clear in the factor $\exp[H''\mathbf{I}_{m.p}\mathbf{I}_{m.(p-1)}]$. Because of this factor there occurs in the matrix representation an exponent of the form $I_{m.p}I_{m.(p-1)}$ and thus again an expression that is non-quadratic in the Γ-matrices, as could be expected.

It is only for completeness that we now consider the formulation starting from C[2]. It is clear that such an analysis will be analogous to that of the formulation B[1], with rows of m elements instead of single elements.

Firstly there will now occur a shifting operator $\mathbf{R}'$ that works on functions of the form $\psi_j(\nu_1,\ldots,\nu_p)$ with $1 \leq j \leq 2^{m.p}$, where each ν indicates the 2^m possibilities for the constellations in a row. Per definition we have

$$\mathbf{R}'\psi_j(\nu_1,\nu_2,\ldots,\nu_p) = \psi_j(\nu_2,\nu_3,\ldots,\nu_p,\nu_1)\ . \qquad \text{(IV,103)}$$

Ordering the m spin variables indicated by a single ν_i we see that

$$\mathbf{R}'\psi_j = \mathbf{R}^m\psi_j\ , \qquad \text{(IV,104)}$$

with the operator $\mathbf{R}$ defined in (IV,91). Secondly there will occur the operator

$$\prod_{j=1}^{m}(e^{H} + e^{-H}\mathbf{C}_{j,p})$$

with $\mathbf{C}_{j,p}$ as the operator that inverts the j-th of the spin variables indicated by the variable on ν the p-th place. Thirdly there will occur the operator

$$\prod_{j=1}^{m}\exp(H''\mathbf{I}_{j,p}\mathbf{I}_{j+1,p}),$$

where $\mathbf{I}_{j,p}$ is an operator that works like the $\mathbf{I}$ operators before in relation to the j-th of the spin variables indicated by the variable ν on the p-th place. An extra factor that occurs in comparison with the problem B in the analogous treatment is the one that represents the interactions in a row namely the operator

$$\prod_{j=1}^{m} \exp(H' \mathbf{I}_{j,p} \mathbf{I}_{j+1,p}) .$$

We now have to deal with the matrix from (IV,22) in the form

$$V_{\nu'_p,\ldots,\nu'_1 ; \nu_p,\ldots,\nu_1} = \exp\left[H \sum_{j=1}^{m} \sigma_{j,1}\sigma'_{j,p} + H' \sum_{j=1}^{m} \sigma'_{j,p}\sigma'_{j+1,p} + \right. \tag{IV,105}$$

$$\left. + H'' \sum_{j=1}^{m} \sigma_{j,p}\sigma'_{j,p} \right] \cdot \prod_{i=1}^{p-1} \delta_{\nu'_i,\nu_{i+1}} ,$$

where $\delta_{\nu'_i,\nu_{i+1}}$ stands for $\prod_{j=1}^{m} \delta_{\sigma'_{j,i},\sigma_{j,i+1}}$. In terms of this matrix the partition function reads

$$Z_{\ell x m x p} = \prod_{i=1}^{\ell \cdot p} V_{\nu_{p+i},\ldots,\nu_{1+i} ; \nu_{p+i-1},\ldots,\nu_i} = \mathrm{Sp}\, V^{\ell \cdot p} . \tag{IV,106}$$

The matrix in (IV,105) is a representation of the operator

$$\mathbf{V} = \exp(H'' \sum_{j=1}^{m} \mathbf{I}_{j,p}\mathbf{I}_{j,p-1}) . \exp(H' \sum_{j=1}^{m} \mathbf{I}_{j,p}\mathbf{I}_{j+1,p}) . \prod_{j=1}^{m} (e^{H} + e^{-H}\mathbf{C}_{j,p}) . \mathbf{R}', \tag{IV,107}$$

for we have

$$\mathbf{V}\psi_j(\nu_1,\ldots,\nu_2) = \exp(H'' \sum_{j=1}^{m} \mathbf{I}_{j,p}\mathbf{I}_{j,p-1}) \exp(H' \sum_{j=1}^{m} \mathbf{I}_{j,p}\mathbf{I}_{j+1,p}) \cdot \tag{IV,108}$$

$$\cdot \prod_{j=1}^{m} (e^{H} + e^{-H}\mathbf{C}_{j,p}) \cdot \psi_j(\nu_2,\ldots,\nu_p,\nu_1) .$$

When now the operator

$$\prod_{j=1}^{m} (e^{H} + e^{-H}\mathbf{C}_{j,p})$$

comes into action 2^m terms appear corresponding to the being or not being inverted of the m variables σ indicated by the symbol ν_1. An operator $\exp(H'\mathbf{I}_{j,p}\mathbf{I}_{j\,p-1})$ gives $\exp(-H'\sigma_{j,1}\sigma_{j,p})$ respectively $\exp(+H'\sigma_{j,1}\sigma_{j,p})$ depending on whether the operator $\mathbf{C}_{j,p}$ has worked on ψ or not. A similar thing is the case with the operator

$$\exp(H'' \sum_{j=1}^{m} I_{j,p} I_{j,p-1}) .$$

So for the matrix elements of the representing matrix we find expressions of the form

$$\prod_{j=1}^{m} \exp(\pm H'' \sigma_{j,1} \sigma_{j,p}) . \prod_{j=1}^{m} \exp(\pm H' \sigma_{j,1} \sigma_{j+1,1}) .$$

(IV, 109)

$$. \prod_{j=1}^{m} \exp(\pm H) . \prod_{j=1}^{p-1} \delta_{\nu'_{i},\nu_{i+1}} . \prod_{j=1}^{m} \delta_{\sigma'_{j,p},\pm\sigma_{j,1}} ,$$

where the sign ± indicates a certain combination of minus signs distributed over the m places of the m variables $\sigma_{j,1}$. The factors $\delta_{\sigma'_{j,p},\pm\sigma_{j,1}}$ now make that this matrix can be written as in formula (IV, 105).

In (IV, 107) we see that again the structure of the layers manifests itself in pairs of operators I whose indices cannot always be consecutive for some ordering of the elements, with the known consequences when one tries to apply the KAUFMAN analysis.

In chapter III we considered the formulation of problem $_{n.n.}B_1$ that appeared to be more complicated, on a higher level of the hierarchy discussed there, than that of problem C_1. In the formulations that we consider now the situation is somewhat different. It will come out that the two problems cannot be solved by KAUFMAN's method for the same reason, be it that with problem $_{n.n.}B_2$ products of only 4 Γ-matrices occur instead of products of Γ-matrices where the number of factors is of the same order of magnitude as the number of elements in one of the three directions of the lattice, which number therefore grows with the dimensions of the lattice. We start from the treatment of the triangular lattice given by HOUTAPPEL [32] in 1950. We do this because firstly this is an apt point to start from and secondly because here we have again a treatment where the form of the matrices is found by inspection*.

Consider a lattice, with toroidal boundary conditions, consisting of 2m×p elements that form p rows of 2m elements that we draw in a special way, viz. as zig-zag lines (drawn lines in figure 32).

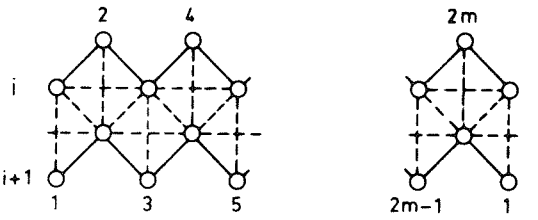


figure 32

The interaction energies are defined as follows

* HUANG and HOUTAPPEL are the only authors I know who , from the way they present the problems, also seem to have abandoned the building-up principle completely.

J_1 for pairs $\sigma_{i,j}$ and $\sigma_{i,j+1}$ with j odd and for pairs $\sigma_{i,j}$ and $\sigma_{i+1,j+1}$ with j even
J_2 for pairs $\sigma_{i,j}$ and $\sigma_{i,j+1}$ with j even and for pairs $\sigma_{i+1,j}$ and $\sigma_{i,j+1}$ with j odd
J_3 for pairs $\sigma_{i,j}$ and $\sigma_{i+1,j}$ with j arbitrary
J_4 for pairs $\sigma_{i,j}$ and $\sigma_{i,j+2}$ with j arbitrary.

With H_1, H_2, H_3 and H_4 for J_1/kT, J_2/kT, J_3/kt and J_4/kT the partition function now reads

$$Z_{px2m} = \sum_{\{\sigma_{i,j}\}} \prod_{i=1}^{p} \Big[H_1 \sum_{j=1}^{m} \sigma_{i,2j-1}\,\sigma_{i,2j} + H_2 \sum_{j=1}^{m} \sigma_{i,2j}\,\sigma_{i,2j+1} + H_3 \sum_{j=1}^{m} \sigma_{i,2j-1}\,\sigma_{i+1,2j-1} + H_1 \sum_{j=1}^{m} \sigma_{i,2j}\,\sigma_{i+1,2j+1} + H_2 \sum_{j=1}^{m} \sigma_{i+1,2j-1}\,\sigma_{i,2j} + H_3 \sum_{j=1}^{m} \sigma_{i,2j}\,\sigma_{i+1,2j+1} + H_4 \sum_{j=1}^{2m} \sigma_{i+1,j}\,\sigma_{i+1,j+2} \Big] . \qquad \text{(IV, 110)}$$

Thus

$$Z_{px2m} = \sum_{\{\sigma_{i,j}\}} \prod_{i=1}^{p} V_{\sigma_{i,1},\sigma_{i,2},\ldots,\sigma_{i,2m};\,\sigma_{i+1,1}\,\sigma_{i+1,2},\ldots,\sigma_{i+1,2m}} = \mathrm{Sp}\; V^p, \qquad \text{(IV, 111)}$$

where V is a matrix of the order 2^{2m}.

Indicating the rows by $(\sigma_1, \ldots, \sigma_{2m})$ and the columns by $(\sigma'_1, \ldots, \sigma'_{2m})$ the general element of V is

$$V_{\sigma_1,\ldots,\sigma_{2m};\,\sigma'_1,\ldots,\sigma'_{2m}} = \exp\Big[\sum_{j=1}^{m} (H_1 \sigma_{2j-1}\,\sigma_{2j} + H_2 \sigma_{2j}\sigma_{2j+1} + H_3 \sigma_{2j-1}\sigma'_{2j-1} + H_2 \sigma'_{2j-1}\sigma_{2j} + H_1 \sigma_{2j}\sigma'_{2j+1} + H_3 \sigma_{2j}\sigma'_{2j+1}) + H_4 \sum_{j=1}^{2m} \sigma'_j \sigma'_{j+2} \Big] . \qquad \text{(IV, 112)}$$

The last factor

$$\exp\Big(H_4 \sum_{j=1}^{2m} \sigma'_j \sigma'_{j+2}\Big)$$

is extra in comparison with the corresponding expression in HOUTAPPEL's analysis. One can verify that the right-hand side of (IV. 112) can be written as

$$\sum_{\{\sigma^{(2)},\sigma^{(3)},\sigma^{(4)},\sigma^{(5)}\}} \exp\Big[\sum_{j=1}^{m} (H_1 \sigma_{2j-1}\,\sigma_{2j} + H_2 \sigma_{2j}\,\sigma_{2j+1}) \Big] \cdot \prod_{j=1}^{m} \delta_{\sigma_j,\sigma_j^{(5)}} \qquad \text{I}$$

$$\cdot \exp\Big[\sum_{j=1}^{m} H_3 \sigma_{2j-1}^{(5)} \sigma_{2j-1}^{(4)} \Big] \cdot \prod_{j=1}^{m} \delta_{\sigma_{2j}^{(5)},\sigma_{2j}^{(4)}} \qquad \text{II} \qquad \text{(IV, 113)}$$

$$\cdot \exp\left[\sum_{j=1}^{m} (H_2 \sigma_{2j-1}^{(4)} \sigma_{2j}^{(4)} + H_1 \sigma_{2j}^{(4)} \sigma_{2j+1}^{(4)}\right] \cdot \prod_{j=1}^{2m} \delta_{\sigma_j^{(4)}, \sigma_j^{(3)}} \qquad \text{III}$$

$$\cdot \exp\left[\sum_{j=1}^{m} H_3 \sigma_{2j}^{(3)} \sigma_{2j}^{(2)}\right] \cdot \prod_{j=1}^{m} \delta_{\sigma_{2j-1}^{(3)}, \sigma_{2j-1}^{(2)}} \qquad \text{IV}$$

$$\cdot \exp\left[\sum_{j=1}^{2m} H_4 \sigma_{2j}^{(2)} \sigma_{2j}^{(2)}\right] \cdot \prod_{j=1}^{2m} \delta_{\sigma_j^{(2)}, \sigma_j'} \qquad \text{V}$$

We should of course note cyclic boundary conditions.

In this rather artificial way, by introducing the auxiliary σ variables, we see that V can be considered to be a product of 5 matrices, corresponding to the factors (I) up to (V) in (IV, 113). We are mainly interested in the fifth matrix. We shall not repeat HOUTAPPEL's further analysis. The first and the third matrix are diagonal and representations of operators that are products of operators of the type $\exp[\text{const.}\, \mathbf{I}_j \mathbf{I}_{j+1}]$. The second and the fourth matrix are representations of operators that represent row-row interactions and that are, one guesses rightly, products of operators of the type $\exp[\text{const.}\, \mathbf{C}_j]$. The fifth matrix finally is diagonal and a representation of the operator

$$\mathbf{V}_v = \exp\left[H_4 \sum_{j=1}^{2m} \mathbf{I}_j \mathbf{I}_{j+2}\right] = \exp\left[H_4 \sum_{j=1}^{2m} \mathbf{I}_j \mathbf{I}_{j+1} \mathbf{I}_{j+1} \mathbf{I}_{j+2}\right], \qquad (IV, 114)$$

where use has been made of the fact that $\mathbf{I}_{j+1}^2$ is the unit operator. Using the form of the representations of the operators $\mathbf{I}$, as given in (IV, 70), we obtain

$$V_v = \prod_{j=1}^{2m} \exp\left[H_4 I_j I_{j+2}\right] = \prod_{j=1}^{2m} \exp\left[H_4 I_j I_{j+1} I_{j+1} I_{j+2}\right]. \qquad (IV, 115)$$

This form one could of course also have had by inspection of the factor (V) in formula (IV, 113)*. Transition to Γ-matrices as in the former paragraph leads to

$$V_v = \prod_{j=1}^{2m-2} \exp\left[-H_4 \Gamma_{2j} \Gamma_{2j+1} \Gamma_{2j+1} \Gamma_{2j+2}\right] \cdot \exp\left[+H_4 U (\Gamma_{4m-2} \Gamma_{4m-1} \Gamma_{4m} \Gamma_1 + \Gamma_{4m} \Gamma_1 \Gamma_2 \Gamma_3)\right], \qquad (IV, 116)$$

where now $U = C_1 . C_2 . \ldots . C_{2m}$. This is the result mentioned before. Thus the KAUFMAN analysis fails in the case of problem ${}_{n.n.}B_2$ as well as in the various formulations of problem C_3.

* Note that we are now dealing with KRONECKER products of 2m matrices of the order 2.

§6. THE CONNECTION WITH QUANTUM - FIELD THEORETIC FORMULATIONS. AN ATTEMPT TO "LINEARIZE"

Recent developments tend in the direction of a translation of the problems into field-theoretic problems. The general idea can be said to be that one may hope to apply the various techniques developed in that field, if only to find sharply approximating solutions. In this connections we refer to the articles written by HURST in 1966 and by GIBBERD and HURST in 1967. In the first named article it is shown what difficulties arise when going over to a Fermion problem from a formulation in terms of Pfaffians. More specifically it is shown that for the problem ${}_{n.n.}B$ expressions occur that are of the fourth order in the Fermion creation and annihilation operators introduced and it is said that for problem C much more complicated expressions in these operators might occur. In the article of SCHULTZ, MATTIS and LIEB it is stated that when between other elements than nearest neighbours occur expressions of higher order than the second in terms of the Fermion operators introduced will turn up. Let us try to discover similar difficulties now that we have various matrix formulations at hand.

In the former paragraphs it has been shown that we can see the matrices in question as representations of operators. Analogously to the way THOMPSON introduces Fermion creation and annihilation operators, we define these operators $a_\alpha^\dagger$ and a_α in terms of their representations accordingly to

$$a_\alpha + a_\alpha^\dagger = \Gamma_{2\alpha}$$

$$a_\alpha - a_\alpha^\dagger = i\,\Gamma_{2\alpha-1} \quad . \tag{IV, 117}$$

The Γ's are of course the same as in (IV, 78). One easily verifies by means of (IV, 79) that the quantities introduced indeed satisfy the anticommutation rules

$$\left[a_\alpha, a_\beta^\dagger\right]_+ = E'\,\delta_{\alpha\beta} \quad . \tag{IV, 118}$$

Now because of

$$\Gamma_{2\alpha}\Gamma_{2\alpha+1} = (a_{\alpha+1} - a_{\alpha+1}^\dagger)(a_\alpha + a_\alpha^\dagger)$$

$$i\,\Gamma_{2\alpha-1}\Gamma_{2\alpha} = 2a_\alpha a_\alpha^\dagger - E' \quad , \tag{IV, 119}$$

we see that forms that are quadratic in the Γ-matrices are also quadratic in the Fermion operators.

We now directly verify from the results in the former paragraph that for problem ${}_{n.n}B_2$ expressions occur that are of the fourth order in the Fermion operators and that indeed for all three formulations of the problem C_3 expressions occur that are of a much higher order in these operators. It is now easy to translate the various formulations into the language of these operators, if one wants to enter field theory. We shall not do this as the most relevant result from our point of view has been obtained. It has been shown what difficulties in these modern formulations correspond to the difficulties in the more "classical" formulations, discussed in this chapter.

So far the ONSAGER method has not been considered at all. We have already quoted in chapter I the remark of NEWELL and MONTROLL about the failure of that method in the unsolved cases. Now when for these problems we generate a LIE algebra with the matrices respectively operators in question, it is now clear that these will have a more complicated structure than in the case of problem B_2. We could of course start an investigation of these algebras, but in view of the fact that the KAUFMAN method and the ONSAGER method are strongly related we may better try to solve the corresponding difficulties in the more transparant formulations that we have given here.

Formulations in which the Γ-matrices, or the Fermion operators, occur in quadratic forms only, but for some special factor like the matrix U in the formulation of problem B_2, are often called "linear" or "non-essentially non-linear". In the case of the problem C_3 and ${}_{n.n.}B_2$ we have to deal then with "essentially non-linear" problems. The natural question to ask now is *"Can we linearize?"* Usually one does not report on failing attempts. The reason to break with this (understandable but bad) habit lies in the fact that linearization in such an obvious thing to try and in the fact that the set-up of this chapter is to have a survey of the impossibilities of the matrix method. The now following attempt to linearize is made along the lines of the KAUFMAN analysis using some concepts developed in chapter II. On one hand we mean abandoning the building-up principle in change for "seeing" a formulation and on the other hand we mean the concept of a counting chain.

Let the lattice consist of ℓ layers of $m \times (m-1)$ elements, in each of which the counting chain for problem B_1 can be indicated. In the former paragraph we met the difficulty in the formulation of problem C_3 that the second time an element was met along the counting chain it could not take care of the numbering of some factor in the KRONECKER products occurring in the matrices V_1 that represent the interactions in some layer. This would lead to matrices of the order $2^{2m(m-1)}$ as each element is met twice, whereas there are only $2^{m(m-1)}$ states possible for a layer. However, when we simply consider the matrix

$$V_1 = \exp\left[H' \sum_{\alpha=1}^{m(m-1)} I_{2\alpha-1} I_{2\alpha} + H'' \sum_{\alpha=1}^{m(m-1)} I_{2\alpha} I_{2\alpha+1} \right] , \qquad (IV,120)$$

where the matrices I_α are now KRONECKER products of $2m(m-1)$ matrices of the order 2 of the same type as the matrices I_i defined in formula (IV,70), we do have a matrix that would permit a KAUFMAN analysis. The price we have to pay for this "linearization" is the following.

An element occurs twice in the counting chain and therefore twice takes care of the numbering of the rows and columns of a factor in the KRONECKER product I_α. The concept that is introduced is that of "non-physical" states. It is possible that an element of the counting chain determined a value +1 for the spin variable of the element it corresponds with whereas the other corresponding element determines a value -1 for that same element. In problem B_2[2] we had to deal with a matrix V for which the counting chain was that for a linear lattice in which element occurs only once with the consequence that each state of a row is a "physical" state. Now for each physical state, in which each pair of elements on the counting chain determines the same value for the spin variable of the corresponding element, there are $2^{m(m-1)}-1$ non-physical states. This is seen by letting one of both elements be characteristic for the "real" value for the spin variable. The other element can then be "right" or "wrong" and this is the case for all

m.(m-1) pairs of elements on the counting chain. The states that take care of the numbering of the rows and columns of the matrix V_1 in (IV, 120) admit a division in $2^{m(m-1)}$ classes each containing $2^{m(m-1)}$ states viz. 1 physical and $2^{m(m-1)}-1$ non-physical states. Let alternately in the counting chain an element determine the value of the spin variable of the corresponding element of the lattice, then m.(m-1) characterizing elements have been defined, the elements that took care of the numbering of the states in the formulation C_3[3] in the former paragraph.

Now consider a certain physical state of a layer. This state will contribute a factor $\prod_{i=1}^{m(m-1)} F_{i,1} F_{i,2}$, where $F_{i,1}$ and $F_{i,2}$ are factor contributions of the bonds of the i-th group of two bonds between successive characterizing elements. Thus $F_{i,1}$ can be $e^{\pm H'}$ and $F_{i,2}$ can be $e^{\pm H''}$. With the special choice of the characteristic elements that we have made one can show that the states of the class to which that physical state belongs give the factor contribution $\prod_{i=1}^{m(m-1)} (F_{i,1} F_{i,2} + F_{i,1}^{-1} F_{i,2}^{-1})$*. We now cut a long story short. It has not been possible to change the constants H' and H'' in such a way that the states of a class all give the same, right, factor contribution nor to throw the ballast of the inverse powers of $F_{i,1}$ and $F_{i,2}$ overboard in another way.

There remains the possibility to let the contributions of the non-physical states compensate each other by making a handy choice of the matrix V_2 that has to represent the interaction between the layers. This matrix will also have to be of a certain structure viz. it has to be a KRONECKER product of matrices of the type $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ or e.g. alternately of two different matrices of that type if one wants to apply the KAUFMAN analysis. One can come quite far by choosing e.g. the matrices

$$\begin{pmatrix} e^{\frac{1}{2}H} & ie^{-\frac{1}{2}H} \\ ie^{-\frac{1}{2}H} & e^{\frac{1}{2}H} \end{pmatrix} \text{and} \begin{pmatrix} e^{\frac{1}{2}H} & -ie^{-\frac{1}{2}H} \\ -ie^{-\frac{1}{2}H} & e^{\frac{1}{2}H} \end{pmatrix}.$$

However, there still is no complete compensation of unwanted terms. The situation is rather analogous to that with the Pfaffian method. One can come to an answer but this answer takes into account various contributions to the partition function that should not be there.

§7. YET ANOTHER MATRIX FORMULATION OF THE PROBLEM FOR THE SIMPLE CUBIC LATTICE

The only formulation of the scheme given in the introduction that has not yet been discussed is that of problem C_2. We now consider a lattice consisting of rows of n elements and assume a counting off of these rows as for the single elements in problem B_1. Thus the lattice is chosen as a block with dimensions $m \times (m-1) \times n$. We arrive at the same formulation whether we

* A sideway remark in this connection is the following. The sum of states (physical and non-physical) of a single layer is $(4\ \mathrm{ch}H'\ \mathrm{ch}H'')^N + (4\ \mathrm{sh}H'\ \mathrm{sh}H'')^N$ because it is considered as a linear chain of 2N independent elements. We can consider problem B as the problem to calculate $\{2^{m(m-1)}\sum_{\text{states}}\} \prod_{i=1}^{m(m-1)} F_{i,1} F_{i,2}$ knowing what $\{2^{m(m-1)}\sum_{\text{states}}\} \prod_{i=1}^{m(m-1)} (F_{i,1}F_{i,2} + F_{i,1}^{-1} F_{i,2}^{-1})$ is. Similar formulations can be given for the problems $C_{n.n.}$ B etc.

generalize from problem B_1 or from problem B_2. The difficulty that arises is again to carry out the summation over the indices, that again occur twice in the array of matrix elements, in the proper way when considering a summation over the spin variables as a summation over the indices of matrix elements. For this we need a lemma analogous to the one given in chapter II for the deduction of problem B_1. As a result of the application of this lemma the partition function will turn out to be a sum of spurs of matrices of the order 2^n. Even if these spurs could be calculated e.g. by means of the KAUFMAN analysis there still remains the summation of these spurs. This rather unpleasant prospect can only be partly compensated by the hope that in the resulting series the main terms might be indicated and consequently an approximate solution.

In conformity with the goal and the character of this chapter we shall still give at least the results of our analysis. Moreover one of the results will be needed in the next chapter.

Lemma V: When A, B, C and D are matrices of the order 4 and E is the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and I is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then for arbitrary indices i, k, l and m

$$\sum_{j=1}^{4} A_{ij} B_{jk} C_{lj} D_{jm} = \frac{1}{4} \sum_{p=1}^{4} (AI_p^{(2)} B)_{ik} (CI_p^{(2)} D)_{lm} , \qquad (IV,121)$$

where $I_p^{(2)}$ is one of the four matrices E x E, E x I, I x E and I x I.

The proof follows by writing out, analogously to the proofs of the corresponding lemma given in the chapters II and III. Lemma V is needed in the next chapter. The following lemma can be proved by induction from this lemma or from lemma I.

Lemma VI: When A, B, C and D are matrices of the order 2^n, E is the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and I the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then for arbitrary indices i, k, l and m

$$\sum_{j=1}^{2^n} A_{ij} B_{jk} C_{lj} D_{jm} = 2^{-n} \sum_{p=1}^{2^n} (AI_p^{(n)} B)_{ik} (CI_p^{(n)} D)_{lm} , \qquad (IV,122)$$

where $I_p^{(n)}$ is one of the 2^n possible KRONECKER products that can be formed by n factors chosen from E and I.

We omit the proof although it is not entirely trivial.

When V_1 is the matrix that represents the interactions in a row and V_2 and V_3 are matrices that represent the interactions between rows in the two directions, whose structure is the same as that of V_1 and V_2 in problem B_2, the partition function becomes

$$Z_{n \times m \times (m-1)} = \sum_{\{\nu_{i,j}\}} (V_1)_{\nu_{1,1};\nu_{1,1}} (V_2)_{\nu_{1,1};\nu_{1,2}} (V_3)_{\nu_{1,2};\nu_{2,2}} (V_1)_{\nu_{2,2};\nu_{2,2}} \cdots \qquad (IV,123)$$

with the same ordering for the indices $\nu_{i,j}$ as in problem B_1 for the indices $\sigma_{i,j} \cdot \nu_{i,j}$ indicates the state of the row (i, j) and $1 \leq \nu_{i,j} \leq 2$. An extra facet in comparison with the problem B_1 is the occurrence of the diagonal matrix V_1. Due to the fact that V_1 is diagonal we may apply lemma VI (IV, 122) and can carry out the summation over the spin variables which comes down to

a summation over the indices $\nu_{i,j}$. Each summation splits the partition function into 2^n terms and a factor 2^{-n} appears in front of each term. The summation over all variables finally leads to $2^{m(m-1)n}$ terms of the form

$$2^{-m(m-1)n}\, Sp(V_1 V_2 V_3 V_1 V_2 V_3 V_1 \cdots) \quad , \qquad \text{(IV, 124)}$$

where between the matrix factors occur pairs of matrices $I_p^{(n)}$, just as in problem B_1 pairs of factors E or I occurred. These factors $I_p^{(n)}$ occur in the array of matrix factors on exactly the same places as the pairs $I_p^{(1)}$ in problem B_1.

The matrices "between" a pair of matrices $I_p^{(n)}$ are KRONECKER products and either diagonal (V_1) or consist of factors

$$\begin{pmatrix} e^{H'} & e^{-H'} \\ e^{-H'} & e^{H'} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{H''} & e^{-H''} \\ e^{-H''} & e^{H''} \end{pmatrix} \quad .$$

Annihilation of two factors $I_p^{(n)}$ again causes infections of the matrices in between. In problem B_1 there were two possibilities: infection or not, here there are 2^n possibilities to infect the matrices. There is one possibility that no infections occur namely when $I_p^{(n)} = E \times E \times \ldots \times E$, the 2^n-dimensional unit matrix. We see what happens when the various groups of infections are combined. Only when two groups of the same type are combined is there total disinfection of the matrices that belong to both groups. So we obtain much more complicated infection-disinfection patterns than in problem B_2. Moreover it is not clear whether after treating this combinatorial aspect the calculation of the spurs can be carried out as "easily" as in problem B_2 as the various matrices do not commute any more. An attack on problem C along this road is therefore not very hopeful.

CHAPTER V

OTHER ISING PROBLEMS

§1. THE ONE-DIMENSIONAL PROBLEM IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD

We consider the problem A^M. Let the interaction of an element $i (1 \leq i \leq N)$ with an external magnetic field M give a contribution $-\sigma_i \mu M$ to the energy, where the magnitude of the magnetic moment of the element is μ and σ_i is the spin variable of the element i. A factor contribution to a term of the partition function will then be either $\exp[\mu M/kT]$ or $\exp[-\mu M/kT]$*. Putting

$$\frac{\mu M}{kT} = B, \tag{V,1}$$

these factors can be seen as the diagonal elements $M_{\sigma_i;\sigma_i}$ of the matrix

$$M = \begin{pmatrix} e^{B} & 0 \\ 0 & e^{-B} \end{pmatrix}, \tag{V,2}$$

where the spin variable σ_i takes care of the numbering of rows and columns. The further specification of the problem is as in chapter II, §1 and the partition function for this model becomes

$$Z_N(M) = \sum_{\{\sigma_i\}} V_{\sigma_1;\sigma_2} M_{\sigma_2;\sigma_2} V_{\sigma_2;\sigma_3} \cdots V_{\sigma_N;\sigma_1} M_{\sigma_1;\sigma_1} = Sp(VM)^N . \tag{V,3}$$

As now

$$VM = \begin{pmatrix} e^{H+B} & e^{-H-B} \\ e^{-H+B} & e^{H-B} \end{pmatrix} \tag{V,4}$$

is a matrix with eigenvalues

* Note that the choice of the sign that is linked to the direction of the field is arbitrary.

$$\lambda_{1,2} = e^{H} chB \pm e^{H} \sqrt{sh^2B + e^{-4H}} \qquad (V,5)$$

and the spur is invariant under similarity transformations we find

$$Z_N(M) = \lambda_1^N + \lambda_2^N . \qquad (V,6)$$

The quantity that we are mainly interested in is the free energy per element, from which all thermodynamical information can be obtained. Leaving out a factor -kT this is the quantity

$$f_N = \frac{1}{N} \log Z_N \qquad (V,7)$$

or, more relevantly, in the thermodynamic limit

$$f = \lim_{N\to\infty} f_N = \log \lambda_{max} = \log \left[e^{H} chB + e^{H} \sqrt{sh^2B + e^{-4H}} \right] , \qquad (V,8)$$

where λ_{max} is the larger of λ_1 and λ_2.

This constitutes the solution to our problem. To investigate whether there is a spontaneous magnetization m(0), so typical for the systems for which ISING developed his model, we consider

$$m(0) = \lim_{M\to 0} m(M) = \lim_{M\to 0} kT \frac{\partial f}{\partial M} = \lim_{M\to 0} \frac{\mu\, shB}{\sqrt{sh^2B + e^{-4H}}} = 0 \qquad (V,9)$$

as $B \to 0$ with $M \to 0$. So there is no spontaneous magnetization for a one-dimensional ISING model. Of course for large fields practically all elements must be directed along the field. Indeed we then do have $m(M) \approx \mu$.

§2. FORMULATIONS OF THE PROBLEMS FOR OTHER MODELS

We consider the quadratic ISING lattice and want to incorporate an external magnetic field in a formulation that is analogous to that of the problems B_1, C_1 and ${}_{n.n.}B_1$ etc. Problem B_1^M is in this respect representative of other problems like C_1^M, ${}_{n.n.}B_1^M$, etc. and we shall consider this problem in some detail. In chapter II we have seen how to come to the formulation of problem B_1. Again we want to let the various factor contributions to a term of the partition function be given by a matrix element and to be able to consider the summation over the values of the spin variables as a summation over the matrix indices. This can be done by ordering these elements along with the bonds they correspond with on the counting chain. The difficulty that we meet is the fact that a certain spin variable $\sigma_{i,j}$ is encountered twice as second and first index of consecutive matrices, between which diagonal matrices may occur that have to take care of the interaction of the elements with the external magnetic field. This can be solved by the following lemma.

Lemma VII: When A, B, C and D are arbitrary matrices of order 2 whereas P and Q are diagonal matrices of that order and $I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then for all combinations of indices i, k, l and m we have

$$\sum_{j=1}^{2} A_{ij} P_{jj} B_{jk} C_{lj} Q_{jj} D_{jm} = \tfrac{1}{2}\left[(APB)_{ik} (CQD)_{lm} + (APIB)_{ik} (CQID)_{lm} \right] . \qquad (V,10)$$

The proof follows from writing out the right-hand side of (V,10), just as in the case of the corresponding lemma I for the problem B_1, given in chapter II. In that case P and Q do not occur.

The matrices P and Q will have to represent the interaction of an element with the external magnetic field by means of their elements. On the two places on the counting chain that correspond to an element of the lattice we can let the matrix element both times give a factor $\exp[B/2]$ or a factor $\exp[-B/2]$ depending on the value of the spin variable. This means that P and Q are both chosen as

$$M^{\frac{1}{2}} = \begin{pmatrix} e^{B/2} & 0 \\ 0 & e^{-B/2} \end{pmatrix} . \qquad (V,11)$$

The partition function now is

$$Z_N(M) = \sum_{\{\sigma_{i,j}\}} (V_1)_{\sigma_{1,1};\sigma_{1,2}} (M^{\frac{1}{2}})_{\sigma_{1,2};\sigma_{1,2}} (V_2)_{\sigma_{1,2};\sigma_{2,2}} (M^{\frac{1}{2}})_{\sigma_{2,2};\sigma_{2,2}} \cdots \qquad (V,12)$$

$$\cdots (M^{\frac{1}{2}})_{\sigma_{1,1};\sigma_{1,1}} ,$$

when the element (1,1) is chosen as the starting point for the counting off procedure of the m.(m-1) bonds. Another possibility is to take the unit matrix one time and the matrix M the other time. An alternative is therefore

$$Z_N(M) = \sum_{\{\sigma_{i,j}\}} (V_1)_{\sigma_{1,1};\sigma_{1,2}} (V_2)_{\sigma_{1,2};\sigma_{2,2}} (M)_{\sigma_{2,2};\sigma_{2,2}} (V_1)_{\sigma_{2,2};\sigma_{2,3}} \cdots \qquad (V,13)$$

$$\cdots (M)_{\sigma_{1,1};\sigma_{1,1}} .$$

Application of lemma VII (V,10) leads to a splitting up of the partition function into 2^N terms of the form

$$2^{-N} \operatorname{Sp}(V_1 M^{\frac{1}{2}} V_2 M^{\frac{1}{2}} V_1 M^{\frac{1}{2}} \cdots V_2 M^{\frac{1}{2}}) \qquad (V,14)$$

respectively

$$2^{-N} \operatorname{Sp}(V_1 V_2 M V_1 V_2 M \cdots V_2 M) , \qquad (V,15)$$

where in the array of matrix factors all kinds of combinations of matrices I occur. These matrices commute with the matrices $M^{1/2}$ or M. The matrices I occur in pairs that by means of commutation can be brought together and then annihilate each other as $I^2 = E$, the unit matrix. The resulting infection

patterns are the same as those in the case of problem B_1. Our problem is now to calculate and sum the various terms like (V, 14) or (V, 15).

A more general formulation can be given where terms like

$$2^{-N}\ \mathrm{Sp}(V_1 M^{s}\ V_2 M^{1-s} V_1 M^{s} V_2 \ldots\ V_2 M^{1-s}) \qquad (V,16)$$

occur. Here the number s can be chosen quite arbitrarily. The choice $s = \frac{1}{2}$ leads to the most "regular" formulation. An obvious thing to try is to make use of this arbitrariness of s. However, no special choice of s has led to any useful result in connection with further calculations.

The problems C_1^M and ${}_{n.n.}B_1^M$ can be formulated in an analogous way by using the concepts developed for the problems C_1 and ${}_{n.n.}B_1$. In the first case terms have to be summed that are e.g. of the type

$$4^{-N}\ \mathrm{Sp}(V_1 M^{\frac{1}{3}} V_2 M^{\frac{1}{3}} V_3 M^{\frac{1}{3}} V_1 M^{\frac{1}{3}} \ldots)\ . \qquad (V,17)$$

In the second case we encounter terms like

$$8^{-N}\ \mathrm{Sp}(V_1 M^{\frac{1}{4}} V_2 M^{\frac{1}{4}} V_3 M^{\frac{1}{4}} V_4 M^{\frac{1}{4}} V_1 M^{\frac{1}{4}} \ldots)\ . \qquad (V,18)$$

We see that also in formulations like ours an external magnetic field can be incorporated quite naturally. We shall now investigate in which way the infection-disinfection problem is changed due to the extra matrices $M^{1/2}$, $M^{1/3}$, $M^{1/4}$ etc. We only consider the problem B_1^M as this problem is representative of the other problems.

§3. CORRELATION FUNCTIONS. DEVELOPMENT OF THE PARTITION FUNCTION IN TERMS OF MULTIPLE CORRELATIONS

For simplicity we assume that the interactions for both directions in the quadratic lattice are the same, i.e. $H_1 = H_2 = H$ and $V_1 = V_2 = V$. We consider a term like that given by (V, 14). The matrices $M^{1/2}$ that stand between the matrices V and the infected matrices V' do not commute with these matrices. Therefore the spur cannot be calculated in the easy way it can be calculated in case the matrices $M^{1/2}$ are lacking. This is an essential difference with problem B as can be verified by assuming that it is allowed to let $M^{1/2}$ commute with V and V' or V $M^{1/2}$ with V' $M^{1/2}$. In both cases one arrives at irrelevant answers for quantities like the magnetization per element m(0) or the susceptibility $\chi = M^{-1} m(M)$. We omit the tedious and unimportant calculations.

Thus we are forced to calculate with the matrices $M^{1/2}$ in the right way. We remark that

$$M^{\frac{1}{2}} = \mathrm{ch}\tfrac{1}{2}B\ E + \mathrm{sh}\tfrac{1}{2}B\ I = \mathrm{ch}\tfrac{1}{2}B\ (E + \mathrm{th}\tfrac{1}{2}B\ I) \overset{\text{p.d.}}{=} \mathrm{ch}\tfrac{1}{2}B\ (E + P_{\frac{1}{2}}\ I). \qquad (V,19)$$

Writing the factors $M^{1/2}$ in this way and writing out the multiplication will lead to 2^{2N} terms, partial spurs, of the type

$$(\mathrm{ch}\tfrac{1}{2}B)^{2N}\, p_{\frac{1}{2}}{}^{k}\ \mathrm{Sp}\ (V.V.\ \ldots\ )\ . \qquad (V,20)$$

Between the matrices V and V', k factors I occur in one of the $\binom{2N}{k}$ possible combinations for the partition of these factors over the 2N places.

Partial spurs with an odd number of factors I are zero. To see this we note that V and V' are of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ as are all products of these matrices. The factors I can be grouped in pairs but for one matrix. Annihilation of these pairs changes the infection pattern and next to the matrix I only factors V and V' occur. As the spur is invariant under cyclic change of the matrix factors such a term vanishes due to

$$\mathrm{Sp}\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Sp}\begin{pmatrix} x & -y \\ y & -x \end{pmatrix} = 0\ . \qquad (V,21)$$

The matrices I that are due to the application of the summation lemma cause "groups" of infections that effect 2m-1 consecutive matrices. Combination of two such groups may cause extra infections or disinfections when a second minus sign appears on the counter diagonal. In problem B_1 the various infection patterns are generated by all possible combinations of such groups of infections. In problem $B_1{}^{M}$ we now see that upon each infection pattern groups of infections are superposed that are due to the matrices I that occur in the matrices $M^{1/2}$. Apart from the factor $(\mathrm{ch}\,\frac{1}{2}B)^{2N}$ each of the 2^{2N-1} remaining partial spurs into which one original spur is splitted possesses a weight factor $p_{1/2}{}^{k}$ when in that partial spur k extra factors I occur. This leads quite naturally to a development in powers of $p_{1/2}$ by taking together from all 2^{N} original spurs the partial spurs with the same weight factor.

To be able to discuss these partial spurs we now introduce the concept of correlation function or simply correlation. The correlation $\langle \sigma_{i,j}.\sigma_{k,l}\rangle$ between two spin variables $\sigma_{i,j}$ and $\sigma_{k,l}$ is defined as the statistical mean of $\sigma_{i,j}\cdot\sigma_{k,l}$ viz.

$$\langle \sigma_{i,j}\cdot\sigma_{k,l}\rangle = \frac{\sum_{n=1}^{2^N} \exp[-E_n/kT]\,(\sigma_{i,j}\cdot\sigma_{k,l})_n}{Z_N}\ , \qquad (V,22)$$

where Z_N is the partition function, here for problem B_1, and $(\sigma_{i,j}\cdot\sigma_{k,l})_n$ denotes the value of the product $\sigma_{i,j}\cdot\sigma_{k,l}$ in the n-th state. When we want to introduce the factor $\sigma_{i,j}$ in our formulation in terms of matrices of the order 2 this can be done by identifying $\sigma_{i,j}$ with the matrix element $(I)\sigma_{i,j}$; $\sigma_{i,j}$ and interjecting this matrix element in each term of the partition function for the problem B_1 on one of the two places where the element (i,j) occurs on the counting chain. The same thing can be done for the factor $\sigma_{k,l}$. As I is diagonal the summation can be carried out in view of lemma VII and we obtain the terms of the partition function for problem B_1 but for two extra factors I in each spur on certain places. Denoting the resulting sum by $Z_N(\sigma_{i,j}\,;\sigma_{k,l})$ we may write

$$< \sigma_{i,j} \cdot \sigma_{k,l} > = \frac{Z_N(\sigma_{i,j};\sigma_{k,l})}{Z_N} \quad . \tag{V,23}$$

In an analogous way we can define multiple correlations like $<\sigma_{i,j} \cdot \sigma_{k,l} \cdot \sigma_{m,n} \cdot \sigma_{o,p}>$ etc.

Annihilation of the extra pair of factors I leads to infection or disinfection of a number of matrices V and V' or, we may say, of links of the counting chain. An infection means an extra factor w=thH, a disinfection means a factor 1 instead of an originally present factor w in the contribution of a certain spur to the partition function of problem B_1. We recall that when n is the number of infections in a pattern and A(n) is the number of patterns with n infections the partition function for problem B_1 could be written as

$$Z_N = (2ch^2H)^N \sum A(n)(w^n + w^{2N-n}) , \tag{II,30}$$

where the summation is over all 2^N patterns. Thus the exponents of w are influenced.

Suppose the two factors I are on a "distance" of c links of the counting chain and that c_1 of these links do not belong to the infection pattern whereas c_2 do.* Then c_1 extra infections will appear whereas c_2 links respectively matrices V' will be disinfected. For a certain infection pattern w^n changes into $w^{n+c_1-c_2}$ and w^{2N-n} into $w^{2N-n-c_1+c_2}$. We recall that the second term corresponds to the "complementary" part of the counting chain. Extracting a factor $w^{c_1+c_2}=w^c$ we obtain

$$w^c(2ch^2H)^N \sum A(n)(w^{n-2c_2} + w^{2N-n-2c_1}) = w^c(2ch^2H)^N$$
$$\sum A(n)\left[w^{n-c_2}(w^{-1})^{c_2} + w^{2N-n-c_1}(w^{-1})^{c_1}\right] , \tag{V,24}$$

which is w^c times the partition function for the problem B_1 where now the c links in question contribute a factor w^{-1} instead of a factor w. This fact plays an important role in the article written by MONTROLL, POTTS and WARD [33] in 1963 in which they show how these correlations can be calculated by the construction of Pfaffians and the calculation of the resulting TOEPLITZ determinants.

A point that asks for our attention in the comparison of the formulation in that article and ours is the factor w^c. The number c depends on the choice that was made for the places of the matrices I that could be placed on either of the two places of the elements (i,j) respectively (k,l) on the counting chain. So in first instance there are four possible values for c (two are the same) in our formulation. For the correlation in question, however, all four are equivalent. It is irrelevant whether the matrix I is placed on one place or the other because the spurs in which the group of infections corresponding to an element does occur with a factor I on one place give the same contributions as the spurs in which that group does not occur with a factor I in the other place. We can thus change the distance of both factors I by 2m-1 links. For the same reason can be left out of consideration any part of the c consecutive links that forms a group of infections corresponding to a pair of elements (a,b) when the element (o,p) does not occur on the

* Meant are the links along which the matrices I are brought together by commutation.

2m-2 places in between. The "road" that connects e.g. the elements (1,1) and (o,p) on the lattice and corresponds with the c links in question, or what is left of them, is not disrupted by this procedure. The only thing that changes in this reduction procedure is the number c of bonds that give w^{-1} instead of w. We may now for example choose c by posing the criterium that the number of links left respectively bonds on the lattice that connect the elements (1,1) and (o,p) should be minimal. As an example let $2(2m-1)<c<3(2m-1)$, then two equivalent roads are drawn in figure 33.

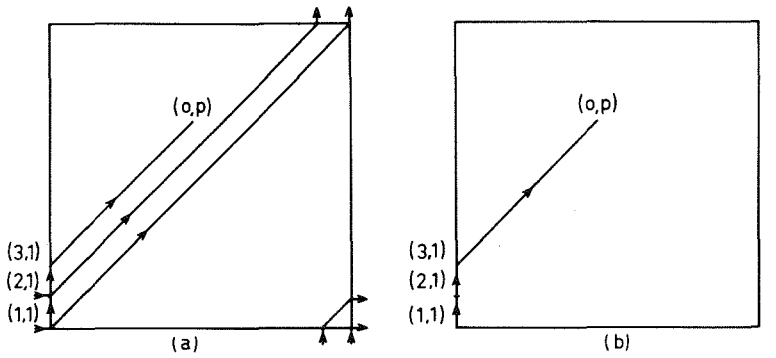


figure 33

The stair-like counting chain on the lattice is drawn as a straight line

The "loops" on (1,1) and (2,1) are left out in (b). Element (o,p) is reached by a breadth step from (o-1,p). We see that o+p-2 bonds connect the element (1,1) and (o,p) in (b). This number does not change when other loops of 2m-1 links are left out, giving rise to other roads from (1,1) to (o,p). This equivalence of roads is also remarked in ref. [33] and used there to obtain a simplest calculation of $<\sigma_{1,1}.\sigma_{o,p}>$. However, also other (longer) roads are equivalent with that consisting of o+p-2 bonds. In formula (15) of ref. [33] it can also be seen easily that the road from (1,1) to (o,p) can be freely chosen. So far the connection with literature. We may also prescribe the place of the extra factors I as the "first" of the two places where they can be interjected in the array of matrix factors. The other, empty, place is then 2m-1 links further on the counting chain. The road in figure 33(a) corresponds to a placing of the two factors I in such a way.

The correlation is now determined by the number c_1 of links between the extra factors I. Let us denote it by $C_1(c_1)$. The four, six, etc. factors I in the multiple correlations form disjoint intervals on the counting chain of c_1, c_2, c_3, etc. links and therefore these correlations can be indicated by $C_2(c_1,c_2)$, $C_3(c_1,c_2,c_3)$ etc. The place of these intervals on the counting chain relative to each other has not yet been specified. We return to the terms of the partition function that are partial spurs as given in (V,20). It is now clear that all partial spurs corresponding to one combination of k extra factors I can be taken together to give

$$(\mathrm{ch}\tfrac{1}{2}B)^{2N} p_{\frac{1}{2}}^{k} Z_N(0)\, C_{\frac{k}{2}} \quad (c_1, c_2, \ldots, c_{\frac{k}{2}}), \qquad (V,25)$$

leading to

$$\frac{Z_N(M)}{Z_N(0)} = (\mathrm{ch}\tfrac{1}{2}B)^{2N}\left\{1 + p_{\frac{1}{2}}^{2} \sum_{\{c_1\}} C_1(c_1) + p_{\frac{1}{2}}^{4} \sum_{\{c_1,c_2\}} C_2(c_1, c_2) + \ldots\right\} \qquad (V,26)$$

The symbols $\{c_1\}$ and $\{c_1, c_2\}$ under the summation signs indicate that the summation is over all possible combinations of intervals of length c_1, c_2, c_3, etc.

In the formulation that we have chosen, starting from (V,12) and (V,14), all combinations of factors I occur. Thus also the combinations in which two factors I occur on the two places corresponding to one element of the lattice. It can be said that then c=2m-1 and can be reduced to zero by the reduction procedure mentioned before. It comes down to a calculation of $Z_N(0)$ but now with a factor $p_{1/2}^2$ for the two factors I and a factor $\binom{N}{1}$ for the number of combinations of these factors. The combinations of four factors I contain analogously $\binom{N}{2}$. $p_{1/2}^4$ times the partition function Z (0) etc. Extracting all these terms and summing leads to a first term $(\mathrm{ch}\frac{1}{2}B)^{2N}(1+\mathrm{th}^2\frac{1}{2}B)^N = (\mathrm{ch}^2\frac{1}{2}B+\mathrm{sh}^2\frac{1}{2}B)^N=(\mathrm{ch}B)^N$. What is left consists of "real" correlations. This phenomenon does not occur when we start from (V,13) and (V,15). With p=thB we are led in an analogous way to

$$\frac{Z_N(M)}{Z_N(0)} = (\mathrm{ch}B)^N\left\{1 + p^2 \sum_{\{c_1\}} C_1(c_1) + p^4 \sum_{\{c_1,c_2\}} C_2(c_1,c_2) + \ldots\right\}. \qquad (V,27)$$

In this form $Z_N(M)$ can be found a.o. in ref. [16] p. 303. Up to now these formulations have only been of formal value. It is evident that also for the problems C^M and ${}_{n.n.}B^M$ similar series developments can be given.

We want to point out here a possibility to find an approximate solution to the problem B_1^M. Starting from (V,14) in which each matrix $M^{1/2}$ is written as $M^{1/4}.M^{1/4}$ we can e.g. infect or disinfect each matrix V_1 or V_2 resp. V_1' or V_2' separately, which involves each time a factor $(\mathrm{th}\frac{1}{4}B)^2$ as $M^{1/4}=\mathrm{ch}\frac{1}{4}B(E+\mathrm{th}\frac{1}{4}BI)$. As a correction on the first approximation $(\mathrm{ch}B)^N.Z_N(0)$ we then obtain an expression consisting of contributions from those correlations that are built up so to say from correlations of nearest neighbours. One can sum this selection of terms to give expressions like that in which the solution for problem B can be given [34].

§4. SOME OTHER FORMULATIONS OF THE PROBLEM FOR THE QUADRATIC LATTICE

We have seen that also problems like B^M and C^M etc. can be formulated as infection-disinfection problems and what extra combinatorial problem is superposed on the basic problem. Next to the one, two, three, etc. groups

of infections of a very specific number of links now all kinds of groups come into play as well, each of which brings along a certain factor depending on the formulation that one choses. The generation of infection patterns is much more complicated than e.g. in the case of problem C_1. Nevertheless, the kind of problem is quite analogous, although not entirely part of the hierarchy outlined in chapter III, due to the factors like $(thB)^2$, etc. That our formulation is quite in between the matrix formulation and the combinatorial formulation of e.g. problem B^M becomes clear from the other ways in which this problem can be formulated.

Firstly there is the formulation B_2^M. In chapter IV the partition function for problem B was given as

$$Z_N(0) = Sp(V_1.V_2)^{\ell} \qquad (IV,66)$$

for a lattice of $N=\ell \times m$ elements. The matrices V_1 and V_2 are of the order 2^m and could be written in terms of KRONECKER products of m matrices of the order 2, which were the well-known PAULI matrices. The interaction of the m elements in a row with the external magnetic field M is incorporated in a matrix V_3 such that

$$Z_N(M) = Sp(V_1.V_2.V_3)^{\ell} \ , \qquad (V,28)$$

where

$$V_3 = \exp\left[B \sum_{i=1}^{m} I_i\right] \qquad (V,29)$$

and

$$I_i = E \times E \times \ldots \times E \times I \times E \times \ldots \times E \ . \qquad (IV,70)$$

The matrix I is the i-th factor in this KRONECKER product.

The matrices V_1 and V_2 can be formulated in terms of 2m ordered matrices Γ_α with the result that essentially only expressions occur that are quadratic in terms of these matrices in the form $\Gamma_\alpha \Gamma_{\alpha+1}$. The fact that this is not the case for the problems C_3 and ${}_{n.n.}B_2$ prevented the application of the KAUFMAN analysis there. Here the matrix V_3 even more strongly sets the KAUFMAN analysis out of action as

$$I_1=\Gamma_1 \ , I_2 = +i\,\Gamma_1\Gamma_2\Gamma_3 \ , I_3 = -\,\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 \ , I_4 = -i\,\Gamma_1 \ldots \Gamma_7 \text{ etc.} \qquad (V,30)$$

For a derivation the reader is referred to chapter IV, §4. Thus not only expressions occur that are not quadratic in the Γ-matrices, but there is also a great variety of such expressions. This is in correspondence with the many types of groups of infections occurring in the formulation B_1^M. On translation into a quantum-field theoretic problem, in the way discussed in chapter IV, §6, there occur, of course, corresponding expressions in the Fermion operators involved.

Secondly there is the formulation B_c^M. Without a magnetic field we have to find the generating function for the number $B(n_1', n_2')$ of all closed figures on the lattice with n_1' resp. n_2' sides in both directions. When there are N elements we have to calculate, as was shown in chapter II, the function

$$L_N(w,w') = \sum B(n_1',n_2')w^{n_1'}w'^{n_2'} , \qquad (II,34)$$

where w=thH and w'=thH'. The complication that occurs on incorporating the magnetic field is the same for this development in hyperbolical tangents, often called high-temperature expansion, as that considered in the former paragraph and leads to a series in p=thB. A similar formulation can be given for problem C_c^M.

On the other hand it is possible to give a development in which the problem is reduced to the determination of the number of ways the elementary rectangles of the lattice can be painted with two different colours such that the number of bonds in both directions that separate the colours is prescribed as well as the total number of rectangles (total surface if one likes) of each colour. This is an extra facet that corresponds to the weight factors occurring in the formulation B_1^M. Such a development is one in terms of a variable exp [-2H] and is, therefore, a "low-temperature expansion" analogous to the one given in chapter IV, §3, in the discussion of the transition point. For a detailed discussion we refer to ref. [16], where it is said that for the cubic lattice a similar approach is possible that, however, is more complicated. So far no progress has been made along this line whereas the other combinatorial approach does not admit application of the Pfaffian method as was also shown in ref. [16].

Next to the formulations B_2^M and B_c^M we pay some attention to the formulation given by SUZUKI [35] in 1965 which is interesting in connection with the formulations given in chapters II and III. SUZUKI showed that the calculation of the partition function for problem B^M is equivalent to the calculation of the partition function for a system consisting of two layers in the limit of infinitely strong interaction in one of both layers. It is obvious that we choose the substitute lattice as one with dimensions m×(m-1)×2 and consider a counting off of pairs of elements (one in each layer) as was done for elements in problem B_1. We now incorporate the interaction energies in elements of matrices of the order 4.

In chapter I, §3, we have said that no more than two values for the interaction energy of two elements would be considered as this would influence the order of the matrices. We considered "spin-$\frac{1}{2}$" systems. For "spin-1" systems there are three values for the spin variables leading to a formulation in terms of matrices of the order 3. For such systems formulations analogous to that of the problems B_1, C_1, etc. can be given, because for the matrices in question summation lemmas, similar to the ones proved in chapters II and III, can be given easily. Here the higher order is a consequence of the fact that we mentally dissect the lattice in strings of two elements. We can now give a formulation for problem B^M analogous to that of problem C_2.

Let H_A characterize the interactions in one layer (A) and H_B the interactions in the other layer (B), whereas H_{AB} characterizes the interaction between both layers. The states $\sigma_i(A,B)$ of a A-B pair are (+,+), (+,-), (-,+) and (-,-) where i indicates the place on the counting chain of a particular pair that, of course, appears twice. Let these states number the rows and the columns of the matrix V_i that represents the interaction between consecutive pairs on the counting chain. This matrix has the form

$(V_1)_{\sigma_i(A,B);\,\sigma_{i+1}(A,B)} =$

$\sigma_i(A,B)$ \ $\sigma_{i+1}(A,B)$	(+,+)	(+,−)	(−,+)	(−,−)
(+,+)	$e^{H_A+H_B}$	$e^{H_A-H_B}$	$e^{-H_A+H_B}$	$e^{-H_A-H_B}$
(+,−)	$e^{H_A-H_B}$	$e^{H_A+H_B}$	$e^{-H_A-H_B}$	$e^{-H_A+H_B}$
(−,+)	$e^{-H_A+H_B}$	$e^{-H_A-H_B}$	$e^{H_A+H_B}$	$e^{H_A-H_B}$
(−,−)	$e^{-H_A-H_B}$	$e^{-H_A+H_B}$	$e^{H_A-H_B}$	$e^{H_A+H_B}$

(V,31)

The interaction between both elements of a pair can be represented by a diagonal matrix V_2 of the form

$$(V_2)_{\sigma_i(A,B);\,\sigma_i(A,B)} = \begin{pmatrix} e^{H_{AB}} & 0 & 0 & 0 \\ 0 & e^{-H_{AB}} & 0 & 0 \\ 0 & 0 & e^{-H_{AB}} & 0 \\ 0 & 0 & 0 & e^{H_{AB}} \end{pmatrix} . \qquad (V,32)$$

Again we order the factors of a term of the partition function that are considered as matrix elements V_1 or V_2 as prescribed by the counting chain. Just as with the matrix M in the formulation B_1^M for this substitute system the elements of the matrix V_2 can be placed as $V_2^{1/2}$ on both places of a pair on the counting chain or on one place in its integral form whereas on the other place no matrix element is present that refers to the interaction between the elements of a pair. Anyhow, the problem is to carry out the summation over the states respectively over the matrix indices correctly. We can use the lemma V of chapter IV, §7. When the matrices $I_p (p=1,2,3,4)$ are

$$I_1 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&-1&0 \\ 0&0&0&1 \end{pmatrix}$$

for arbitrary matrices A, B, C and D and indices i, k, l and m we have

$$\sum_{i=1}^{4} A_{ij} B_{jk} C_{lj} D_{jm} = \frac{1}{4} \sum_{p=1}^{4} (AI_pB)_{ik} (CI_pD)_{mn} . \qquad (IV,121)$$

This result makes it possible to carry out the summation over the indices and we obtain 4^N terms of the type

$$4^{-N}\ \mathrm{Sp}(V_1 V_2^{\frac{1}{2}} V_1 V_2^{\frac{1}{2}} V_1 \dots)\ , \qquad (V,33)$$

where in the same way as was the case with the matrices $I=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ in problem B_1 now all combinations of pairs of matrices I_p occur. Annihilation of these matrices again gives infections, now of three different types and therefore much more complicated.

The important point is that here too the resulting spurs cannot be calculated in a simple way because the matrices do not commute. This is the same reason why we had to make a halt in the formulation of problem B_1^M. It is hardly useful to give more details concerning the various ways of infecting V_1. Let us only remark that a similar formulation can be given for the problems C^M and ${}_{n.n.}B^M$ because (IV,121) can be generalized. For problem C^M we then consider two identical three-dimensional lattices in a four-dimensional space where A-B pairs can be defined by corresponding places in the counting chains of both lattices.

We conclude that this reduction to a substitute system does not result into an essentially simpler problem as far as the formulations as infection-disinfection problems are concerned. So far our discussion of alternative formulations.

§5. ON "DECORATED" LATTICES

In this paragraph we shall consider from the new point of view some problems that have been solved even in the presence of an external magnetic field. These problems are of some variety but we shall consider only two examples as an illustration of the general principle. We reconsider problem B for which a solution is known. In the formulation B_1 the partition function is a sum of 2 terms of the type

$$2^{-N}S_p(V_1 V_2 V_1 V_2 V_1 \dots), \qquad (II,20)$$

Each matrix corresponds to a bond of the lattice. The system can be changed causing changes in the contribution of an original bond and thus changes in the matrices V_1 and V_2. When we see to it that the matrices are changed into matrices V_1^* and V_2^* that are again of the form

$$\begin{pmatrix} p_{1,2} & q_{1,2} \\ q_{1,2} & p_{1,2} \end{pmatrix}$$

for all bonds of the two types, then the problem will remain solvable. Indeed, we have

$$\begin{pmatrix} p_{1,2} & q_{1,2} \\ q_{1,2} & p_{1,2} \end{pmatrix} = \sqrt{p_{1,2}q_{1,2}} \begin{pmatrix} \sqrt{\frac{p_{1,2}}{q_{1,2}}} & \sqrt{\frac{q_{1,2}}{p_{1,2}}} \\ \sqrt{\frac{q_{1,2}}{p_{1,2}}} & \sqrt{\frac{p_{1,2}}{q_{1,2}}} \end{pmatrix} \overset{\text{p.d.}}{=} \sqrt{p_{1,2}q_{1,2}} \begin{pmatrix} e^{H_{1,2}^+} & e^{-H_{1,2}^+} \\ e^{-H_{1,2}^+} & e^{H_{1,2}^+} \end{pmatrix} \qquad (V,34)$$

and the partition function of the new system is $(p_1 q_1 p_2 q_2)^{\frac{1}{2}N}$ times the partition function for problem B in terms of the variables H_1^+ and H_2^+ instead of H and H'.

Before giving an application we remark that we have even more possibilities. With the toroidal boundary conditions that we have assumed we may write

$$\sum_{j=1}^{m} K_1(\sigma_{i,j} - \sigma_{i,j+1}) = 0 \quad \text{for all } i,$$

$$\sum_{i=1}^{m-1} K_2(\sigma_{i,j} - \sigma_{i+1,j}) = 0 \quad \text{for all } j. \qquad (V,35)$$

Adding the left-hand sides of these equations to the exponent of a term of the partition function means that e.g. for the matrix corresponding to the bond between the elements (i,j) and (i,j+1) we may also write

$$V_1 = \begin{pmatrix} e^{H} & e^{2K_1}.e^{-H} \\ e^{-2K_1}.e^{-H} & e^{H} \end{pmatrix}, \qquad (V,36)$$

where K_1 can still be chosen arbitrarily. A similar thing is valid for the matrix V_2 and the parameter K_2. This gives us the possibility to "correct" the elements on the counter diagonal.

We now consider the following problem that was treated in ref. [16]. The lattice is "decorated" by adding an element to each bond that has a ferromagnetic interaction with one of the two elements that form the bond and an antiferromagnetic interaction with the other element. Moreover these extra elements can interact with the magnetic field, whereas the original elements cannot. Summation over the values of the spin variables of the added elements shows us that the factor contribution to a term of the partition function due to an original bond is now given by an element of

$$\begin{pmatrix} e^{H_{1,2}} & e^{-H_{1,2}} \\ e^{-H_{1,2}} & e^{H_{1,2}} \end{pmatrix} \begin{pmatrix} e^{B_{1,2}} & 0 \\ 0 & e^{-B_{1,2}} \end{pmatrix} \begin{pmatrix} e^{-H_{1,2}} & e^{H_{1,2}} \\ e^{H_{1,2}} & e^{-H_{1,2}} \end{pmatrix} = 2 \begin{pmatrix} \mathrm{ch}B_{1,2} & \mathrm{ch}(B_{1,2}+2H_{1,2}) \\ \mathrm{ch}(B_{1,2}-2H_{1,2}) & \mathrm{ch}B_{1,2} \end{pmatrix}. \qquad (V,37)$$

$H_{1,2}$ characterize the interactions for bonds in both directions and $B_{1,2}$ the interactions of the extra elements with the magnetic field. The terms on the counter diagonal are not yet equal and this can be corrected by the factors $\exp[2K_{1,2}]$ and $\exp[-2K_{1,2}]$. Omitting the indices 1,2 for the two directions we have to calculate spurs of products of matrices of the type

$$2 \begin{pmatrix} \mathrm{ch}B & e^{2K}\mathrm{ch}(B+2H) \\ e^{-2K}\mathrm{ch}(B-2H) & \mathrm{ch}B \end{pmatrix}. \qquad (V,38)$$

The choice

$$e^{4K} = \frac{\mathrm{ch}(B-2H)}{\mathrm{ch}(B+2H)} \qquad (V,39)$$

leads to a problem that is solvable according to what has been said before. Note that we have written $B_{1,2}$. The decorating elements need not interact in the same way with the external field. The decorations of length bonds may also differ from those of breadth bonds. Nobody asks for the relevance of this model anyhow. One thing and another can, of course, immediately be carried out for an exact or approximate solution of e.g. problem C too.

There are no doubt other systems where one can come to an exact solution along this road. Letting out of consideration the solvability rather general transformations of problems for decorated lattices to problems for equivalent non-decorated lattices are possible as was shown by FISHER [36] in 1959. FISHER [37] also gave the following example of a solvable problem.

The decorating element has a ferromagnetic interaction with the elements of the bonds it decorates when this is e.g. a length bond and an antiferromagnetic interaction when this is a breadth bond. Again only the added elements interact with the magnetic field. These situations are represented per original bond respectively by the matrix

$$A = \begin{pmatrix} e^{H_1} & e^{-H_1} \\ e^{-H_1} & e^{H_1} \end{pmatrix} \begin{pmatrix} e^{B_1} & 0 \\ 0 & e^{-B_1} \end{pmatrix} \begin{pmatrix} e^{H_1} & e^{-H_1} \\ e^{-H_1} & e^{H_1} \end{pmatrix} = 2 \begin{pmatrix} \mathrm{ch}(B_1 + 2H_1) & \mathrm{ch}B_1 \\ \mathrm{ch}B_1 & \mathrm{ch}(B_1 - 2H_1) \end{pmatrix} \qquad (V,40)$$

and the matrix

$$B = \begin{pmatrix} e^{-H_2} & e^{H_2} \\ e^{H_2} & e^{-H_2} \end{pmatrix} \begin{pmatrix} e^{B_2} & 0 \\ 0 & e^{-B_2} \end{pmatrix} \begin{pmatrix} e^{-H_2} & e^{H_2} \\ e^{H_2} & e^{-H_2} \end{pmatrix} = 2 \begin{pmatrix} \mathrm{ch}(B_2 - 2H_2) & \mathrm{ch}B_2 \\ \mathrm{ch}B_2 & \mathrm{ch}(B_2 + 2H_2) \end{pmatrix}. \qquad (V,41)$$

We see that A and B do not have the wanted structure. However, in this case we have

$$AB = 4 \begin{pmatrix} \mathrm{ch}(B_1+2H_1)\mathrm{ch}(B_2-2H_2) + \mathrm{ch}B_1\mathrm{ch}B_2 & \mathrm{ch}B_1\,\mathrm{ch}(B_2+2H_2) + \mathrm{ch}B_2\,\mathrm{ch}(B_1+2H_1) \\ \mathrm{ch}B_1\mathrm{ch}(B_2-2H_2) + \mathrm{ch}B_2\mathrm{ch}(B_1-2H_1) & \mathrm{ch}(B_1-2H_1)\mathrm{ch}(B_2+2H_2) + \mathrm{ch}B_1\mathrm{ch}B_2 \end{pmatrix} \qquad (V,42)$$

In case that $B_1=B_2=B$ and $H_1=H_2=H$ we read

$$AB = 4 \begin{pmatrix} \mathrm{ch}(B+2H)\ \mathrm{ch}(B-2H) + \mathrm{ch}^2B & 2\,\mathrm{ch}B\ \mathrm{ch}(B+2H) \\ 2\,\mathrm{ch}B\ \mathrm{ch}(B-2H) & \mathrm{ch}(B+2H)\mathrm{ch}(B-2H) + \mathrm{ch}^2B \end{pmatrix}. \qquad (V,43)$$

The elements on the diagonal are equal. The elements on the counter diagonal can be "corrected" by remarking that analogous to the formula (V,35) we have

$$\sum_{i=1}^{2N} K(\sigma_i - \sigma_{i+1}) = 0 , \qquad (V,44)$$

where i counts off the elements of the counting chain. After correction of A and B the element of AB in the first row and the second column is multiplied by $\exp[2K]$ and that in the second row and the first column is multiplied by $\exp[-2K]$. Fitting K, which can also be done by (V, 39), makes AB go over into a matrix of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \tfrac{1}{2}b \begin{pmatrix} 2a/b & 2 \\ 2 & 2a/b \end{pmatrix} . \qquad (V,45)$$

Apart from the factor $\frac{1}{2}b$ this matrix can be considered as

$$\begin{pmatrix} 2\ ch2G & 2 \\ 2 & 2\ ch2G \end{pmatrix} = \begin{pmatrix} e^{G} & e^{-G} \\ e^{-G} & e^{G} \end{pmatrix} \begin{pmatrix} e^{G} & e^{-G} \\ e^{-G} & e^{G} \end{pmatrix} . \qquad (V,46)$$

Identifying a/b with ch 2G shows that this problem might be equivalent to the problem B with equal interactions in both directions characterized by G. To establish the equivalence we have only to remark the following. We have assumed that two matrices were concerned between which in the array of matrices after the summation over the spin variables none of the matrices I occurred. If this is the case then we have to do with the matrix

$$AIB = 4 \begin{pmatrix} ch(B+2H)\ ch(B-2H) - ch^2B & 0 \\ 0 & -ch(B+2H)\ ch(B-2H) + ch^2B \end{pmatrix} . \qquad (V,47)$$

For the equivalence mentioned above we must have consistency in that this matrix is equal to

$$\tfrac{1}{2}b \begin{pmatrix} e^{G} & e^{-G} \\ e^{-G} & e^{G} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{G} & e^{-G} \\ e^{-G} & e^{G} \end{pmatrix} = \tfrac{1}{2}b \begin{pmatrix} 2sh2G & 0 \\ 0 & -2sh2G \end{pmatrix} . \qquad (V,48)$$

This is easily verified.

Thus this problem too is solvable and other problems can probably be treated in the same way. Although it is important that the basic problems are representative for a large group of problems, the foregoing has a major drawback. It is carefully avoided to let the original elements of the lattice interact with the external magnetic field. Possibilities for this more interesting problem are discussed in the following paragraph.

§6. ON A RESULT OF YANG AND LEE

It sometimes happens that a result is given without a proof. These results are even more challenging than conjectures. A famous example is FERMAT's theorem. In the theory of the ISING problem an example is the formula for the magnetization of a two-dimensional ISING lattice announced by ONSAGER in 1948. His formula was confirmed in 1952 by YANG who, in the same year, in an article on phase transitions together with LEE [38] announced a result for which only as late as 1965 a derivation was given by BAXTER [39]. The problem in question is the following.

The problem B^M is considered with an external magnetic field such that

$$B = \tfrac{1}{2}\pi i \ . \tag{V,49}$$

BAXTER solved this problem, just as YANG and LEE said they did, by a method related to the combinatorial approach of KAC and WARD. We shall consider this problem in terms of the new formulation.

We start from the formulation (V,15). For $B=\pi i$ the matrices M read

$$M = \begin{pmatrix} e^{\pi i} & 0 \\ 0 & e^{-\pi i} \end{pmatrix} = -E. \tag{V,50}$$

So for this value of B (mod. πi) the partition function is exactly the same as for B=0. For $B=\frac{1}{2}\pi i$, however, we find

$$M = \begin{pmatrix} e^{\frac{1}{2}\pi i} & 0 \\ 0 & e^{-\frac{1}{2}\pi i} \end{pmatrix} = e^{\frac{1}{2}\pi i} I \ . \tag{V,51}$$

Thus each spur contains a factor $\exp[\frac{1}{2}N\pi i]$ and extra factors I. These extra factors can be annihilated in pairs, each of which sandwiches two factors V that become infected or disinfected. The relevant quantity is

$$f = \tfrac{1}{2}\pi i + \lim_{N\to\infty} \tfrac{1}{N} \log Z_N'(0) \ , \tag{V,52}$$

where the accent in $Z_N'(0)$ indicates the fact that the matrices in the array of matrix factors were infected in groups of two before the usual infections are imposed in groups of 2m-1 consecutive links respectively matrices. We simply have the problem B_1 but with altered contributions from half the bonds.

We now remark that

$$V_1' = \begin{pmatrix} e^{H} & -e^{-H} \\ -e^{-H} & e^{H} \end{pmatrix} = e^{-\frac{1}{2}\pi i} \begin{pmatrix} e^{(H+\frac{1}{2}\pi i)} & e^{-(H+\frac{1}{2}\pi i)} \\ e^{-(H+\frac{1}{2}\pi i)} & e^{(H+\frac{1}{2}\pi i)} \end{pmatrix} \overset{\text{p.d.}}{=} e^{-\frac{1}{2}\pi i} \begin{pmatrix} e^{\overline{H}} & e^{-\overline{H}} \\ e^{-\overline{H}} & e^{\overline{H}} \end{pmatrix} \tag{V,53}$$

and analogously

$$V_2' = e^{-\frac{1}{2}\pi i} \begin{pmatrix} e^{\overline{H}'} & e^{-\overline{H}'} \\ e^{-\overline{H}'} & e^{\overline{H}'} \end{pmatrix} , \qquad (V,54)$$

where $\overline{H}=H+\frac{1}{2}\pi i$ and $\overline{H}'=H'+\frac{1}{2}\pi i$. The factor $(e^{-\frac{1}{2}\pi i})^N = (-i)^N = \pm 1$ according to whether $\frac{1}{2}N$ is even or odd and has to be retained in order to secure the positivity of the spurs. However, we may assume that $\frac{1}{2}N$ is even to avoid difficulties. Thus we have the problem B_1 for a substitute system in which on the counting chain, apart from the factors I, the matrices are ordered as

$$V_1 V_2 \overline{V}_1 \overline{V}_2 V_1 V_2 \overline{V}_1 \overline{V}_2 V_1 V_2 \overline{V}_1 \ldots \quad . \qquad (V,55)$$

The matrices $\overline{V}_1$ and $\overline{V}_2$ are equal to V_1 and V_2 but for the fact that the variables H and H' have been changed into $\overline{H}$ and $\overline{H}'$.

We have no method to solve the problem B_1 yet. However, let us indicate the bonds corresponding to the matrices V_1 and V_2 by drawn lines and those corresponding to the matrices $\overline{V}_1$ and $\overline{V}_2$ by dotted lines. Then the following picture appears for a lattice with m odd, e.g. m=7,

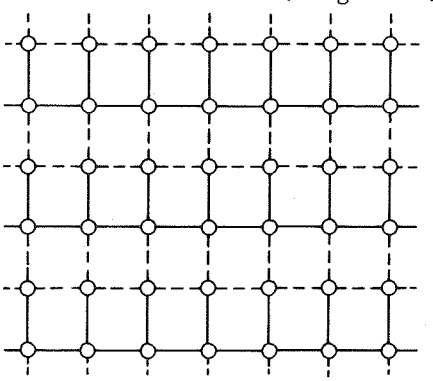


figure 34

The picture is that of alternating drawn and dotted "combs". Now the crux is that this "comb"-problem is solvable e.g. by the KAUFMAN method. We mentally dissect the lattice in figure 34 into horizontal strings consisting of m elements. The interactions in the first string can be represented by a matrix V_1 that is diagonal and of the order 2^m. The interaction between the first and the second string can be represented by a matrix V_2. The interactions in the second string by a diagonal matrix $\overline{V}_1$ and the interaction between the second and the third string by a matrix $\overline{V}_2$, etc. The partition function then becomes*

$$Z_N'(0) = \mathrm{Sp}\ (V_1 V_2 \overline{V}_1 \overline{V}_2 V_1 V_2 \overline{V}_1 \overline{V}_2 V_1 \ldots\ ) = \mathrm{Sp}\ (V_1 V_2 \overline{V}_1 \overline{V}_2)^{\frac{1}{2}(m-1)} , \qquad (V,56)$$

where for simplicity's sake $\frac{1}{2}(m-1)$ has been taken even. The matrices are strictly analogous to those in the formulation B_2. It is a matter of technique to determine the eigenvalues of the matrix

$$V = V_1 V_2 \overline{V}_1 \overline{V}_2 , \qquad (V,57)$$

* See chapter IV for an account on the derivation of a formulation like this.

that has a strong resemblance to the corresponding matrix* in HOUTAPPEL's treatment of the triangular lattice.

Carrying out the KAUFMAN analysis with some minor alterations one finds after lengthy calculations the following results [34]. The relevant quantity f becomes

$$f=\tfrac{1}{2}\pi i+\log\ 2+\frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\log[\mathrm{sh}^2 2H\ \mathrm{sh}^2 2H'+\mathrm{sh}^2 2H\sin^2\alpha+\mathrm{sh}^2 2H'\sin^2\beta]\,d\alpha\,d\beta \qquad (V,58)$$

This should be compared with ONSAGER's solution for problem B of the form

$$f=\log\ 2+\frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi}\log\,[\mathrm{ch}2H\ \mathrm{ch}2H'-\mathrm{sh}2H\cos\alpha-\mathrm{sh}2H'\cos\beta]\,d\alpha\,d\beta \qquad (V,59)$$

The result given by YANG and LEE is based on an interaction energy that is -2J for neighbouring elements with equal values for the spin variables and that is 0 for such that have opposite values for the spin variables. This comes down to an extraction of a factor $\exp[N(H+H')]$ from the partition function. When $x=\exp[-2H]$ and $y=\exp[-2H']$ one obtains instead of (V,58)

$$f=\tfrac{1}{2}\pi i+\tfrac{1}{4}\pi^2\int_0^{\pi}\int_0^{\pi}\log[(1-x^2)^2(1-y^2)^2+4x^2(1-y^2)^2\sin^2\alpha+4y^2(1-x^2)^2\sin^2\beta]\,d\alpha\,d\beta\ , \qquad (V,60)$$

the result given by BAXTER but for the constant, that is not given in his paper. Putting x=y one obtains the result of YANG and LEE but for the constant, that is slightly erroneous in their paper. In the same normalization (V,59) reads

$$f=\tfrac{1}{2}\pi^2\int_0^{\pi}\int_0^{\pi}\log[(1+x^2)(1+y^2)-2y(1-x^2)\cos\alpha-2x(1-y^2)\cos\beta]\,d\alpha\,d\beta. \qquad (V,61)$$

The result (V,58) can of course also be obtained by the Pfaffian method and that method is probably most apt for calculating the magnetization, as the root of the long-range correlation of two spin variables, along the way indicated e.g. in ref. [33]. For B=0 YANG found for $T\leq T_c$

$$m(0)\ =\ \left[\frac{1+x^2}{(1-x^2)^2}\ (1-6x^2+x^4)^{\frac{1}{2}}\right]^{\frac{1}{4}}\ . \qquad (V,62)$$

This in the case that H=H'. The same analysis was carried out for $H\neq H'$ by CHANG [40], who found

$$m(0)\ =\left[1-\left(\frac{2x}{1-x^2}\right)^2\ \left(\frac{2y}{1-y^2}\right)^2\right]^{\frac{1}{8}}\ =\ \left[1-\mathrm{sh}^{-2}2H\ \mathrm{sh}^{-2}2H'\right]^{\frac{1}{8}}\ . \qquad (V,63)$$

For $B=\frac{1}{2}\pi i$ and H=H' YANG and LEE give

$$m\left(\frac{\pi i k T}{2\mu}\right)\ =\ \left[\frac{(1+x^2)^2}{(1-x^2)}\ (1+6x^2+x^4)^{-\frac{1}{2}}\right]^{\frac{1}{8}} \qquad (V,64)$$

* See chapter IV formulae (IV,113).

and for $H \neq H'$ the result is probably

$$m\left(\frac{\pi i KT}{2\mu}\right) = \left[1-\left(\frac{2x}{1+x^2}\right)^2 \left(\frac{2y}{1+y^2}\right)^2\right]^{-\frac{1}{8}} = \left[1-\operatorname{ch}^{-2}2H \operatorname{ch}^{-2}2H'\right]^{-\frac{1}{8}} \quad (V,65)$$

The reason that it is rather useless to carry out the very tedious calculation necessary for the derivation of (V,65) lies a.o. in the following remarks that should be made in connection with these problems. We considered a $m \times (m-1)$ lattice that has an even number of elements. Lattices with an odd number of elements that permit a counting chain of the same type are e.g. $(2m-1)\times(2m+1)$ lattices with m natural. For these lattices one immediately sees from (V,50) that for $B = \pi i \pmod{2\pi i}$ one obtains minus the partition function for zero magnitude field, whereas for $B = \frac{1}{2}\pi i \pmod{\pi i}$ an odd number of factors I occur with the result that, according to (V,21), the partition function is zero, as has already been remarked by SHERMAN. These results can be shown to hold for the C_1^M problem, the B_1^M problem, etc. as well.

This is indeed a remarkable behaviour of the partition functions in question. Strictly speaking there is no limit for f_N as $N \to \infty$ along the natural numbers and the result (V,58) as well as, for similar reasons, the results (V,64) and (V,65) are invalidated to that extent. Nevertheless (V,65) should be taken seriously. For $H \to 0$ and $H' \to 0$ this expression diverges. This is in accordance with the magnetization of a system of independent elements, for which $f = \log 2 \operatorname{ch} B$ and $m(M) = \mu \operatorname{th} B$ so that $m(M)$ diverges for $B \to \frac{1}{2}\pi i$.

§7. IMPERFECT LATTICES

Up to now we have considered the complications that occur when an external magnetic field is incorporated in our problems. This resulted in a change of the way the infection patterns are generated. The last type of problem we want to discuss is that for so-called imperfect lattices. Here we shall see that not the generating principle but the contributions of the various bonds are influenced. Thus here too the principle remains the same. We distinguish between two kinds of problems for imperfect lattices (a) the "bond" problem, (b) the "site" problem.

(a) The "bond" problem

So far we have assumed that the interaction strengths of the bonds in one of the various directions that can be distinguished in the lattices are the same. For the quadratic lattice, to which we shall restrict our considerations, all length bonds are characterized by a variable H and all breadth bonds by a variable H'. The bond problem is arrived at when a certain fraction of the bonds is characterized by another value of these variables. In the

simplest case we can take $H=H'=0$ for such bonds. More generally these bonds are characterized by $\tilde{H}$ and $\tilde{H}'$. In this case these bonds can be stronger as well as weaker than the others.

The partition function that one would like to consider is one that includes not only a summation over all 2^N combinations of the N spin variables, but also over all $\binom{2N}{b}$ partitions of the b influenced bonds over the 2N places. One of the few articles on this kind of problem is that written by ELCOCK [41] in 1957 in which $\tilde{H}=\tilde{H}'=0$ is taken for the influenced bonds and no summation over the partitions is considered. The fraction $q= 2N-b/2N$ of "active" bonds is the probability that a bond is active. The formulation B_c in which

$$Z_N = (2\ \mathrm{ch}H\ \mathrm{ch}H')^N \sum B(n_1',n_2')\ w^{n_1'} w'^{n_2'} ,$$

is changed into $^{\mathrm{imp.}}B_c$

$$Z_N(q) = 2^N(\mathrm{ch}H\ \mathrm{ch}H')^{qN} \sum B(n_1',n_2')(qw)^{n_1'}\ (qw')^{n_2'} . \qquad (V,66)$$

An analogous formulation $^{\mathrm{imp.}}B_1$ can, of course, be given immediately. As the essential problem, the calculation of the generating function $L_N(w,w')$ is known with this formulation the solution is in principle given as well. The relevant quantity f becomes

$$f = \log 2 + q \log \mathrm{ch}H\ \mathrm{ch}H' + \lim_{N\to\infty} \frac{1}{N} \log L_N(qw,qw') . \qquad (V,67)$$

We have to do with a simple change of variables. In this ELCOCK approximation* each bond is weighed with a factor qw respectively qw' instead of a factor w respectively w'. When $\tilde{H}$ and $\tilde{H}'$ are not zero a more general problem is that in which each bond is weighed with a factor $qw+(1-q)\tilde{w}$ respectively $qw'+(1-q)\tilde{w}'$, where $\tilde{w}$=th $\tilde{H}$ and $\tilde{w}'$=th $\tilde{H}'$. For $\tilde{w}=\tilde{w}'=0$ we have the former problem, whereas for $\tilde{w}=w$ and $\tilde{w}'=w'$ we return to the original "pure" problem B. It is clear that this problem is approximately solved in principle too. However, this is only of academic interest as recently LUSHNIKOV [42] announced a rigorous treatment of the problem in which the summation over the partitions has been taken into account as well. This analysis can probably also be generalized to the problem in which $\tilde{H}$ and $\tilde{H}'$ are not zero.

That problem is one for a model that is more realistic than the model considered by ELCOCK and LUSHNIKOV. In the last few years salts have been considered that are very well described by ISING models. It seems indeed possible when not to neutralize bonds completely then at least to weaken respectively to strengthen bonds. Cs_2 Co Cl_4 an anti-ferromagnetic salt with a fraction of fluor or bromine atoms instead of chloric atoms, might be a physical realization of a model as was considered above **.

(b) The "site" problem

More relevant than the bond problem is the problem in which a certain number of elements are inactive or active in another way. Then each element effects a number of bonds that is e.g. four for the quadratic lattice. The

* ELCOCK'S treatment is not rigorous, as he claims.
** MIEDEMA. Private communication.

resulting problem is called the site problem. It differs essentially from the bond problem in that the influenced bonds are not entirely random but are coupled. The problem is completely open as far as exact results are concerned. However, approximate treatments, by means of series developments, give some information about relevant properties like the critical concentration of active elements at which the transition temperature becomes zero and the interesting transition phenomena disappear. In this connection we should mention the article written by MORGAN and RUSHBROOKE [43] in 1959*.

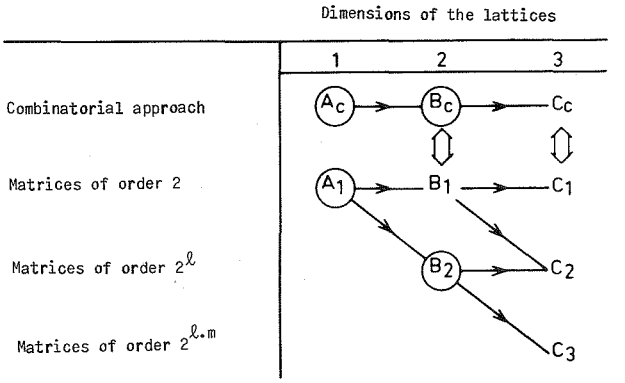
The bond problem was considered mainly to show that the formulation as an infection-disinfection problem remained essentially the same. Only the bond strengths were influenced. The same is the case here and we may close the discussion of this problem, that is only given for completeness, after having remarked the following.

Again we have to sum over all partitions as well as over all possible combinations of the spin variables. For really inactive elements, say b in number, there are now, for each partition, only 2^{N-b} of such combinations instead of 2^N. However, there is no objection to a summation over some auxiliary spin variables for these elements as well. This only mounts up to a factor 2^b for the partition function. Apart from this factor the partition function is the same as that for the problem B in which certain bonds are characterized by the value zero of the variables H and H'.

* The other elements are taken as completely inactive.

DISCUSSION AND OUTLOOK

We have come to the end of the first parts of our investigations that form the contents of this thesis. Let us again draw the scheme given in §3 of chapter I. It now looks like this.



The sign ⇳ indicates the correspondence between the problems B_1 and B_c respectively the problems C_1 and C_c that was treated in the chapters II and III. The ways that lead to C_c, C_2 and C_3, as far as the formulation of these problems is concerned, are not blocked, but the methods developed for the problems B_c and B_2 do not apply. The reasons for this are all in some way modifications of the basic fact that the structure of the C-lattice (and of the $_{n.n.}$B-lattice) is essentially different from that of the B-lattice. In its clearest form we can see this from the reason why the Pfaffian method does not apply. The B-lattice is planar whereas the C-lattice and the $_{n.n.}$B-lattice are not.

Let us now consider the only road that is left in our scheme viz. $A_1 \rightarrow B_1 \rightarrow C_1 \rightarrow {}_{n.n.}B_1$ etc. The ordering of the problems is according to the hierarchy outlined in chapter III. All of them are combinatorial problems on a one-dimensional aggregate. However, this is not the important point. Not only does a generating principle come forward, but the differences in structure of the lattices come forward in the number of groups of infections and in their sizes in comparison with the total length of the counting chain. Both the problems B_1 and $_{n.n.}A_1$ are representative of the problems on the second level of the hierarchy, where only one type of group is involved. A method

specific for these two problems will have to incorporate the difference, in size of the groups, automatically*. This is the reason why we may have some hope that such a method will also apply to e.g. the problem C where two groups are involved that have different sizes in comparison with the total lenth of the counting-chain.

It may turn out that the fact that now two types of groups instead of one type of group occur plays the same rôle as e.g. the non-planarity of the cubic lattice in the case of the Pfaffian method. All in all optimists will find it important to try and find a specific solution to infection-disinfection problems.

However, there are some other problems that are interesting too and that are directly related to the main theme and the results of this thesis. The most interesting amongst them are given here.

1) Both the matrix formulations and the combinatorial formulations have been translated into field-theoretic problems in the last few years. Due to the Fermion character of the links of the counting chain, that can be infected or not, a transformation of the new formulations into such problems is probably possible. How is this transformation effectuated?

2) Much work has been done on series developments by simply counting closed polygons on lattices. Now that a generating principle for the corresponding infection patterns is known the bookkeeping might be simpler. Can one obtain in an easy way more terms of the various series by counting infection patterns?

3) It was pointed out in §4 of chapter IV that the matrices in the case of the problems C_3 and ${}_{n.n.}B_2$ were representations of other transformations than rotations in the space of the so-called Γ-matrices for which a result of BRAUER and WEYL is known in connection with the eigenvalues of both representation and rotation. The transformations concerned were in a space of Γ's and certain products of Γ's and were no rotations. Can for the transformations that are involved a background be created similar to that for the rotations that turn up in problem B_2?

* In this respect these formulations compare favorably with the low-temperature developments where also structure differences manifest themselves but where one cannot generalize to more complex lattices in a natural way and where no generating principle comes forward.

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SUMMARY

This thesis is a contribution to the theory of ISING problems. In the first instance it concerns the investigation and the extension of a part of the old theory (Chapters I and IV). The results lead in the second instance to building-up a new theory (Chapters II, III and V).

In CHAPTER I we give a concise review of the development of the theory of ISING problems in §1, followed by an exposition of the philosophy of this investigation in §2. The central point is the remark that it is not necessary for the derivation of formulations of ISING problems in terms of matrices to make use of a so-called building-up principle, as is mostly the case in literature. The formulations can also be found by inspection. This fact has already been recognized by other authors, as can be seen from some presentations of derivations of the usual matrix formulations. However, it also gives rise to the possibility to derive formulations in terms of matrices whose order is different from that in the usual formulations. This forms the starting-point of the second part of our investigation. In order to have a good survey of the various problems and formulations a classification is given in §3. Throughout the investigation our attention is mainly directed towards the linear chain, the quadratic lattice and the simple cubic lattice.

In CHAPTER II a treatment is given in §1 of the problem for the one-dimensional lattice. This as an example for the treatment of the quadratic lattice given in the following paragraphs. The formulation is in terms of matrices of the order 2. For such a formulation the concept of counting chain is introduced in §2. This necessitates the incorporation of cyclic boundary conditions in the models and a special choice of the lattices concerned. A useful lemma concerning matrix multiplication is also proved and is called summation lemma. In §3 the partition function for the quadratic lattice is now written as a sum of spurs of matrices of the order 2 by using the counting chain and the summation lemma. These spurs are calculated and it appears that the problem has been transformed into the problem of finding a generating function of a certain form for the number of "infection patterns" on the counting chain. This problem is called an infection-disinfection problem. The relevance of the choice of the counting chain is discussed briefly. The formulation thus found is set in correspondence with the usual combinatorial formulation of VAN DER WAERDEN in §4.

In CHAPTER III the procedure developed in chapter II is generalized. In §1 a choice is made for the simple cubic lattice in order to indicate a counting chain for this lattice. A summation lemma, analogous to that in chapter II, is also given. In §2 the partition function is reduced to a sum

of spurs, which again leads to an infection-disinfection problem. Then the correspondence with the usual combinatorial formulation is discussed in § 3. In § 4 we give the main points of a derivation of an analogous formulation of the problem for the lattice "with next-nearest neighbour interactions". The counting chains for the various lattices are investigated and compared in §5. In § 6 a general summation lemma is proved and it is pointed out that one can recognize a hierarchy in the resulting infection-disinfection problems. This hierarchy is investigated more closely in § 7 along with the restrictions that we have posed ourselves by the choice of the lattices. En passant the derivation is given of the formulation of the problem for the one-dimensional lattice with next-nearest neighbour interactions.

In the two foregoing chapters we investigated the possibilities for generalizing the matrix formulation for the one-dimensional problem in another way than has been done up to now. The resulting combinatorial problems differ from the usual combinatorial problems in that a generating principle clearly comes forward. Moreover, it comes forward in which way the various problems differ with respect to the generation of the so-called infection patterns that are concerned here. These differences are directly related to the differences in structure of the various lattices.

In CHAPTER IV a review is given of the other matrix formulations to which the formulation for the one-dimensional lattice can be generalized and the solvability of the resulting problems is investigated. Stress has been laid on the spinor-analytic method of KAUFMAN. In §1 a review is given of the various formulations that can be derived by the method of KRAMERS and WANNIER, using the building-up principle. As an illustration of the derivations a complete treatment of the problem for the one-dimensional lattice with next-nearest neighbour interactions is given in § 2. As next to the matrix formulations and the combinatorial formulations a third kind of formulation is now known, in § 3 the various possibilities to make a conjecture about the transition temperature are investigated. In § 4 the essential points of the KAUFMAN method for the treatment of the problem for the quadratic lattice are brought forward and an alternative treatment is shortly discussed. Three alternative matrix formulations for the simple cubic lattice and the formulation of the quadratic lattice with next-nearest neighbour interactions are derived in § 5, which furnishes at the same time the reason for the failure of the KAUFMAN method for these problems. In §6 the possibility to transform the derived matrix formulations into problems for many-Fermion systems is briefly discussed. As a result there is a.o. a confirmation of what was found by HURST along the combinatorial approach. It concerns the form of the expressions in the Fermion creation and annihilation operators that occur when along that way the problem is translated into a field-theoretic problem. In this paragraph also a possibility to obtain an answer for the problem for the three-dimensional lattice is pointed out. However, this answer takes into account some wrong contributions to the partition function as well. Finally in § 7 for reasons of completeness a formulation of the problem for the simple cubic lattice is discussed, that also is different from the usual matrix formulations. The partition function is written as the sum of spurs of matrices of the same order as those in case of the problem for the quadratic lattice. This road too must be considered as being dead. For the derivation two summation lemmas are given.

In CHAPTER V the new formulations are generalized further to include also other ISING problems like those in which an external magnetic field is present. In §1 the one-dimensional model including an external magnetic field is treated. In §2 it is discussed in what way a magnetic field can be incorporated in the formulations of the various models. The expansion of the partition function in terms of multiple correlations is then discussed in §3 from the point of view of these formulations. It is also indicated how an approximate solution of the problem for the quadratic lattice can be found. Some alternative formulations for this problem are investigated in §4. In §5 problems for so-called decorated lattices are dealt with. Transformations of certain partition functions into such for which it is known that a solution can be found are considered. The same idea is worked out in §6 for the quadratic lattice with certain imaginary values of the magnetic field. Amongst these is a problem for which a result of YANG and LEE is known and that is shown to be solvable e.g. by the KAUFMAN method. In §7 a short discussion is given of the problems for imperfect lattices, another class of ISING problems that can also be formulated as infection-disinfection problems. In a discussion, finally, an outlook is given on further development of the theory.

SAMENVATTING

Dit proefschrift is een bijdrage tot de theorie van ISING-problemen. Het betreft in eerste instantie het onderzoek en de uitbouw van een gedeelte van de oude theorie (Hoofdstukken I en IV). De resultaten hiervan leiden in tweede instantie tot de opbouw van een nieuwe theorie (Hoofdstukken II, III en V).

In HOOFDSTUK I wordt, na een beknopt historisch overzicht van de ontwikkeling van de theorie in §1, in §2 de filosofie van dit onderzoek uiteengezet. De kern van het betoog is, dat het voor de formulering van ISING-problemen in termen van matrices niet noodzakelijk is van het z.g. opbouwprincipe gebruik te maken, zoals dat in de literatuur in dit verband meestal het geval is. De formulering kan ook door inspectie gevonden worden. Deze mogelijkheid is reeds door andere schrijvers onderkend, zoals blijkt uit sommige presentaties van de afleiding van de gebruikelijke matrixformuleringen. Het geeft echter ook de mogelijkheid formuleringen af te leiden in termen van matrices, waarvan de orde een andere is dan die in de gebruikelijke formuleringen. Dit vormt het uitgangspunt van het tweede deel van ons onderzoek. Voor een goed overzicht wordt in §3 een classificatie gegeven van de verschillende problemen en formuleringen. In het gehele onderzoek wordt de aandacht vrijwel uitsluitend gericht op de lineaire keten, het kwadratisch rooster en het eenvoudige kubische rooster.

In HOOFDSTUK II wordt in §1 een behandeling gegeven van het probleem voor het een-dimensionale rooster. Deze dient als voorbeeld voor de in de verdere paragrafen gegeven behandeling van het kwadratische rooster. De formulering is in termen van matrices van de orde 2. Voor een dergelijke formulering wordt in §2 het begrip aftelketen ingevoerd. Dit noopt tot het incorporeren van cyclische randcondities in de modellen en tot een speciale keuze van de roosters in kwestie. Tevens wordt een nuttig lemma bewezen betreffende matrixvermenigvuldiging. Dit wordt sommatielemma genoemd. In §3 wordt nu de toestandssom voor het kwadratische rooster met behulp van de aftelketen en het sommatielemma geschreven als een som van sporen van matrices, die producten zijn van matrices van de orde 2. Deze sporen worden berekend en het blijkt dat het probleem is getransformeerd in het probleem een genererende functie van een bepaalde vorm te vinden voor het aantal "infectiepatronen" op de aftelketen. Dit probleem wordt een infectie-disinfectieprobleem genoemd. De relevantie van de keuze van de aftelketen wordt kort besproken. De gevonden formulering wordt in §4 in correspondentie gebracht met de gebruikelijke combinatorische formulering van VAN DER WAERDEN.

In HOOFDSTUK III wordt de in hoofdstuk II ontwikkelde procedure gegeneraliseerd. In §1 wordt voor de afmetingen van het eenvoudige kubische rooster een keuze gemaakt teneinde voor dit rooster een aftelketen te kunnen aangeven. Tevens wordt een sommatielemma gegeven, dat het analogon is van het lemma in hoofdstuk II. In § 2 wordt de toestandssom herleid tot een som van sporen, hetgeen weer leidt tot een infectie-desinfectieprobleem. Daarop wordt in §3 de correspondentie met de gebruikelijke combinatorische formulering besproken. In §4 worden de voornaamste punten gegeven van de afleiding van een analoge formulering van het probleem voor het kwadratische rooster "met overnaaste buren wisselwerkingen". De aftelketens voor de verschillende roosters worden onderzocht en vergeleken in §5. In §6 wordt een algemeen sommatielemma bewezen en wordt erop gewezen, dat er een hierarchie te onderkennen is in de infectie-desinfectieproblemen. Deze hierarchie wordt nader onderzocht in §7. Tevens wordt hierbij kort ingegaan op de beperkingen, die wij ons hebben opgelegd bij de keuze van de roosters. En passant wordt de formulering van het probleem voor het een-dimensionale rooster met overnaaste buren wisselwerkingen afgeleid.

In de twee voorgaande hoofdstukken zijn de mogelijkheden onderzocht om de matrixformulering voor het een-dimensionale probleem op een andere wijze te generaliseren dan tot dusver gebruikelijk was. De resulterende combinatorische problemen onderscheiden zich van de gebruikelijke combinatorische problemen daarin, dat een genererend principe duidelijk naar voren komt. Bovendien komt in de afgeleide formuleringen naar voren op welke wijze de diverse problemen verschillen met betrekking tot de generatie van de zogenaamde infectiepatronen waarom het hier gaat. Deze verschillen zijn direct gelieerd aan de structuurverschillen van de diverse roosters.

In HOOFDSTUK IV wordt een overzicht gegeven van de andere matrixformuleringen waartoe de formulering van het een-dimensionale rooster gegeneraliseerd kan worden en wordt de oplosbaarheid onderzocht van de resulterende problemen. Hierbij is het accent gelegd op de spinor-analytische methode van KAUFMAN. In §1 wordt een overzicht gegeven van de diverse formuleringen, die af te leiden zijn met de methode van KRAMERS en WANNIER onder gebruikmaking van het opbouwprincipe. Ter illustratie van de afleidingen wordt in § 2 het probleem voor het een-dimensionale rooster met overnaaste buren wisselwerkingen volledig behandeld. Waar nu naast de matrixformuleringen en de combinatorische formuleringen een derde soort formulering bekend is, worden in §3 de verschillende mogelijkheden onderzocht om een vermoeden uit te spreken betreffende de ligging van het overgangspunt. In §4 worden de essentiele punten uit de KAUFMAN-methode voor de behandeling van het probleem voor het kwadratische rooster naar voren gehaald en wordt zeer kort ingegaan op een alternatieve behandeling. Drie alternatieve matrixformuleringen voor het eenvoudige kubische rooster en de formulering voor het kwadratische rooster met overnaaste buren wisselwerkingen worden in § 5 afgeleid, waarmee tevens wordt getoond waarom de KAUFMAN-analyse mislukt voor deze problemen. In § 6 wordt kort ingegaan op de mogelijkheid om de afgeleide matrixformuleringen te transformeren in problemen voor Fermionen-systemen. Hierbij worden o.a. de resultaten van HURST bevestigd, die met betrekking tot de combinatorische formuleringen tot een uitspraak komt over de vorm van de uitdrukkingen in de Fermion scheppings- en vernietigingsoperatoren in kwestie,

indien langs deze weg het ISING-probleem vertaald wordt in een velden-theoretisch probleem. Tevens wordt in deze paragraaf gewezen op de mogelijkheid om tot een antwoord te komen voor het drie-dimensionale probleem. Dit antwoord verrekent echter ook enige onjuiste bijdragen tot de toestandssom. In § 7 tenslotte wordt voor de volledigheid een formulering van het probleem voor het eenvoudige kubische rooster besproken, die eveneens afwijkt van de gebruikelijke matrixformuleringen. De toestandssom wordt hierbij geschreven als som van sporen van matrices van dezelfde orde als die voor het probleem voor het kwadratische rooster. Ook deze weg moet als een doodlopende weg beschouwd worden. Voor de afleiding worden twee sommatielemmas gegeven.

In HOOFDSTUK V worden de nieuwe formuleringen verder gegeneraliseerd zodat ook andere ISING-problemen, zoals die waarin een uitwendig magnetisch veld aanwezig is, inbegrepen zijn. In §1 wordt het een-dimensionale model bij aanwezigheid van een uitwendig magnetisch veld behandeld. In §2 wordt besproken op welke wijze een uitwendig magnetisch veld in de formuleringen voor de diverse modellen geïncorporeerd kan worden. Daarop wordt in §3 vanuit deze formuleringen de ontwikkeling van de toestandssom naar meervoudige correlaties besproken. Tevens wordt kort aangeduid langs welke weg een benaderende oplossing van het probleem voor het kwadratische rooster gevonden kan worden. Enige alternatieve formuleringen voor dit probleem worden in §4 onderzocht. In §5 komen problemen voor zogenaamde gedecoreerde roosters aan de orde. Het betreft hier een transformatie van bepaalde toestandssommen in zulke waarvoor bekend is, dat een oplossing gevonden kan worden. Dit zelfde idee wordt in §6 uitgewerkt voor het kwadratische rooster bij bepaalde imaginaire waarden van het magneetveld. Een van deze problemen is een probleem waarvoor een resultaat van YANG en LEE bekend is en waarvan de oplosbaarheid met bijvoorbeeld de methode van KAUFMAN aangetoond wordt. In §7 wordt kort ingegaan op problemen voor imperfecte roosters, een andere klasse van ISING-problemen, die eveneens als infectie-desinfectie-problemen te formuleren zijn. In een discussie wordt tenslotte een uitzicht gegeven op verdere ontwikkeling van de theorie.

CURRICULUM VITAE

The author of this thesis was born in Amsterdam in 1936. After visiting the grammar school there, he started studying physics and chemistry with mathematics and crystallography as minors at the Municipal University of Amsterdam in 1954. In 1957 he passed the candidate examination and supplemented in mathematics in order to study theoretical physics with mathematics and mechanics as minors. Amongst others he attended the lectures of Prof. Dr. J. DE BOER and Prof. Dr. S.A. WOUTHUYSEN. In the academical year 1959 he received a stipendium from the Federal Republic of Germany that was used to study at the University of Muenster. He passed doctoral examinations in 1962 and was a teacher of mathematics and physics in Haarlem and Amsterdam from 1961 till 1965. From 1962 till 1965 he also worked at the Mathematical Centre in Amsterdam in the department of applied mathematics that was supervised by Prof. Dr. H.A. LAUWERIER. Since 1965 he has been a scientific officer at the Twente Technological University where he was joined to the chair of Prof. Dr. I.W. VAN SPIEGEL.

STELLINGEN

I

De aanname van PARMAKIAN, dat de expansie van lucht in een windketel gekarakteriseerd kan worden door een Poisson-constante 1,2 is zelfs niet bij benadering juist.

J. PARMAKIAN, Waterhammer Analysis (1963).
C. HOEDE , De warmteoverdracht in een windketel, Math. Centrum, T.N.26 (1962).

II

Voor de warmte, opgenomen respectievelijk afgestaan in een halve periode door een homogene metalen bol bij een sinusoidaal wisselende temperatuur aan het oppervlak, is een exact resultaat af te leiden.

C. HOEDE , Warmteopname door een homogene metalen bol bij wisselende temperatuur, Math. Centrum, T.N. 31 (1963).

III

Voor de hoeveelheid materie, die middels diffusie door een vlak dubbelgelaagd medium per oppervlakte-eenheid in een bepaalde tijd wordt opgenomen kan een nuttige benaderingsformule worden gegeven.

C. HOEDE , Diffusion in layered media, Math. Centrum, T.N. 37 (1964).

IV

Voor de absorptie van materiaal middels diffusie door bolvormige partikels, die zich bevinden in een oplossing van dat materiaal van een eindig volume kan, bij voortdurende homogene verdeling in de oplossing, voor de totale hoeveelheid geabsorbeerd materiaal een goede benadering worden afgeleid.

C. HOEDE , On the absorption by homogeneous spherical particles, Math. Centrum, T.N. 38 (1964).

V

Voor de diffusie van materiaal met concentratie $C(x,t)$ in combinatie met langzame absorptie door het medium waarin het diffusieproces plaatsvindt, aanleiding gevend tot een concentratie $C_{abs}(x,t)$, is

$$\frac{\partial C_{abs}(x,t)}{\partial t} = \alpha \left[KC(x,t) - C_{abs}(x,t) \right]$$

een consistente aanname voor de beschrijving.
Naast de plaats- en tijd-variabelen x en t komen hierin voor de evenwichtsconstante K voor het absorptieproces en een evenredigheidsconstante α.

E.K. DUURSMA and C. HOEDE, Theoretical, experimental and field studies concerning molecular diffusion of radioisotopes in sediments and suspended solid particles of the sea, Neth. Journ. of Sea Res. 3, 423 (1967).

VI

Ten onrechte beweert HURST, dat de oplosbaarheid van een Ising model door de Pfaffianenmethode ook de oplosbaarheid via de methode van Onsager inhoudt.

C.A. HURST , New approach to the Ising problem. Journ. Math. Phys. 7, 305 (1966).

VII

Tegen de behandeling door ELCOCK van het zogenaamde "bond problem" voor een imperfect rooster, zijn bezwaren van verschillende aard aan te voeren. Bovendien zijn op eenvoudige wijze verbeteringen aan te brengen.

E.W. ELCOCK , The cooperative behaviour of a two-dimensional defect lattice, Proc. Cambridge Phil. Soc. 53, 863 (1957).

VIII

De bindingen tussen 2m+1 elementen zijn in m typen in te delen. Het is mogelijk om voor deze bindingen een Euler-circuit aan te geven, dat bestaat uit herhaalde aftellingen van m bindingen, waarin elk type binding precies eenmaal voorkomt en wel steeds op dezelfde plaats in de volgorde van de aftelling.

IX

Er zijn redenen aan te voeren waarom een studie van zogenaamde deltafunctie benaderende rijen van belang zou kunnen zijn voor de opbouw van een niet-lineaire distributietheorie.

X

Bij het gymnasiale onderwijs zijn de waarderingen voor de wiskunde en de andere natuurwetenschappelijke vakken niet met elkaar in evenwicht.

XI

De in de baccalaureaatsnota van de TH Twente gegeven beschrijving van de studie voor baccalaureus technische wetenschappen biedt te weinig waarborg voor het academisch karakter van deze studie.

Nota over het baccalaureaat technische wetenschappen van de Technische Hogeschool Twente. Drienerlo 1967.

XII

Tegen het "rapport Van Os" zijn zowel wat betreft de inhoud als de vorm fundamentele bezwaren aan te voeren.

"Structuur van het wetenschappelijk corps". Publicatie nr. 9 van de Academische Raad (juni 1968).

ERRATA

The text contains some printing errors, partly due to the way of reproduction. Some non-trivial errors are the following:

page 27 - formula (III, 11) read: $4^{-N}\mathrm{Sp}\ V_1^{n_1} V_2^{n_2} V_3^{n_3} V_1'^{n_1'} V_2'^{n_2'} V_3'^{n_3'}$

page 48 - 2nd line after formula (IV, 27) read: DOMB's article [23]

page 62 - 1st line read: $Y_i = Ex \ldots xYx \ldots xE$

page 77 - 4th line from below read: $1 \leq \nu_{i,j} \leq 2^n$.

page 83 - formula (V, 21) read: $\mathrm{Sp}\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ etc.

page 96 - formula (V, 60) read: $\frac{1}{2}\pi i + \frac{1}{4\pi^2}$. integral

- formula (V, 61) read: $\frac{1}{2\pi^2}$. integral