Shearer's Measure and Stochastic Domination of Product Measures

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Abstract Let $G = (V, E)$ be a locally finite graph. Let $\vec{p} \in [0, 1]^V$. We show that Shearer's measure, introduced in the context of the Lovász Local Lemma, with marginal distribution determined by $\vec{p}$, exists on G if and only if every Bernoulli random field with the same marginals and dependency graph G dominates stochastically a non-trivial Bernoulli product field. Additionally, we derive a non-trivial uniform lower bound for the parameter vector of the dominated Bernoulli product field. This generalises previous results by Liggett, Schonmann, and Stacey in the homogeneous case, in particular on the k-fuzz of $\mathbb{Z}$. Using the connection between Shearer's measure and a hardcore lattice gas established by Scott and Sokal, we transfer bounds derived from cluster expansions of lattice gas partition functions to the stochastic domination problem.

1 Introduction

The question under which conditions a Bernoulli random field (short BRF) stochastically dominates a Bernoulli product field (short BPF) is of interest in probability and percolation theory. Knowledge of this kind allows the transfer of results from the independent case to more general settings. Of particular interest are BRFs with a

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dependency structure described by a graph G and prescribed common marginal parameter p, as they often arise from rescaling arguments [11], dependent models [6] or particle systems [14]. In this setting, an interesting question is to find lower bounds on p which guarantee stochastic domination for every such BRF.

This question has been investigated in the setting of bootstrap percolation [1, Sect. 2] and super-critical Bernoulli percolation [2, Sect. 2]. Finally, Liggett, Schonmann, and Stacey [15] derived a generic lower bound for dependency graphs with uniformly bounded degree. Of particular interest is the k-fuzz of $\mathbb{Z}$ (short $\mathbb{Z}_{(k)}$, that is, the graph with vertex set $\mathbb{Z}$ and edges between all integers at distance less than or equal to k), which is the dependency graph of k-dependent BRFs on $\mathbb{Z}$. In this case, they determined the minimal p for which stochastic domination of a non-trivial BPF holds for each such BRF on $\mathbb{Z}_{(k)}$. Even more, they showed that in this case the parameter of the dominated BPF is uniformly bounded from below and nonzero for this minimal p and made a conjecture about the size of the jump of the value of the parameter of the dominated BPF at this minimal p.

Their main tools have been a sufficient condition highly reminiscent of the Lovász Local Lemma [8] (short LLL, also known as the Dobrushin condition [7] in statistical mechanics) and the explicit use of Shearer's measure [19] on $\mathbb{Z}_{(k)}$ to construct a series of probability measures dominating only trivial BPFs. Recall that Shearer's measure is the uniform minimal probability measure in the context of the LLL. It is also related to the grand canonical partition function of a lattice gas with both hard-core interaction and hard-core self-repulsion [5, 18].

Extending the work of Liggett, Schonmann, and Stacey in a natural way, we demonstrate that the use of Shearer's measure and the overall similarity between their proof and those concerning only Shearer's measure is not coincidence, but part of a larger picture. We show that there is a non-trivial uniform lower bound on the parameter vector of the BPF dominated by a BRF with marginal parameter vector $\vec{p}$ and dependency graph G iff Shearer's measure with prescribed marginal parameter vector $\vec{p}$ exists on G.

After reparametrisation, the set of admissible vectors $\vec{p}$ is equivalent to the polydisc of absolute and uniform convergence of the cluster expansion of the partition function of a hard-core lattice gas around fugacity $\vec{0}$ [5, 18] allowing a high-temperature expansion [7]. This opens the door to a reinterpretation of results from cluster expansion techniques [4, 9, 12] or tree equivalence techniques [18, Sects. 6 and 8], leading to improved estimates on admissible $\vec{p}$ for the domination problem. Possible future lines of research include the search for probabilistic interpretations of these combinatorial and analytic results.

The layout of this paper is as follows: We formulate the stochastic domination problem in Sect. 2 and give a short introduction to Shearer's measure in Sect. 3. Section 4 contains our new results, followed by examples of reinterpreted bounds in Sect. 4.1. Finally, Sect. 6 deals with the weak invariant case and we refute the conjecture by Liggett, Schonmann, and Stacey concerning the minimality of Shearer's measure for the dominated parameter on $\mathbb{Z}_{(k)}$ in Sect. 7.

2 Setup and Problem Statement

Let $G := (V, E)$ be a locally finite graph. Denote by $\mathcal{N}(v)$ the *set of neighbours* of v and by $\mathcal{N}_1(v) := \mathcal{N}(v) \uplus \{v\}$ the *neighbourhood of v including v itself*. For every $W \subseteq V$, denote by $G(W)$ the *subgraph of G induced by W*.

Vectors are indexed by V, i.e. $\vec{x} := (x_v)_{v \in V}$. Multiplication of vectors acts coordinate-wise. We have the natural partial order $\leq$ on real-valued vectors. We emphasize the important notion of strict inequality in all coordinates by writing $\vec{x} \ll \vec{y}$. For $W \subseteq V$, let $\vec{x}_W := (x_v)_{v \in W}$, where needed for disambiguation. We otherwise ignore superfluous coordinates. If we use a scalar x in place of a vector, then we mean to use $x\vec{1}$ and call this the *homogeneous setting*. We *always assume the relation* $q = 1 - p$, also in vectorised form and when having corresponding subscripts. Denote by $\mathcal{X}_V := \{0, 1\}^V$ the compact *space of binary configurations* indexed by V. Equip $\mathcal{X}_V$ with the natural partial order induced by $\vec{x} \leq \vec{y}$ (isomorph to the partial order induced by the subset relation in $\mathcal{P}(V)$).

A *Bernoulli random field* (short BRF) $Y := (Y_v)_{v \in V}$ on G is a rv taking values in $\mathcal{X}_V$, seen as a collection of Bernoulli rvs Y_v indexed by V. A *Bernoulli product field* (short BPF) X is a BRF where $(X_v)_{v \in V}$ is a collection of independent Bernoulli rvs. We write its law as $\Pi_{\vec{x}}^V$, where $x_v := \Pi_{\vec{x}}^V(X_v = 1)$.

A subset A of the space $\mathcal{X}_V$ or the space $[0, 1]^V$ is an *up-set* iff

$$\forall \vec{x} \in A, \vec{y} \in \mathcal{X}_V : \quad \vec{x} \leq \vec{y} \quad \Rightarrow \quad \vec{y} \in A. \tag{1}$$

Replacing $\leq$ by $\geq$ in (1) we define a *down-set*.

We recall the definition of *stochastic domination* [14]. Let Y and Z be two BRFs on G. Denote by $\text{Mon}(V)$ the set of *monotone continuous functions* from $\mathcal{X}_V$ to $\mathbb{R}$, that is, $\vec{s} \leq \vec{t}$ implies $f(\vec{s}) \leq f(\vec{t})$. We say that Y dominates Z stochastically iff they respect monotonicity in expectation:

$$Y \overset{\text{st}}{\geq} Z \quad \Leftrightarrow \quad \big(\forall f \in \text{Mon}(V) : \mathbb{E}\big[f(Y)\big] \geq \mathbb{E}\big[f(Z)\big]\big). \tag{2}$$

Equation (2) actually refers to the laws of Y and Z. We abuse notation and treat a BRF and its law as interchangeable. Stochastic domination is equivalent to the existence of a coupling of Y and Z with $\mathbb{P}(Y \geq Z) = 1$ [20].

The *set of all dominated Bernoulli parameter vectors* (short: set of dominated vectors) by a BRF Y is

$$\Sigma(Y) := \big\{\vec{c} : Y \overset{\text{st}}{\geq} \Pi_{\vec{c}}^V\big\}. \tag{3a}$$

It describes all the different BPFs minorating Y stochastically. The set $\Sigma(Y)$ is a closed down-set. The definition of dominated vector extends to a non-empty class C of BRFs by

$$\Sigma(C) := \bigcap_{Y \in C} \Sigma(Y) = \big\{\vec{c} : \forall Y \in C : Y \overset{\text{st}}{\geq} \Pi_{\vec{c}}^V\big\}. \tag{3b}$$

For a class C of BRFs, denote by $C(\vec{p})$ the subclass consisting of BRFs with marginal parameter vector $\vec{p}$. We call a BPF with law $\Pi_{\vec{c}}^V$, respectively the vector $\vec{c}$, *non-trivial* iff $\vec{c} \gg 0$. Our *main question* is under which conditions all BRFs in a class

C dominate a non-trivial BPF. Even stronger, we ask whether they all dominate a common non-trivial BPF. Hence, given a class C, we investigate the *set of parameter vectors guaranteeing non-trivial domination*

$$\mathcal{P}^C_{\text{dom}} := \left\{ \vec{p} \in [0,1]^V : \forall Y \in C(\vec{p}) : \exists \vec{c} \gg \vec{0} : \vec{c} \in \Sigma(Y) \right\} \tag{3c}$$

and the *set of parameter vectors guaranteeing uniform non-trivial domination*

$$\mathcal{P}^C_{\text{udom}} := \left\{ \vec{p} \in [0,1]^V : \exists \vec{c} \gg \vec{0} : \vec{c} \in \Sigma\left(C(\vec{p})\right) \right\}. \tag{3d}$$

We have the obvious inclusion

$$\mathcal{P}^C_{\text{udom}} \subseteq \mathcal{P}^C_{\text{dom}}. \tag{3e}$$

The main contribution of this paper is the characterisation and description of certain properties of the sets (3d) and (3c) for some classes of BRFs.

A first class of BRFs is the so-called *weak dependency class* [15, (1.1)] with marginal parameter $\vec{p}$ on G:

$$\mathcal{C}^{\text{weak}}_G(\vec{p}) := \left\{ \text{BRF } Y : \forall v \in V : \mathbb{P}(Y_v = 1 | Y_{V \setminus \mathcal{N}_1(v)}) \geq p_v \right\}. \tag{4}$$

In this context, G is a *weak dependency graph* of Y. We say that G is a *strong dependency graph* of a BRF Y iff

$$\forall W_1, W_2 \subset V : \quad d(W_1, W_2) > 1 \quad \Rightarrow \quad Y_{W_1} \text{ is independent of } Y_{W_2}. \tag{5}$$

In both cases, adding edges does not change G's status as dependency graph of Y. It is possible that Y has multiple minimal dependency graphs [18, Sect. 4.1]. The second class is the so-called *strong dependency class* [15, Sect. 0] with marginal parameter $\vec{p}$ on G:

$$\mathcal{C}^{\text{strong}}_G(\vec{p}) := \left\{ \text{BRF } Y : \begin{array}{l} \forall v \in V : \mathbb{P}(Y_v = 1) = p_v \\ G \text{ is a strong dependency graph of } Y \end{array} \right\}. \tag{6}$$

In particular,

$$\mathcal{C}^{\text{strong}}_G(\vec{p}) \subseteq \mathcal{C}^{\text{weak}}_G(\vec{p}). \tag{7}$$

In all but some trivial cases, the inclusion (7) is strict (see after Theorem 19).

3 A Primer on Shearer's Measure

This section contains an introduction to and overview of Shearer's measure. The following construction is due to Shearer [19]. Let $G := (V, E)$ be finite and $\vec{p} \in [0,1]^V$. Recall that an *independent set of vertices* (in the graph theoretic sense) contains no adjacent vertices. Create a signed measure $\mu_{G,\vec{p}}$ on $\mathcal{X}_V$ with strong dependency graph G by setting the marginals

$$\forall W \subseteq V : \quad \mu_{G,\vec{p}}(Y_W = \vec{0}) := \begin{cases} \prod_{v \in W} q_v & W \text{ independent,} \\ 0 & W \text{ not independent.} \end{cases} \tag{8a}$$

Use the *inclusion–exclusion principle* to complete $\mu_{G,\vec{p}}$:

$$\forall W \subseteq V : \quad \mu_{G,\vec{p}}(Y_W = \vec{0}, Y_{V\setminus W} = \vec{1}) := \sum_{\substack{W \subseteq T \subseteq V \\ T \text{ indep}}} (-1)^{|T|-|W|} \prod_{v \in T} q_v. \tag{8b}$$

Define the *critical function* of Shearer's signed measure on G by

$$\varXi_G : [0,1]^V \to \mathbb{R}, \qquad \vec{p} \mapsto \varXi_G(\vec{p}) := \mu_{G,\vec{p}}(Y_V = \vec{1}) = \sum_{\substack{T \subseteq V \\ T \text{ indep}}} \prod_{v \in T} (-q_v). \tag{9}$$

In graph theory, (9) is also known as the *independent set polynomial* of G [10, 13] and in lattice gas theory as the grand *canonical partition function* at negative fugacity $-\vec{q}$ [18, Sect. 2]. It satisfies a *fundamental identity* (an instance of a *deletion–contraction identity*)

$$\forall v \in V, \vec{p} \in [0,1]^V : \quad \varXi_G(\vec{p}) = \varXi_{G(V\setminus\{v\})}(\vec{p}) - q_v \varXi_{G(V\setminus\mathcal{N}_1(v))}(\vec{p}), \tag{10}$$

derived from (9) by discriminating between independent sets containing v and those which do not.

The *set of admissible parameters for Shearer's measure* is

$$\mathcal{P}_{\text{sh}}^G := \left\{ \vec{p} \in [0,1]^V : \mu_{G,\vec{p}} \text{ is a probability measure} \right\}$$
$$= \left\{ \vec{p} \in [0,1]^V : \forall W \subseteq V : \varXi_{G(W)}(\vec{p}) \geq 0 \right\}. \tag{11}$$

The set $\mathcal{P}_{\text{sh}}^G$ is closed, strictly decreasing when adding edges and an up-set [18, Proposition 2.15(b)], hence connected. It always contains the vector $\vec{1}$ and, unless $E = \emptyset$, never the vector $\vec{0}$. Therefore, it is a non-trivial subset of $[0,1]^V$ (see also Sect. 4.1). The function $\varXi_G$ is strictly increasing on $\mathcal{P}_{\text{sh}}^G$. It is convenient to subdivide $\mathcal{P}_{\text{sh}}^G$ further into its *boundary*

$$\partial \mathcal{P}_{\text{sh}}^G := \left\{ \vec{p} : \varXi_G(\vec{p}) = 0 \text{ and } \mu_{G,\vec{p}} \text{ is a probability measure} \right\} \tag{12}$$

and *interior* (both seen as subsets of the space $[0,1]^V$)

$$\mathring{\mathcal{P}}_{\text{sh}}^G := \mathcal{P}_{\text{sh}}^G \setminus \partial \mathcal{P}_{\text{sh}}^G$$
$$= \left\{ \vec{p} : \varXi_G(\vec{p}) > 0 \text{ and } \mu_{G,\vec{p}} \text{ is a probability measure} \right\}$$
$$= \left\{ \vec{p} : \varXi_H(\vec{p}) > 0 \text{ for all subgraphs } H \text{ of } G \right\}. \tag{13}$$

Finally, we see that, for $\vec{p} \in \mathcal{P}_{\text{sh}}^G$, the probability measure $\mu_{G,\vec{p}}$

$$\text{has dependency graph } G, \tag{14a}$$

$$\text{has marginal parameter } \vec{p}, \text{ i.e., } \forall v \in V : \mu_{G,\vec{p}}(Y_v = 1) = p_v, \tag{14b}$$

$$\text{and forbids neighbouring 0s, i.e., } \forall (v,w) \in E : \mu_{G,\vec{p}}(Y_v = Y_w = 0) = 0. \tag{14c}$$

Properties (14a) and (14b) are equivalent to $\mu_{G,\vec{p}} \in \mathcal{C}_G^{\text{strong}}(\vec{p})$. Every probability measure ν on $\mathcal{X}_V$ fulfilling (14a)–(14c) can be constructed by (8a)–(8b) and thus coincides with $\mu_{G,\vec{p}}$. Hence (14a)–(14c) *characterises* $\mu_{G,\vec{p}}$.

The importance of Shearer's measure is due to its *uniform minimality* with respect to certain conditional probabilities:

Lemma 1 ([19, Theorem 1]) *Let* $\vec{p} \in \mathcal{P}_{\text{sh}}^G$ *and* $Z \in \mathcal{C}_G^{\text{weak}}(\vec{p})$. *Then* $\forall W \subseteq V$,

$$\mathbb{P}(Z_W = \vec{1}) \geq \mu_{G,\vec{p}}(Y_W = \vec{1}) = \varXi_{G(W)}(\vec{p}) \geq 0 \qquad (15a)$$

and $\forall W \subseteq U \subseteq V$, *if* $\varXi_{G(W)}(\vec{p}) > 0$, *then*

$$\mathbb{P}(Z_U = \vec{1}|Z_W = \vec{1}) \geq \mu_{G,\vec{p}}(Y_U = \vec{1}|Y_W = \vec{1}) = \frac{\varXi_{G(U)}(\vec{p})}{\varXi_{G(W)}(\vec{p})} \geq 0. \qquad (15b)$$

The cost of isolating 0s drives and is equivalent to the above minimality.

If G is *infinite* we have the well-defined [18, (8.4)] extension

$$\mathcal{P}_{\text{sh}}^G := \bigcap_{E' \subseteq E, |E'| < \infty} \mathcal{P}_{\text{sh}}^{(V,E')} \quad \text{and} \quad \mathring{\mathcal{P}}_{\text{sh}}^G := \bigcap_{E' \subseteq E, |E'| < \infty} \mathring{\mathcal{P}}_{\text{sh}}^{(V,E')}. \qquad (16)$$

The set $\mathring{\mathcal{P}}_{\text{sh}}^G$ is not the interior of the closed set $\mathcal{P}_{\text{sh}}^G$ (discussed in detail in [18, Theorem 8.1]). For $\vec{p} \in \mathcal{P}_{\text{sh}}^G$, the family of marginals $\{\mu_{G(W),p} : W \subsetneq V, W \text{ finite}\}$ forms a consistent family à la Kolmogorov [3, (36.1) and (36.2)]. Hence Kolmogorov's existence theorem [3, Theorem 36.2] establishes the existence of an extension of this family, which we call $\mu_{G,\vec{p}}$. The $\pi-\lambda$ theorem [3, Theorem 3.3] asserts the uniqueness of this extension. Furthermore, $\mu_{G,\vec{p}}$ has all the properties listed in (14a)–(14c) on the infinite graph G. Conversely, let ν be a probability measure having the properties (14a)–(14c). Then all its finite marginals have them, too, and they coincide with Shearer's measure. Hence by the uniqueness of the Kolmogorov extension ν coincides with $\mu_{G,\vec{p}}$ and (14a)–(14c) *characterises* $\mu_{G,\vec{p}}$ also on infinite graphs.

4 Main Results and Discussion

Our main result is

Theorem 2 *For every locally finite graph* G, *we have*

$$\mathcal{P}_{\text{dom}}^{\mathcal{C}_G^{\text{weak}}} = \mathcal{P}_{\text{udom}}^{\mathcal{C}_G^{\text{weak}}} = \mathcal{P}_{\text{dom}}^{\mathcal{C}_G^{\text{strong}}} = \mathcal{P}_{\text{udom}}^{\mathcal{C}_G^{\text{strong}}} = \mathring{\mathcal{P}}_{\text{sh}}^G. \qquad (17)$$

Its proof is in Sect. 5. Theorem 2 consists of two a priori unrelated statements: The first one consists of the left three equalities in (17): uniform and non-uniform domination of a non-trivial BPF are the same, and even taking the smaller class $\mathcal{C}_G^{\text{strong}}$ does not admit more $\vec{p}$. The second one is that these sets are equivalent to the set of parameters for which Shearer's measure exists. The minimality of Shearer's measure

(see Lemma 1) lets us construct BRFs dominating only trivial BPFs for $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^G$ (see Sect. 5.2) and clarifies the role Shearer's measure played as a counterexample in the work of Liggett, Schonmann, and Stacey [15, Sect. 2]. Even more, this minimality implies an explicit lower bound for the non-trivial uniform dominated vector:

Theorem 3 *For $\vec{p} \in \mathring{\mathcal{P}}_{sh}^G$, define the vector $\vec{c}$ component-wise by*

$$c_v := \begin{cases} 1 & \text{if } p_v = 1, & (18a) \\ 1 - (1 - \varXi_{G_v}(\vec{p}))^{1/|V_v|} & \text{if } p_v < 1 \text{ and } |V_v| < \infty, & (18b) \\ q_v \min\{q_w : w \in \mathcal{N}(v) \cap V_v\} & \text{if } p_v < 1 \text{ and } |V_v| = \infty, & (18c) \end{cases}$$

where V_v are the vertices of the connected component of v in the subgraph of G induced by all vertices v with $p_v < 1$. Then $\vec{0} \ll \vec{c} \in \Sigma(\mathcal{C}_G^{weak}(\vec{p}))$.

The proof of Theorem 3 is in Sect. 5.4. For infinite, connected G, we have a *discontinuous transition* in $\vec{c}$ as $\vec{p}$ approaches the boundary of $\mathring{\mathcal{P}}_{sh}^G$ (18c), while in the finite case it is continuous (18b). On the other hand, there are classes of BRFs having a continuous transition also in the infinite case, for example, the class of 2-factors on $\mathbb{Z}$ [15, Theorem 3.0].

Our proof trades accuracy in capturing all of $\mathring{\mathcal{P}}_{sh}^G$ against accuracy in the lower bound for the parameter of the dominated BPF. Intuitively it is clear that $\Sigma(\mathcal{C}_G^{weak}(\vec{p}))$ should increase with $\vec{p}$ (29), but our explicit lower bound (18c) decreases in $\vec{p}$. There is an explicit growing lower bound already shown by Liggett, Schonmann, and Stacey [15, Corollary 1.4], although only on a restricted set of parameters (19).

Equation (15a) does not imply that $\mu_{G,\vec{p}} \overset{st}{\leq} Y$, for all $Y \in \mathcal{C}_G^{weak}(\vec{p})$: for a finite $W \subsetneq V$, take $f := 1 - \mathbb{I}_{\{\vec{0}\}} \in \text{Mon}(W)$ and see that $\varPi_{\vec{p}}^W \overset{st}{\not\geq} \mu_{G(W),\vec{p}}$. Furthermore, $\Sigma(\mu_{G,\vec{p}})$ is neither minimal nor maximal (with respect to set inclusion) in the class $\mathcal{C}_G^{weak}(\vec{p})$. The maximal law is $\varPi_{\vec{p}}^W$ itself, as $[\vec{0}, \vec{p}] = \Sigma(\varPi_{\vec{p}}^W)$. We give a counterexample to the minimality of $\Sigma(\mu_{G,\vec{p}})$ in Sect. 7.

4.1 Reinterpretation of Bounds

Theorem 2 allows the application of conditions for admissible $\vec{p}$ for $\mathcal{P}_{udom}^{\mathcal{C}_G^{weak}}$ to $\mathring{\mathcal{P}}_{sh}^G$ and vice-versa. Hence we can play questions about the existence of a BRF dominating only trivial BPFs or the existence of Shearer's measure back and forth. In the following, we list known necessary or sufficient conditions for $\vec{p}$ to lie in $\mathring{\mathcal{P}}_{sh}^G$, most of them previously unknown for the domination problem. We assume that G contains no isolated vertices. The classical sufficient condition for the existence of Shearer's measure has been established independently several times and is known as either the "Lovász Local Lemma" [8] in graph theory or the "Dobrushin condition" [7, Theorem 6.1] in statistical mechanics:

Theorem 4 (Version of [9, (2.13)]) *Let $\vec{p} \in [0, 1]^V$. If there exists $\vec{s} \in]0, \infty[^V$ such that*

$$\forall v \in V : \quad q_v \prod_{w \in \mathcal{N}_1(v)} (1 + s_w) \le s_v, \tag{19}$$

then $\vec{p} \in \mathring{\mathcal{P}}_{\mathrm{sh}}^G$.

In the homogeneous case, there has been again a parallel and independent improvement on Theorem 4 by Liggett, Schonmann, and Stacey in probability theory and Scott and Sokal in statistical mechanics. Here p_{sh}^G is identified with the endpoint of the interval $[p_{\mathrm{sh}}^G, 1]$ corresponding to $\mathring{\mathcal{P}}_{\mathrm{sh}}^G$.

Theorem 5 ([15, Theorem 1.3], [18, Corollary 5.7]) *If G is uniformly bounded with degree D, then*

$$p_{\mathrm{sh}}^G \le 1 - \frac{(D-1)^{(D-1)}}{D^D}. \tag{20}$$

This leads to the only two cases of infinite graphs where p_{sh}^G is exactly known, namely the D-regular tree $\mathbb{T}_D$ with $p_{\mathrm{sh}}^{\mathbb{T}_D} = 1 - \frac{(D-1)^{(D-1)}}{D^D}$ and $\mathbb{Z}_{(k)}$, the k-fuzz of $\mathbb{Z}$, with $p_{\mathrm{sh}}^{\mathbb{Z}_{(k)}} = 1 - \frac{k^k}{(k+1)^{(k+1)}}$. The complementary inequality is [19, before Theorem 2] and [15, Corollary 2.2] for $\mathbb{T}_d$ and $\mathbb{Z}_{(k)}$, respectively. In these cases, explicit constructions of Shearer's measure are possible. See, for example, the construction as a $(k+1)$-factor in the case of $\mathbb{Z}_{(k)}$ [16, Sect. 4.2].

Fernández and Procacci derived another more recent and elaborate sufficient condition for a vector $\vec{p}$ to lie in $\mathring{\mathcal{P}}_{\mathrm{sh}}^G$:

Theorem 6 ([9, Theorem 1]) *Let $\vec{p} \in [0, 1]^V$. If there exists $\vec{s} \in]0, \infty[^V$ such that*

$$\forall v \in V : \quad q_v \Xi_{G(\mathcal{N}_1(v))}(-\vec{s}) \le s_v, \tag{21}$$

then $\vec{p} \in \mathring{\mathcal{P}}_{\mathrm{sh}}^G$.

The minus in (21) stems from their cluster expansion technique and assures that $\Xi_{G(\mathcal{N}_1(v))}(-\vec{s}) \ge 1$, whence $q_v \le 1$. The condition takes into account the local structure of G, via the triangles in $\mathcal{N}_1(v)$. It thus improves upon the LLL, which only considers the degree of v.

We present an example of a necessary condition by Scott and Sokal in the homogeneous case. Define the *upper growth rate* of a tree $\mathbb{T}$ rooted at o by

$$\overline{\mathrm{gr}}(\mathbb{T}) := \limsup_{n \to \infty} |V_n|^{1/n}, \tag{22}$$

where V_n are the vertices of $\mathbb{T}$ at distance n from o. Then we have

$$\mathring{\mathcal{P}}_{\mathrm{sh}}^{G} \overset{(\mathrm{UD})}{\subseteq} \mathcal{P}_{\mathrm{udom}}^{\mathcal{C}_G^{\mathrm{weak}}} \subseteq \mathcal{P}_{\mathrm{dom}}^{\mathcal{C}_G^{\mathrm{weak}}}$$

$$\cup| \qquad\qquad \cup|$$

$$\mathcal{P}_{\mathrm{udom}}^{\mathcal{C}_G^{\mathrm{strong}}} \subseteq \mathcal{P}_{\mathrm{dom}}^{\mathcal{C}_G^{\mathrm{strong}}} \overset{(\mathrm{ND})}{\subseteq} \mathring{\mathcal{P}}_{\mathrm{sh}}^{G}$$

Fig. 1 Inclusions in the proof of (17)

Theorem 7 ([18, Proposition 8.3]) *Let G be infinite. Then*

$$p_{\mathrm{sh}}^{G} \geq 1 - \frac{\overline{\mathrm{gr}}(\mathbb{T})^{\overline{\mathrm{gr}}(\mathbb{T})}}{(\overline{\mathrm{gr}}(\mathbb{T}) + 1)^{(\overline{\mathrm{gr}}(\mathbb{T})+1)}}. \tag{23}$$

Here $\mathbb{T}$ is a particular pruned subtree of the SAW (self-avoiding-walk) tree of G defined in [18, Sect. 6.2].

The pruned subtree $\mathbb{T}$ referred to above stems from a recursive expansion of the critical function via the fundamental identity (10) and the subsequent identification of this calculation with the one on $\mathbb{T}$. It is a subtree of the SAW tree of G, which not only avoids revisiting previously visited nodes, but also some of their neighbours. An example demonstrating this result is the following statement [18, (8.53)]:

$$p_{\mathrm{sh}}^{\mathbb{Z}^d} \geq 1 - \frac{d^d}{(d+1)^{(d+1)}}. \tag{24}$$

It follows from the fact that one can embed a regular rank d rooted tree in the pruned SAW tree $\mathbb{T}$ of $\mathbb{Z}^d$, whence $d \leq \overline{\mathrm{gr}}(\mathbb{T})$. For the full details, we refer the reader to [18, Sects. 6 and 8].

5 Proofs

We prove Theorem 2 by showing all inclusions outlined in Fig. 1. The four centre inclusions follow straight from (3e) and (7). The core part are two inclusions marked (UD) and (ND) in Fig. 1. The second inclusion (ND) generalises an idea of Liggett, Schonmann, and Stacey in Sect. 5.2. The key is the usage of Shearer's measure on finite subgraphs H for suitable $\vec{p} \in \partial \mathcal{P}_{\mathrm{sh}}^{H}$ to create BRFs dominating only trivial BPFs. Our novel contribution is the inclusion (UD). It replaces the LLL style proof for restricted parameters employed in [15, Proposition 1.2] by an optimal bound reminiscent of the optimal bound presented in [18, Sect. 5.3], using the fundamental identity (10) to full extent. After some preliminary work on Shearer's measure in Sect. 5.3, we prove the inclusion (UD) in Sect. 5.4.

5.1 Tools for Stochastic Domination

We list useful statements related to stochastic domination between BRFs.

Lemma 8 ([14, Chap. II, p. 79]) *Let Y, Z be two BRFs indexed by V, then*

$$Y \overset{\text{st}}{\geq} Z \quad \Leftrightarrow \quad \left(\forall \text{ finite } W \subseteq V : \quad Y_W \overset{\text{st}}{\geq} Z_W\right). \tag{25}$$

We build on the following technical result, inspired by [17, Lemma 1].

Proposition 9 *If $Z := \{Z_n\}_{n \in \mathbb{N}}$ is a BRF with*

$$\forall n \in \mathbb{N}, \vec{s}_{[n]} \in \mathcal{X}_{[n]} : \quad \mathbb{P}(Z_{n+1} = 1 | Z_{[n]} = \vec{s}_{[n]}) \geq p_n, \tag{26}$$

then $Z \overset{\text{st}}{\geq} \Pi_{\vec{p}}^{\mathbb{N}}$.

Proof Essentially the same inductive proof as in [17, Lemma 1].   □

If Y and Z are two independent BRFs with marginal vectors $\vec{p}$ and $\vec{r}$, then we denote by

$$Y \wedge Z := (Y_v \wedge Z_v)_{v \in V} \tag{27}$$

the *vertex-wise minimum* with marginal vector $\vec{p}\vec{r}$. Coupling shows that, for every two BRFs Y and Z, we have

$$Y \wedge Z \overset{\text{st}}{\leq} Y, \tag{28a}$$

and if X is a third BRF independent of (Y, Z) also

$$Y \overset{\text{st}}{\geq} Z \quad \Rightarrow \quad (Y \wedge X) \overset{\text{st}}{\geq} (Z \wedge X). \tag{28b}$$

Proposition 10 *For each dependency class C used in this paper and all $\vec{p}$ and $\vec{r}$, we have*

$$\Sigma\big(C(\vec{p}\vec{r})\big) \subseteq \Sigma\big(C(\vec{p})\big). \tag{29}$$

Proof Let $\vec{c} \in \Sigma(C(\vec{p}\vec{r}))$. Let $Y \in C(\vec{p})$ and X be $\Pi_{\vec{r}}^V$-distributed independently of Y. Using (28a)–(28b) we get $\Pi_{\vec{c}}^V \overset{\text{st}}{\leq} Y \wedge X \overset{\text{st}}{\leq} Y$, whence $\vec{c} \in \Sigma(Y)$. As this holds for every $Y \in C(\vec{p})$ we have $\vec{c} \in \Sigma(C(\vec{p}))$.   □

5.2 Non-domination

In this section, we prove inclusion (ND) from Fig. 1, that is, $\mathcal{P}_{\text{dom}}^{C_G^{\text{strong}}} \subseteq \mathring{\mathcal{P}}_{\text{sh}}^G$. The plan is as follows: in Lemma 11 we recall a coupling involving Shearer's measure on a finite graph H [19, proof of Theorem 1], which creates a BRF dominating only trivial BPFs for every $\vec{p} \notin \mathring{\mathcal{P}}_{\text{sh}}^H$. In Proposition 12, we generalise an approach used by

Liggett, Schonmann, and Stacey [15, Theorem 2.1] to arbitrary graphs and inhomogeneous parameters. For infinite G and $\vec{p} \notin \mathring{\mathcal{P}}_{\text{sh}}^G$, we find a suitable finite subgraph H of G on which to effectuate the above mentioned coupling and extend it with an independent BPF on the complement. The resulting BRF dominates only trivial BPFs.

Lemma 11 ([19, proof of Theorem 1]) *Let G be finite. If $\vec{p} \notin \mathring{\mathcal{P}}_{\text{sh}}^G$, then there exists a BRF $Z \in \mathcal{C}_G^{\text{strong}}(\vec{p})$ with $\mathbb{P}(Z_V = \vec{1}) = 0$.*

Proof As $\vec{p} \notin \mathring{\mathcal{P}}_{\text{sh}}^G$ and $\vec{1} \in \mathring{\mathcal{P}}_{\text{sh}}^G$, the line segment $[\vec{p}, \vec{1}]$ crosses $\partial \mathcal{P}_{\text{sh}}^G$ at the vector $\vec{r}$ (unique because $\mathring{\mathcal{P}}_{\text{sh}}^G$ is an up-set [18, Proposition 2.15(b)]). Let $\vec{x}$ be the solution of $\vec{p} = \vec{x}\vec{r}$. Let Y be $\mu_{G,\vec{r}}$-distributed and X be $\Pi_{\vec{x}}^V$-distributed independently of Y. Set $Z := Y \wedge X$. Then $Z \in \mathcal{C}_G^{\text{strong}}(\vec{p})$ and

$$\mathbb{P}(Z_V = \vec{1}) = \mathbb{P}(X_V = \vec{1})\mu_{G,\vec{r}}(Y_V = \vec{1}) = 0. \qquad \square$$

Proposition 12 *We have $\mathcal{P}_{\text{dom}}^{\mathcal{C}_G^{\text{strong}}} \subseteq \mathring{\mathcal{P}}_{\text{sh}}^G$.*

Proof Let $\vec{p} \notin \mathring{\mathcal{P}}_{\text{sh}}^G$. Then there exists a finite set $W \subseteq V$ with $\vec{p}_W \notin \mathring{\mathcal{P}}_{\text{sh}}^{G(W)}$. Using Lemma 11, create a $Y_W \in \mathcal{C}_{G(W)}^{\text{strong}}(\vec{p})$ with $\mathbb{P}(Y_W = \vec{1}) = 0$. Extend this to a $Y \in \mathcal{C}_G^{\text{strong}}(\vec{p})$ by letting $Y_{V \setminus W}$ be $\Pi_{\vec{p}_{V \setminus W}}^{V \setminus W}$-distributed independently of Y_W. Suppose that $Y \overset{\text{st}}{\geq} X$, where X is $\Pi_{\vec{x}}^V$-distributed. Then Lemma 8 implies that $Y_W \overset{\text{st}}{\geq} X_W$ and, using $f := \mathbb{I}_{\{\vec{1}\}} \in \text{Mon}(W)$, that

$$0 = \mathbb{P}(Y_W = \vec{1}) = \mathbb{E}\big[f(Y_W)\big] \geq \mathbb{E}\big[f(X_W)\big] = \mathbb{P}(X_W = \vec{1}) = \prod_{v \in W} x_v \geq 0.$$

Hence there exists a $v \in W$ with $x_v = 0$, whence $\vec{x} \not\gg \vec{0}$ and $\vec{p} \notin \mathcal{P}_{\text{dom}}^{\mathcal{C}_G^{\text{strong}}}$. $\qquad \square$

5.3 One Vertex Open Extension Probabilities

In this section, we re-encode our knowledge of Shearer's measure from the critical functions as ratios of critical functions, that is, conditional probabilities of the form "open on some vertices | open on some other vertices". These are exactly the ones Shearer's measure is minimal for (15b). This viewpoint admits a more succinct formulation of the fundamental identity (10) and bounds in Proposition 14. The notion of "escaping" pair introduced in this section is inspired by [19, Theorem 2]. It allows us to push the mass of unwanted conditional events away. We obtain lower bounds on conditional events of the above form, which are independent of size of the condition, as long as the escape persists.

For finite $W \subseteq V$ with $v \notin W$ and when $\varXi_{G(W)}(\vec{p}) > 0$, define the *one vertex open extension probability* of (W, v) by

$$\alpha_W^v(\vec{p}) := \mu_{G,\vec{p}}(Y_v = 1 | Y_W = \vec{1}). \tag{30}$$

Reformulate the *fundamental identity* (10) as

$$\alpha_W^v(\vec{p}) = 1 - \frac{q_v}{\prod_{i=1}^m \alpha_{W\setminus\{w_i,\ldots,w_m\}}^{w_i}(\vec{p})}, \tag{31}$$

where $W \cap \mathcal{N}(v) =: \{w_1, \ldots, w_m\}$.

Definition 13 Call the pair (W, v), respectively α_W^v, *escaping* iff $\mathcal{N}(v) \setminus W \neq \emptyset$ and call every vertex $w \in \mathcal{N}(v) \setminus W$ an *escape* of (W, v).

Proposition 14 Let $\vec{p} \in \mathcal{P}_{\text{sh}}^G$, then

$$\forall (W, v): \quad \alpha_W^v(\vec{p}) \le p_v \tag{32a}$$

and

$$\forall (W, v), w \in \mathcal{N}(v) \setminus W: \quad q_w \le \alpha_W^v(\vec{p}). \tag{32b}$$

Proof We use the fundamental identity (31) to see that

$$\alpha_W^v(\vec{p}) = 1 - \frac{q_v}{\prod \alpha_\star^\star(\vec{p})} \le 1 - q_v = p_v.$$

Likewise, if (W, v) is escaping with escape $w \in \mathcal{N}(v) \setminus W$, then (31) yields

$$0 \le \alpha_{W \uplus \{v\}}^w(\vec{p}) = 1 - \frac{q_w}{\alpha_W^v(\vec{p}) \prod \alpha_\star^\star(\vec{p})} \le 1 - \frac{q_w}{\alpha_W^v(\vec{p})},$$

hence $q_w \le \alpha_W^v(\vec{p})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 15 Let $\vec{p} \in \mathcal{P}_{\text{sh}}^G$. Then $\alpha_W^v(\vec{p})$ decreases, as W increases.

Proof If $p_v = 0$, then $\alpha_W^v(\vec{p}) = 0$ for all W. If $p_w = 0$ for $w \in W$ and v connected to w, then $\alpha_W^v(\vec{p})$ is not defined. Hence for the remainder of this proof assume $\vec{0} \ll \vec{p}$. We prove the statement by simultaneous induction for all v over the cardinality of W. The base case is

$$\alpha_\emptyset^v(\vec{p}) = 1 - q_v \begin{cases} \ge \frac{1 - q_v - q_w}{1 - q_w} = \alpha_{\{w\}}^v(\vec{p}) & \text{if } v \frown w, \\ = 1 - q_v = \alpha_{\{w\}}^v(\vec{p}) & \text{if } v \not\frown w. \end{cases}$$

For the induction step, we add just one vertex w to W and set $U := W \uplus \{w\}$. Let $\{w_1, \ldots, w_m\} := \mathcal{N}(v) \cap U$. First, assume that $w \not\frown v$. Using the fundamental identity (31), we have

$$\alpha_U^v(\vec{p}) = 1 - \frac{q_v}{\prod_{i=1}^m \alpha_{U\setminus\{w_i,\ldots,w_m\}}^{w_i}(\vec{p})} \le 1 - \frac{q_v}{\prod_{i=1}^m \alpha_{W\setminus\{w_i,\ldots,w_m\}}^{w_i}(\vec{p})} = \alpha_W^v(\vec{p}).$$

Secondly, assume that $v \frown w = w_m$. Hence

$$\alpha_U^v(\vec{p}) = 1 - \frac{q_v}{\prod_{i=1}^m \alpha_{U\setminus\{w_i,\ldots,w_m\}}^{w_i}(\vec{p})} \le 1 - \frac{q_v}{\prod_{i=1}^{m-1} \alpha_{W\setminus\{w_i,\ldots,w_{m-1}\}}^{w_i}(\vec{p})} = \alpha_W^v(\vec{p}). \qquad$$ □

5.4 Domination

In this section, we prove inclusion (UD) from Fig. 1, that is, $\mathring{\mathcal{P}}^G_{\mathrm{sh}} \subseteq \mathcal{P}^{\mathcal{C}^{\mathrm{weak}}_G}_{\mathrm{udom}}$. We split the proof in two and deal with finite and infinite G separately in Proposition 16 and 17, respectively. Additionally (33) and (34) combined yield a proof of (18a)–(18c) from Theorem 3.

On a finite graph, our approach is direct: Proposition 16 uses the minimality of $\mu_{G,\vec{p}}$ to construct a homogeneous nontrivial dominated vector $\vec{0} \ll \vec{c} \in \Sigma(\mathcal{C}^{\mathrm{weak}}_G(\vec{p}))$. For an infinite graph, the situation is more involved, and we use a technique of Antal and Pisztora [2, p. 1040–1041]: Suppose you have a $Y \in \mathcal{C}^{\mathrm{weak}}_G(\vec{p})$ with $\vec{0} \ll \vec{y} \in \Sigma(Y)$. Let X be $\Pi^{\vec{x}}_V$ with $\vec{0} \ll \vec{x}$ independently of Y and set $Z := X \wedge Y$. Then $\vec{0} \ll \vec{x}\vec{y} \in \Sigma(Z) \subseteq \Sigma(Y)$, that is, a coordinate-wise independent perturbation independent of X does not change the quality of Y's domination behaviour.

Proposition 18 uses this perturbation to blame adjacent 0 realisations of Z on X instead of Y, leading to the uniform technical minorisation (35):

$$\mathbb{P}(Z_v = 1 | Z_W = \vec{s}_W) \ge q_v \alpha^v_W(\vec{p}),$$

connecting the domination problem with Shearer's measure. Finally, in Proposition 17, we ensure to look at only escaping (W, v)s, hence getting rid of the $\alpha^v_W(\vec{p})$ term. This allows us to apply Proposition 9 and guarantee stochastic domination of a non-trivial BPF.

Proposition 16 *Let G be finite and $\vec{p} \in \mathring{\mathcal{P}}^G_{\mathrm{sh}}$. Let X be Π^V_c-distributed with*

$$c := 1 - \left(1 - \Xi_G(\vec{p})\right)^{1/|V|} > 0. \tag{33}$$

Then every $Y \in \mathcal{C}^{\mathrm{weak}}_G(\vec{p})$ fulfils $Y \overset{\mathrm{st}}{\ge} X$, hence $\vec{p} \in \mathcal{P}^{\mathcal{C}^{\mathrm{weak}}_G}_{\mathrm{udom}}$.

Proof The choice of $\vec{p}$ implies that $\Xi_G(\vec{p}) > 0$, therefore $c > 0$, too. Let $f \in \mathrm{Mon}(V)$ and $Y \in \mathcal{C}^{\mathrm{weak}}_G(\vec{p})$. Then

$$
\begin{aligned}
&\mathbb{E}\big[f(X)\big] \\
&= \sum_{\vec{s} \in \mathcal{X}_V} f(\vec{s})\mathbb{P}(X = \vec{s}) \\
&\le f(\vec{0})\mathbb{P}(X = \vec{0}) + f(\vec{1})\mathbb{P}(X \ne \vec{0}) \quad \text{monotonicity of } f \\
&= f(\vec{0})(1-c)^{|V|} + f(\vec{1})\big[1 - (1-c)^{|V|}\big] \\
&= f(\vec{0})\big[1 - \Xi_G(\vec{p})\big] + f(\vec{1})\Xi_G(\vec{p}) \\
&\le f(\vec{0})\mathbb{P}(Y \ne \vec{1}) + f(\vec{1})\mathbb{P}(Y = \vec{1}) \quad \text{minimality of Shearer's measure (15a)} \\
&\le \sum_{\vec{s} \in \mathcal{X}_V} f(\vec{s})\mathbb{P}(Y = \vec{s}) \qquad\qquad \text{monotonicity of } f \\
&= \mathbb{E}\big[f(Y)\big].
\end{aligned}
$$

Hence $X \overset{\mathrm{st}}{\le} Y$. As $\vec{0} \ll c\vec{1}$ we have $\vec{p} \in \mathcal{P}^{\mathcal{C}^{\mathrm{weak}}_G}_{\mathrm{udom}}$. $\square$

Proposition 17 *Let G be infinite and connected. Let $\vec{1} \gg \vec{p} \in \mathring{\mathcal{P}}^G_{\mathrm{sh}}$. Define the vector $\vec{c}$ by*

$$\forall v \in V : \quad c_v := q_v \min \{q_w : w \in \mathcal{N}(v)\}. \tag{34}$$

Then $\vec{c} \gg \vec{0}$ and every $Y \in \mathcal{C}^{\mathrm{weak}}_G(\vec{p})$ fulfils $Y \overset{\mathrm{st}}{\geq} \Pi^V_{\vec{c}}$, whence $\vec{p} \in \mathcal{P}^{\mathcal{C}^{\mathrm{weak}}_G}_{\mathrm{udom}}$.

Remark Proposition 17 motivated the definition of "escaping" pairs. It allows for non-trivial lower bounds for escaping $\alpha^v_W(\vec{p})$, in a correctly chosen ordering of a finite subgraph. Arbitrary $\alpha^v_W(\vec{p})$ defy control at the boundary of $\mathring{\mathcal{P}}^G_{\mathrm{sh}}$.

Proof We show that $Y_W \overset{\mathrm{st}}{\geq} \Pi^W_{\vec{c}_W}$, for every finite $W \subsetneq V$. Admitting this momentarily, Lemma 8 asserts that $Y \overset{\mathrm{st}}{\geq} \Pi^V_{\vec{c}}$. Conclude as $\vec{p} \ll \vec{1}$ implies that $\vec{c} \gg \vec{0}$.

Choose a finite $W \subsetneq V$ and let $|W| =: n$. As G is connected and infinite, there is a vertex $v_n \in W$ which has a neighbour w_n in $V \setminus W$. It follows that $(W \setminus \{v_n\}, v_n)$ is escaping with escape $w_n \in \mathcal{N}(v_n) \setminus W$. Apply this argument recursively to $W \setminus \{v_n\}$ and thus produce a total ordering $v_1 \prec \cdots \prec v_n$ of W, where, setting $W_i := \{v_1, \ldots, v_{i-1}\}$, every (W_i, v_i) is escaping with escape $w_i \in \mathcal{N}(v_i) \setminus W_i$.

Let X be $\Pi^V_{\vec{q}}$-distributed independently of Y. Set $Z := Y \wedge X$. Then (35) from Proposition 18 and the minoration for escaping pairs (32b) combine to

$$\forall i \in [n], \forall \vec{s}_{W_i} \in \mathcal{X}_{W_i} : \quad \mathbb{P}(Z_{v_i} = 1 | Z_{W_i} = \vec{s}_{W_i}) \geq \alpha^{v_i}_{W_i}(\vec{p}) q_{v_i} \geq q_{w_i} q_{v_i} \geq c_{v_i}.$$

This is sufficient for Proposition 9 to construct a coupling with $Z_W \overset{\mathrm{st}}{\geq} \Pi^W_{\vec{c}_W}$. Apply (28a) to get

$$Y_W \overset{\mathrm{st}}{\geq} Y_W \wedge X_W = Z_W \overset{\mathrm{st}}{\geq} \Pi^W_{\vec{c}_W}$$

and extend this to all of V with the help of Lemma 8. $\qquad\square$

Proposition 18 *Let $\vec{1} \gg \vec{p} \in \mathring{\mathcal{P}}^G_{\mathrm{sh}}$ and $Y \in \mathcal{C}^{\mathrm{weak}}_G(\vec{p})$. Let X be $\Pi^V_{\vec{q}}$-distributed independently of Y and set $Z := X \wedge Y$. We claim that, for all admissible (W, v),*

$$\forall \vec{s}_W \in \mathcal{X}_W : \quad \mathbb{P}(Z_v = 1 | Z_W = \vec{s}_W) \geq q_v \alpha^v_W(\vec{p}). \tag{35}$$

Remark This generalises [15, Proposition 1.2], the core of Liggett, Schonmann, and Stacey's proof, in the following ways: we localise the parameters α and r they used and assume no total ordering of the vertices yet. Furthermore, $r_v = q_v$ follows from a conservative bound of the form

$$r_v := 1 - \sup \{\alpha^v_W(\vec{p}) : (W, v) \text{ escaping}\} = 1 - p_v = q_v,$$

where the sup is attained in $\alpha^v_\emptyset(\vec{p}) = p_v$.

Proof Recall that $\vec{p} \in \mathring{\mathcal{P}}^G_{\mathrm{sh}}$ implies that $\vec{p} \gg \vec{0}$. Whence $\vec{q} \ll \vec{1}$ and (35) is well defined because

$$\forall \text{ finite } W \subseteq V, \vec{s}_W \in \mathcal{X}_W : \quad \mathbb{P}(Z_W = \vec{s}_W) > 0.$$

For every decomposition $N_0 \uplus N_1 := \mathcal{N}(v) \cap W$ with $N_0 =: \{u_1, \ldots, u_l\}$, $N_1 =: \{w_1, \ldots, w_m\}$ and $M := W \setminus \mathcal{N}(v)$, the fundamental identity (31) implies the inequality

$$\left[1 - \alpha_W^v(\vec{p})\right] \left(\prod_{j=1}^{l} p_{u_j}\right) \prod_{i=1}^{m} \alpha_{M \uplus \{w_1, \ldots, w_{i-1}\}}^{w_i}(\vec{p}) \geq q_v, \tag{36}$$

where $p_{u_j} \geq \alpha_{M \uplus N_1 \uplus \{u_1, \ldots, u_{j-1}\}}^{u_j}(\vec{p})$ follows from (32a).

We prove (35) inductively over the cardinality of W. The induction base $W = \emptyset$ is easy as $\mathbb{P}(Z_v = 1) = q_v \mathbb{P}(Y_v = 1) \geq q_v p_v = q_v \alpha_{\emptyset}^v(\vec{p})$. For the induction step, fix $\vec{s}_W \in \mathcal{X}_W$ and the decomposition

$$N_0 := \{w \in W \cap \mathcal{N}(v) : s_w = 0\} =: \{u_1, \ldots, u_l\}$$

and

$$N_1 := \{w \in W \cap \mathcal{N}(v) : s_w = 1\} =: \{w_1, \ldots, w_m\}.$$

We write

$$\mathbb{P}(Y_v = 0 | Z_W = \vec{s}_W)$$

$$= \mathbb{P}(Y_v = 0 | Z_{N_0} = \vec{0}, Z_{N_1} = \vec{1}, Z_M = \vec{s}_M)$$

$$= \frac{\mathbb{P}(Y_v = 0, Z_{N_0} = \vec{0}, Z_{N_1} = \vec{1}, Z_M = \vec{s}_M)}{\mathbb{P}(Z_{N_0} = \vec{0}, Z_{N_1} = \vec{1}, Z_M = \vec{s}_M)}$$

$$\leq \frac{\mathbb{P}(Y_v = 0, Z_M = \vec{s}_M)}{\mathbb{P}(X_{N_0} = \vec{0}, Y_{N_1} = \vec{1}, Z_M = \vec{s}_M)} \tag{37a}$$

$$= \frac{\mathbb{P}(Y_v = 0 | Z_M = \vec{s}_M)\mathbb{P}(Z_M = \vec{s}_M)}{\mathbb{P}(X_{N_0} = \vec{0})\mathbb{P}(Y_{N_1} = \vec{1}, Z_M = \vec{s}_M)} \tag{37b}$$

$$\leq \frac{q_v}{\mathbb{P}(X_{N_0} = \vec{0})\mathbb{P}(Y_{N_1} = \vec{1} | Z_M = \vec{s}_M)} \tag{37c}$$

$$= \frac{q_v}{\prod_{j=1}^{l}(1 - q_{u_j}) \prod_{i=1}^{m} \mathbb{P}(Y_{w_i} = 1 | Y_{w_1} = \cdots = Y_{w_{i-1}} = 1, Z_M = \vec{s}_M)}$$

$$\leq \frac{q_v}{\prod_{j=1}^{l} p_{u_j} \prod_{i=1}^{m} \alpha_{M \uplus \{w_1, \ldots, w_{i-1}\}}^{w_i}(\vec{p})} \tag{37d}$$

$$\leq 1 - \alpha_W^v(\vec{p}). \tag{37e}$$

The key steps in (37a)–(37e) are:

(37a) increasing the numerator by dropping $Z_{N_0} = \vec{0}$ and $Z_{N_1} = \vec{1}$ while decreasing the denominator by using the definition of Z,

(37c) as $d(v, M) \geq 1$ and $Y \in \mathcal{C}_G^{\text{weak}}(\vec{p})$,

(37b) using the independence of X_{N_0} from (Y_{N_1}, Z_M),

(37d) applying the induction hypothesis (35) to the factors of the rhs product in the denominator, which have strictly smaller cardinality,

([37e](#)) applying inequality ([36](#)).

Hence

$$\mathbb{P}(Z_v = 1 | Z_W = \vec{s}_W) \geq q_v \mathbb{P}(Y_v = 1 | Z_W = \vec{s}_W) \geq q_v \alpha_W^v(\vec{p}). \qquad \square$$

6 The Weak Invariant Case

In this section, we extend our characterisation to the case of BRFs with weak dependency graph, which are invariant under a group action. Let Γ be a subgroup of $\mathrm{Aut}(G)$. A BRF Y is Γ-*invariant* iff

$$\forall \gamma \in \Gamma : \quad (\gamma Y) := (Y_{\gamma(v)})_{v \in V} \text{ has the same law as } Y. \qquad (38)$$

For a given Γ and Γ-invariant $\vec{p}$, we denote by $\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}(\vec{p})$ the *weak, Γ-invariant dependency class*, that is, Γ-invariant BRFs with weak dependency graph G, and by $\mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}(\vec{p})$ the corresponding strong version.

We call a pair (G, Γ) *partition exhaustive* iff there exists a sequence of partitions $(P_n)_{n \in \mathbb{N}}$ of V with $P_n := (V_i^{(n)})_{i \in \mathbb{N}}$ such that

$$\forall n, i, j \in \mathbb{N} : \quad G(V_i^{(n)}) \text{ is isomorph to } G(V_1^{(n)}) =: G_n, \qquad (39a)$$

$$\forall n \in \mathbb{N} : \quad \text{the orbit of } P_n \text{ under } \Gamma \text{ is finite,} \qquad (39b)$$

$$V_1^{(n)} \xrightarrow[n \to \infty]{} V, \quad \text{that is } (G_n)_{n \in \mathbb{N}} \text{ exhausts } G. \qquad (39c)$$

The kind of graphs we have in mind are regular infinite trees and tree-like graphs, $\mathbb{Z}^d$ and other regular lattices (triangular, hexagonal, etc.). We think of the group Γ as generated by some of the natural shifts and rotations of the graph. An example are increasing regular rectangular decompositions of $\mathbb{Z}^d$ together with translations of $\mathbb{Z}^d$.

Theorem 19 *Let (G, Γ) be partition exhaustive. Then*

$$\mathcal{P}_{\text{udom}}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}} = \mathcal{P}_{\text{dom}}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}} = \mathring{\mathcal{P}}_{\text{sh}}^{\Gamma\text{-inv}} := \left\{ \vec{p} \in \mathring{\mathcal{P}}_{\text{sh}}^G : \vec{p} \text{ is } \Gamma\text{-invariant} \right\}. \qquad (40)$$

Remark It follows from ([39a](#))–([39c](#)) that Γ acts quasi-transitively on G. Hence $\mathring{\mathcal{P}}_{\text{sh}}^{\Gamma\text{-inv}}$ can be seen as a subset of a finite-dimensional space.

The mixing in ([41](#)) destroys strong independence even in simple cases like $G = \mathbb{Z}$ and Γ the group of translations of $\mathbb{Z}$ [15, end of Sect. 2]. The easiest way to see this is to let $G := (\{v, w\}, \emptyset)$, $X^{(1)}, X^{(2)} \in \mathcal{C}_G^{\text{strong}}(\vec{p})$ and Y be Bernoulli$(\frac{1}{2})$-distributed, all independent of each other. Define $Z := X^{(Y)}$ and ask if $\mathbb{P}(Z_v = Z_w = 1) = \mathbb{P}(Z_v = 1)\mathbb{P}(Z_w = 1)$. This fails for most choices of $\vec{p}$. Calculations on slightly more complex graphs as $G = (\{u, v, w\}, \{(u, v)\})$ show that Z from ([41](#)) has no strong dependency graph. Thus the present approach, inspired by [15, p. 89], does not allow to characterise $\mathcal{P}_{\text{udom}}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}}$ and $\mathcal{P}_{\text{dom}}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}}$.

Proof As $\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}(\vec{p})$ is a subclass of $\mathcal{C}_G^{\text{weak}}(\vec{p})$, Theorem (2) implies that $\mathring{\mathcal{P}}_{\text{sh}}^{\Gamma\text{-inv}} \subseteq \mathcal{P}_{\text{udom}}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}} \subseteq \mathcal{P}_{\text{dom}}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}}$. We show $\mathcal{P}_{\text{dom}}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}} \subseteq \mathring{\mathcal{P}}_{\text{sh}}^{\Gamma\text{-inv}}$ by constructing a counterexample. If $\vec{p} \notin \mathring{\mathcal{P}}_{\text{sh}}^{\Gamma\text{-inv}}$, then by (39c) there exists a $n \in \mathbb{N}$ such that $\vec{p} \notin \mathring{\mathcal{P}}_{\text{sh}}^{G_n,\Gamma}$ (the intersection of the projections of Γ-invariant parameters on G with $\mathring{\mathcal{P}}_{\text{sh}}^{G_n}$). Let $P := P_n$ and let $(P^{(1)}, \ldots, P^{(k)})$ be its finite orbit under the action of Γ (39b). By (39a), each class $V_{(i,j)} \in P^{(j)}$ has a graph $G(V_{(i,j)})$ isomorph to G_n. Use Lemma 11 to construct i.i.d. BPFs $Z^{(i,j)} \in \mathcal{C}_{G_n}^{\text{strong}}(\vec{p})$ with $\mathbb{P}(Z^{(i,j)} = \vec{1}) = 0$. For $j \in [k]$, collate the $Z_{(i,j)}$ to a BPF $Z^{(j)}$. This works, as $P^{(j)}$ is a partition of G. By definition, $Z^{(j)} \in \mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}(\vec{p})$. Finally, let U be Uniform($[k]$)-distributed and independent of everything else. Define the final BPF Z by

$$Z := \sum_{j=1}^{k} [U = j] Z^{(j)}. \tag{41}$$

We claim that $Z \in \mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}(\vec{p})$. The mixing in (41) keeps $Z \in \mathcal{C}_G^{\text{weak}}(\vec{p})$. To see its Γ-invariance, let $\gamma \in \Gamma$. The automorphism γ acts injectively on $(P^{(1)}, \ldots, P^{(k)})$ and thus also on $[k]$. Therefore, using the fact that U is uniform and everything is constructed independently, we have

$$\gamma Z = \sum_{j=1}^{k} [U = j] \gamma Z^{(j)} = \sum_{j=1}^{k} [U = \gamma^{-1} j] Z^{(j)} = \sum_{j=1}^{k} [U = j] Z^{(j)} = Z. \qquad \square$$

7 The Asymptotic Size of the Jump on $\mathbb{Z}_{(k)}$

Liggett, Schonmann, and Stacey formulated the following conjecture about the size of the jump at the critical value on $\mathbb{Z}_{(k)}$, the k-fuzz of $\mathbb{Z}$:

Conjecture 20 ([15, after Corollary 2.2])

$$k \in \mathbb{N}_0: \quad \sigma\left(\mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{weak}}\left(p_{\text{udom}}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{weak}}}\right)\right) = \frac{k}{k+1}. \tag{42}$$

We think that Liggett, Schonmann, and Stacey were led by the intuition that the extra randomness used in obtaining the above lower bound (see the Y in [15, Proposition 1.2] or the X in the proof of Proposition 18) can be ignored in a suitable transitive setting. This would, in general, yield $\sigma(\mu_{G,p_{\text{sh}}^G}) = \sigma(\mathcal{C}_G^{\text{weak}}(p_{\text{udom}}^{\mathcal{C}_G^{\text{weak}}}))$, and, in the particular case of $\mathbb{Z}_{(k)}$, $\sigma(\mu_{\mathbb{Z}_{(k)},p_{\text{sh}}^{\mathbb{Z}_{(k)}}}) = \frac{k}{k+1}$ [16, Sect. 4.2], with $p_{\text{sh}}^{\mathbb{Z}_{(k)}} = 1 - \frac{k^k}{(k+1)^{(k+1)}}$ [16, Sect. 4.2].

Proposition 21 shows that asymptotically $\sigma(\mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{weak}}(p_{\text{sh}}^{\mathbb{Z}_{(k)}}))$ is much closer to the lower bound of $\frac{k}{(k+1)^2}$ from [15, Corollary 2.5]. This is caused by the increasing range of dependence, as $k \to \infty$, which allows for extreme correlations on the same order as the extra randomness used to decorrelate them.

Proposition 21 *For $k \in \mathbb{N}$ and Γ_k the translations of $\mathbb{Z}$, let C_k be either $\mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{strong}}$ or $\mathcal{C}_{\Gamma_k\text{-inv}}^{\text{weak}}$. We have*

$$\forall \varepsilon > 0 : \exists K(\varepsilon) : \forall k \geq K : \quad \sigma\left(C_k\left(p_{\text{sh}}^{\mathbb{Z}_{(k)}}\right)\right) \leq \frac{1 + (1 + \varepsilon)\ln(k+1)}{k+1}. \quad (43)$$

Proof Let $\mathcal{N}_1(0)^+ := \{0, \ldots, k\}$ be the non-negative closed half-ball of radius k centred at 0. Define a BRF Y on $\mathbb{Z}$ by setting $\mathbb{P}(Y_{\mathcal{N}_1(0)^+} = \vec{1}) := p_{\text{dom}}^{\mathbb{Z}_{(k)}}$, $\mathbb{P}(Y_{\mathcal{N}_1(0)^+} = \vec{0}) := q_{\text{dom}}^{\mathbb{Z}_{(k)}}$ and letting $Y_{\mathbb{Z}\setminus\mathcal{N}_1(0)^+}$ be $\Pi_{p_{\text{dom}}^{\mathbb{Z}_{(k)}}}^{\mathbb{Z}\setminus\mathcal{N}_1(0)^+}$-distributed independently of $Y_{\mathcal{N}_1(0)^+}$. As $Y \in \mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{strong}}(p_{\text{dom}}^{\mathbb{Z}_{(k)}})$, [15, Corollary 2.5] applies and $Y \overset{\text{st}}{\geq} X$, where X is $\Pi_\sigma^{\mathbb{Z}}$-distributed with $\sigma \in [\frac{k}{(k+1)^2}, \frac{k}{k+1}]$. Lemma 8 implies $X_{\mathcal{N}_1(\vec{0})^+} \overset{\text{st}}{\leq} Y_{\mathcal{N}_1(\vec{0})^+}$ and in particular the inequality

$$1 - (1 - \sigma)^{(k+1)} = \mathbb{P}(X_{\mathcal{N}_1(\vec{0})^+} \neq \vec{0}) = \mathbb{P}(Y_{\mathcal{N}_1(\vec{0})^+} \neq \vec{0}) = 1 - \frac{k^k}{(k+1)^{(k+1)}}.$$

Rewrite it into

$$\sigma \leq 1 - \frac{k^{\frac{k}{k+1}}}{k+1}$$
$$= \frac{1}{k+1} + \frac{k}{k+1}\left(1 - k^{-\frac{1}{k+1}}\right)$$
$$\leq \frac{1}{k+1} + \left(1 - (k+1)^{-\frac{1}{k+1}}\right).$$

For every $\varepsilon > 0$ and z close enough to 0 we know that $1 - e^{-z} \leq (1 + \varepsilon)z$. The statement for $\mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{strong}}$ follows from $z_k := \frac{\ln(k+1)}{k+1} \xrightarrow{k \to \infty} 0$. The result for $\mathcal{C}_{\Gamma_k\text{-inv}}^{\text{weak}}$ follows from a mixing construction similar to (41). $\qquad\square$

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