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by

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Asymptotic Expansions*

Part III. THE SUM FORMULA OF EULER

Chapter I. REAL VARIABLES

Section 1. FIRST FUNDAMENTAL IDENTITY

The purpose of this chapter is to write the sum

$$S = f(A+1) + f(A+2) + \dots + f(B) \quad ,$$

where A and B are real numbers such that B - A is a positive integer, in another form involving integrals. To that end I let $\varphi_1(x) = x - \frac{1}{2}$ and I introduce the periodic function $\psi_1(x)$ with period 1 which is equal to $\varphi_1(x) = x - \frac{1}{2}$ in the open interval (0,1) and which has the value zero for integral values of x.

THEOREM 1. Let α and β be real and assume $0 \leqq \theta \leqq 1$. If f(x) is continuously differentiable in the interval

$$\min(A+\alpha, B+\beta, A+\theta) \leqq x \leqq \max(A+\alpha, B+\beta, B+\theta) \quad ,$$

then

$$(1.1)\quad S = \int_{A+\alpha}^{B+\beta} f(x)dx + \varphi_1(1-\beta)f(B+\beta) - \varphi_1(1-\alpha)f(A+\alpha) - R_1 \quad ,$$

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*Part I (Enveloping Series) and Part II (Transformation of an enveloping series into a convergent series) consist of notes made by John H. Gay and Thomas E. Kurtz of lectures given by J.G. van der Corput at National Bureau of Standards, Los Angeles, California, Summer Session 1951 (U.S. Department of Commerce, National Bureau of Standards, INA 51-8; June 28,1951).

where

$$R_1 = \int_{A+\alpha}^{A+\theta} \varphi_1(A + 1 - x)f'(x)dx + \int_{A+\theta}^{B+\theta} \psi_1(B + 1 - x)f'(x)dx +$$

$$+ \int_{B+\theta}^{B+\beta} \varphi_1(B + 1 - x)f'(x)dx \quad .$$

I call (1.1) the first fundamental identity.

Remark: It is true that the definition of R_1 involves θ, but in reality the remainder is independent of θ, as long as θ lies in the interval $0 \leqq \theta \leqq 1$. In fact, in the open interval $0 < \theta < 1$ the derivative of R_1 with respect to θ is equal to

$$\varphi_1(1 - \theta)f'(A + \theta) + \psi_1(1 - \theta)f'(B + \theta) - \psi_1(B + 1 - A - \theta)f'(A + \theta) -$$

$$- \varphi_1(1 - \theta)f'(B + \theta) = 0 \quad ,$$

by the definition of the periodic function $\psi_1(x)$. The remainder R_1, which is a continuous function of θ in the closed interval (0,1) is therefore in that whole interval independent of θ.

In many cases the parameters α and β can be chosen in such a way that the integral and the two following terms on the right hand side of (1.1) possess simple values.

In the special case that α and β belong to the open interval (0,1), the formula takes the form

$$S = \int_{A+\alpha}^{B+\beta} f(x)dx + \psi_1(1 - \beta)f(B + \beta) - \psi_1(1 - \alpha)f(A + \alpha) - R_1 \, ,$$

where

$$R_1 = \int_{A+\alpha}^{B+\beta} \psi_1(B + 1 - x)f'(x)dx \quad .$$

In this case the formula does not involve the linear polynomial $\varphi_1(x)$, but only the periodic function $\psi_1(x)$.

Proof: Since R_1 is independent of θ in the interval $0 \leqq \theta \leqq 1$, we may assume in the proof that $0 < \theta < 1$. Integrating by parts we obtain, since $\varphi_1'(x) =$

$$R_1 = \Big|_{A+\alpha}^{A+\theta} \varphi_1(A + 1 - x)f(x) + \int_{A+\alpha}^{A+\theta} f(x)dx$$

$$+ \Big|_{A+\theta}^{B+\theta} \psi_1(B + 1 - x)f(x) + \int_{A+\theta}^{B+\theta} f(x)dx$$

$$+ \Big|_{B+\theta}^{B+\beta} \varphi_1(B + 1 - x)f(x) + \int_{B+\theta}^{B+\beta} f(x)dx \quad .$$

The three integrals together furnish

$$\int_{A+\alpha}^{B+\beta} f(x)dx \quad .$$

Since $\psi_1(B + 1 - x)$ makes a jump 1 if x passes a point $B + 1 - m$, where m is an integer, we obtain (See note on page I,1,4)

$$\Big|_{A+\theta}^{B+\theta} \psi_1(B + 1 - x)f(x) = \psi_1(1 - \theta)f(B + \theta) - \psi_1(1 - \theta)f(A + \theta) - T \quad ,$$

where

$$T = \Sigma f(B + 1 - m) \quad ;$$

the last sum is extended over the integers m such that

$$A + \theta < B + 1 - m < B + \theta \quad ,$$

so that $B + 1 - m$ runs through the values $A + 1, A + 2, \cdots, B$. This shows that $T = S$. Evaluating the two other integrated parts and using the relation

$$\psi_1(1 - \theta) = \varphi_1(1 - \theta) \quad ,$$

the result after cancellation is

$$R_1 = \int_{A+\alpha}^{B+\beta} f(x)\,dx - \varphi_1(1 - \alpha)f(A + \alpha) + \varphi_1(1 - \beta)f(B + \beta) - S \quad ,$$

which implies (1.1).

Note to page I,1,3. In fact, we have for each integer m

$$\psi_1\,(B + 1 - x) \to -\frac{1}{2} \quad ,$$

as $x < B + 1 - m$ tends to $B + 1 - m$, and

$$\psi_1\,(B + 1 - x) \to \frac{1}{2} \quad ,$$

as $x > B + 1 - m$ tends to $B + 1 - m$.

Section 2. THE SECOND FUNDAMENTAL IDENTITY

The next problem is to write the remainder R_1, occurring in the first fundamental identity, in another form. To that end I introduce the polynomials $\varphi_h(x)$ ($h \geqq 2$) uniquely defined by

$$\varphi_h'(x) = \varphi_{h-1}(x) \quad \text{and} \quad \int_0^1 \varphi_h(x)dx = 0 \quad .$$

Then

$$\varphi_h(0) = \varphi_h(1) \qquad \text{for } h = 2,3, \cdots ; \tag{2.1}$$

since

$$\varphi_2(1) - \varphi_2(0) = \int_0^1 \varphi_1(x)dx = \int_0^1 (x - \tfrac{1}{2})dx = 0$$

and for $h \geqq 3$

$$\varphi_h(1) - \varphi_h(0) = \int_0^1 \varphi_{h-1}(x)dx = 0 \quad .$$

Let $\Psi_h(x)$ be the periodic function with period 1 which is equal to $\varphi_h(x)$ in the closed interval $(0,1)$; it follows from (2.1) that such a periodic function exists.

The first few polynomials $\varphi_h(x)$ are given by the following table.

$1!\varphi_1(x) = x - \frac{1}{2}.$

$2!\varphi_2(x) = x^2 - x + \frac{1}{6}.$

$3!\varphi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$

$4!\varphi_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$

$5!\varphi_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$

$6!\varphi_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.$

$7!\varphi_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x.$

$8!\varphi_8(x) = x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}.$

$9!\varphi_9(x) = x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x.$

$10!\varphi_{10}(x) = x^{10} - 5x^9 + \frac{15}{2}x^8 - 7x^6 + 5x^4 - \frac{3}{2}x^2 + \frac{5}{66}.$

$11!\varphi_{11}(x) = x^{11} - \frac{11}{2}x^{10} + \frac{55}{6}x^9 - 11x^7 + 11x^5 - \frac{11}{2}x^3 + \frac{5}{6}x.$

$12!\varphi_{12}(x) = x^{12} - 6x^{11} + 11x^{10} - \frac{33}{2}x^8 + 22x^6 - \frac{33}{2}x^4 + 5x^2 - \frac{691}{2730}.$

The second fundamental identity is as follows:

THEOREM 2. *Let* α *and* β *be real and* $0 \leqq \theta \leqq 1$. *Suppose that* $f(x)$ *is continuously differentiable in the interval*

$$\min(A + \alpha, B + \beta, A + \theta) \leqq x \leqq \max(A + \alpha, B + \beta, B + \theta)$$

and let

$$R_h(f) = \int_{A+\alpha}^{A+\theta} \varphi_h(A + 1 - x)f(x)dx + \int_{A+\theta}^{B+\theta} \Psi_h(B + 1 - x)f(x)dx +$$

$$+ \int_{B+\theta}^{B+\beta} \varphi_h(B + 1 - x)f(x)dx \quad .$$

Then

$$R_h(f) = -\varphi_{h+1}(1-\beta)f(B+\beta) + \varphi_{h+1}(1-\alpha)f(A+\alpha) + R_{h+1}(f') \quad .$$

<u>Proof</u>: $R_{h+1}(f') - R_h(f)$ can be written as a sum of three terms. The first of these three terms is

$$\int_{A+\alpha}^{A+\theta} \left\{ \varphi_{h+1}(A+1-x)f'(x) + \varphi'_{h+1}(A+1-x)f(x) \right\} dx =$$

$$= \varphi_{h+1}(1-\theta)\,f\,(A+\theta) - \varphi_{h+1}(1-\alpha)\,f\,(A+\alpha) \quad .$$

The second of these three terms is

$$\int_{A+\theta}^{B+\theta} \left\{ \psi_{h+1}(B+1-x)f'(x) + \psi'_{h+1}(B+1-x)f(x) \right\} dx =$$

$$= \psi_{h+1}(1-\theta)\,f\,(B+\theta) - \psi_{h+1}(1-\theta)\,f\,(A+\theta)$$

and finally the last of these three terms is

$$\int_{B+\theta}^{B+\beta} \left\{ \varphi_{h+1}(B+1-x)f'(x) + \varphi'_{h+1}(B+1-x)f(x) \right\} dx =$$

$$= \varphi_{h+1}(1-\beta)\,f\,(B+\beta) - \varphi_{h+1}(1-\theta)\,f\,(B+\theta) \quad .$$

Consequently, since

$$\psi_{h+1}(1-\theta) = \varphi_{h+1}(1-\theta) \quad ,$$

we get the required result

$$R_{h+1}(f') - R_h(f) = \varphi_{h+1}(1-\beta)f(B+\beta) - \varphi_{h+1}(1-\alpha)f(A+\alpha) \quad .$$

Section 3. THE SUM FORMULA OF EULER

THEOREM 3. *Let* A *and* B *be arbitrary real numbers such that* B - A *is a positive integer. Let* α *and* β *be real and* $0 \leqq \theta \leqq 1$. *Then the sum*

$$S = f(A+1) + f(A+2) + \dots + f(B) ,$$

where f(x) *is* h *times* ($h \geqq 1$) *continuously differentiable in the interval*

$$\min(A+\alpha , B+\beta , A+\theta) \leqq x \leqq \max(A+\alpha , B+\beta , B+\theta)$$

can be written as

$$S = \int_{A+\alpha}^{B+\beta} f(x)dx + \sum_{s=0}^{h-1} \varphi_{s+1}(1-\beta)f^{(s)}(B+\beta) \tag{3.1}$$
$$- \sum_{s=0}^{h-1} \varphi_{s+1}(1-\alpha)f^{(s)}(A+\alpha) - R_h ,$$

where

$$R_h = \int_{A+\alpha}^{A+\theta} \varphi_h(A+1-x)f^{(h)}(x)dx + \int_{A+\theta}^{B+\theta} \psi_h(B+1-x)f^{(h)}(x)dx +$$

$$+ \int_{B+\theta}^{B+\beta} \varphi_h(B+1-x)f^{(h)}(x)dx .$$

(sum formula of Euler*)

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*The sum formula of Euler in this general form can be found in the doctor's thesis of Duncan, Stanford 1952.

In many cases the parameters α and β can be chosen in such a way that the terms, occurring in the expansion for S, take a simple form. In the special case that α and β belong to the closed interval (0,1), the sum formula of Euler takes the form

$$
\begin{aligned}
S = \int_{A+\alpha}^{B+\beta} f(x)dx &+ \sum_{s=o}^{h-1} \varphi_{s+1}(1-\beta)f^{(s)}(B+\beta) - \\
&- \sum_{s=o}^{h-1} \varphi_{s+1}(1-\alpha)f^{(s)}(A+\alpha) - R_h \quad ,
\end{aligned} \tag{3.2}
$$

in which the remainder term

$$R_h = \int_{A+\alpha}^{B+\beta} \psi_h(B+1-x)f^{(h)}(x)dx$$

does not involve the polynomial $\varphi_h(x)$, but only the periodic function $\psi_h(x)$

The special case h = 1 of the Euler sum formula has already been proved in §1. The sum formula follows for $h \geqq 2$ from §2, since theorem 2 of that section furnishes for $s = 1,2, \cdots, h-1$

$$R_s - R_{s+1} = -\varphi_{s+1}(1-\beta)f^{(s)}(B+\beta) + \varphi_{s+1}(1-\alpha)f^{(s)}(A+\alpha) \quad ,$$

so that

$$R_1 - R_h = \sum_{s=1}^{h-1}(R_s - R_{s+1}) =$$

$$-\sum_{s=1}^{h-1}\varphi_{s+1}(1-\beta)f^{(s)}(B+\beta) + \sum_{s=1}^{h-1}\varphi_{s+1}(1-\alpha)f^{(s)}(A+\alpha) \quad .$$

Solving this for R_1 and substituting it into (1.1) we get (3.1).

From the sum formula of Euler it follows immediately that the remainder R_h is independent of Θ in the interval $0 \leqq \Theta \leqq 1$.

It is easy to find an upper bound for the absolute value of the remainder. The polynomial $\varphi_h(x)$ is uniquely defined by h and its absolute value possesses therefore between $1 - \Theta$ and $1 - \alpha$ an upper bound which depends only on h and α, and not on Θ, since Θ is bounded. Thus $|\varphi_h(A + 1 - x)|$ has between $A + \alpha$ and $A + \Theta$ an upper bound which depends only on h and α. Similarly $|\varphi_h(B + 1 - x)|$ possesses between $B + \Theta$ and $B + \beta$ an upper bound which depends only on h and β. Finally the absolute value of the periodic function $\psi_h(x)$ is less than a suitably chosen number which depends only on h. It is therefore possible to find a number C_h depending only on h, α and β such that in the remainder R_h each factor $\varphi_h(A + 1 - x)$, $\varphi_h(B + 1 - x)$ and $\psi_h(B + 1 - x)$ is in absolute value $\leqq C_h$. Consequently

$$|R_h| \leqq C_h \int_{\min(A+\alpha,\ B+\beta,\ A+\Theta)}^{\max(A+\alpha,\ B+\beta,\ B+\Theta)} |f^{(h)}(x)|dx \quad .$$

This result is often sharp enough, if we are only interested in the order of magnitude of the remainder. In the following section we deduce sharper inequalities for the remainder term.

The polynomial $\varphi_h(x)(h = 1,2,\cdots)$ has the following property:

$$\varphi_h(1 - x) = (-)^h \varphi_h(x) \quad . \tag{3.3}$$

This is evident for $\varphi_1(x) = x - \frac{1}{2}$. If $h \geqq 2$ and if the formula has already been proved with $h - 1$ instead of h, then the two sides of (3.3) have the same derivative, so that their difference is a constant and this constant is equal to zero, since both $\varphi_h(1 - x)$ and $\varphi_h(x)$, integrated from $x = 0$ to $x = 1$, yield zero by the definition of $\varphi_h(x)$ given on P. (III.I.2.1).

For $h \geqq 2$ we have by (2.1) and (3.3)

$$\varphi_h(1) = \varphi_h(0) \quad \text{and} \quad \varphi_h(1) = (-)^h \varphi_h(0) \quad ,$$

so that $\varphi_h(0) = 0$ for each odd $h > 1$.

The polynomials $h!\varphi_h(x)(h \geqq 1)$ can be written as

$$(3.4) \qquad h!\varphi_h(x) = x^h - \tfrac{1}{2}\binom{h}{1}x^{h-1} + \binom{h}{2}B_1x^{h-2} - \binom{h}{4}B_2x^{h-4} +$$

$$+ \binom{h}{6}B_3x^{h-6} - \binom{h}{8}B_4x^{h-8} + \cdots ,$$

where the last term is a constant or linear in x and where the coefficients B_1, B_2, ··· denote suitably chosen numbers, called the numbers of Bernoulli. In fact, the special case $h = 1$ follows from $\varphi_1(x) = x - \frac{1}{2}$. Suppose that $h \geqq 2$ and that the formula has already been proved for $h - 1$ instead of h.

The derivative of the right hand side of (3.4), divided by h, is equal to

$$x^{h-1} - \tfrac{1}{2}\binom{h-1}{1}x^{h-2} + \binom{h-1}{2}B_1x^{h-3} \cdots = (h-1)!\varphi_{h-1}(x) = (h-1)!\varphi_h'(x) \quad ,$$

so that the two sides of (3.4) possess the same derivative. Their difference is therefore a constant. If h is an odd number > 1, both sides vanish for $x = 0$, so that they are equal for all x. If h is an even number, w choose the number of Bernoulli $B_{\frac{1}{2}h}$ such that the two sides of (3.4) assume the same value at $x = 0$, so that they also possess in this case the same value for all x.

From (3.4) it follows that

$$(3.5)\qquad \varphi_{2k}(0) = \varphi_{2k}(1) = (-)^{k-1}\frac{B_k}{(2k)!} \quad .$$

Taking in theorem 3 $\alpha = \beta = 0$ and choosing for A and B integers, we get the following result for the sum

$$S' = {\sum_{n=A}^{B}}' f(n) \quad ,$$

where the prime indicates that the terms with $n = A$ and with $n = B$ are counted only half.

THEOREM 4. *If* A *and* B *denote integers with* $A < B$ *and if* $f(x)$ *is* 2k *times* $(k \geqq 1)$ *continuously differentiable in the interval* $A \leqq x \leqq B$, *then*

$$S' = \int_A^B f(x)dx + \sum_{s=1}^{k} (-)^{s-1}\frac{B_s}{(2s)!}\left\{f^{(2s-1)}(B) - f^{(2s-1)}(A)\right\} - r_k \quad ,$$

where

$$r_k = \int_A^B \psi_{2k}(-x)f^{(2k)}(x)dx \quad .$$

If $f(x)$ is $2k + 1$ times continuously differentiable in the interval $A \leqq x \leqq B$, we can write r_k also in the form

$$r_k = \int_A^B \psi_{2k+1}(-x)f^{(2k+1)}(x)dx \quad .$$

These results follow immediately from (3.1), if we choose $\alpha = \beta = 0$, for then the contribution to the right hand side of (3.1) furnished by the terms with $s = 0$ is equal to

$$\psi_1(1)f(B) - \psi_1(1)f(A) = \tfrac{1}{2}f(B) - \tfrac{1}{2}f(A)$$

The first few Bernoulli numbers are*

$B_1 = \frac{1}{6}$; $B_2 = \frac{1}{30}$; $B_3 = \frac{1}{42}$;

$B_4 = \frac{1}{30}$; $B_5 = \frac{5}{66}$; $B_6 = \frac{691}{2730}$;

$B_7 = \frac{7}{6}$; $B_8 = \frac{3617}{510}$; $B_9 = \frac{43867}{798}$;

$B_{10} = \frac{174611}{330}$; $B_{11} = \frac{854513}{138}$; $B_{12} = \frac{236364091}{2730}$;

$B_{13} = \frac{8553103}{6}$; $B_{14} = \frac{23749461029}{870}$;

$B_{15} = \frac{8615841276005}{14322}$; $B_{16} = \frac{7709321041217}{510}$;

$B_{17} = \frac{2577687858367}{6}$; $B_{18} = \frac{26315271553053477373}{1919190}$;

$B_{19} = \frac{2929993913841559}{6}$; $B_{20} = \frac{261082718496449122051}{13530}$;

$B_{21} = \frac{1520097643918070802691}{1806}$; $B_{22} = \frac{27833269579301024235023}{690}$;

$B_{23} = \frac{596451111593912163277961}{282}$; $B_{24} = \frac{5609403368997817686249127547}{46410}$;

$B_{25} = \frac{495057205241079648212477525}{66}$;

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*See Tables of the Higher Mathematical Functions, computed and compiled under the direction of Harold T. Davis, II, p. 230, The Principia Press, Inc.; Bloomington, Indiana.

$$B_{26} = \frac{8011657181354899573479249\cdot1853}{1590} \; ;$$

$$B_{27} = \frac{2914996363488486242141812381269\text{1}}{798} \; ;$$

$$B_{28} = \frac{2479392929313226753685415739663229}{870} \; ;$$

$$B_{29} = \frac{84483613348880041862046775994036021}{354} \; ;$$

$$B_{30} = \frac{1215233140483755572040304994079820246041491}{56786730} \; .$$

Let us show now that the periodic functions $\Psi_h(x)$ $(h = 1,2,\cdots)$ possess the property

$$\Psi_h(x) + \Psi_h(x + \tfrac{1}{2}) = 2^{1-h}\Psi_h(2x) \quad , \tag{3.6}$$

in particular

$$\Psi_1(x) + \Psi_1(x + \tfrac{1}{2}) = \Psi_1(2x) \quad .$$

The last identity is obvious for $x = 0$ and for $x = \frac{1}{2}$, since $\Psi_1(0)$, $\Psi_1(\frac{1}{2})$ and $\Psi_1(1)$ vanish. In the interval $0 < x < \frac{1}{2}$ we have

$$\Psi_1(x) + \Psi_1(x + \tfrac{1}{2}) - \Psi_1(2x) = (x - \tfrac{1}{2}) + x - (2x - \tfrac{1}{2}) = 0$$

and in the interval $\frac{1}{2} < x < 1$ we obtain

$$\Psi_1(x) + \Psi_1(x + \tfrac{1}{2}) - \Psi_1(2x) = (x - \tfrac{1}{2}) + (x - 1) - (2x - \tfrac{3}{2}) = 0 \quad .$$

The periodic function $\Psi_1(x)$ therefore satisfies the required relation.

If $h \geqq 2$ and if (3.6) has been proved with $h - 1$ instead of h, then the two sides of (3.6) have the same derivatives, so that their difference is a constar and this constant is equal to zero, since each of the functions $\Psi_h(x)$, $\Psi_h(x + \frac{1}{2})$ and $\Psi_h(2x)$, integrated from $x = 0$ to $x = 1$, yield zero by the definition of $\Psi_h(x)$ and $\varphi_h(x)$. This establishes the proof of (3.6).

Letting $x = 0$ in that identity we obtain

$$\varphi_h(\tfrac{1}{2}) = \Psi_h(\tfrac{1}{2}) = (2^{1-h} - 1)\ \Psi_h(0) \quad .$$

Therefore, $\Psi_h(\frac{1}{2}) = 0$ for odd values of h and it follows from (3.5) that

$$\varphi_{2k}(\tfrac{1}{2}) = \Psi_{2k}(\tfrac{1}{2}) = (-)^k(1 - \frac{1}{2^{2k-1}})\ \frac{B_k}{(2k)!} \quad . \tag{3.7}$$

Taking $\alpha = \beta = \frac{1}{2}$ in (3.2) we obtain therefore,

THEOREM 5. <u>If</u> A <u>and</u> B <u>are integers with</u> $A < B$ <u>and if</u> $f(x)$ <u>is</u> $2k$ <u>times</u> $(k \geqq 1)$ <u>continuously differentiable in the interval</u> $A + \frac{1}{2} \leqq x \leqq B + \frac{1}{2}$, <u>then</u>

$$\sum_{n=A+1}^{B} f(n) = \int_{A+\frac{1}{2}}^{B+\frac{1}{2}} f(x)dx +$$

$$+ \sum_{s=1}^{k} (-)^s \left(1 - \frac{1}{2^{2s-1}}\right) \frac{B_s}{(2s)!} \left\{f^{(2s-1)}(B + \tfrac{1}{2}) - f^{(2s-1)}(A + \tfrac{1}{2})\right\} - \rho_k \ ,$$

<u>where</u>

$$\rho_k = \int_{A+\frac{1}{2}}^{B+\frac{1}{2}} \Psi_{2k}(-x)f^{(2k)}(x)dx \quad .$$

If $f(x)$ is $(2k + 1)$ times continuously differentiable, ρ_k may be written as

$$\rho_k = \int_{A+\frac{1}{2}}^{B+\frac{1}{2}} \Psi_{2k+1}(-x)f^{(2k+1)}(x)dx \quad .$$

Section 4. SOME PROPERTIES OF THE FUNCTIONS $\varphi_h(x)$ AND $\psi_h(x)$

In the interval $0 < x < \frac{1}{2}$ the polynomial $\varphi_{2k-1}(x)$ is negative for odd values of $k \geq 1$ and positive for even values of $k \geq 2$. To begin with, $\varphi_1(x) = x - \frac{1}{2}$ is negative in that interval. Suppose that $k \geq 2$ and that we have proved the property already for $k - 1$ instead of k. Let us consider first the case in which k is odd. Then the function $\varphi_{2k-1}(x)$ has a positive second derivative $\varphi_{2k-3}(x)$, so that its first derivative is monotonically increasing. Furthermore, since $\varphi_{2k-1}(x)$ takes the value zero at $x = 0$ and at $x = \frac{1}{2}$, the derivative $\varphi'_{2k-1}(x)$ is negative at $x = 0$ and positive at $x = \frac{1}{2}$, so that $\varphi_{2k-1}(x)$ itself is negative between 0 and $\frac{1}{2}$.

In the case that k is even, the function $\varphi_{2k-1}(x)$ has a negative second derivative and therefore a monotonically decreasing first derivative which is positive at $x = 0$ and negative at $x = \frac{1}{2}$, so that $\varphi_{2k-1}(x)$ is positive between 0 and $\frac{1}{2}$.

Since $\varphi_{2k-1}(x)$ is the derivative of $\varphi_{2k}(x)$, we have in the interval $0 \leq x \leq \frac{1}{2}$

$$\varphi_{2k}(\tfrac{1}{2}) \leq \varphi_{2k}(x) \leq \varphi_{2k}(0) \quad \text{if } k \text{ is odd}$$

and

$$\varphi_{2k}(0) \leq \varphi_{2k}(x) \leq \varphi_{2k}(\tfrac{1}{2}) \quad \text{if } k \text{ is even} \quad .$$

Consequently, it follows from (3.5) and (3.7) that we have in the interval $0 \leq x \leq \frac{1}{2}$

(4.1) $$-\left(1-\frac{1}{2^{2k-1}}\right)\frac{B_k}{(2k)!} \leqq (-)^{k-1}\varphi_{2k}(x) \leqq \frac{B_k}{(2k)!}$$

and

(4.2) $$0 \leqq (-)^k\left\{\varphi_{2k}(x) - \varphi_{2k}(0)\right\} \leqq \left(2-\frac{1}{2^{2k-1}}\right)\frac{B_k}{(2k)!} \quad .$$

These results imply that the numbers of Bernoulli are positive. From the formula

(4.3) $$\varphi_{2k}(x) = \varphi_{2k}(1-x) \quad ,$$

proved in (3.3) it follows that the formulas (4.1) and (4.2) hold in the whole interval $0 \leqq x \leqq 1$.

In this way we find an upper bound for the absolute value of the polynomial $\varphi_{2k}(x)$ in the interval $0 \leqq x \leqq 1$. To obtain an upper bound on the whole real axis, we prove for $h \geqq 1$

(4.4) $$\varphi_h(x+1) - \varphi_h(x) = \frac{x^{h-1}}{(h-1)!} \quad .$$

This identity is obvious for $\varphi_1(x) = x - \frac{1}{2}$. If $h \geqq 2$ and if the identity has already been proved for $h - 1$ instead of h, then the two sides of (4.4) have the same derivative, so that their difference is a constant. This difference is equal to zero at $x = 0$ and therefore identically equal to zero.

This result shows that for each $x \geqq 0$

(4.5) $$|\varphi_{2k}(x)| \leqq \frac{1}{(2k)!}(B_k + x^{2k}) \quad .$$

This formula follows in the interval $0 \leqq x \leqq 1$ from (4.1). If $x \geqq 1$ and if the inequality has already been proved for $x - 1$ instead of x, then

$$|\varphi_{2k}(x-1)| \leqq \frac{1}{(2k)!}(B_k + (x-1)^{2k})$$

and therefore by (4.3)

$$|\varphi_{2k}(x)| \leqq |\varphi_{2k}(x-1)| + \frac{(x-1)^{2k-1}}{(2k-1)!}$$

$$\leqq \frac{1}{(2k)!}(B_k + (x-1)^{2k}) + \frac{(x-1)^{2k-1}}{(2k-1)!}$$

$$\leqq \frac{1}{(2k)!}(B_k + x^{2k}) \quad ,$$

since

$$x^{2k} - (x-1)^{2k} = 2k \int_{x-1}^{x} u^{2k-1}\, du \geqq 2k(x-1)^{2k-1}$$

Combining (4.5) and (4.3) we find for $x \leqq 1$

$$|\varphi_{2k}(x)| \leqq \frac{1}{(2k)!}(B_k + (1-x)^{2k}) \quad . \tag{4.6}$$

It is easy to write the periodic functions $\psi_h(x)$ as sums of Fourier Series. We have namely for each real x and for $k \geqq 1$

$$\psi_{2k-1}(x) = (-)^k \sum_{n=1}^{\infty} \frac{2 \sin 2\pi nx}{(2\pi n)^{2k-1}} \tag{4.7}$$

and

$$\psi_{2k}(x) = (-)^{k-1} \sum_{n=1}^{\infty} \frac{2 \cos 2\pi nx}{(2\pi n)^{2k}} \quad . \tag{4.8}$$

For k = 1 the first formula takes the form

$$(4.9) \qquad \psi_1(x) = - \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{\pi n} \quad .$$

Since $\psi_1(x)$ has the period 1 and since $\psi_1(0) = 0$, it is sufficient to prove this identity in the open interval $0 < x < 1$. The right hand side of (4.9) is the imaginary part of

$$-\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{z^n}{n} \quad , \qquad \text{where } z = e^{2\pi ix} \quad .$$

This series converges at all points $z \neq 1$ lying on or inside the unit circle and represents there the branch of the function $\frac{1}{\pi} \log (1 - z)$ which assumes the value zero at $z = 0$. For $z = e^{2\pi ix}$ and $0 < x < 1$ the imaginary part of $\frac{1}{\pi} \log (1 - z)$ is equal to

$$\frac{1}{\pi} \arg (1 - z) = x - \frac{1}{2} \quad ,$$

which yields (4.9).

Integrating this identity repeatedly we obtain (4.8) and (4.7), since all occurring expressions, integrated from zero to 1, give zero.

Combining (4.8) for x = 0 and (3.5) we obtain

$$(4.10) \qquad B_k = \frac{2}{(2\pi)^{2k}} (2k)! \; \zeta(2k) \quad .$$

In this report $\zeta(s)$ denotes always the zeta function of Riemann, which is defined in the half plane Re $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad .$$

This result shows that B_k is large for large k, namely greater than $2\frac{(2k)!}{(2\pi)^{2k}}$, since $\zeta(2k) > 1$. On the other hand we have for real $s > 1$

$$\zeta(s) < 1 + \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{du}{u^s} = 1 + \int_{1}^{\infty} u^{-s}\, du = 1 + \frac{1}{s-1} = \frac{s}{s-1} ,$$

so that

$$B_k < 2\frac{(2k)!}{(2\pi)^{2k}} \cdot \frac{2k}{2k-1} \quad \text{for } k \geqq 1 . \tag{4.11}$$

From (4.7) and (4.8) it follows that for $h \geqq 2$ and for $0 \leqq x \leqq 1$

$$|\varphi_h(x)| \leqq \frac{2\zeta(h)}{(2\pi)^h} . \tag{4.12}$$

Precisely as (4.5) and (4.6) were proved, we find therefore for $h \geqq 1$

$$|\varphi_h(x)| \leqq \frac{2\zeta(h)}{(2\pi)^h} + \frac{x^h}{h!} \quad \text{for } x \geqq 0 \tag{4.13}$$

and

$$|\varphi_h(x)| \leqq \frac{2\zeta(h)}{(2\pi)^h} + \frac{(1-x)^h}{h!} \quad \text{for } x \leqq 1 ; \tag{4.14}$$

the case that h is even gives the formulas (4.5) and (4.6).

Section 5. ON THE REMAINDER TERM

In this section we deduce bounds for the remainder term occurring in the sum formula of Euler.

THEOREM 6. If α and β lie in the closed interval $(0,1)$ and if the $(2k)^{th}$ derivative $(k \geqq 1)$ of $f(x)$ is continuous and either always $\geqq 0$ or always $\leqq 0$ in the interval $A + \alpha \leqq x \leqq B + \beta$, then the sum

$$S = f(A + 1) + f(A + 2) + \cdots + f(B)$$

can be written as

$$S = \int_{A+\alpha}^{B+\beta} f(x)dx + \sum_{s=0}^{2k-1} \varphi_{s+1}(1-\beta)f^{(s)}(B + \beta) -$$

$$- \sum_{s=0}^{2k-1} \varphi_{s+1}(1 - \alpha)f^{(s)}(A + \alpha) + R_k \quad ,$$

where*)

$$R_k = (-)^k \, \theta_k \frac{B_k}{(2k)!} \left(f^{(2k-1)}(B + \beta) - f^{(2k-1)}(A + \alpha)\right)$$

and

(5.1) $$- \left(1 - \frac{1}{2^{2k-1}}\right) \leqq \theta_k \leqq 1 \quad .$$

Remark: The condition that $f^{(2k)}(x)$ is in the interval $A + \alpha \leqq x \leqq B + \beta$ either always $\geqq 0$ or always $\leqq 0$ may be replaced by the weaker condition that in the interval $0 \leqq u < 1$ the sum

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*) R_k in this formula corresponds to $-R_{2k}$ in (3.2).

$$\sum_n f^{(2k)}(n+u) \quad ,$$

extended over the integers n such that $A + \alpha \leqq n + u < B + \beta$, does not change its sign. Note that the range of n depends on u.

<u>Proof:</u> In (3.2) we found for S the required expression with the remainder

$$R_k = -\int_{A+\alpha}^{B+\beta} \psi_{2k}(B+1-x)f^{(2k)}(x)dx = -\int_0^1 \psi_{2k}(B-u) \sum_n f^{(2k)}(n+u)du$$

Since the sum $\sum_n$ is always $\geqq 0$ or always $\leqq 0$, there exists a number ξ such that

$$R_k = -\psi_{2k}(\xi) \int_0^1 \sum_n f^{(2k)}(n+u)du$$

$$= -\psi_{2k}(\xi) \int_{A+\alpha}^{B+\beta} f^{(2k)}(x)dx$$

$$= -\psi_{2k}(\xi)\left\{ f^{(2k-1)}(B+\beta) - f^{(2k-1)}(A+\alpha)\right\} \quad .$$

According to (4.1) we can write

$$\psi_{2k}(\xi) = (-)^{k-1}\theta_k \frac{B_k}{(2k)!} \quad ,$$

where θ_k satisfies the inequalities (5.1). This completes the proof.

If at least one of the numbers α and β lies outside the closed interval (0,1), the remainder R_{2k}, occurring in theorem 3 of section 3, contains at least one of the integrals

$$\int_{A+\alpha}^{A+\theta} \varphi_{2k}(A+1-x)f^{(2k)}(x)dx \text{ and } \int_{B+\theta}^{B+\beta} \varphi_{2k}(B+1-x)f^{(2k)}(x)dx \quad ,$$

which do not involve the periodic function Ψ_{2k}, but the polynomial φ_{2k}. To obtain an upper bound for the absolute values of these integrals, we may apply the inequalities

$$|\varphi_{2k}(x)| \leqq \frac{1}{(2k)!}(B_k + x^{2k}) \qquad \text{for } x \geqq 0$$

and

$$|\varphi_{2k}(x)| \leqq \frac{1}{(2k)!}(B_k + (1-x)^{2k}) \qquad \text{for } x \leqq 1 \quad ,$$

obtained in (4.5) and (4.6).

Let us now consider special cases in which $\alpha = \beta = 0$ or $\alpha = \beta = \frac{1}{2}$. Choosing $\alpha = \beta = 0$ we obtain

THEOREM 7. <u>Let</u> A <u>and</u> B <u>be integers with</u> $A < B$ <u>and let the</u> $(2k)^{th}$ <u>derivative of</u> $f(x)$ <u>be continuous and either always</u> $\geqq 0$ <u>or always</u> $\leqq 0$ <u>in the interval</u> $A \leqq x \leqq B$. <u>Let</u>

$$S' = \sum_{n=A}^{B}{}' f(n) \quad ,$$

<u>the prime indicating that the terms with</u> $n = A$ <u>and</u> $n = B$ <u>are counted only half</u>. <u>Then</u>

$$(5.2)\qquad S' = \int_A^B f(x)dx + \sum_{s=1}^{k-1} (-)^{s-1} \frac{B_s}{(2s)!}\left(f^{(2s-1)}(B) - f^{(2s-1)}(A)\right) + r_k$$

<u>and</u>

$$(5.3)\qquad r_k = (-)^{k-1}\Theta'_k \frac{B_k}{(2k)!}\left(f^{(2k-1)}(B) - f^{(2k-1)}(A)\right) \quad ,$$

where

$$0 \leqq \theta_k' \leqq 2 - \frac{1}{2^{2k-1}} \; . \tag{5.4}$$

Remark: For the proof it is sufficient to apply the preceding theorem with

$$\alpha = \beta = 0 \; ; \qquad r_k = \varphi_{2k}(0) \left(f^{(2k-1)}(B) - f^{(2k-1)}(A) \right) + R_k \; ,$$

so that we obtain (5.3) with $\theta_k' = 1 - \theta_k$.

The remainder has in theorem 7 the same sign as the first neglected term and is in absolute value at most equal to twice that term.

If we know moreover that $f^{(2k+2)}(x)$ is continuous and definite in the interval $A \leqq x \leqq B$ and that this derivative has the same sign as the $(2k)^{th}$ derivative, then

$$r_k = (-)^{k-1} \frac{B_k}{(2k)!} \left(f^{(2k-1)}(B) - f^{(2k-1)}(A) \right) + r_{k+1} \; , \tag{5.5}$$

where r_{k+1} has the same sign as

$$(-)^k \frac{B_{k+1}}{(2k+2)!} \left(f^{(2k+1)}(B) - f^{(2k+1)}(A) \right) ,$$

so that the two terms occurring on the right hand side of (5.5) possess opposite signs. From (5.3) it follows therefore that $\theta_k' = 1 - p$, where $p \geqq 0$, so that

$$0 \leqq \theta_k' \leqq 1 \; .$$

In this case the remainder term r_k has therefore the same sign as the first neglected term and is a fraction of that term.

The condition that $f^{(2k)}(x)$, respectively $f^{(2k+2)}(x)$ is either always $\geqq 0$ or always $\leqq 0$ in the interval $A \leqq x \leqq B$ may be replaced by the weaker condition that in the interval $0 \leqq u < 1$ the sum

$$\sum_{n=A}^{B-1} f^{(2k)}(n+u) \quad , \quad \text{respectively} \sum_{n=A}^{B-1} f^{(2k+2)}(n+u)$$

is either always $\geqq 0$ or always $\leqq 0$.

The choice $\alpha = \beta = \frac{1}{2}$ yields

THEOREM 8. If A and B are integers with $A < B$ and if the $(2k)^{th}$ derivative of $f(x)$ is continuous and either always $\geqq 0$ or always $\leqq 0$ in the interval $A + \frac{1}{2} \leqq x \leqq B + \frac{1}{2}$, then

$$\sum_{n=A+1}^{B} f(n) = \int_{A+\frac{1}{2}}^{B+\frac{1}{2}} f(x)dx + \sum_{s=1}^{k-1} (-)^s \left(1 - \frac{1}{2^{2s-1}}\right) \frac{B_s}{(2s)!} \left(f^{(2s-1)}(B + \frac{1}{2}) - f^{(2s-1)}(A + \frac{1}{2})\right) + \rho_k$$

and

$$(5.6) \qquad \rho_k = (-)^k \theta_k^* \frac{B_k}{(2k)!} \left(f^{(2k-1)}(B + \frac{1}{2}) - f^{(2k-1)}(A + \frac{1}{2})\right) \quad ,$$

where

$$0 \leqq \theta_k^* \leqq 2 - \frac{1}{2^{2k-1}} \quad .$$

Remark: For the proof it is sufficient to apply theorem 6 with

$$\alpha = \beta = \frac{1}{2} \;\; ; \;\; \rho_k = \varphi_{2k}(\tfrac{1}{2}) . \left(f^{(2k-1)}(B + \frac{1}{2}) - f^{(2k-1)}(A + \frac{1}{2})\right) + R_k \quad ,$$

so that we obtain (5.6) with $\theta_k^* = \theta_k + 1 - \frac{1}{2^{2k-1}}$.

The remainder has also in theorem 8 the same sign as the first neglected term and is in absolute value at most equal to that term multiplied by $2 - \frac{1}{2^{2k-1}}$.

If we know moreover that $f^{(2k+2)}(x)$ is continuous and has definite sign in the interval $A \leqq x \leqq B$ and that this derivative has the same sign as the $(2k)^{th}$ derivative, then

$$(5.7) \quad \rho_k = (-)^k(1 - \frac{1}{2^{2k-1}}) \frac{B_k}{(2k)!} \left(f^{(2k-1)}(B + \frac{1}{2}) - f^{(2k-1)}(A + \frac{1}{2})\right) + \rho_{k+1} \;;$$

where ρ_{k+1} has the same sign as

$$(-)^{k+1}(1 - \frac{1}{2^{2k+1}}) \frac{B_{k+1}}{(2k+2)!} \left(f^{(2k+1)}(B + \frac{1}{2}) - f^{(2k+1)}(A + \frac{1}{2})\right) \quad ,$$

so that the two terms occurring on the right hand side of (5.7) possess opposite signs. From (5.6) it follows therefore that

$$\theta_k^* = 1 - \frac{1}{2^{2k-1}} - p \quad ,$$

where $p \geqq 0$, so that

$$0 \leqq \theta_k^* \leqq 1 - \frac{1}{2^{2k-1}} \quad .$$

In this case the remainder term ρ_k has therefore the same sign as the first neglected term and is a fraction of that term.

Of course also in this theorem the condition that $f^{(2k)}(x)$, and $f^{(2k+2)}(x)$ have definite and identical signs, can be replaced by a weaker condition.

The preceding theorems in this section contain the condition that a certain sum is either always $\geqq 0$ or always $\leqq 0$. If we do not know whether this condition is satisfied or not, we can often apply the following theorem, which gives however in general weaker results.

THEOREM 9. Suppose that A and B are real, that B - A is a positive integer, that $0 \leqq \alpha \leqq 1$ and that $f(x)$ is $2k$ times $(k \geqq 1)$ continuously differentiable in the interval $A + \alpha \leqq x \leqq B + \alpha$. Let

$$S = f(A + 1) + f(A + 2) + \dots + f(B) \quad .$$

Then

$$S = \int_{A+\alpha}^{B+\alpha} f(x)dx + \sum_{s=0}^{2k-1} \varphi_{s+1}(1 - \alpha) \left(f^{(s)}(B + \alpha) - f^{(s)}(A + \alpha)\right)$$

$$+ (-)^{k-1} \frac{B_k}{(2k)!} \left(f^{(2k-1)}(B + \alpha) - f^{(2k-1)}(A + \alpha)\right) \tag{5.8}$$

$$+ (-)^k (B - A) \frac{B_k}{(2k)!} f^{(2k)}(\xi_1) \quad ,$$

where ξ_1 denotes a suitably chosen number lying between $A + \alpha$ and $B + \alpha$.

Moreover we have

$$S = \int_{A+\alpha}^{B+\alpha} f(x)dx + \sum_{s=0}^{2k-1} \varphi_{s+1}(1 - \alpha)\left(f^{(s)}(B + \alpha) - f^{(s)}(A + \alpha)\right)$$

$$+ (-)^k\left(1 - \frac{1}{2^{2k-1}}\right) \frac{B_k}{(2k)!} \left(f^{(2k)}(B + \alpha) - f^{(2k)}(A + \alpha)\right) \tag{5.9}$$

$$+ (-)^{k-1}\left(1 - \frac{1}{2^{2k-1}}\right)(B - A) \frac{B_k}{(2k)!} f^{(2k)}(\xi_2) \quad ,$$

where ξ_2 denotes also a suitably chosen number lying between $A+\alpha$ and $B+\alpha$.

Proof. According to (3.2) we have

$$S = \int_{A+\alpha}^{B+\beta} f(x)dx + \sum_{s=0}^{2k-1} \varphi_{s+1}(1-\alpha) \left\{ f^{(s)}(B+\alpha) - f^{(s)}(A+\alpha) \right\} - R_{2k} ,$$

where

$$R_{2k} = \int_{A+\alpha}^{B+\alpha} \psi_{2k}(B+1-x) f^{(2k)}(x) .$$

Letting $\lambda = 0$ or $\frac{1}{2}$ we find therefore

$$R_{2k} = \psi_{2k}(\lambda) \left\{ f^{(2k-1)}(B+\alpha) - f^{(2k-1)}(A+\alpha) \right\}$$
$$+ \int_{A+\alpha}^{B+\alpha} \left\{ \psi_{2k}(B+1-x) - \psi_{2k}(\lambda) \right\} f^{(2k)}(x) .$$

The factor $\psi_{2k}(B+1-x) - \psi_{2k}(\lambda)$ is always $\geqq 0$ or always $\leqq 0$ by (4.2), so that

$$\int_{A+\alpha}^{B+\alpha} \left\{ \psi_{2k}(B+1-x) - \psi_{2k}(\lambda) \right\} f^{(2k)}(x)dx$$

$$= f^{(2k)}(\xi) \int_{A+\alpha}^{B+\alpha} \left\{ \psi_{2k}(B+1-x) - \psi_{2k}(\lambda) \right\} dx$$

$$= - f^{(2k)}(\xi)(B-A) \psi_{2k}(\lambda) ,$$

where ξ denotes a suitably chosen number lying between $A + \alpha$ and $B + \alpha$. Consequently the required identities follow from

$$\Psi_{2k}(0) = (-)^{k-1} \frac{B_k}{(2k)!} \quad \text{and} \quad \Psi_{2k}(\tfrac{1}{2}) = (-)^k (1 - \frac{1}{2^{2k-1}}) \frac{B_k}{(2k)!} .$$

Section 6. ON THE ASYMPTOTIC BEHAVIOR OF A SUM; CONSIDERED AS A FUNCTION OF THE NUMBER OF TERMS

Consider a sum

$$S = f(A + 1) + f(A + 2) + \cdots + f(B) \quad ,$$

where A is a fixed real number and where B is a variable such that B - A is a large positive integer. We want to know the behavior of S for very large values of B - A.

THEOREM 10. Let A and B be real such that B - A is a positive integer. Let $0 \leqq \beta \leqq 1$. Suppose that f(x) is 2k times ($k \geqq 1$) continuously differentiable for $x \geqq A$ such that $f^{(2k)}(x)$ is for $x \geqq B + \beta$ either always $\geqq 0$ or always $\leqq 0$ and that $f^{(2k-1)}(x)$ tends to zero as x approaches infinity. Then

$$S = \int_A^{B+\beta} f(x)dx + c + \sum_{s=0}^{2k-1} \varphi_{s+1}(1 - \beta) f^{(s)}(B + \beta) \tag{6.1}$$
$$+ (-)^{k-1}\theta_k \frac{B_k}{(2k)!} f^{(2k-1)}(B + \beta) \quad ,$$

where

$$-\left(1 - \frac{1}{2^{2k-1}}\right) \leqq \theta_k \leqq 1 \tag{6.2}$$

and where c is a suitably chosen number which is independent of B and β (it may depend on A and on the choice of the function f). If there exists a positive integer k_o such that the conditions hold for each integer $k \geqq k_o$ then the number c is also independent of k for $k \geqq k_o$.

Remark: The condition that $f^{(2k)}(x)$ is for $x \geqq B + \beta$ either always $\geqq 0$ or always $\leqq 0$ may be replaced by the weaker condition that in the interval $0 \leqq u < 1$ the sum

$$\sum_n f^{(2k)}(n + u)$$

extended over the integers $n \geqq B + \beta$ does not change its sign.

Proof. Precisely as in the proof of theorem 6 we find for $Q > P \geqq B + \beta$

$$\int_P^Q \psi_{2k}(B + 1 - x) f^{(2k)}(x)\,dx = (-)^{k-1} \theta'_k \frac{B_k}{(2k)!} \left(f^{(2k-1)}(Q) - f^{(2k-1)}(P) \right)$$

where θ'_k satisfies the inequalities (6.2). The right hand side tends to zero, as P and Q approach infinity, so that the integral

$$\int_{B+\beta}^{\infty} \psi_{2k}(B + 1 - x) f^{(2k)}(x)\,dx$$

exists by the Cauchy criterion and can be written in the form

$$(-)^k \theta_k \frac{B_k}{(2k)!} f^{(2k-1)}(B + \beta) \quad ,$$

where θ_k satisfies the inequalities (6.2).

Formula (3.2), occurring in theorem 3, applied with $\alpha = 0$, yields therefore the required result, where

$$(6.3) \quad c = - \sum_{s=0}^{2k-1} \varphi_{s+1}(1 - \alpha) f^{(s)}(A) - \int_A^{\infty} \psi_{2k}(A + 1 - x) f^{(2k)}(x)\,dx \quad .$$

Finally we must prove that this number is independent of k for $k \geqq k_0$, if the conditions hold for each fixed integer $k \geqq k_0$. It follows from the second fundamental identity, proved in theorem 2 of section 2, that

$$\int_A^\infty \psi_{2k}(A+1-x)f^{(2k)}(x)dx$$

$$= \varphi_{2k+1}(1-\alpha)f^{(2k)}(A) + \int_A^\infty \psi_{2k+1}(A+1-x)f^{(2k+1)}(x)dx$$

$$= \varphi_{2k+1}(1)f^{(2k)}(A) + \varphi_{2k+2}(1)f^{(2k+1)}(A)$$

$$+ \int_A^\infty \psi_{2k+2}(A+1-x)f^{(2k+2)}(x)dx \quad .$$

Substituting this result into (6.3) we find that c does not change its value, if k is replaced by k + 1.

<u>Remark</u>: The constant c can be calculated by means of (6.3). It can also be calculated by means of (6.1), for if B is sufficiently large, the remainder term in that formula is very small.

Taking $\beta = 0$ we obtain

THEOREM 11. <u>Let</u> A <u>and</u> $B > A$ <u>be integers and suppose that</u> f(x) <u>is</u> 2k <u>times</u> ($k \geqq 1$) <u>continuously differentiable for</u> $x \geqq A$ <u>such that</u> $f^{(2k)}(x)$ <u>is for</u> $x \geqq B$ <u>either always</u> $\geqq 0$ <u>or always</u> $\leqq 0$ <u>and that</u> $f^{(2k-1)}(x)$ <u>tends to zero as</u> x <u>approaches infinity. Then</u>

$$\sum_{n=A}^{B} f(n) = \int_A^B f(x)dx + c + \tfrac{1}{2}f(B) + \sum_{s=1}^{k-1} (-)^{s-1} \frac{B_s}{(2s)!} f^{(2s-1)}(B)$$

$$+ (-)^{k-1} \theta'_k \frac{B_k}{(2k)!} f^{(2k-1)}(B) \quad ,$$

where $\theta_k' = 1 + \theta_k$ and therefore

$$\frac{1}{2^{2k-1}} \leq \theta_k' \leq 2 \quad .$$

Remark: The remainder has the same sign as the first neglected term and is in absolute value at most equal to twice that term. If we know moreover that $f^{(2k+2)}(x)$ is for $x \geq B$ continuous and definite with the same sign as the $(2k)^{th}$ derivative, then

$$\frac{1}{2^{2k-1}} \leq \theta_k' \leq 1 \quad ,$$

so that in that case the remainder is a fraction of the first neglected term.

The condition that $f^{(2k)}(x)$, and $f^{(2k+2)}(x)$ are of definite and the same sign in the interval $A \leq x \leq B$ may be replaced by the weaker condition that in the interval $0 \leq u < 1$ the function

$$\sum_{n=B}^{\infty} f^{(2k)}(n + u) \quad , \qquad \text{and} \qquad \sum_{n=B}^{\infty} f^{(2k+2)}(n + u)$$

are definite and of the same sign.

Example: For large positive integers B the sum

$$S = \sum_{n=1}^{B} \sqrt{n} \log^2 n$$

possesses the asymptotic expansion

$$c + \frac{1}{2} \sqrt{B} \log^2 B + \sum_{s=0}^{\infty} B^{3/2-2s}(\alpha_s \log^2 B + \beta_s \log B + \gamma_s) \quad ,$$

where c, α_s, β_s, γ_s denote suitably chosen constants. The assertion means that for each positive integer k

$$S = c + \frac{1}{2}\sqrt{B}\log^2 B + \sum_{s=0}^{k-1} B^{3/2-2s}(\alpha_s \log^2 B + \beta_s \log B + \gamma_s) + R_k \quad ,$$

where R_k is for large integers B at most of the same order of magnitude as $B^{3/2-2k}\log^2 B$.

This result follows immediately from the preceding theorem, since

$$f(x) = x^{\frac{1}{2}} \log^2 x$$

has the property that

$$\int_1^B f(x)dx = \frac{2}{9} B^{3/2}(3 \log^2 B - 4 \log B + \frac{8}{3}) - \frac{16}{27} \tag{6.4}$$

and for $h \geq 1$

$$f^{(h)}(x) = x^{\frac{1}{2}-h}(\rho_h \log^2 x + \sigma_h \log x + \tau_h) \quad ,$$

which satisfies the assumptions of theorem 11 for sufficiently large x.

This last formula holds for h = 1, that is

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} (\log^2 x + 4 \log x) \tag{6.5}$$

and can be proved by mathematical induction.

Formula (6.4) shows that

$$\alpha_o = \frac{2}{3} \quad ; \qquad B_o = -\frac{8}{9} \quad ; \qquad \gamma_o = \frac{16}{27} \quad .$$

From (6.5) and $\frac{B_1}{2} = \frac{1}{12}$ it follows that

$$\alpha_1 = \frac{1}{24} \quad ; \qquad \beta_1 = \frac{1}{6} \quad ; \qquad \gamma_1 = 0 \quad .$$

Choosing $\beta = \frac{1}{2}$ we find

THEOREM 12. Let A and B > A be integers and suppose that f(x) is 2k times ($k \geqq 1$) continuously differentiable for $x \geqq A$ such that $f^{(2k)}(x)$ is for $x \geqq B + \frac{1}{2}$ either always $\geqq 0$ or always $\leqq 0$ and that $f^{(2k-1)}(x)$ tends to zero as x approaches infinity. Then

$$\sum_{n=A}^{B} f(n) = \int_A^{B+\frac{1}{2}} f(x)dx + c + \sum_{s=1}^{k-1} (-)^s \left(1 - \frac{1}{2^{2s-1}}\right) f^{(2s-1)}(B + \tfrac{1}{2})$$

$$+ (-)^k \, \theta_k^* \, \frac{B_k}{(2k)!} f^{(2k-1)}(B + \tfrac{1}{2}) \quad ,$$

where

$$0 \leqq \theta_k^* \leqq 2 - \frac{1}{2^{2k-1}} \quad .$$

Remark: The remainder has the same sign as the first neglected term and is in absolute value at most equal to twice that term. If we know moreover that $f^{(2k+2)}(x)$ is for $x \geqq B + \frac{1}{2}$ continuous and definite with the same sign as the $(2k)^{th}$ derivative, then

$$0 \leqq \theta_k^* \leqq 1 - \frac{1}{2^{2k-1}} \quad ,$$

so that in that case the remainder term is a fraction of the first neg-

Section 7. ON THE SUM OF CONSECUTIVE INTEGERS RAISED TO THE SAME POWER

THEOREM 13. *For each positive integer* n *and each positive integer* h *we have*

$$\frac{1}{(h-1)!}\sum_{m=0}^{n-1} m^{h-1} = \varphi_h(n) - \varphi_h(0)$$

and more generally

$$\frac{1}{(h-1)!}\sum_{m=0}^{n-1} (m+w)^{h-1} = \varphi_h(n+w) - \varphi_h(w) \quad .$$

Proof. We have proved in (4.4)

$$\frac{x^{h-1}}{(h-1)!} = \varphi_h(x+1) - \varphi_h(x) \quad .$$

Applying this formula with $x = 0, 1, \cdots, n-1$ and adding we obtain the first required result. Using the formula with $x = w, w+1, \cdots, w+n-1$ and adding, we find the second required result.

Section 8. ON THE INITIAL SEGMENT OF THE HARMONIC SERIES

THEOREM 14. For each positive integer n

$$\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}=\log n+\gamma+\frac{1}{2n}-\frac{B_1}{2}\frac{1}{n^2}+\frac{B_2}{4}\frac{1}{n^4}-\dots$$

$$+(-)^{k-1}\frac{B_{k-1}}{2k-2}\frac{1}{n^{2k-2}}+R_k \quad ,$$

where the remainder R_k has the same sign as and is a fraction of the first neglected term. Here γ denotes a suitably chosen constant, called the constant of Euler.

This result follows immediately from the remark added to theorem 11, applied with

$$A=1 \quad , \qquad B=n \quad , \qquad f(x)=\frac{1}{x} \quad ,$$

since

$$\frac{1}{(2s)!}f^{(2s-1)}(x)=-\frac{(2s-1)!}{(2s)!\,x^{2s}}=-\frac{1}{2sx^{2s}}$$

satisfies the requirements in that remark.

The remainder is in absolute value at most equal to

$$\frac{B_k}{2k}\frac{1}{n^{2k}}<\frac{2}{2k-1}\frac{(2k)!}{(2\pi)^{2k}}\frac{1}{n^{2k}}$$

according to (4.11). For instance, taking n = 4 and k = 4 the remainder is in absolute value less than

$$\frac{2}{7}\frac{8!}{(2\pi)^8}\frac{1}{4^8}<10^{-6} \quad ,$$

so that this choice of n and k gives the value of the constant of Euler with an error $< 10^{-6}$.

In the same way we obtain

THEOREM 15. *If* $0 < w \leq 1$ *we have for each integer* n

$$\frac{1}{w} + \frac{1}{w+1} + \cdots + \frac{1}{w+n} = \log (w + n) + C + \frac{1}{2(w+n)}$$

$$- \frac{B_1}{2} \frac{1}{(w+n)^2} + \frac{B_2}{4} \cdot \frac{1}{(w+n)^4} - \cdots + (-)^{(k-1)} \frac{B_{k-1}}{2k-2} \frac{1}{(w+n)^{2k-2}} + R_k ,$$

where the remainder R_k *has the same sign as and is a fraction of the first neglected term. Here* C *denotes a suitably chosen number which depends on* w *but is independent of* n. *This constant can be calculated as follows.*

From the Weierstrass' canonical form of the gamma function

$$w\Gamma(w) = \Gamma(w + 1) = e^{-\gamma w} \prod_{h=1}^{\infty} (1 + \frac{w}{h})^{-1} e^{w/h} ,$$

where γ denotes the constant of Euler, it follows, taking the logarithmic derivative, that

$$\frac{1}{w} + \frac{\Gamma'(w)}{\Gamma(w)} = - \gamma + \sum_{h=1}^{\infty} \left(\frac{1}{h} - \frac{1}{h+w}\right) ,$$

so that

$$\sum_{h=0}^{n-1} \frac{1}{h+w} - \log n = \left(\sum_{h=1}^{n-1} \frac{1}{h} - \log n\right) + \left\{ \frac{1}{w} - \sum_{h=1}^{n-1} \left(\frac{1}{h} - \frac{1}{h+w}\right)\right\}$$

tends as $n \to \infty$ to

$$\gamma + \left(- \gamma - \frac{\Gamma'(w)}{\Gamma(w)}\right) = - \frac{\Gamma'(w)}{\Gamma(w)}$$

since it follows from theorem 14 that

$$\lim_{n \to \infty} \left(\sum_{h=1}^{n-1} \frac{1}{h} - \log n\right) = \gamma .$$

Consequently the number C, occurring in theorem 15, has the value $\frac{-\Gamma'(w)}{\Gamma(w)}$.
Of course, in the special case w = 1 this constant is equal to $-\Gamma'(1)$ and therefore equal to the constant of Euler.

Section 9. ON THE FORMULA OF STIRLING

THEOREM 16. For each positive integer n

$$\log n! = (n + \tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log 2\pi + \frac{B_1}{1\cdot 2} \frac{1}{n} - \frac{B_2}{3\cdot 4} \frac{1}{n^3}$$

$$+ \cdots + (-1)^k \frac{B_{k-1}}{(2k-3)(2k-2)} \frac{1}{n^{2k-3}} + R_k \quad ,$$

where the remainder R_k has the same sign as and is a fraction of the first neglected term.

<u>Proof</u>. Applying the remark added to theorem 11 with

$$A = 1 \quad , \qquad B = n \qquad \text{and } f(x) = \log x \quad ,$$

we obtain

$$\int_1^n f(x)dx = n \log n - n + 1$$

and

$$\frac{1}{(2s)!} f^{(2s-1)}(x) = \frac{(2s-2)!}{(2s)!\, x^{2s-1}} = \frac{1}{(2s-1)2s\, x^{2s-1}} \quad ,$$

so that

(9.1)
$$\log n! = (n + \tfrac{1}{2}) \log n - n + c + \frac{B_1}{1\cdot 2} \frac{1}{n} - \frac{B_2}{3\cdot 4} \frac{1}{n^3}$$

$$+ \cdots + (-1)^k \frac{B_{k-1}}{(2k-3)(2k-2)} \frac{1}{n^{2k-3}} + R_k \quad ,$$

possesses the property that its derivatives of the order 2k and 2k + 2 have the same sign for $x \geqq 0$, if

$$(s + 2k)(s + 2k + 1) \geqq 0 \quad .$$

Consequently, if we choose the positive integer k either $\geqq -\frac{1}{2}s$ or $\leqq -\frac{1}{2}s - \frac{1}{2}$, these two derivatives have the same sign, so that, according to the remark added to theorem 7, the remainder r_k in (11.2) has the same sign as the first neglected term and is a fraction of that term.

If $s = -p$, where p denotes an integer $\geqq 0$, then the remainder r_k in (11.2) vanishes for $k \geqq \frac{1}{2}(p + 1)$. In that case we find for $\zeta(-p,w)$ a polynomial in n + w. Since $\zeta(-p,w)$ is independent of n, we can choose n = 0, so that

$$-(p + 1)\zeta(-p,w) = w^{p+1} - \tfrac{1}{2}(p + 1)w^p$$

$$+ (p + 1) \sum_{1 \leqq q \leqq \frac{1}{2}(p-1)} (-)^{q-1} \frac{B_q}{2q} \frac{p(p-1)\cdots(p-2q+2)}{(2q-1)!} w^{p-2q-1}$$

$$= w^{p+1} - \tfrac{1}{2}(p + 1)w^p + \sum_{1 \leqq q \leqq \frac{1}{2}(p-1)} (-)^{q-1} B_q \binom{p+1}{2q} w^{p-2q-1}$$

$$= (p + 1)!\, \varphi_{p+1}(w)$$

according to (3.4). In this way we find for each positive $w \leqq 1$ and for each integer $p \geqq 0$

$$\zeta(-p,w) = -p!\, \varphi_{p+1}(w) \quad . \tag{11.3}$$

Choosing $w = 1$ we obtain in particular that $\zeta(s)$ vanishes for $s = -2, -4, \cdots$ by the first formula on (III,I,3,4), that $\zeta(0) = -\frac{1}{2}$ and that for each positive integer k

$$\zeta(1-2k) = -(2k-1)!\ \varphi_{2k}(1) = (-)^k \frac{B_k}{2k}$$

according to (3.5).

To prove that (11.3) holds not only for $0 < w \leqq 1$, but for all positive w, we note that according to (4.4)

$$p!\,\varphi_{p+1}(w+1) - p!\,\varphi_{p+1}(w) = w^p$$

and that for Re $s > 1$

$$\zeta(s,w) - \zeta(s,w+1) = \sum_{n=o}^{\infty} \frac{1}{(n+w)^s} - \sum_{n=o}^{\infty} \frac{1}{(n+w+1)^s} = \frac{1}{w^s} ,$$

so that the formula

$$\zeta(s,w) - \zeta(s,w+1) = w^{-s}$$

holds in the whole complex s-plane, the point $s = 1$ excepted, in particular

$$\zeta(-p,w) - \zeta(-p,w+1) = w^p .$$

Therefore, if (11.3) holds for a given value of w, it also holds for $w+1$, $w+2$, etc.

Chapter II

SOME GENERAL REMARKS ON ASYMPTOTIC EXPANSIONS

Section 1. DEFINITION OF AN ASYMPTOTIC SERIES

Let ω be a real or complex number belonging to an unbounded set Ω such that $|\omega| \geqq 1$ for each element ω of Ω. Each number and each function, occurring in this section, may depend on ω. If they do not depend on ω, they are said to be fixed. Each number and each function is supposed to be defined for all elements ω of Ω. Two numbers are called asymptotically equal, in symbols $a \sim b$, if for each fixed real number q the product $|\omega|^q|a - b|$ is bounded for all elements ω of Ω. For instance

$$e^{-|\omega|} + e^{-\sqrt{|\omega|}} \sin \omega \sim 0 \quad .$$

If a and b depend not only on ω but also on other parameters and the product $|\omega|^q|a - b|$ is bounded uniformly in these parameters, then a and b are said to be asymptotically equal uniformly in these parameters. For instance

$$e^{-\log^2|\omega|} \sin x \sim 0$$

uniformly in the real variable x. We have also

$$e^{-\omega^2} x \sim 0$$

uniformly in x in the interval $-10^{10} \leqq x \leqq 10^{10}$, but the relation does not hold uniformly in x, if x runs through the whole real axis.

The notion of asymptotical equality is reflexive (each number is asymptotically equal to itself), commutative ($a \sim b$ implies $b \sim a$) and transitive (if $a \sim b$ and $b \sim c$, then $a \sim c$).

An asymptotic series is a series $a_o + a_1 + \cdots$, for which it is possible to find a sequence of fixed numbers $q_o, q_1, \cdots$, such that $q_h \to \infty$ as $h \to \infty$ and that for each fixed integer $h \geqq 0$ the product $|\omega|^{q_h}|a_h|$ is bounded for all elements ω of Ω. Examples:

$$\sum_{h=o}^{\infty} \frac{h!}{\omega^h} \ ; \qquad \sum_{h=o}^{\infty} \frac{(h \log |\omega|)^h}{\omega\sqrt{h}}$$

I call $q_o, q_1, \cdots$ the exponents belonging to the asymptotic series. It is always possible to choose these exponents in such a way that $q_o \leqq q_1 \leqq \cdots$. In fact, the number m_h, defined as the smallest of the numbers $q_h, q_{h+1}, \cdots$ is a fixed number which tends to infinity as $h \to \infty$, whereas $|\omega|^{m_h}|a_h|$ is bounded, so that we can choose as exponents the numbers m_h which possess the property $m_o \leqq m_1 \leqq \cdots$.

Let $a_o + a_1 + \cdots$ be an asymptotic series with monotonic non-decreasing exponents $q_o, q_1, \cdots$. A number s is called the asymptotic sum of that series if for each fixed integer $h \geqq 0$ the product

$$|\omega|^{q_h} \,|\, s - (a_o + a_1 + \cdots + a_{h-1})|$$

is bounded for all elements ω of Ω.

This sum is not uniquely defined, for if s is the asymptotic sum of an asymptotic series, then each number which is asymptotically equal to s, is also an asymptotic sum of that series. Conversely, two numbers, which are the asymptotic sums of the same asymptotic series are asymptotically equal. Therefore an asymptotic series does not define a single number, but only a certain class of numbers which are all asymptotically equal.

Section 2. SOME PROPERTIES OF ASYMPTOTIC SERIES

THEOREM 1. Each asymptotic series possesses an asymptotic sum.

Proof: Consider an asymptotic series with monotonic non-decreasing exponents $q_o, q_1, \cdots,$ so that

$$|a_h| \leqq c_h\,|\omega|^{-q_h} \qquad (h = 0, 1, \cdots)$$

for conveniently chosen fixed number c_h.

I choose a positive integer $H \geqq 0$, depending on ω, such that H tends to infinity as $|\omega| \to \infty$, but so slowly that

$$\sum_{h=0}^{H-1} |c_h| \leqq |\omega| .$$

To show that

$$s = a_o + a_1 + \cdots + a_{H-1}$$

is the asymptotic sum of the series, I must show for each fixed integer $h \geqq 0$ that

$$|s - (a_o + \cdots + a_{h-1})| \leqq C_h\,|\omega|^{-q_h}$$

where C_h denotes a suitably chosen fixed number.

There exists a fixed number $k > h$ such that $q_k \geqq q_h + 1$ and there exists a fixed number γ such that $H > k$ for each element ω of Ω with $|\omega| \geqq \gamma$. For the elements ω of Ω with $1 \leqq |\omega| < \gamma$ I have

$$|s| \leqq c_o\,|\omega|^{-q_o} + c_1|\omega|^{-q_1} + \cdots + c_{H-1}\,|\omega|^{-q_{H-1}}$$

$$\leqq (c_o + c_1 + \cdots + c_{H-1})\,|\omega|^{-q_o} \leqq |\omega|^{1-q_o} \leqq \tfrac{1}{2}\,C_h\,|\omega|^{-q_h}$$

and

$$|a_o + a_1 + \cdots + a_{h-1}| \leqq (c_o + c_1 + \cdots + c_{h-1})|\omega|^{-q_o} \leqq \tfrac{1}{2}\,C_h\,|\omega|^{-q_h}$$

for suitably chosen fixed number C_h. For the elements ω of Ω with $|\omega| \geqq \gamma$ I have $H > k > h$, so that

$$|s - (a_o + a_1 + \cdots + a_{h-1})| = |a_h + a_{h+1} + \cdots + a_{H-1}|$$

$$\leqq |a_h + \cdots + a_{k-1}| + |a_k + \cdots + a_{H-1}|$$

$$\leqq (c_h + \cdots + c_{k-1})\,|\omega|^{-q_h} + (c_k + \cdots + c_{H-1})\,|\omega|^{-q_h-1}$$

$$\leqq (c_h + \cdots + c_{k-1} + 1)\,|\omega|^{-q_h}\,..$$

This completes the proof.

Conversely, a series with an asymptotic sum is an asymptotic series.

More precisely:

THEOREM 2. Suppose that a series $a_o + a_1 \cdots$ has the property that it is possible to find a number s and moreover fixed exponents $q_o, q_1, \cdots$ such that q_h tends to infinity as $h \longrightarrow \infty$ and that for each fixed integer $h \geqq 0$* the product

$$|\omega|^{q_h}\,|s - (a_o + a_1 + \cdots + a_{h-1})|$$

is bounded for all elements ω of Ω. Then the series $a_o + a_1 + \cdots$ is an asymptotic series.

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* [illegible]

Proof: If m_h denotes the smallest of the two numbers q_h and q_{h+1}, the products

$$|\omega|^{m_h} |s - (a_o + \cdots + a_{h-1})| \quad \text{and} \quad |\omega|^{m_h} |s - (a_o + \cdots + a_h)|$$

are bounded, so that, subtracting, also $|\omega|^{m_h}|a_h|$ is bounded. Here m_h is fixed and tends to infinity as $h \to \infty$, so that the series $a_o + a_1 + \cdots$ is an asymptotic series.

THEOREM 3. (Sum theorem). If

$$s \sim a_o + a_1 + \cdots$$

and

$$t \sim b_o + b_1 + \cdots \quad ,$$

then

$$s + t \sim (a_o + b_o) + (a_1 + b_1) + \cdots \quad .$$

Proof: Let $q_o, q_1, \ldots$ be the exponents belonging to the first asymptotic series and let $p_o, p_1, \ldots$ be the exponents belonging to the second series. Let m_h be the smallest of the two numbers q_h and p_h. Then

$$s - (a_o + a_1 + \cdots + a_{h-1}) \quad \text{and} \quad t - (b_o + b_1 + \cdots + b_{h-1}) \quad ,$$

and therefore also

$$(s + t) - (a_o + b_o) - (a_1 + b_1) - \cdots - (a_{h-1} + b_{h-1})$$

are at most of the same order of magnitude as $|\omega|^{-m_h}$. This completes the proof.

THEOREM 4. The asymptotic sum of an asymptotic series is independent of the order of the terms of the series *).

Proof: Let s be the asymptotic sum of an asymptotic series $a_o + a_1 + \cdots$. Without loss of generality we may suppose that the exponents $q_o, q_1, \cdots$ belonging to that series are monotonic non-decreasing. Suppose that $b_o + b_1 + \cdots$ contains the same terms as $a_o + a_1 + \cdots$, but in another order. We must show that $b_o + b_1 + \cdots$ is an asymptotic series with the same asymptotic sum.

*) Let $b_h = a_{n_h}$. If $h \to \infty$, then $n_h \to \infty$. Since

$$|\omega|^{q_{n_h}} |b_h| = |\omega|^{q_{n_h}} |a_{n_h}|$$

is bounded for each fixed integer $h \geqq 0$ and for each element ω of Ω, the series $b_o + b_1 + \cdots$ is asymptotic. If k denotes the largest integer $\geqq 0$, depending on h, such that the system $n_1, n_2, \cdots, n_{h-1}$ contains the integers $0, 1, \cdots, k-1$, then $k \to \infty$ as $h \to \infty$. The difference

$$(b_o + \cdots + b_{h-1}) - (a_o + \cdots + a_{k-1})$$

can be written as a sum of at most h terms a_m with $m \geqq k$, so that each of these terms is at most of the same order of magnitude as $|\omega|^{-q_k}$. We know that

$$s - (a_o + \cdots + a_{k-1})$$

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*) We assume that n_h is a fixed integer $\geqq 0$ for each fixed integer $h \geqq 0$.

is also at most of the same order of magnitude as $|\omega|^{-q_k}$; therefore that is also the case with

$$s - (b_o + \cdots + b_{h-1}) =$$

$$\left\{s - (a_o + \cdots + a_{k-1})\right\} - \left\{(b_o + \cdots + b_{h-1}) - (a_o + \cdots + a_{k-1})\right\} .$$

This completes the proof.

The double series $\sum_{h=o}^{\infty} \sum_{k=o}^{\infty} a_{hk}$ is said to be asymptotic if the series

$$a_{00} + a_{01} + a_{10} + a_{02} + a_{11} + a_{20} + a_{03} + \cdots ,$$

ordered according to non-decreasing values of $h + k$, is asymptotic and the asymptotic sum s of the last series is called the asymptotic sum of the double series, in notation

$$s \sim \sum_{h=o}^{\infty} \sum_{k=o}^{\infty} a_{hk} .$$

A similar definition of course is possible for triple series, and so on.

The following theorem is immediately clear.

THEOREM 5. A double series

$$\sum_{h=o}^{\infty} \sum_{k=o}^{\infty} a_{hk}$$

is asymptotic if and only if it is possible to find fixed numbers q_{hk} with the property that q_{hk} tends to infinity as at least one of the integers h and k approaches infinity and with the property that

$$|\omega|^{-q_{hk}} \; |a_{hk}|$$

is bounded for any pair of fixed integers $h \geqq 0$ and $k \geqq 0$ and for each element ω of Ω.

If the double series is asymptotic, it is possible to choose these exponents q_{hk} such that the inequality

$$q_{hk} \leqq q_{H,K}$$

holds (1) if $h + k < H + K$

(2) if $h + k = H + K$ and $h \leqq H$.

THEOREM 6. If

$$s \sim \sum_{h=0}^{\infty} a_h$$

and if for each fixed integer h

$$a_h \sim \sum_{k=0}^{\infty} b_{hk} \quad ,$$

then

$$s \sim \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} b_{hk} \quad ,$$

provided that the double series is asymptotic.

Remark: This last condition is not superfluous. For instance for $\omega \geqq 2$

$$\frac{1}{\omega - 1} \sim \sum_{h=1}^{\infty} \frac{1}{\omega^h}$$

and

$$\frac{1}{\omega^h} = 1 - 1 + \frac{1}{\omega^h} \quad ,$$

but the double series

$$\sum_{h=1}^{\infty} \sum_{k=o}^{2} b_{hk} \quad ,$$

where

$$b_{h0} = 1 \quad ; \quad b_{h1} = -1 \quad ; \quad b_{h2} = \frac{1}{\omega^h} \quad ,$$

is not asymptotic.

Proof: If q_o, q_1, ... denote the monotonic non-decreasing exponents belonging to the asymptotic series $a_o + a_1 + \cdots$, then the remainder u_H, defined by

$$s = \sum_{h=o}^{H-1} a_h + u_H \quad ,$$

is at most of the same order of magnitude as $|\omega|^{-q_H}$.

If q_{h0}, q_{h1}, ... denote the monotonic non-decreasing exponents belonging to the asymptotic series $b_{h0} + b_{h1} + \cdots$, then the remainder v_{hK}, defined by

$$a_h = {\sum_{k=o}^{K}}' b_{hk} + v_{h,K} \quad ,$$

is at most of the same order of magnitude as $|\omega|^{-q_{hK}}$; the prime indicates that the term with $k = K$ may be omitted. The choice allows us to write any initial sum of the original double series as

$$\sum_{h=o}^{H-1} {\sum_{k=o}^{H-h}}' b_{hK}$$

In this way we find that

$$s - \sum_{h=o}^{H-1} {\sum_{k=o}^{H-h}}' b_{hk} = u_H + \sum_{h=o}^{H-1} v_{h,H-h}$$

is at most of the same order of magnitude as $|\omega|^{-r_H}$, where r_H is the smallest of the numbers q_H and $q_{h,H-h}$($h = 0,1, \cdots , H - 1$). This number r_H tends to infinity as $H \to \infty$, which completes the proof.

THEOREM 7. (Product Theorem). If

(2.1) $$s \sim \sum_{h=o}^{\infty} a_h \quad \text{and} \quad t \sim \sum_{k=o}^{\infty} b_k \ ,$$

then

$$s\,t \sim \sum_{h=o}^{\infty} \sum_{k=o}^{\infty} a_h b_k \ .$$

Proof: From the first of the relations (2.1) it follows that

$$s\,t \sim \sum_{h=o}^{\infty} a_h t \ ,$$

since there exists a fixed number q_o such that t is at most of the same order of magnitude as $|\omega|^{-q_o}$. In the same way it follows from the second of the relations (2.1) that

$$a_h t \sim \sum_{k=o}^{\infty} a_h b_k \ .$$

The assertion follows now from the preceding theorem, since the double series is asymptotic. For, if p_h and q_h denote the exponents belonging to $a_o + a_1 + \cdots$ and $b_o + b_1 + \cdots$, then $a_h b_k$ is at most of the same order of magnitude as $|\omega|^{-p_h-q_k}$, where $p_h + q_k$ tends to infinity, as at least one of the integers h and k approaches infinity.

A series may be convergent and at the same time asymptotic, but in that case its sum is not necessarily its asymptotic sum. For instance, let

$$a_n = \frac{1}{(n+1)(n+2)} \min\left(1, \frac{e^{n^2}}{|\omega|^n}\right) \ ,$$

so that

$$\sum_{n \geq \log|\omega|} a_n = \sum_{n \geq \log|\omega|} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{N+1} ,$$

where N is the smallest integer $\geq \log |\omega|$. In that case

$$s = \sum_{0 \leq n < \log|\omega|} a_n + \frac{1}{N+1}$$

is the sum of the convergent series $a_o + a_1 + \cdots$. If s were the asymptotic sum of this asymptotic series, then, since the exponent q_n belonging to this series is > 1 for sufficiently large h, there would exist a fixed integer h such that $s - (a_o + \cdots + a_{h-1})$ is at most of the same order of magnitude as $\frac{1}{|\omega|}$. But for sufficiently large $|\omega|$ we have $\log |\omega| > h$, so that

$$s - \sum_{n=o}^{h-1} a_n = \sum_{h \leq n < \log|\omega|} a_n + \frac{1}{N+1} \geq \frac{1}{N+1} .$$

This gives a contradiction, since $\frac{1}{N+1}$ has the same order of magnitude as $\frac{1}{\log|\omega|}$, therefore larger than the order of $\frac{1}{|\omega|}$.

The following theorem, however, shows that the sum of a convergent asymptotic series is under general conditions also its asymptotic sum.

THEOREM 8. Suppose the bounded function s of ω can be written as the sum of a convergent series

$$s = \sum_{n=o}^{\infty} a_n .$$

Suppose that there exists four numbers $\varepsilon < 0$, $\lambda > 0$, $c > 0$, and γ, independent of ω and n such that

$$|a_n| \leqq c\,\lambda^n\,|\omega|^{\gamma - n\varepsilon}.$$

Then s is also the asymptotic sum of the asymptotic series.

Proof: If $1 \leqq |\omega| < (2\lambda)^{2/\varepsilon}$, then

$$s - (a_o + a_1 + \cdots + a_{h-1})$$

is bounded for each fixed integer $h \geqq 0$ and therefore in absolute value $\leqq C\,|\omega|^{\gamma - \frac{1}{2}h\varepsilon}$, where C denotes a suitably chosen fixed number. If $|\omega| \geqq (2\lambda)^{2/\varepsilon}$, then

$$\left|s - \sum_{n=o}^{h-1} a_n\right| = \left|\sum_{n=h}^{\infty} a_n\right|$$

$$\leqq c \sum_{n=h}^{\infty} \lambda^n\,|\omega|^{\gamma - \frac{1}{2}n\varepsilon}|\omega|^{-\frac{1}{2}n\varepsilon}$$

$$\leqq c\,|\omega|^{\gamma - \frac{1}{2}h\varepsilon} \sum_{n=h}^{\infty} \lambda^n\,|\omega|^{-\frac{1}{2}n\varepsilon}$$

$$\leqq c\,|\omega|^{\gamma - \frac{1}{2}h\varepsilon} \sum_{n=h}^{\infty} 2^{-n}$$

$$< 2c\,|\omega|^{\gamma - \frac{1}{2}h\varepsilon}$$

so that we get an asymptotic series with exponents $q_n = \frac{1}{2}h\varepsilon - \gamma$.

This completes the proof.

THEOREM 9. If a bounded function s of ω can be written as the sum of a convergent power series

$$s = \sum_{h=o}^{\infty} \gamma_h\,u^h$$

with fixed coefficients and if there exists a fixed positive number ε such that $|\omega|^{\varepsilon}\,|u|$ is bounded for all elements ω of Ω, then s is also the asymptotic sum of the asymptotic power series.

Proof: The assertion is evident, if u = 0 for each element ω of Ω. Suppose therefore that u assumes a value $v \neq 0$ for at least one element ω of Ω. The power series in question converges for u = v, so that

$$|\gamma_h| \leqq c|v|^{-h}$$

for suitably chosen number c which is independent of ω and h. Consequently, since

$$|u| \leqq C\ |\omega|^{-\varepsilon}\ , \quad \text{we get } |\gamma_h u^h| \leqq c\ C^h\ |v|^{-h} |\omega|^{-h\varepsilon}$$

so that the required result follows from the preceding theorem.

THEOREM 10. Suppose that a bounded function s of ω can be written as the sum of the convergent power series

$$s = \sum_{h=0}^{\infty} \gamma_h\ u^h$$

with fixed coefficients. Suppose moreover that u possesses an asymptotic expansion

$$u \sim \beta_0 + \beta_1 + \beta_2 + \cdots \quad ,$$

with positive exponents so that u^h possesses, according to the product theorem (theorem 7), for each positive integer h an asymptotic expansion

$$u^h \sim \sum_{k=0}^{\infty} \beta_{hk} \quad .$$

Then

$$s \sim \sum_{h=0}^{\infty}\ \sum_{k=0}^{\infty} \gamma_h \beta_{hk} \quad .$$

Proof: Let q_o, q_1, $\cdots$ be the monotonic non-decreasing exponents of the series $\beta_o + \beta_1 + \cdots$. Then q_o is positive and according to the definition of an asymptotic sum, applied with $h = 0$, the product $|\omega|^{q_o} |u|$ is bounded, so that we can apply the preceding theorem with $\varepsilon = q_o$. Consequently

$$s \sim \sum_{h=o}^{\infty} \gamma_h u^h .$$

According to theorem 6 this gives the required result, provided that the double series

$$\sum_{h=o}^{\infty} \sum_{k=o}^{\infty} \gamma_h \beta_{hk}$$

is asymptotic. In the expansion

$$u \sim \beta_{10} + \beta_{11} + \cdots$$

the exponents are $q_{1k} = q_k$; in

$$u^2 \sim \beta_{20} + \beta_{21} + \cdots , \quad \text{where } \beta_{2k} = \sum_{n=o}^{k} \beta_{1n} \beta_{1,k-n} ,$$

the exponents are therefore

$$q_{2k} = \min (q_{k_1} + q_{k_2}) , \quad \text{where } k_1 + k_2 = k ;$$

in

$$u^3 \sim \beta_{30} + \beta_{31} + \cdots$$

the exponents are

$$q_{3k} = \min (q_{k_1} + q_{k_2} + q_{k_3}) , \quad \text{where } k_1 + k_2 + k_3 = k$$

and for any fixed positive integer h the exponents in

$$u^h \sim \beta_{h0} + \beta_{h1} + \beta_{h2} + \cdots$$

are

$$q_{hk} = \min\,(q_{k_1} + q_{k_2} + \cdots + q_{k_h}) \quad , \quad \text{where } k_1 + k_2 + \cdots + k_h = k \ .$$

Therefore, since γ_h is fixed,

$$|\omega|^{q_{hk}}|\gamma_h\, u^h|$$

is bounded. It still remains to show that q_{hk} tends to infinity, as h + k approaches infinity. To that end I prove for any fixed number t

$$q_{hk} \geqq t \quad ,$$

if at least one of the two subscripts h and k is large enough. From the above definition of q_{hk} it follows that

$$q_{hk} \geqq hq_o \quad ,$$

so that the required inequality holds if $h \geqq \frac{t}{q_o}$. If $h < \frac{t}{q_o}$, then at least one of the h numbers $k_1, \cdots, k_n$, whose sum is equal to k, is equal to χ, where χ is the smallest integer $\geqq \frac{q_o}{t}\,k$. Then

$$q_{hk} \geqq q_\chi \geqq t \quad ,$$

if k, and therefore also χ, is large enough.

<u>Example</u>: Suppose u possesses an asymptotic expansion

$$u \sim \beta_o + \beta_1 + \cdots$$

with positive exponents. Then e^u also possesses an asymptotic expansion

$$e^u \sim \alpha_o + \alpha_1 + \cdots \quad .$$

As we have seen this expansion for $e^u = \sum_{h=o}^{\infty} \frac{u^h}{h!}$ can be found in a formal way, so that I can define the terms of that expansion by the formal identity

$$e^{\beta_o \lambda + \beta_1 \lambda^2 + \cdots} = \alpha_o + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots \quad .$$

This gives first $\alpha_o = 1$.

Taking the formal derivative with respect to λ, we obtain

$$\alpha_1 + 2\alpha_2 \lambda + 3\alpha_3 \lambda^2 + \cdots = (\beta_o + 2\beta_1 \lambda + \cdots)(\alpha_o + \alpha_1 \lambda + \cdots) \quad ,$$

so that we obtain for $h \geqq 0$

$$(h+1)\alpha_{h+1} = (h+1)\alpha_o \beta_h + h\alpha_1 \beta_{h-1} + \cdots + 1\alpha_h \beta_o \quad .$$

Thus we find

$$\alpha_1 = \beta_o \quad ; \quad \alpha_2 = \tfrac{1}{2}\beta_o^2 + \beta_1 \quad ; \quad \alpha_3 = \tfrac{1}{6}\beta_o^3 + \beta_o\beta_1 + \beta_2 \quad .$$

This is not the only, and even not always the best, asymptotic expansion for e^u. We get another expansion if we define its terms by means of the formal identity

$$e^{\beta_o \lambda + \beta_1 \lambda^3 + \beta_2 \lambda^5 + \cdots} = \alpha_o + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots$$

In this case we obtain the relation

$$\alpha_1 + 2\alpha_2 \lambda + 3\alpha_3 \lambda^2 + \cdots =$$

$$(\beta_o + 3\beta_1 \lambda^2 + 5\beta_2 \lambda^4 + \cdots)(\alpha_o + \alpha_1 \lambda + \cdots) \quad ,$$

which gives

$$\alpha_0 = 1 \;;\; \alpha_1 = \beta_0 \;;\; \alpha_2 = \tfrac{1}{2}\beta_0^2 \;;\; \alpha_3 = \tfrac{1}{6}\beta_0^3 + \beta_1 \,,$$

and so on.

THEOREM 11. *If*

$$s \sim a_0 + a_1 + \cdots$$

and

$$a_h \sim b_h \quad \textit{for each fixed integer } h \geqq 0 \,,$$

then

$$s \sim b_0 + b_1 + \cdots$$

We may therefore replace each term a_h of an asymptotic series by a term which is asymptotically equal to a_h.

Proof: By the definition of the asymptotic sum of an asymptotic series it is possible to find fixed numbers q_h which tend to infinity as $h \to \infty$ such that

$$|\omega|^{q_h} \, |s - (a_0 + a_1 + \cdots + a_{h-1})|$$

is bounded. Since a_k and b_k $(0 \leqq k \leqq h - 1)$ are asymptotically equal, the products

$$|\omega|^{q_h} \, (a_k - b_k)$$

are also bounded, so that

$$|\omega|^{q_h} \, |s - (b_0 + b_1 + \cdots + b_{h-1})|$$

is bounded. Consequently s is the asymptotic sum of the asymptotic series $b_0 + b_1 + \cdots$.

THEOREM 12. Let g be a fixed positive integer. If $a_h(z)$ $(h = 0,1,2, \cdots)$ denotes a function of z which is at least g times differentiable in a given interval j or in a given region j (j may depend on ω) such that the series

$$\sum_{h=0}^{\infty} a_h^{(n)}(z) \qquad (n = 0,1, \cdots , g)$$

are asymptotic, uniformly in z, then there exists a function $s(z)$ which is g times differentiable in j such that for $n = 0,1, \cdots , g$

$$s^{(n)}(z) \sim \sum_{h=0}^{\infty} a_h^{(n)} (z)$$

uniformly in z.

Remark: Therefore not only does the function $s(z)$ itself possess an asymptotic expansion, but also its 1st, 2nd, $\cdots$, g-th derivatives, and the expansions for the derivatives are obtained by differentiating the original expansion term by term.

Proof: The proof is similar to that of theorem 1. Since $\sum_{h=0}^{\infty} a_h^{(n)}(z)$ is asymptotic, we know for $h \geqq 0$ and $0 \leqq n \leqq g$

$$(2.2) \qquad |a_h^{(n)} (z)| \leqq c_h^{(n)} \, |\omega|^{-q_{h,n}} ,$$

where $c_h^{(n)}$ and $q_{h,n}$ are independent of ω and z; if n is given, the numbers $q_{h,n}$ tend to infinity monotonically as $h \to \infty$.

I choose an integer $H \geqq 0$, dependent on ω but not on z, which tends to infinity as $|\omega| \to \infty$ but so slowly that

$$\sum_{h=0}^{H-1} c_h^{(n)} \leqq |\omega| \qquad \text{for } n = 0,1, \cdots , g .$$

It is clear that the sum

$$s(z) = a_o(z) + a_1(z) + \cdots + a_{H-1}(z)$$

is g times differentiable. To show that s(z) also possesses the other required property, we must prove for $n = 0,1, \cdots, g$ and for each fixed integer $h \geqq 0$ that

$$(2.3) \quad \left| s^{(n)}(z) - \left(a_o^{(n)}(z) + \cdots + a_{h-1}^{(n)}(z)\right) \right| \leqq C_h^{(n)} |\omega|^{-q_{h,n}} ,$$

where $C_h^{(n)}$ denotes a suitably chosen number which is independent of ω and z.

From the fact that $q_{kn} \to \infty$ as $k \to \infty$ it follows that there exists a fixed number $k > h$ such that $q_{kn} \geqq q_{hn} + 1$. From the fact that $H \to \infty$ as $|\omega| \to \infty$ it follows that there exists a fixed number γ such that $H > k$ for each element ω of Ω with $|\omega| \geqq \gamma$.

For the elements ω of Ω with $|\omega| \leqq \gamma$ we have that

$$|s^{(n)}(z)| \leqq c_o^{(n)} |\omega|^{-q_{on}} + \cdots + c_{H-1}^{(n)} |\omega|^{-q_{H-1,n}}$$

$$(2.4) \qquad \leqq \left(c_o^{(n)} + \cdots + c_{H-1}^{(n)}\right) |\omega|^{-q_{on}}$$

$$\leqq \tfrac{1}{2} C_h^{(n)} |\omega|^{-q_{h,n}}$$

and

$$\left| a_o^{(n)}(z) + \cdots + a_{h-1}^{(n)}(z) \right| \leqq \left(c_o^{(n)} + \cdots + c_{h-1}^{(n)}\right) |\omega|^{-q_{on}}$$

$$\leqq \tfrac{1}{2} C_h^{(n)} |\omega|^{-q_{h,n}}$$

for suitably chosen fixed number $C_h^{(n)}$. In this case we obtain therefore the required inequality (2.3).

For the elements ω of Ω with $|\omega| \geqq \gamma$ we have $H > k > h$, so that

$$(2.5)\quad \begin{cases} \left| s^{(n)}(z) - \left(a_o^{(n)}(z) + \cdots + a_{h-1}^{(n)}(z)\right) \right| = \left| a_h^{(n)}(z) + \cdots + a_{H-1}^{(n)}(z \right. \\ \leqq \left| a_h^{(n)}(z) + \cdots + a_{k-1}^{(n)}(z) \right| + \left| a_k^{(n)}(z) + \cdots + a_{H-1}^{(n)}(z) \right| \\ \leqq \left(c_h^{(n)} + \cdots + c_{k-1}^{(n)} \right) |\omega|^{-q_{hn}} + \left(c_k^{(n)} + \cdots + c_{H-1}^{(n)} \right) |\omega|^{-q_{hn}-} \\ \leqq \left(c_h^{(n)} + \cdots + c_{k-1}^{(n)} + 1 \right) |\omega|^{-q_{hn}} \quad . \end{cases}$$

Consequently the inequality (2.3) holds also in this case. This establishes the proof.

THEOREM 13. _If_ $a_h(z) (h = 0,1, \cdots)$ _is indefinitely differentiable with respect to_ z _in a given interval_ j _or in a given region_ j (j _may depend on_ ω) _and if for each fixed integer_ $n \geqq 0$ _the series_

$$\sum_{h=o}^{\infty} a_h^{(n)}(z)$$

is asymptotic uniformly in z, _then there exists a function_ s(z) _which is indefinitely differentiable in_ j _such that for each fixed integer_ $n \geqq 0$

$$s^{(n)}(z) \sim \sum_{h=o}^{\infty} a_h^{(n)}(z) \quad ,$$

uniformly in z.

Proof: The inequality (2.2) holds for each fixed integer $h \geqq 0$ and each fixed integer $n \geqq 0$, where $c_h^{(n)}$ and q_{hn} are independent of ω and z; if n is given, the numbers q_{hn} tend monotonically to infinity as $h \to \infty$.

I choose an integer $H \geqq 0$, dependent on ω but not on z, which tends to infinity as $|\omega| \to \infty$, but so slowly that*)

$$(2.6) \qquad \sum_{h=0}^{H-1} c_h^{(n)} \leqq |\omega| \qquad \text{for } n = 0, 1, \ldots, H-1 .$$

It is clear that the sum

$$s(z) = a_0(z) + a_1(z) + \ldots + a_{H-1}(z)$$

is indefinitely differentiable. To show that $s(z)$ also possesses the other required property, we must prove that the inequality (2.3) holds for each fixed integer $h \geqq 0$ and each fixed integer $n \geqq 0$, where $C_h^{(n)}$ denotes a suitably chosen number which is independent of ω and z. Note that in the rest of the proof h and n are fixed.

There exists a fixed number $k > h$ satisfying the inequality $q_{kn} \geqq q_{hn} + 1$ and there exists a fixed number γ such that $H > k$ and $H > n$ for each element ω of Ω with $|\omega| \geqq \gamma$. We find the

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*) In the proof of the preceding theorem the range of n was from 0 to g. It is not allowed to replace here g by ∞, since $c_h^{(n)}$ may tend to infinity as $n \to \infty$ and so we have no assurance of the existence of an integer $H \geqq 0$ such that (2.6) holds for all integers $n \geqq 0$.

inequalities (2.4), and therefore also (2.3) for the elements ω of Ω with $|\omega| \leqq \gamma$. For the elements ω of Ω with $|\omega| > \gamma$ we have $H > k > h$ and $H > n$, so that the formula (2.5) and therefore also (2.3) are true. This completes the proof.

THEOREM 14. Consider the sum

$$S = \sum_{n=A}^{B-1} f(n)$$

where A and B are integers with $B > A$; the integer A is assumed to be finite. B may be infinite, but in that case the series S is assumed to be convergent. Assume that it is possible to find positive numbers $p(n)$ (for $A \leqq n < B$) such that

$$\frac{f(n)}{p(n)} \sim \frac{a_{no}}{p(n)} + \frac{a_{n1}}{p(n)} + \cdots \quad . \qquad (A \leqq n < B) \tag{2.7}$$

uniformly in n, and that

$$\sum_{n=A}^{B-1} p(n) \leqq \gamma |\omega|^m \quad ,$$

where γ and m denote suitably chosen positive fixed numbers.

Under these conditions

$$S \sim \sum_{k=o}^{\infty} s_k \quad , \qquad \text{where } s_k = \sum_{n=A}^{B-1} a_{nk} \quad ;$$

if B is infinite, we suppose that this series converges.

Proof: Let $q_o, q_1, \ldots$ be the monotonic non-decreasing exponents belonging to the asymptotic series occurring on the right hand side of (2.7). Then we have for $A \leqq n < B$ and for each fixed integer $h \geqq 0$

$$f(n) = a_{no} + a_{n1} + \cdots + a_{n,h-1} + p(n) r_{n,h} \quad ,$$

$$|r_{n,h}| \leqq c_h \, |\omega|^{-q_h} \quad ,$$

c_h and q_h indicating numbers independent of ω and n. Then

$$\left| S - \sum_{k=o}^{h-1} s_k \right| = \left| \sum_{n=A}^{B-1} f(n) - \sum_{k=o}^{h-1} \sum_{n=A}^{B-1} a_{nk} \right|$$

$$= \left| \sum_{n=A}^{B-1} \left(f(n) - \sum_{k=o}^{h-1} a_{nk} \right) \right|$$

$$\leqq \sum_{n=A}^{B-1} p(n) \left| r_{n,h} \right| \leqq$$

$$\leqq c_h |\omega|^{-q_h} \sum_{n=A}^{B-1} p(n) \leqq \gamma c_h |\omega|^{-(q_h - m)} \quad ,$$

where $q_h - m \rightarrow \infty$ as $h \rightarrow \infty$. Consequently S is the asymptotic sum of the series $s_o + s_1 + \cdots$, which therefore is an asymptotic series.

Of course a similar theorem holds for integrals instead of sums.

Example. Let us show that for large positive values of ω

$$\sum_{n=o}^{\infty} e^{-n} \log(1 + \tfrac{n}{\omega}) \sim \frac{1}{\omega} \sum_{o}^{\infty} ne^{-n} - \frac{1}{2\omega^2} \sum_{o}^{\infty} n^2 e^{-n}$$

$$+ \frac{1}{3\omega^3} \sum_{o}^{\infty} n^3 e^{-n} \cdots \quad ,$$

where the coefficients can easily be calculated by the formula

$$\sum_{n=o}^{\infty} n^h e^{-n} = (-)^h \left(\frac{d^h}{dt^h} \frac{1}{1-e^{-t}} \right)_{t=1} \quad .$$

We know

$$\log(1 + \frac{n}{\omega}) = \frac{n}{\omega} - \frac{n^2}{2\omega^2} + \cdots \pm \frac{n^{h-1}}{(h-1)n^{h-1}} + R_h(n) \quad ,$$

where

$$|R_h(n)| \leqq c_h \frac{n^h}{\omega^h} \quad :$$

here c_h is a suitably chosen number depending on h but not on n and ω.

Then

$$e^{-\frac{1}{2}n} \log(1 + \frac{n}{\omega}) \sim e^{-\frac{1}{2}n} \left(\frac{n}{\omega} - \frac{n^2}{2\omega^2} + \frac{n^3}{3\omega^3} + \cdots\right) \quad ,$$

uniformly in n for $n = 0,1,2, \cdots$. Applying theorem 14 with

$$B = \infty \quad , \quad p(n) = e^{-\frac{1}{2}n} \quad , \quad \text{fixed } m > 0 \quad ,$$

we obtain the required result.

Section 3: On certain analytic functions

Let $a_o + a_1 (z - z_o) + a_2(z - z_o)^2 + \cdots$ be a formal power series in $z - z_o$ and let R be a positive number; the coefficients a_h and the number R may depend on ω. If it is possible to find for each fixed integer $h \geqq 0$ a fixed number c_h such that

$$(3.1) \qquad |a_h| \leqq c_h R^{-h} \qquad (h = 0,1, \cdots) ,$$

then the power series is said to possess an <u>asymptotic radius</u> $\geqq$ R. Note that the asymptotic radius itself is not defined. If the numbers c_h can be chosen independently of certain parameters, then the power series is said to possess an <u>asymptotic radius</u> $\geqq$ R <u>uniformly in these parameters</u>. If a power series possesses an asymptotic radius $\geqq$ R, it possesses also an asymptotic radius $\geqq$ C R, where C denotes an arbitrary fixed number, since (3.1) implies

$$|a_h| \leqq c_h C^h (C R)^{-h} \qquad (h = 0,1, \cdots) .$$

where $c_h C^h$ is again fixed.

A power series in $z - z_o$ with asymptotic radius $\geqq$ R is asymptotic for all number z which satisfy the inequality

$$(3.2) \qquad |z - z_o| \leqq C |\omega|^{-\gamma} R ,$$

where C and γ denote arbitrary fixed positive numbers, since for these values of z

$$|a_h (z - z_o)^h| \leqq c_h R^{-h} C^h |\omega|^{-h\gamma} R^h$$
$$= c_h C^h |\omega|^{-h\gamma} ,$$

where $h_\gamma \to \infty$ as $h \to \infty$.

The formal derivatives of the power series are

$$a_1 + 2a_2 (z - z_o) + 3a_3 (z - z_o)^2 + \dots ,$$

$$2a_2 + 3\cdot 2a_3 (z - z_o) + 4\cdot 3a_4(z - z_o)^2 + \dots ,$$

etc. and are obviously also asymptotic for the values of z, satisfying (3.2). According to theorem 13 of the preceding section it is therefore possible to construct a function s(z) which is analytic at the points z satisfying (3.2) such that

$$(3.3) \quad \begin{cases} s(z) \sim \sum_{n=o}^{\infty} a_n (z - z_o)^n ; \\ s'(z) \sim \sum_{n=1}^{\infty} n a_n (z - z_o)^{n-1} ; \\ s''(z) \sim \sum_{n=2}^{\infty} n(n-1) a_n (z - z_o)^{n-2} , \text{ etc.} \end{cases}$$

A function s(z) with these properties is said to be generated by the power series. The analytic functions s(z) is not uniquely defined by the power series, for if s(z) possesses the required properties, then so does also for instance $s(z) + e^{-\omega} t(z)$, where t(z) is an arbitrary fixed function of z, which is analytic at z_o.

If the coefficients a_h in the power series are fixed, the asymptotic radius is ≥ 1. In this case it is useful to distinguish two cases, according to whether the radius of convergence of the power series is positive or zero. If the power series has a positive radius of convergence, it possesses a sum s(z) which, according to theorem 9 in the preceding section, satisfies the relations (3.3) at the points z subject to (3.2), so that the power series generates this function s(z) with asymptotic radius $\geqq 1$. This function is the sum of a convergent power series with fixed coefficients and is therefore independent of ω, that is, the value s(z) is independent of ω if z is given. Therefore each function s(z), which is independent of ω and analytic at a given point z_o, is generated with an asymptotic radius $\geqq 1$, by the corresponding power series $a_o + a_1(z - z_o) + \cdots$ of which s(z) is the sum.

Let us now consider a power series $a_o + a_1(z - z_o) + \cdots$ with fixed coefficients whose radius of convergence is equal to zero (the asymptotic radius is of course $\geqq 1$). This power series generates a function s(z) which is analytic at z_o. It is impossible that this function s(z) be independent of ω. For let us suppose that a fixed function s(z) satisfies the relations (3.3) for the points z subject to (3.2). Choosing $z = z_o$, we would obtain

$$s(z_o) \sim 0!a_o \quad , \quad s'(z_o) \sim 1!a_1 \quad , \quad s''(z_o) \sim 2!a_2 \; , \; \cdots \quad .$$

Since these numbers are fixed, we get $s^{(h)}(z_o) = h!a_h$ $(h = 0,1, \cdots)$, so that the power series

$$\sum_{h=o}^{\infty} a_h(z - z_o)^h = \sum_{h=o}^{\infty} \frac{s^{(h)}(z_o)}{h!} (z - z_o)^h$$

would represent the function s(z) which is analytic at z_o. In this case the power series would have a positive radius of convergence, contrary to the hypothesis.

For instance the power series

$$\sum_{n=0}^{\infty} (-)^n \, n! \, z^n$$

has an asymptotic radius $\geqq 1$ and a radius of convergence $= 0$, so that it generates a function of z which is analytic at the origin; this function is necessarily dependent on ω. But in the sector

$$(3.4) \qquad z \neq 0 \;; \quad -\pi + \varepsilon < \arg z < \pi - \varepsilon \;,$$

where ε denotes an arbitrary fixed positive number $< \frac{\pi}{2}$, the integral

$$(3.5) \qquad \psi(z) = \int_0^{\infty} \frac{e^{-u}}{1+zu} \, du$$

represents a function of z which is analytic in the above mentioned sector. Moreover the formula

$$(3.6) \quad \psi^{(h)}(z) \sim \sum_{n=h}^{\infty} (-)^n \, n! \, n(n-1) \cdots (n-h+1) \, z^{n-h}$$

holds for each fixed integer $h \geqq 0$ and for each point z which satisfies the inequalities (3.2) with R = 1, and (3.4). The power series $\sum_{n=0}^{\infty} (-)^n \, n! z^n$ generates the function $\psi(z)$ in the considered sector with asymptotic radius $\geqq 1$. As we see, this generated function is independent of ω.

To prove (3.6) we note that

$$\psi^{(h)}(r) = (-)^h \, h! \int_0^{\infty} \frac{u^h e^{-u}}{(1+zu)^{h+1}} \, du$$

in the specified sector and we use the following lemma.

LEMMA. *Suppose*

$$-\pi + \epsilon < \arg w < \pi - \epsilon$$

and let α *be real. Put*

$$(1+w)^{\alpha} = 1 + \binom{\alpha}{1} w + \cdots + \binom{\alpha}{m-1} w^{m-1} + R_m .$$

Then we have for each integer $m \geq 0$

$$|R_m| \leq c_m(\alpha) \left(|w|^{\alpha} + |w|^m\right) \qquad \text{if } \alpha \geq m$$

and

$$|R_m| \leq c_m(\alpha) (\sin \epsilon)^{\alpha - m} |w|^m \qquad \text{if } \alpha \leq m ,$$

where $c_m(\alpha)$ *denotes a suitably chosen number, depending on* m *and* α, *but not on* w *and* ϵ.

Proof. For any function $f(t)$ which is m times continuously differentiable on the segment $(0, w)$ we have

$$f(w) = f(0) + \frac{f'(0)}{1!} w + \cdots + \frac{f^{(m-1)}(0)}{(m-1)!} w^{m-1} + R_m ,$$

where

$$R_m = \frac{1}{(m-1)!} \int_0^w (w-t)^{m-1} f^{(m)}(t)\, dt .$$

The formula for the remainder term can be verified by integration by parts.

The particular case $f(t) = (1+t)^{\alpha}$ gives

$$f(0) = 1 \; ; \; \frac{f'(0)}{1!} = \binom{\alpha}{1}, \; \cdots , \; \frac{f^{(m-1)}(0)}{(m-1)!} = \binom{\alpha}{m-1}$$

and

$$R_m = \binom{\alpha}{m} m \int_0^w (w - t)^{m-1} (1 + t)^{\alpha - m} dt =$$

$$= \binom{\alpha}{m} m w^m \int_0^1 (1 - u)^{m-1} (1 + uw)^{\alpha - m} du$$

The argument of w and therefore also the argument of t lies between $-\pi + \epsilon$ and $\pi - \epsilon$. This implies that $|1 + t|$ is at least equal to $\sin \epsilon$, for if t lies on or to the right of the imaginary axis, the distance $|1 + t|$ between -1 and t is $\geqq 1 \geqq \sin \epsilon$; otherwise the distance between -1 and t is at least equal to the length of the perpendicular drawn from -1 to the halfline formed by the points with argument $= -\pi + \epsilon$, so that $|1 + t| \geqq \sin \epsilon$.

In the case that $\alpha \leqq m$ we find therefore

$$|1 + uw|^{\alpha - m} \leqq (\sin \epsilon)^{\alpha - m} ,$$

hence

$$|R_m| \leqq \left|\binom{\alpha}{m}\right| m |w|^m (\sin \epsilon)^{\alpha - m} \int_0^1 (1 - u)^{m-1} du$$

$$= \left|\binom{\alpha}{m}\right| (\sin \epsilon)^{\alpha - m} |w|^m .$$

In the case that $\alpha \geqq m$ we use the fact that

$$|1 + uw| \leqq 1 + |uw| \leqq 1 + |w| ,$$

hence

$$|1 + uw|^{\alpha - m} \leqq (1 + |w|)^{\alpha - m} .$$

The last side is

$$\leqq 2^{\alpha-m} \qquad \text{if } |w| \leqq 1$$

and

$$\leqq 2^{\alpha-m} |w|^{\alpha-m} \qquad \text{if } |w| \geqq 1 \ ,$$

so that

$$|1 + uw|^{\alpha-m} < 2^{\alpha-m} \left(1 + |w|^{\alpha-m}\right) \ .$$

Consequently

$$|R_m| \leqq \left|\binom{\alpha}{m}\right| m\, 2^{\alpha-m} \left(1 + |w|^{\alpha-m}\right) |w|^m \int_0^1 (1-u)^{m-1}\, du$$

$$= \left|\binom{\alpha}{m}\right| 2^{\alpha-m} (1 + |w|^{\alpha-m}) |w|^m \ .$$

This completes the proof of the lemma.

Now we return to the proof of (3.6). If z satisfies the inequalities (3.4), then using the preceding lemma, applied with $w = zu$ and with $\alpha = -h-1$, we find that

$$\psi^{(h)}(z) = (-)^h h! \int_0^\infty \frac{u^h e^{-u}}{(1+zu)^{h+1}}\, du$$

is equal to

$$(-)^h h! \int_0^\infty u^h e^{-u} \left\{ \sum_{k=0}^{m-1} \binom{-h-1}{k} z^k u^k + R_m(u) \right\} du \ ,$$

where

$$|R_m(u)| \leq c_m(-h-1)\ (\sin\ \epsilon)^{-h-1-m}\ |z|^m\ u^m$$

since $\alpha = -h-1 < m$.

From

$$h!\binom{-h-1}{k} = (-)^k\ h!\ \frac{(h+1)(h+2)\ \cdots\ (h+k)}{k!} = (-)^k\ \frac{(h+k)!}{k!}$$

it follows that

$$\psi^{(h)}(z) = \sum_{k=o}^{m-1}\left\{(-)^{h+k}\ \frac{(h+k)!}{k!}\ z^k \int_o^{\infty} u^{h+k}e^{-u}\ du\right\} + R_m^*$$

$$= \sum_{k=o}^{m-1}\left\{(-)^{h+k}\ \frac{(h+k)!}{k!}\ z^k\ (h+k)!\right\} + R_m^*$$

$$= \sum_{n=h}^{m+h-1} (-)^n\ n!\ n(n-1)\ \cdots\ (n-h+1)\ z^{n-h} + R_m^* ,$$

where

$$|R_m^*| \leq h!\ c_m(-h-1)\ (\sin\ \epsilon)^{-h-1-m}\ |z|^m \int_o^{\infty} u^{h+m}\ e^{-u}\ du$$

$$= h!\ (h+m)!\ c_m(-h-1)\ (\sin\ \epsilon)^{-h-1-m}\ (z)^m .$$

This implies (3.6) for each fixed integer $h \geq 0$ and for each point z in the sector $-\pi + \epsilon < \arg < \pi - \epsilon$ which satisfies the inequality (3.2), so that the power series $\Sigma\ (-)^n\ n!\ z^n$ generates the fixed function $\psi(z)$ in the considered sector with asymptotic radius ≥ 1. But $\psi(z)$ is not the only analytic fixed function which is generated by the power series in the said sector with asymptotic radius ≥ 1. To show this, we consider the function $e^{-z^{-\frac{1}{2}}}$ which is analytic in the region in question; the nth derivative can be written as $z^{-3/2n}\, e^{-z^{-\frac{1}{2}}}\, p(z^{\frac{1}{2}})$, where $p(w)$ is a polynomial in w. For all z in the considered sector such that $|z| \leq 1$, the factor $e^{-z^{-\frac{1}{2}}}$ has an absolute value which is so small that it is possible to find for each fixed real q a fixed number c such that

$$\left| \frac{d^n}{dz^n}\, e^{-z^{-\frac{1}{2}}} \right| \leq c\ |z|^q$$

For the points z in the considered sector which satisfy (3.2) the nth derivative of $e^{-z^{-\frac{1}{2}}}$ is therefore asymptotically equal to zero, so that formula (3.6) remains true if $\Psi(z)$ is replaced by $\Psi(z) + e^{-z^{-\frac{1}{2}}}$.

Nevertheless, sometimes we may define uniquely an analytic function by means of a power series with fixed coefficients and radius of convergence = 0 and by using an additional condition, for example, the condition that the function satisfy a differential equation or a Laplace integral. For instance the formal derivative of the power series $\Sigma_0^{\infty}(-)^n\, n!\, z^n$, multiplied by z^2 is

$$\sum_0^{\infty} (-)^n\, n!\, nz^{n+1} = -\sum_0^{\infty} (-)^{n+1}\, (n+1)!\, z^{n+1} - z \sum_0^{\infty} (-)^n\, n!\, z^n$$

$$= 1 - \sum_0^{\infty} (-)^n\, n!\, z^n - z \sum_0^{\infty} (-)^n\, n!\, z^n \quad ,$$

so that it is natural to introduce the inhomogeneous linear differential equation

$$z^2\, \chi'(z) + (1 + z)\, \chi(z) = 1 \quad .$$

The solutions of this differential equation have the form

$$\chi(z) = \psi(z) + \frac{a}{z} e^{1/z} ,$$

where a is an arbitrary constant and where $\psi(z)$ denotes the functions defined by (3.5). If $a \neq 0$, the function $\chi(z)$ is not generated in the considered sector by the power series $\Sigma_0^{\infty}(-)^n n! z^n$. Consequently $\psi(z)$ is the only solution of the inhomogeneous differential equations which is generated in the considered sector by the power series in question.

Up till now we have always assumed that z_0 lies in the finite z-plane; however we can also take z_0 at infinity. In that case we consider the power series $\Sigma_0^{\infty} a_n z^{-n}$. If it is possible to find for each fixed integer $n \geqq 0$ a fixed integer c_n such that

$$|a_n| \leqq c_n R^n \qquad (n = 0,1, \cdots) ,$$

then we say that the <u>power series</u> $\Sigma_0^{\infty} a_n z^{-n}$ <u>possesses an asymptotic radius</u> $\leqq R$. If the coefficient c_n can be chosen independently of one or more parameters, then we say that the <u>power series possesses an asymptotic radius</u> $\leqq R$, <u>uniformly in these parameters</u>.

Suppose $\Sigma_0^{\infty} a_n z^{-n}$ possesses an asymptotic radius R. For the points z with

$$(3.7) \qquad |z| \geqq C\, |\omega|^{\gamma}\, R ,$$

where C and γ denote arbitrary fixed positive numbers, we can construct, according to theorem 13 in the preceding section, an analytic function $s(z)$ such that

$$(3.8) \quad s^{(h)}(z) \sim (-)^h \sum_{n=0}^{\infty} n(n+1) \cdots (n+h-1)\, a_n\, z^{-n-h}$$

for each fixed integer $h \geqq 0$. We say that the power series $\Sigma_0^\infty a_n z^{-n}$ <u>generates this analytic function</u> $s(z)$ <u>with asymptotic radius</u> $\leqq R$.

Sometimes we restrict ourselves to a sector defined by $\alpha < \arg z < \beta$. In that case we construct a function $s(z)$ which is analytic for the points z lying in that sector and satisfying (3.7) such that (3.8) holds for each fixed integer $h \geqq 0$. It is possible that the coefficients a_h are fixed, that the power series $\Sigma_0^\infty a_n z^{-n}$ diverges everywhere and that nevertheless this function $s(z)$ is independent of ω. For instance, replacing in (3.5) z by z^{-1}, and dividing both sides by z, we see that the power series

$$\sum_{n=0}^{\infty} \frac{(-)^n n!}{z^{n+1}}$$

generates the function

$$\int_0^\infty \frac{e^{-u}}{u+z}\, du$$

with asymptotic radius $\leqq 1$ in the sector $-\pi + \varepsilon < \arg z < \pi - \varepsilon$.

Section 4. Rules of calculations with respect to the asymptotic radius

THEOREM 15. <u>If both the power series</u> $\Sigma_0^{\infty} a_n(z - z_0)^n$ <u>and</u> $\Sigma_0^{\infty} b_n(z - z_0)$ <u>have an asymptotic radius</u> $\geqq R$, <u>then the sum series</u> $\Sigma_0^{\infty}(a_n + b_n)(z - z_0)^n$ <u>and the product series</u> $\Sigma_0^{\infty} p_n (z - z_0)^n$, <u>when</u> $p_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$ <u>also have an asymptotic radius</u> $\geqq R$.

<u>Proof:</u> We know

$$|a_n| \leqq c_n R^{-n} \quad \text{and} \quad |b_n| \leqq c_n' \ R^{-n} ,$$

when c_n and c_n' are suitably chosen fixed numbers. Then

$$|a_n + b_n| \leqq (c_n + c_n') R^{-n}$$

and

$$|p_n| \leqq (c_0 c_n' + c_1 c_{n-1}' \dots + \dots + c_n c_0') R^{-n} ,$$

which establishes the proof.

<u>Remark:</u> It is clear that the corresponding result holds for the power series

$$\sum_0^{\infty} a_n z^{-n} \quad \text{and} \sum_0^{\infty} b_n z^{-n} .$$

THEOREM 16. <u>If</u> $\Sigma_{m=0}^{\infty} a_m z^m$ <u>had an asymptotic radius</u> $\geqq 1$ <u>and the formal series</u> $z = \Sigma_{n=1}^{\infty} b_n w^n$ <u>has an asymptotic radius</u> $\geqq R$, <u>then the formal expansion of</u>

$$\sum_{m=0}^{\infty} a_m \left(\sum_{n=1}^{\infty} b_n w^n\right)^m \tag{4.1}$$

<u>gives a power series in</u> w <u>with asymptotic radius</u> $\geqq R$.

Proof: We know that

$$|a_m| \leqq c_m \ (m \geqq 0) \qquad \text{and} \qquad |b_n| \leqq c'_n R^{-n} \ (n \geqq 1) \ ,$$

where c_m and c'_n are fixed. The formal power series in w, given by (4.1) has therefore the majorant

$$\sum_{m=o}^{\infty} c_m \left(\sum_{n=1}^{\infty} c'_n \ R^{-n} \ |w|^n \right) = \sum_{h=o}^{\infty} C_h \ R^{-h} \ |w|^h \ ,$$

where C_h is fixed. The formal power series in w therefore possesses an asymptotic radius $\geqq$ R.

THEOREM 17. If s(z) is generated by the power series $\Sigma_0^\infty a_n z^n$ with asymptotic radius $\geqq 1$ and if $z = z(w)$ is generated in a sector $\alpha < \arg w < \beta$ by the power series $\Sigma_1^\infty b_n w^n$ with asymptotic radius $\geqq$ R, then the formal expansion of

$$\sum_{m=o}^{\infty} a_m \left(\sum_{n=1}^{\infty} b_n \ w^n \right)^m \tag{4.2}$$

according to ascending powers of w gives a power series in w which generates the function s(z(w)) in the interval $\alpha < \arg w < \beta$ with asymptotic radius $\geqq$ R.

Proof: Let $\Sigma_0^\infty d_h w^n$ be the formal power series given by (4.2). According to the preceding theorem, this power series possesses an asymptotic radius $\geqq$ R, so that the power series is asymptotic for the points w satisfying the inequality

$$|w| \leqq C|\omega|^{-\gamma} \ R \ , \tag{4.3}$$

where C and γ denote arbitrary fixed positive numbers. According to theorem 6 of section 2 of this chapter we obtain

$$s(z(w)) \sim \sum_{h=o}^{\infty} d_h w^h$$

for the points w which lie in the sector $\alpha < \arg w < \beta$ and satisfy inequality (4.3).

Since the formal derivative of $\Sigma_o^\infty d_h w^h$ also possesses an asymptotic radius $\geqq R$, we obtain in the same manner as above

$$\frac{d}{dw} s(z(w)) \sim \sum_{h=1}^{\infty} hd_h w^{h-1}$$

for the points w which lie in the sector $\alpha < \arg w < \beta$ and satisfy inequality (4.3).

In the same way we obtain the similar result for the n-th derivative of s(z(w)) with respect to w, where n denotes a fixed integer $\geqq 0$. This completes the proof.

THEOREM 18. <u>If</u> $a_o + a_1 z + \cdots$ <u>is a power series in</u> z <u>with asymptotic radius</u> $\geqq R$, <u>then the substitution</u> $z = w^k$, <u>where</u> k <u>is a fixed positive integer, transforms the power series into the power series</u> $\Sigma_{h=o}^\infty a_h w^{kh}$ <u>in</u> w <u>with asymptotic radius</u> $\geqq R^{1/k}$.

<u>Proof</u>: Let $\Sigma_{h=o}^\infty a_h w^{kh} = \Sigma_{n=o}^\infty b_n w^n$, and therefore

$$b_n = \begin{cases} \frac{a_n}{k}, & \text{if } n \text{ is divisible by } k \\ 0 \text{ otherwise} & . \end{cases}$$

For each fixed integer $n \geqq 0$ there exists a fixed number C_n such that

$$|b_n| \leqq C_n R^{-n/k} .$$

That follows immediately from the definition of the numbers b_n, if n is not divisible by k. If n is divisible by k, we obtain by (3.1)

$$|b_n| = |a_{\frac{n}{k}}| \leq c_{\frac{n}{k}} R^{-\frac{n}{k}} = C_n R^{-\frac{n}{k}} ,$$

where we choose $C_n = c_{\frac{n}{k}}$. This establishes the proof.

THEOREM 19. If k is a fixed positive integer, if λ denotes a number $\neq 0$ which is independent of z (it may depend on w) and if the power series

$$\sum_{n=k}^{\infty} a_n (z - z_o)^n$$

has an asymptotic radius $\geq$ R, then

$$\sum_{n=k}^{\infty} \lambda a_n (z - z_o)^n$$

has an asymptotic radius $\geq$ R if $|\lambda| \leq 1$ and an asymptotic radius $\geq |\lambda|^{-1/k} R$ if $|\lambda| \geq 1$.

Proof: From (3.1) it follows that

$$|\lambda a_n| \leq |\lambda| c_n R^{-n}$$

and therefore

$$\leq c_n R^{-n} \qquad \text{if } |\lambda| \leq 1$$

and

$$\leq c_n (|\lambda|^{-1/k} R)^{-n} \qquad \text{if } |\lambda| \geq 1 \quad \text{and } n \geq k .$$ Q.E.D.

Let us give an example. The same example will be treated in Part IV, Chapter I, Section 2 as an application of the method of asymptotically enveloping series.

Put

$$A(z) = \log (1 + z) - z + \frac{1}{2} z^2$$

and

$$\Lambda(z,u) = (1 + u) A\left(\frac{z}{1+u}\right) + (1 - u) A\left(\frac{-z}{1-u}\right) - A(z) - A(-z)$$

where $-1 < u < 1$. Let m be positive. Then

$$(1 + u)(1 + u + z)^{-1} e^{m \Lambda(z,u)}$$

is generated by a power series

$$\sum_{k=0}^{\infty} (1 + u)\gamma_k (u) z^k$$

with an asymptotic radius $\geqq R$, uniformly in m and u, where

$$(4.4) \qquad R = \begin{cases} 1 - u^2 & \text{if} \quad mu(1 - u^2) \leqq 1 \\ m^{-1/3} |u|^{-1/3} (1 - u^2)^{2/3} & \text{if} \quad mu(1 - u^2) \geqq 1 \end{cases} .$$

The coefficients $\gamma_h(u)$ are determined by $\gamma_0(u) = (1 - u)T$, where $T = (1 - u^2)^{-1}$ and by the recurrence relations, valid for $k \geqq 0$,

$$(4.5)\quad \begin{aligned}(k+1)\gamma_{k+1} = &\Big\{(2k+1)u - 1\Big\}\, T\gamma_k + \Big\{kT + (k-1)\Big\}\gamma_{k-1}\\ &+ \Big\{-4mu\,T + 1 - (2k-3)u\Big\}\, T\gamma_{k-2}\\ &+ \Big\{2(1-T^2)m - (k-2)\,T\Big\}\gamma_{k-3}\\ &+ 4u(T-1)T\,m\gamma_{k-4} + 2(T-1)T\,m\gamma_{k-5}\ ;\end{aligned}$$

where $\gamma_{-1} = \gamma_{-2} = \gamma_{-3} = \gamma_{-4} = \gamma_{-5} = 0$.

Remark: From the recurrence relations it follows that $\gamma_k(u)$ can be written as a polynomial in T, Tu and m. The numbers γ_0, γ_1, and γ_2 are independent of m; γ_3, γ_4 and γ_5 are linear polynomials in m; γ_6, γ_7, γ_8 are polynomial in m at most of the second degree; and so on. In general, γ_k is a polynomial in m of a degree which is at most 1/3 k. From

$$T\,u^2 = \frac{u^2}{1-u^2} = \frac{1-(1-u^2)}{1-u^2} = T - 1$$

it follows that $\gamma_k(u)$ can be written as $\rho_k + u\sigma_k$, where ρ_k and σ_k are polynomials in T and m; consequently $\gamma_k(u) + \gamma_k(-u) = 2\rho_k$ is a polynomial in T and m, therefore a polynomial in $T - 1 = \frac{u^2}{1-u^2}$ and m; the degree in m is at most equal to 1/3 k.

Proof: The function A(z) possesses the expansion $\frac{z^3}{3} - \frac{z^4}{4} + \cdots$, so that $\Lambda(z,u)$ has an expansion of the form

$$\sum_{h=3}^{\infty} \frac{z^h}{h} \left\{ \frac{(-)^{h-1}}{(1+u)^{h-1}} - \frac{1}{(1-u)^{h-1}} - (-)^{h-1} + 1 \right\} .$$

The expression between the braces can be written as a fraction whose denominator is equal to $(1 - u^2)^{h-1}$ and whose numerator is a polynomial in u. We see that this polynomial vanishes for $u = 0$, so that $\Lambda(z,u)$ has an expansion of the form

$$(4.6) \qquad u(1 - u^2) \sum_{h=3}^{\infty} \frac{\lambda_h(u)}{(1-u^2)^h} z^h ,$$

where $\lambda_h(u)$ is a polynomial in u. From the fact that u lies between -1 and 1 it follows that $|\lambda_h(u)|$ is less than a suitably chosen number which depends only on h. The power series, occurring in (4.6), has therefore an asymptotic radius $\geqq 1 - u^2$, uniformly in u.

According to theorem 19, applied with $k = 3$ and $\lambda = mu(1 - u^2)$ the power series

$$\sum_{h=3} mu(1 - u^2) \frac{\lambda_h(u)}{(1-u^2)^h} z^h$$

possesses an asymptotic radius $\geqq R$, uniformly in m and u, where R is defined by (4.4); this power series generates the function $m\,\Lambda(z,u)$. According to theorem 17, applied with

$$\sum_{m=0}^{\infty} a_m z^m = e^z = \sum_{m=0}^{\infty} \frac{z^m}{m!}$$

the function $e^{m\,\Lambda(z,u)}$ is also generated by the corresponding power series with asymptotic radius $\geqq R$, uniformly in u.

The function $(1 + u)(1 + u + z)^{-1}$ is generated by the power series $1 - \frac{z}{1+u} + \cdots$ with asymptotic radius $\geqq 1 + u$, therefore with asymptotic radius $\geqq 2(1 + u) \geqq 1 - u^2 \geqq R$. Consequently the product

$$(1 + u)(1 + u + z)^{-1} \cdot e^{m\,\Lambda(z,u)}$$

is generated by the corresponding power series with asymptotic radius $\geqq R$, uniformly in m and u.

Finally we must deduce the recurrence relations between the coefficients $\gamma_k = \gamma_k(u)$. To that end we use the formal relation

$$e^{m\,\Lambda(z,u)} = (1 + u + z) \sum_{k=0}^{\infty} \gamma_k z^k \quad . \tag{4.7}$$

We have

$$A'(z) = \frac{1}{1+z} - 1 + z = \frac{z^2}{1+z} \quad ,$$

so that the derivative of $\Lambda(z,u)$ with respect to z is equal to

$$\frac{\partial \Lambda}{\partial z} = A'\left(\frac{z}{1+u}\right) - A'\left(\frac{-z}{1-u}\right) - A'(z) + A'(-z)$$

$$= \frac{z^2}{(1+u)(1+u+z)} - \frac{z^2}{(1-u)(1-u-z)} - \frac{z^2}{1+z} + \frac{z^2}{1-z}$$

$$= - \frac{(4u+2z)z^2}{(1-u^2)(1-(u+z)^2)} + \frac{2z^3}{1-z^2} = \frac{Z}{(1-u^2)(1-z^2)(1-(u+z)^2)} \quad ,$$

where

$$Z = - 4uz^2 + (-4u^2 + 2u^4)z^3 + 4u^3z^4 + 2u^2z^5 \quad .$$

Taking the formal derivative with respect to z of both sides of (4.7) we get the formal relations

$$m\, e^{m\,\Lambda(z,u)} \frac{\partial \Lambda}{\partial z} = (1 + u) \sum_{k=0}^{\infty} k \gamma_k z^{k-1} + \sum_{k=0}^{\infty} (k + 1) \gamma_k z^k \quad ,$$

hence

$$m\left(\sum_{k=0}^{\infty}\gamma_k z^k\right) Z = (1-u^2)\, Y \sum_{k=0}^{\infty} (k+1)\left((1+u)\gamma_{k+1}+\gamma_k\right) z^k$$

where

$$Y = \frac{(1-z^2)(1-(u+z)^2)}{1+u+z} =$$

$$= (1-z^2)(1-u-z) = (1-u) - z - (1-u)z^2 + z^3 \quad .$$

Comparing the coefficients of z^k on both sides we obtain for $k \geqq 0$

$$m\left\{-4u\gamma_{k-2} + (-4u^2+2u^4)\gamma_{k-3} + 4u^3\gamma_{k-4} + 2u^2\gamma_{k-5}\right\}$$

$$= (1-u^2)\Big\{(1-u)(k+1)\left((1+u)\gamma_{k+1}+\gamma_k\right) - k\left((1+u)\gamma_k + \gamma_{k-1}\right)$$

$$- (1-u)(k-1)\left((1+u)\gamma_{k-1}+\gamma_{k-2}\right) + (k-2)\left((1+u)\gamma_{k-2}+\gamma_{k-3}\right)\Big\}$$

In this way we find for $(k+1)(1-u^2)^2\gamma_{k+1}$ a linear combination of γ_k, γ_{k-1}, γ_{k-2}, γ_{k-3}, γ_{k-4} and γ_{k-5}. In this linear combination the coefficient of γ_k is

$$-(1-u^2)\left\{(k+1)(1-u) - k(1+u)\right\} = +(1-u^2)\left((2k+1)u-1\right) \quad ;$$

the coefficient of γ_{k-1} is

$$-(1-u^2)\left\{-k-(k-1)(1-u^2)\right\} = (1-u^2)\left(k+(k-1)(1-u^2)\right) \quad ;$$

the coefficient of γ_{k-2} is

$$- 4\,m\,u - (1 - u^2)\Big(- (1 - u)(k - 1) + (k - 2)(1 + u)\Big)$$

$$= - 4\,m\,u + (1 - u^2)\Big(1 - (2k - 3)\,u\Big) \quad ;$$

the coefficient of γ_{k-3} is

$$(-4u^2 + 2u^4)m - (1 - u^2)(k - 2) = 2\left((1 - u^2)^2 - 1\right) m - (k - 2)(1 - u^2) \quad ;$$

the coefficient of γ_{k-4} is $4u^3m = 4u\left\{1 - (1 - u^2)\right\} m$ and the coefficient of γ_{k-5} is $2u^2m = 2\left\{1 - (1 - u^2)\right\} m$. In this way we know the required recurrence relations between the coefficients γ_k.

Chapter III. ON LIMITS MODULO A GIVEN CLASS OF FUNCTIONS

Section 1. DEFINITION OF LIMITS MODULO A GIVEN CLASS OF FUNCTIONS

The theorems in the first chapter have the disadvantage that they involve a function which is supposed to possess in a given interval a derivative of certain order which is very small in that interval. This condition is satisfied only in special cases. For instance, even in a simple sum such as

$$\sum_{1\leq n\leq\frac{1}{2}\omega} \cot \frac{\pi n}{\omega} \quad ,$$

where ω denotes a large positive number, this condition is not satisfied, since the function $\cot \frac{\pi x}{\omega}$ and also its derivatives are large in the neighborhood of the origin. It is true that we can write the sum in question as

$$\sum_{1\leq n\leq\frac{1}{2}\omega} \left(\cot \frac{\pi n}{\omega} - \frac{\omega}{\pi n}\right) + \frac{\omega}{\pi} \sum_{1\leq n\leq\frac{1}{2}\omega} \frac{1}{n} \quad ,$$

in which the last sum has been calculated in Section 8 of the first chapter, whereas the first sum satisfies the required condition, since the h^{th} derivative of $\cot \frac{\pi x}{\omega} - \frac{\omega}{\pi x}$ is at most of the same order of magnitude as ω^{-h} *) in the interval $0 \leq x \leq \frac{1}{2}\omega$ and therefore very small for sufficiently large ω. It is also true that a similar device can be applied in many other cases. But, apart from the simplest cases, the calculations become so complicated, that this method is practically inapplicable. To avoid these complications I introduce a generalized concept of limit, namely that of the limit modulo a given class of functions.

Let T be a point set lying on the real axis or in the complex plane. Let a be a given limit point of that set (a may be infinite). Let M(a) be a set of functions g(t) which are defined at all points t of T in the neighborhood of a such that, if g(t) belongs to M(a), then M (a) contains all functions c g(t), where c denotes a constant, and if two functions belong to M(a), then M(a) also contains their sums. Such a set of functions M(a) is called a <u>modulus</u>. It follows from this definition that a modulus which contains g(t) also contains -g(t) and their sum which is identically equal to zero, so that each modulus M(a) contains the function which is identically equal to zero.

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*) Since the function $\cot \pi u - \frac{1}{\pi u}$ is analytic at the origin, the derivatives $\frac{d^n}{du^n}\left(\cot \pi u - \frac{1}{\pi u}\right)$ are bounded near the origin. But then if $u = \frac{x}{\omega}$

$$\left|\frac{d^h}{dx^h}\left(\cot \frac{\pi x}{\omega} - \frac{\omega}{\pi x}\right)\right| = \frac{1}{\omega^h}\left|\frac{d^h}{du^h}\left(\cot \pi u - \frac{1}{\pi u}\right)\right| \leq \frac{c_h}{\omega^h} ,$$

which proves our statement.

We call the limit point a of T an <u>ordinary point with respect to M(a)</u> under the following conditions: if M(a) contains a function g(t) such that g(t) tends to a finite limit λ, in the ordinary sense of "limit", as t in T tends to a in an arbitrary way, then this limit is equal to zero. If the real point a is an ordinary point with respect to a modulus M(a) and the point set T contains only real numbers $t > a$, then we call a + an ordinary point with respect to the modulus; in that case we denote the modulus by M(a+). If the real point a is an ordinary point with respect to a modulus M(a) and T contains only real numbers $t < a$, then we call a− an ordinary point with respect to the modulus; in that case we denote the modulus by M(a−).

Let us now consider the definition of the limit of a function f(t) as $t \to a$, with respect to a given modulus M(a). Suppose that a is an ordinary point with respect to that modulus and suppose that the modulus contains at least one function g(t) such that f(t) − g(t) tends to a finite limit λ, in the ordinary sense of "limit", as t in T tends to a in an arbitrary way. In that case we call λ the <u>limit of f(t), modulo M(a)</u> as $t \to a$, and we write

$$\lambda = \lim_{t \to a} f(t) \qquad (M(a))$$

or

$$f(t) \to \lambda \qquad (M(a)) \qquad \text{as } t \to a \ .$$

This limit, modulo M(a), if it exists, is uniquely defined; for if M(a) contains the functions g(t) and h(t) such that

$f(t) - g(t)$ tends to a finite limit λ as $t \rightarrow a$ and $f(t) - h(t)$ tends to a finite limit λ' as $t \rightarrow a$ (where "limit" is defined in the ordinary way), then the function $g(t) - h(t)$, belonging to $M(a)$, tends to the finite limit $\lambda' - \lambda$ as $t \rightarrow a$. This limit is equal to zero, since a is an ordinary point with respect to $M(a)$; consequently $\lambda = \lambda'$.

If $f(t)$ tends to a finite limit λ as $t \rightarrow a$ in the ordinary sense of "limit", it tends to λ with respect to any modulus, for which a is an ordinary point, since that modulus contains a function $g(t)$ which is identically equal to zero and $f(t) - g(t) \rightarrow \lambda$, as t in T tends to a. Conversely, if $f(t) \rightarrow \lambda$ with respect to any modulus for which a is an ordinary point then $f(t) \rightarrow \lambda$ with respect to the modulus whose functions are identically equal to zero and therefore $f(t)$ in the ordinary sense of "limit" tends to a finite limit λ as $t \rightarrow a$. We see that the previous definition of limit modulo a certain modulus is a generalization of the usual concept.

A modulus with at least one ordinary point does not contain a function which is identically equal to a non-zero content; for example, a modulus with at least one ordinary point can therefore not contain both log t and log 2t, for then it would also contain $\log 2t - \log t = \log 2$.

If $f(t)$ and $f^*(t)$ possess the limits λ and λ^* modulo $M(a)$, as t in T tends to a, then $c\, f(t) \rightarrow c\, \lambda$ and $f(t) + f^*(t) \rightarrow \lambda + \lambda^*$ modulo $M(a)$, for $M(a)$ contains two functions $g(t)$ and $g^*(t)$ such that $f(t) - g(t) \to \lambda$ and $f^*(t) - g^*(t) \to \lambda^*$ in the ordinary sense, so that $c\, f(t) - c\, g(t) \to c\lambda$ and $(f(t) + f^*(t)) - (g(t) + g^*(t)) \to \lambda + \lambda^*$, (all limits taken in the ordinary sense) where $c\, g(t)$ and $g(t) + g^*(t)$ belong to the modulus $M(a)$.

It is not certain, however, that the product $f(t)\, f^*(t)$ tends to a finite limit modulo $M(a)$. Furthermore, even if $f(t)\, f^*(t)$ tends to a finite limit modulo $M(a)$, this limit is not necessarily equal to the product $\lambda\lambda^*$. For instance, let $M(\infty)$ be the modulus formed by the functions $c\, t$, where c is an arbitrary constant. Then infinity is an ordinary point with respect to this modulus, for if ct tends to a finite limit λ, as t approaches infinity, then $c = 0$ and therefore $\lambda = 0$. Then we have as $t \to \infty$ (in the ordinary sense), the functions t, $\sin \frac{1}{t}$ and $\frac{1}{t}$ tend to 0 $M(\infty)$. However, $t \sin \frac{1}{t}$ has no finite limit, modulo $M(\infty)$, since it is impossible to find a constant c such that $t \sin \frac{1}{t} - ct$ tends to a finite limit, as $t \to \infty$. Furthermore the product $t \cdot \frac{1}{t} = 1$ has the limit 1 and not zero.

<u>Example</u>: If $M(\infty)$ consists of the functions $c \log t$, where c is an arbitrary constant, then

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x} = 0 \qquad (M(\infty)) \quad .$$

I write in this case

$$\int_1^\infty \frac{dx}{x} = 0 \quad , \qquad (M(\infty)) \quad ,$$

or, simply

$$\int_1^{\infty} \frac{dx}{x} = 0 \quad ,$$

recognizing, of course, that this is with respect to the modulus $M(\infty)$.

If $M^*(\infty)$ is the modulus formed by the functions $c \log 2t$, where c is an arbitrary constant, then we have modulo $M^*(\infty)$

$$\int_1^{\infty} \frac{dx}{x} = - \log 2 \qquad \text{and} \qquad \int_{\frac{1}{2}}^{\infty} \frac{dx}{x} = 0 \quad ,$$

since

$$\log t - \log 2t = - \log 2 \qquad \text{and} \qquad \log t - \log \tfrac{1}{2} - \log 2t = 0 \quad .$$

The limit with respect to a certain modulus may therefore depend on the modulus.

Suppose that

$$\lambda = \lim_{t \to \infty} f(t)$$

exists with respect to a certain modulus $M(\infty)$, for which infinity is an ordinary point. Then the limit of $f(t)$, as $t \to \infty$, exists with respect to any modulus $M^*(\infty)$, which contains all functions belonging to $M(\infty)$ and for which infinity is an ordinary point; this limit modulo $M^*(\infty)$ is also equal to λ. For $M(\infty)$ and therefore also

$M^*(\infty)$ contains at least one function g(t) such that f(t) - g(t) tends to λ, in the ordinary sense, as t in T approaches infinity. Of course, the existence of the limit of f(t), as $t \to \infty$, modulo $M^*(\infty)$ does not guarantee the existence of the limit modulo $M(\infty)$. Therefore, if we have constructed a certain modulus $M(\infty)$, for which infinity is an ordinary point, then it is useful to add to this modulus as many functions as possible, provided that infinity is also an ordinary point for the new modulus $M^*(\infty)$, which we construct in the described way.

Section 2. CONSTRUCTION OF CERTAIN MODULI

Let $L(\infty)$ be the modulus formed by the functions $g(t)$ which can be written as a linear combination of products

$$t_o^{\alpha_o} \; t_1^{\alpha_1} \ldots t_h^{\alpha_h} \,, \tag{2.1}$$

where h denotes an arbitrary integer $\geqq 0$, where the exponents are real such that at least one of them is different from zero and where finally

$$t_o = t \qquad \text{and} \qquad t_{k+1} = \log t_k \qquad (k \geqq 0) \ .$$

To show that infinity is an ordinary point with respect to this modulus, we consider two products

$$t_o^{\beta_o} \; t_1^{\beta_1} \ldots t_h^{\beta_h} \qquad \text{and} \qquad t_o^{\gamma_o} \; t_1^{\gamma_1} \ldots t_h^{\gamma_h} \ .$$

Suppose $\beta_o \geqq \gamma_o$; if $\beta_o = \gamma_o$, we suppose $\beta_1 \geqq \gamma_1$; if $\beta_o = \gamma_o$ and $\beta_1 = \gamma_1$, then we assume $\beta_2 \geqq \gamma_2$, and so on; finally we assume that if $\beta_o = \gamma_o$, ... , $\beta_{h-1} = \gamma_{h-1}$, then $\beta_h > \gamma_h$. Then the first product is of higher order of magnitude than the second product; that means that the first product divided by the second product tends to infinity as $t \rightarrow \infty$.

An arbitrary function $g(t)$ of $L(\infty)$ is a linear combination of a finite number of terms of the form (2.1). If at least one of the coefficients occurring in that linear combination is different from zero, then the result of the preceding paragraph implies that the combination contains a term u of highest order of magnitude. This term u tends to infinity or to zero as t approaches infinity. If $u \to 0$, then g(t) tends to zero as $t \to \infty$. If $u \to \infty$, then g(t) also tends to infinity as $t \to \infty$. Consequently, in this case, if g(t) tends to a finite limit, this limit is equal to zero. On the other hand, if each coefficient in the linear combination g(t) is equal to zero, then g(t) is identically equal to zero, which implies that the limit to which g(t) tends is also equal to zero. Consequently infinity is an ordinary point with respect to $L(\infty)$.

Consider a modulus $M(\infty)$ whose functions are defined on a set T, which contains arbitrarily large values of t, but such that each element t is positive; let infinity be an ordinary point of $M(\infty)$. This modulus generates a modulus $M(-\infty)$ (for which $-\infty$ is an ordinary point), a modulus M(a+) (for which a+ is an ordinary point) and a modulus M(a−), (for which a− is an ordinary point); here a denotes an arbitrary real finite number. We define $M(-\infty)$ as the modulus formed by the functions g(−t), defined for −t in T. We define M(a+) as the modulus formed by the functions $g(\frac{1}{t-a})$, defined for $\frac{1}{t-a}$ in T; finally we define M(a−) as the modulus formed by the functions $g(\frac{1}{a-t})$, defined for $\frac{1}{a-t}$ in T. Here g(t) denotes an arbitrary function belonging to $M(\infty)$. All the moduli, generated in this way by the original modulus $M(\infty)$ are said to be equivalent. For instance all the logarithmic moduli $L(\infty)$, $L(-\infty)$, L(a+) and L(a−), where a is an arbitrary real number, are equivalent.

Example: To prove that the integral

$$I = \int_1^{\infty} \sqrt{x+1}\ \log(x+2)\, dx$$

modulo $L(\infty)$ exists, we may write by expanding the integrand

$$\sqrt{x+1}\ \log(x+2) = x^{\frac{1}{2}} \log x + \frac{\log x}{2\ x^{\frac{1}{2}}} + \frac{2}{x^{\frac{1}{2}}} + r(x) \quad ,$$

where $r(x)$ is integrable in the ordinary way from 1 to ∞.

We have modulo $L(\infty)$

$$\int_1^{\infty} x^{\frac{1}{2}} \log x\, dx = \lim_{t\to\infty} \left\{ \frac{2}{3}\, t^{\frac{3}{2}} \log t - \frac{4}{9}\, t^{\frac{3}{2}} + \frac{4}{9} \right\} = \frac{4}{9} \quad ,$$

$$\int_1^{\infty} x^{-\frac{1}{2}} \log x\, dx = \lim_{t\to\infty} \left\{ 2\, t^{\frac{1}{2}} \log t - 4\, t^{\frac{1}{2}} + 4 \right\} = 4$$

and

$$\int_1^{\infty} x^{-\frac{1}{2}}\, dx = \lim_{t\to\infty} \left\{ 2\, t^{\frac{1}{2}} - 2 \right\} = -2 \quad ,$$

so that

$$I = -\frac{14}{9} + \int_1^{\infty} r(x)\, dx \quad .$$

We see that the introduction of the logarithmic moduli enables us to generalize considerably the notion of integrals. Let us consider for instance the integral

$$\int_{\alpha}^{\beta} (x + x^2)^{-\frac{4}{3}} \log |x| \, dx \quad ,$$

where $-1 < \alpha < \beta$. There is a difficulty if the closed interval (α , β) contains the origin. If $\beta > 0$, in the interval $0 < x \leqq \beta$, we can write

$$(x + x^2)^{-\frac{4}{3}} \log |x| = x^{-\frac{4}{3}} \log x + r(x) \quad ,$$

where $r(x)$ is integrable in the ordinary way from zero to β, so that we obtain modulo $L(0+)$

$$\int_0^{\beta} (x + x^2)^{-\frac{4}{3}} \log |x| \, dx = \int_0^{\beta} x^{-\frac{4}{3}} \log x \, dx + \int_0^{\beta} r(x) \, dx \quad ;$$

the first integral on the right hand side is modulo $L(0+)$ equal to

$$- 3 \, \beta^{-\frac{1}{3}} \log \beta - 9 \, \beta^{-\frac{1}{3}} \quad .$$

If $\alpha < 0$, then we have in the interval $\alpha \leqq x < 0$

$$(x + x^2)^{-\frac{4}{3}} \log |x| = x^{-\frac{4}{3}} \log (-x) + r(x) \quad ,$$

where $r(x)$ is integrable from α to zero. Then we obtain modulo $L(0-)$

$$\int_{\alpha}^{0} (x + x^2)^{-\frac{4}{3}} \log |x| \, dx = 3 \, \alpha^{-\frac{1}{3}} \log (-\alpha) + 9 \, \alpha^{-\frac{1}{3}} + \int_{\alpha}^{0} r(x) \, dx \quad .$$

In the case $-1 < \alpha < 0 < \beta$ we find thus

$$\int_{\alpha}^{\beta} (x + x^2)^{-\frac{4}{3}} \log |x| \, dx = -3\beta^{-\frac{1}{3}} \log \beta - 9 \beta^{-\frac{1}{3}} +$$

$$+ 3\alpha^{-\frac{1}{3}} \log(-\alpha) + 9\alpha^{-\frac{1}{3}} + \int_{\alpha}^{\beta} r(x) \, dx \;;$$

the integral on the left hand side is taken with respect to the moduli $L(0+)$ and $L(0-)$.

Similarly we can calculate the integral in question also for $\alpha \leq -1$, but then we must use the moduli $L(-1+)$ and $L(-1-)$.

The remainder of this section is devoted to the construction of an important new modulus, and to that end we first introduce the concept of "hyperpolynomials".

The functions $f(x) = x^{3/2}$ has the property that

$$f(x + h) = x^{3/2}\left(1 + \frac{h}{x}\right)^{3/2} = x^{3/2} + \frac{3}{2} x^{\frac{1}{2}} h + \epsilon(x,h) \quad ,$$

where $\epsilon(x,h)$ tends for zero, if h is fixed and x tends to infinity. There exists therefore a polynomial in h

$$p_x(h) = x^{3/2} + \frac{3}{2} x^{\frac{1}{2}} h \quad ,$$

of which the coefficients (but not the degree) depend on x such that $f(x + h) - p_x(h)$ tends to zero as $x \to \infty$, provided that h is fixed. We call a function with this property a hyperpolynomial; more precisely: a hyperpolynomial is a function $f(x)$ which is defined for each sufficiently large positive x with the property that it is possible to find a polynomial $p_x(h)$ in h, of which the coefficients but not the degree may depend on x, such that for each fixed h

$$f(x+h) - p_x(h)$$

tends to zero as x approaches infinity.

The following properties are immediately clear:

1. A polynomial is a hyperpolynomial.

2. A function which tends to zero as x approaches infinity, is a hyperpolynomial.

3. A hyperpolynomial, multiplied by a constant, is a hyperpolynomial. The sum of two hyperpolynomials is a hyperpolynomial.

4. If there exists an integer $k \geq 0$ such that the k-th derivative of $f(x)$ exists for sufficiently large x and

$$f^{(k)}(x) \to 0 \qquad \text{as } x \to \infty ,$$

then $f(x)$ is a hyperpolynomial.

The proof is easy. We have

$$f(x+h) = f(x) + \frac{h}{1} f'(x) + \cdots + \frac{h^{k-1}}{(k-1)!} f^{(k-1)}(x) + \frac{h^k}{k!} f^{(k)}(\xi) ,$$

where ξ lies between x and $x + h$. If h is fixed and x tends to infinity, then ξ also approaches infinity, so that the last term tends to zero. This shows that $f(x)$ is a hyperpolynomial.

As particular case we find that each function belonging to the logarithmic modulus $L(\infty)$ defined by (2.1) is a hyperpolynomial.

Remark: The product of two hyperpolynomials is not always a hyperpolynomial. For example, let $f(x) = \frac{1}{x}$ if x is rational and $f(x) = 0$ if x is irrational. Then x and $f(x)$ are hyperpolynomials.

If their product were a hyperpolynomial, then there would exist a polynomial $p_x(h)$ in h such that $(x + h)\ f(x + h) - p_x(h)$ would tend to zero as x approaches infinity. Suppose now that x is a rational number tending to infinity. If h is a fixed irrational number, $(x + h)\ f(x + h) = 0$, so that $p_x(h) \to 0$. Therefore, for each fixed irrational number h, each coefficient of $p_x(h)$ would tend to zero, as the rational number x approaches infinity. Then $p_x(h)$ would tend to zero, not only for each fixed irrational h but also for each fixed rational number h. This is impossible since for each rational number h $(x + h)\ f(x + h) = 1$, so that $1 - p_x(h)$ tends to 0 and therefore $p_x(h)$ to 1.

Consider a set K formed by periodic functions with the same period and satisfying the following conditions: if $p_1(x), \cdots, p_s(x)$ denote functions belonging to K and if it is possible to find constants $c_1, \cdots, c_s$ such that

$$c_1\ p_1(x) + \cdots + c_s\ p_s(x)$$

is equal to a constant for all x, then this constant is equal to zero.

Let $N(\infty)$ be the set formed by all functions of the form

$$\chi(x) + p_1(x)\ g_1(x) + \cdots + p_s(x)\ g_s(x) \quad ,$$

where $s \geq 0$, where $\chi(x)$ denotes a function belonging to $L(\infty)$, the logarithmic modulus at infinity, where $p_1(x), \cdots, p_s(x)$ are periodic functions belonging to K and where $g_1(x), \cdots\ g_s(x)$ are hyperpolynomials. We shall prove that $N(\infty)$ is a modulus for which infinity is an ordinary point.

Proof. It is clear that $N(\infty)$ is a modulus. To prove that ∞ is an ordinary point of this modulus we assume

(2.2) $$\chi(x) + p_1(x)\, g_1(x) + \cdots + p_s(x)\, g_s(x) \rightarrow \lambda \quad ,$$

as $x \rightarrow \infty$, where λ denotes a finite number. Under this assumption we must know that $\lambda = 0$.

Without loss of generality we may suppose that the periodic function $p_1(x), \cdots, p_s(x)$ are linearly independent, for otherwise it is possible to write at least one of those functions, say $p_s(x)$ as a linear combination of the $s - 1$ other functions,

$$p_s(s) = \sum_{\sigma=1}^{s-1} c_\sigma p_\sigma(x) \quad ,$$

whence $c_1, \cdots c_{s-1}$ are constants; in this case we have

$$\chi(x) + \sum_{\sigma=1}^{s-1} p_\sigma(x) \left[g_\sigma(x) + c_\sigma g_s(x) \right] \rightarrow \lambda \quad ,$$

where $g_\sigma(x) + c_\sigma g_s(x)$ ($\sigma = 1, \cdots, s - 1$) are hyperpolynomials, so that the above expression may be written in the form of (2.2) with s replaced by $s - 1$. Therefore, applying the principle of mathematical induction we may assume for each $s \geq 0$ that all the periodic functions of (2.2) are linearly independent.

All the functions belonging to the set K possess the same positive period a. From (2.2) it follows for each fixed integer m

(2.3) $\chi(x + ma) + p_1(x)\ g_1(x + ma) + \cdots + p_s(x)\ g_s(x + ma) \to \lambda$

as $x \to \infty$.

Since $\chi(x)$, $g_1(x)$, $\cdots$, $g_s(x)$ are hyperpolynomials, we can find polynomials $\chi_x(m)$, $g_{1x}(m)$, $\cdots$, $g_{sx}(m)$ in m whose coefficients (but not the degrees) depend on x, such that

(2.4) $$\chi(x + ma) - \chi_x(m) \to 0$$

and

(2.5) $$g_\sigma(x + ma) - g_{\sigma x}(m) \to 0 \qquad (\sigma = 1, \cdots , s) \quad ,$$

as $x \to \infty$. These limit relations hold for each real fixed m.

In this way we find for each fixed integer m

(2.6) $$\Psi_x(m) \to \lambda \qquad \text{as } x \to \infty$$

where

$$\Psi_x(m) = \chi_x(m) + p_1(x)\ g_{1x}(m) + \cdots + p_s(x)\ g_{sx}(m) \quad .$$

This function $\Psi_x(m)$ is a polynomial in m, of which the degree q is independent of x, so that we can write

$$\Psi_x(m) = a_o(x) + a_1(x)m + \cdots + a_q(x)\ m^q \quad .$$

In particular $\Psi_x(0) = a_o(x)$ tends to λ as $x \to \infty$, so that

$$y(m) = a_1(x)m + \cdots + a_q(x)m^q$$

tends to zero as $x \to \infty$ for each fixed positive integer m. It is possible to write each coefficient $a_h(x)$ $(1 \leq h \leq q)$ as a linear combination of $y(1)$, $y(2)$, $\cdots$, $y(q)$, in which the coefficients are independent of x. Since $y_h(m) \to 0$ as $x \to \infty$ we find therefore

$$a_h(x) \to 0 \quad \text{for} \quad h = 1, 2, \cdots, q \ .$$

This result implies that formula (2.6) holds for each fixed real number m. Combining this with (2.4) and (2.5) we see that formula (2.3) is true for each fixed real number m. Replacing x by x − ma and letting ma = −u, we obtain for each real fixed u

$$\chi(x) + p_1(x+u)\, g_1(x) + \cdots + p_s(x+u)\, g_s(x) \to \lambda \quad \text{as } x \to \infty \ .$$

It is sufficient to show that it is possible to let x tend to infinity in such a way that

$$(2.7) \qquad v(x) = |\chi(x) - \lambda| + |g_1(x)| + \cdots + |g_s(x)|$$

tends to zero. For in that case the function $\chi(x)$, belonging to the logarithmic modulus $L(\infty)$ tends to the finite limit λ so that this limit, as we have previously seen, is equal to zero. It is therefore sufficient to deduce a contradiction from the assumption that $v(x)$ possesses a positive lower bound of x tends to infinity in a certain way. With this assumption, we have for each real fixed u

$$(2.8) \qquad \frac{\chi(x) - \lambda}{v(x)} + p_1(x+u)\,\frac{g_1(x)}{v(x)} + \cdots + p_s(x+u)\,\frac{g_s(x)}{v(x)} \to 0$$

$$\text{as } x \to \infty \ .$$

From the definition of $v(x)$ it follows that each of the numbers

$$\frac{\chi(x)-\lambda}{v(x)} \quad \text{and} \quad \frac{g_\sigma(x)}{v(x)} \qquad (\sigma = 1, \cdots, s)$$

is in absolute value ≤ 1 and therefore bounded. Consequently, it is possible to let x tend to infinity in such a way that

$$(2.9) \quad \frac{\chi(x)-\lambda}{v(x)} \to c \quad \text{and} \quad \frac{g_\sigma(x)}{v(x)} \to c_\sigma \qquad (\sigma = 1, \cdots, s)$$

and that

$$x - a\left[\frac{x}{a}\right] \to \xi \ ,$$

where $c, c_1, \cdots, c_s$ and ξ denote suitably chosen constants, which are of course independent of u. Now it follows from (2.8) since $p_o(x + u) = p_o(x - a\left[\frac{x}{a}\right] + u)$ that

$$c + c_1 p_1 (\xi + u) + \cdots + c_s p_s (\xi + u) = 0$$

and from the definition of $v(x)$ and (2.9), that

$$|c| + |c_1| + \cdots + |c_s| =$$

$$(2.10) \quad = \lim_{x\to\infty} \frac{|\chi(x)-\lambda|}{v(x)} + \lim_{x\to\infty} \frac{|g_1(y)|}{v(x)} + \cdots + \lim_{x\to\infty} \frac{|g_s(y)|}{v(x)} = 1 \ .$$

Since this result holds for each real u we have for each real x

$$c_1 p_1 (x) + \cdots + c_s p_s (x) = - c \ .$$

According to the conditions imposed on the set K to which the periodic functions $p_1(x), \cdots, p_s(x)$ belong, this constant $-c$ is equal to zero. From (2.10) it would follow that at least one of the coefficients $c_1, c_2, \cdots, c_s$ is different from zero, contrary to the hypothesis that the periodic functions $p_1(x), \cdots, p_s(x)$ are linearly independent. This contradicts the assumption that $v(x)$ has a positive lower bound if x tends to infinity in a certain way. Consequently, it is possible to let x tend to infinity in such a way that $v(x) \to 0$ so that according to (2.7) $\chi(x) \to \lambda$ and therefore $\lambda = 0$.

This completes the proof.

I denote by $P(\infty)$ the modulus formed by the functions of the form

$$f(x) = \chi(x) + p_1(x)\ g_1(x) + \cdots + p_s(x)\ g_s(x) \quad ,$$

where $\chi(x)$ is a function belonging to the logarithmic modulus $L(\infty)$ at infinity, where $g_1(x), \cdots, g_s(x)$ are hyperpolynomials and where $p_1(x), \cdots, p_s(x)$ are functions with the period 1, which are integrable from 0 to 1 such that

$$\int_0^1 p_\sigma(x)\ dx = 0 \qquad (\sigma = 1 \cdots s) \quad .$$

Infinity is an ordinary point for this modulus $P(\infty)$, for if we choose the constants $c_1, \cdots, c_s, c$ such that for all x

$$c_1 p_1 (x) + \cdots + c_s p_s (x) = c \quad ,$$

then we find by integrating with respect to x from zero to 1, that c = 0, so that the modulus $P(\infty)$ is only a submodulus of the modulus $N(\infty)$, treated above.

Section 3. ON MODULI CONSTRUCTED BY INTEGRATION, SUMMATION AND PASSING TO THE LIMIT

In the preceding section we have constructed a modulus $P(\infty)$ of which $L(\infty)$ is a subset and for which infinity is an ordinary point. To show that there exist other extensions of the logarithmic modulus $L(\infty)$ with the same property we give first a simple example.

Example: The functions $v(t)$ of the form

$$(3.1) \quad v(t) = \chi(t) + c_1 \int_2^t t^{\frac{1}{2}}\sqrt{\log t}\, dt + c_2 \int_2^t t^{-\frac{1}{2}} \sqrt{\log t}\, dt ,$$

where $\chi(t)$ denotes an arbitrary function belonging to $L(\infty)$ and where c_1 and c_2 are arbitrary constants, form a modulus $I(\infty)$, for which infinity is an ordinary point.

To prove this we must show that, if a function $v(t)$ of the form (3.1) tends to a finite limit in the ordinary sense as $t \to \infty$, this limit is equal to zero. Integrating by parts we obtain

$$v(t) = \chi(t) + c_1 t^{3/2} \left\{ \sum_{h=o}^{m-1} a_h (\log t)^{1/2-h} + r_m (t) \right\}$$

$$+ c_2 t^{1/2} \left\{ \sum_{h=o}^{m-1} b_h (\log t)^{\frac{1}{2}-h} + r_m^{*} (t) \right\} + C_m ,$$

where a_h and b_h denote constants $\neq 0$, where C_m is a constant and where $r_m(t)$ and $r_m^{*}(t)$ are at most of the order $(\log t)^{1/2-m}$.

The function $\chi(t)$, which belongs to $L(\infty)$, can be written for sufficiently large t as a finite sum of terms of the form

$$(3.2) \quad c\, t^{\alpha_0} (\log t)^{\alpha_1} (\log \log t)^{\alpha_2} \cdots (\log \cdots \log t)^{\alpha_h} .$$

Since the number of these terms is finite we can choose m so large that their sum $\chi(t)$ contains neither a term of the form $\gamma_1 t^{3/2} (\log t)^{3/2-m+1}$ nor a term of the form $\gamma_2 t^{1/2} (\log t)^{3/2-m+1}$, where γ_1 and γ_2 are constants $\neq 0$. For this choice of m we obtain

$$v(t) = \chi_1(t) + c_1 t^{3/2} r_m(t) + c_2 t^{1/2} r_m^*(t) + C_m ,$$

where the function $\chi_1(t)$, belonging to $L(\infty)$, can be written as a sum of terms of the form (3.2), which certainly contains the terms

$$c_1 a_{m-1} t^{3/2} (\log t)^{3/2-m+1} \quad \text{and} \quad c_2 b_{m-1} t^{1/2} (\log t)^{3/2-m+1} .$$

Therefore $v(t)$ is for large values of t unbounded if $c_1 \neq 0$ and also if $c_1 = 0$, $c_2 \neq 0$. Since $v(t)$ tends to a finite limit, we have therefore $c_1 = c_2 = 0$, so that $v(t) = \chi(t)$ belongs to the modulus $L(\infty)$. As infinity is an ordinary point for this modulus, the limit to which $v(t)$ tends as $t \to \infty$, is equal to zero. This shows that infinity is an ordinary point for the modulus $I(\infty)$.

Remark: Replacing t by $\frac{1}{t}$ we obtain that the functions $v(t)$ of the form

$$v(t) = \chi(t) + c_1 \int_t^{\frac{1}{2}} t^{-3/2} \sqrt{\log \frac{1}{t}}\, dt + c_2 \int_t^{\frac{1}{2}} t^{-\frac{1}{2}} \sqrt{\log \frac{1}{t}}\, dt \quad ,$$

where $\chi(t)$ denotes an arbitrary function belonging to $L(0+)$ and where c_1 and c_2 are arbitrary constants, form a modulus $I(0+)$, for which $0+$ is an ordinary point.

We apply the modulus $I(0+)$ to the following integration. If $f(x)$ is continuous in the interval $0 \leqq x \leqq 1$ and twice differentiable at the origin, then the integral

$$\int_0^1 x^{-5/2} \sqrt{\log \frac{1}{x}}\, f(x)\, dx$$

exists with respect to this modulus $I(0+)$.

For we can write

$$f(x) = f(0) + x\, f'(0) + x^2\, r(x)$$

where $r(x)$ is bounded and we have modulo $I(0+)$, as $\delta \to 0$,

$$\int_\delta^1 x^{-5/2} \sqrt{\log \frac{1}{x}}\, f(x)\, dx = f(0) \int_\delta^1 x^{-5/2} \sqrt{\log \frac{1}{x}}\, dx$$

$$+ f'(0) \int_\delta^1 x^{-3/2} \sqrt{\log \frac{1}{x}}\, dx + \int_\delta^1 x^{-1/2} \sqrt{\log \frac{1}{x}}\, r(x)\, dx$$

$$\to f(0) \int_{\frac{1}{2}}^1 x^{-5/2} \sqrt{\log \frac{1}{x}}\, dx + f'(0) \int_{\frac{1}{2}}^1 x^{-3/2} \sqrt{\log \frac{1}{x}}\, dx$$

$$+ \int_0^1 x^{-\frac{1}{2}} \sqrt{\log \frac{1}{x}}\, r(x)\, dx \quad .$$

$$v(t) = \chi(t) + c_1 \int_t^{\frac{1}{2}} t^{-3/2} \sqrt{\log \frac{1}{t}}\, dt + c_2 \int_t^{\frac{1}{2}} t^{-\frac{1}{2}} \sqrt{\log \frac{1}{t}}\, dt \quad ,$$

where $\chi(t)$ denotes an arbitrary function belonging to $L(0+)$ and where c_1 and c_2 are arbitrary constants, form a modulus $I(0+)$, for which $0+$ is an ordinary point.

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$$+ f'(0) \int_\delta^1 x^{-3/2} \sqrt{\log \frac{1}{x}}\, dx + \int_\delta^1 x^{-1/2} \sqrt{\log \frac{1}{x}}\, r(x)\, dx$$

$$\to f(0) \int_{\frac{1}{2}}^1 x^{-5/2} \sqrt{\log \frac{1}{x}}\, dx + f'(0) \int_{\frac{1}{2}}^1 x^{-3/2} \sqrt{\log \frac{1}{x}}\, dx$$

$$+ \int_0^1 x^{-\frac{1}{2}} \sqrt{\log \frac{1}{x}}\, r(x)\, dx \quad .$$

We can generalize considerably the obtained result by introducing a set $U(\infty)$ formed by one or more (possibly infinitely) many functions of the form

$$(3.3)\quad t^{\alpha_0} (\log t)^{\alpha_1} (\log\log t)^{\alpha_2} \cdots (\log \cdots \log t)^{\alpha_k} ,$$

that is, of the form

$$t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_k^{\alpha_k} ,$$

where $t_0 = t$ and $t_{h+1} = \log t_h$ and where k is the same integer $\geqq 0$ for all functions belonging to $U(\infty)$. We assume that in each function belonging to $U(\infty)$ the exponents $\alpha_0, \alpha_1, \cdots, \alpha_k$ satisfy the following inequalities:

$$(3.4)\quad \begin{cases} \alpha_0 \geqq -1 ; \text{ if } \alpha_0 = -1, \text{ then } \alpha_1 \geqq -1 ; \\ \text{if } \alpha_0 = \alpha_1 = -1 , \text{ then } \alpha_2 \geqq -1 ; \text{ and so on } ; \\ \text{finally, if } \alpha_0 = \alpha_1 = \cdots = \alpha_{k-1} = -1 , \text{ then } \alpha_k \geqq -1 . \end{cases}$$

Let p be a positive number such that t_k exists and is positive for $t \geqq p$. Let $I(\infty)$ be the set formed by all functions of the form

$$(3.5)\qquad v(t) = \chi(t) + \sum_{\sigma=1}^{s} c_\sigma \int_p^t u_\sigma(x)\, dx ,$$

where s denotes an arbitrary integer $\geqq 0$, where $u_\sigma(t)(\sigma = 1, \cdots, s)$ denotes an arbitrary function belonging to $U(\infty)$ and where

c_σ ($\sigma = 1, \cdots, s$) are arbitrary constants; we assume that $\gamma(t)$ is an arbitrary linear combination of terms of the form

$$(3.6) \quad t^{\beta_0}(\log t)^{\beta_1}(\log\log t)^{\beta_2} \cdots (\log \cdots \log t)^{\beta_r} ,$$

which tend to infinity as $t \to \infty$.

Infinity is not necessarily an ordinary point for this modulus $I(\infty)$, for if $U(\infty)$ consists only of the function $t^{3/2}$ and if $p > 0$, then the function

$$-\frac{2}{5} t^{5/2} + \int_p^t x^{3/2}\, dx = -\frac{2}{5} p^{5/2} ,$$

which is a constant $\neq 0$, can not belong to a modulus for which infinity is an ordinary point. Therefore we introduce the additional condition that $I(\infty)$ does not contain a function which is equal to a constant $\neq 0$. In this way we obtain

THEOREM 1. _Suppose that each function belonging to $I(\infty)$ which is identically equal to a constant is equal to zero. Then infinity is an ordinary point for the modulus $I(\infty)$._

Proof. We must show: if a function $\gamma(t)$ of the form (3.5) tends to a finite limit λ, in the usual sense, as $t \to \infty$, then $\lambda = 0$. We know that $\gamma(t)$ is a linear combination of terms of the form (3.6), which tend to infinity, as $t \to \infty$.

Consequently $\frac{d\gamma}{dt}$ is a linear combination of terms of the form

$$(3.7) \qquad t_0^{\gamma_0}\, t_1^{\gamma_1} \cdots t_{r+1}^{\gamma_{r+1}} ,$$

where

$$\gamma_0 \geqq -1 \;;\quad \text{if } \gamma_0 = -1 \;,\quad \text{then } \gamma_1 \geqq -1 \;;$$

$$\text{if } \gamma_0 = \gamma_1 = -1 \;,\quad \text{then } \gamma_2 \geqq -1 \;;\; \ldots \;;$$

$$\text{finally, if } \gamma_0 = \gamma_1 = \cdots = \gamma_r = -1 \;,\quad \text{then } \gamma_{r+1} \geqq -1 \;.$$

In this way we find that

$$v'(t) = \chi'(t) + \sum_{\sigma=1}^{s} c_\sigma \, u_\sigma (t)$$

can be written as a linear combination of terms of the form

$$t_0^{\gamma_0} \, t_1^{\gamma_1} \cdots t_\tau^{\gamma_\tau} \;,$$

where the integer τ is chosen $\geqq k$ and $\geqq r + 1$; the exponents $\gamma_0, \gamma_1, \ldots, \gamma_\tau$ satisfy the inequalities

$$(3.8) \quad \begin{cases} \gamma_0 \geqq -1 \;;\; \text{if } \gamma_0 = -1 \;,\; \text{then } \gamma_1 \geqq -1 \;; \\ \text{if } \gamma_0 = \gamma_1 = -1 \;,\; \text{then } \gamma_2 \geqq -1 \;;\; \ldots \;; \\ \text{finally, if } \gamma_0 = \gamma_1 = \cdots = \gamma_{\tau-1} = -1, \; \text{then } \gamma_\tau \geqq -1 \;. \end{cases}$$

Choosing a number $q \geqq p$ such that t_τ is defined and positive for $t \geqq q$, we obtain for $t \geqq q$

$$\int_q^t v'(t)\, d\tau = \chi(t) - \chi(q) + \sum_{\sigma=1}^{s} c_\sigma \int_q^t u_\sigma(\tau)\, d\tau$$

$$= \chi(t) + \sum_{\sigma=1}^{s} \int_p^t u_\sigma(\tau)\, d\tau - \mu = v(\tau) - \mu \quad ,$$

where

$$\mu = \chi(q) + \sum_{\sigma=1}^{s} c_\sigma \int_p^q u_\sigma(\tau)\, d\tau \quad .$$

We have supposed that $v(t)$ tends to a finite limit λ, in the usual sense, as $t \to \infty$. Therefore

$$(3.9) \qquad \int_q^t v'(\tau)\, d\tau \to \lambda - \mu \quad \text{as } t \to \infty \quad .$$

This implies that $v'(\tau)$ is identically equal to zero. For otherwise the linear combination which represents $v'(t)$ contains a term of highest order of the form

$$c\, t_0^{\gamma_0}\, t_1^{\gamma_1} \cdots t_\tau^{\gamma_\tau} \quad ,$$

where the constant coefficient c is different from zero and where the exponents satisfy the inequalities (3.8). Then

$$v'(t) = c\, t_0^{\gamma_0}\, t_1^{\gamma_1} \cdots t_\tau^{\gamma_\tau} + \rho(t) \quad ,$$

where $\rho(t)$ is of smaller order of magnitude than (3.7). In that case the integral

$$\int_q^t x_o^{\gamma_o} x_1^{\gamma_1} \cdots x_\tau^{\gamma_\tau} \, dx \quad ,$$

where $x_o = x$ and $x_{h+1} = \log x_h$ would increase indefinitely according to the inequalities (3.8), whereas

$$\int_q^t \rho(x) \, dx$$

would be of smaller order of magnitude, so that

$$\int_q^t v'(x) \, dx$$

would be an unbounded function, contrary to (3.9).

Consequently $v'(t)$ is identically equal to zero, so that $v(t)$ is a constant. Since this function belongs to $I(\infty)$, it is by hypothesis equal to zero, so that also its limit λ is equal to zero.

This completes the proof.

This set $u(\infty)$ of terms of the form (3.3) yields, not only the modulus $I(\infty)$ containing integrals, but also a certain modulus $S(\infty)$ which involves sums. To that end we choose a positive integer p such that t_k exists and is positive for each integer $t \geqq p$. Let the modulus $S(\infty)$ be formed by the functions $v(t)$, defined for all integers $t \geqq p$, of the form

$$v(t) = \chi(t) + \sum_{\sigma=1}^{s} c_\sigma \sum_{n=p}^{t} u_\sigma(n) \quad ,$$

where $\chi(t)$ is an arbitrary linear combination of terms of the form (3.6) which tend to infinity as $t \to \infty$, where s denotes an arbitrary integer ≥ 0, where $u_\sigma(t)$ ($\sigma = 1,2, \cdots, s$) denote arbitrary functions belonging to $U(\infty)$ and where c_σ ($\sigma = 1, \cdots, s$) are arbitrary constants. In the same way as above we prove

THEOREM 2. *Suppose that each function belonging to* $S(\infty)$ *which is identically equal to a constant is equal to zero. Then infinity is an ordinary point for the modulus* $S(\infty)$.

For instance the functions

$$v(t) = \chi(t) + c_1 \sum_{n=p}^{t} n^{\frac{1}{2}} \sqrt{\log n} + c_2 \sum_{n=p}^{t} n^{-\frac{1}{2}} \sqrt{\log n} \quad ,$$

where $\chi(t)$ denotes an arbitrary function belonging to $L(\infty)$ and where c_1 and c_2 are arbitrary constants, form a modulus $S(\infty)$ for which infinity is an ordinary point. The series

$$\sum_{n=2}^{\infty} n^{\frac{1}{2}} \sqrt{\log n}\, f(n)$$

converges with respect to this modulus if $f\left(\frac{1}{x}\right)$ is twice differentiable at the points $x \geq 0$ in the neighborhood of the origin.

To construct other moduli for which infinity is an ordinary point, we consider a set T of which infinity is a limit point. Furthermore we introduce a set $V(\infty)$ formed by one or more (possibly infinitely many) functions $v(t)$ which are defined for each element t of the given set T and which possess the two following properties:

1. The set $V(\infty)$ does not contain a function which tends to a finite limit $\neq 0$.

2. For any two different functions belonging to $V(\infty)$ the absolute value of the quotient tends either to infinity or to zero as t in T approaches infinity.

THEOREM 3. Consider each function n(t) which can be written as a linear combination of a finite number of functions belonging to $V(\infty)$. These functions form a modulus $N(\infty)$ for which infinity is an ordinary point.

Proof. If n(t) is not identically equal to zero, we can write n(t) in the form

$$n(t) = \sum_{\sigma=1}^{s} c_\sigma \; v_\sigma(t) \quad ,$$

where

$$c_1 \neq 0 \quad \text{and} \quad \frac{v_\sigma(t)}{v_1(t)} \rightarrow 0 \quad (2 \leq s \leq s) \quad ,$$

as t in T tends to infinity. Therefore

$$n(t) = c_1 \, v_1(t) \left\{ 1 + \sum_{\sigma=2}^{s} c_\sigma \; \frac{v_\sigma(t)}{v_1(t)} \right\} \quad ,$$

where the expression between braces tends to 1 as t in T approaches infinity. If n(t) tends to a finite limit λ, as t in T approaches infinity, then also $c_1 \, v_1(t) \rightarrow \lambda$. In that case $v_1(t)$ tends to a finite limit $\frac{\lambda}{c_1}$ and this limit is equal to zero according to the condition 1 imposed on the set $V(\infty)$. Consequently $\lambda = 0$, so that infinity is an ordinary point with respect to the modulus $N(\infty)$.

We obtain a much more general result by introducing the set $W(t)$ formed by the functions $w(t)$ which can be written for each element t of T as the sum of a convergent series

$$w(t) = \sum_{h=0}^{\infty} c_h v_h (t) \quad ,$$

where the functions $v_h(t)$ $(h = 0,1, \cdots)$ belong to $V(\infty)$; we assume that for each fixed integer $h \geqq 0$

$$\frac{v_{h+1}(t)}{v_h(t)} \to 0 \qquad \text{as } t \to \infty$$

and that

$$w(t) - c_0 v_0 (t) - c_1 v_1 (t) - \cdots - c_{h-1} v_{h-1} (t)$$

is for large t at most of the same order of magnitude as $v_h(t)$.

THEOREM 4. Let $M(\infty)$ be the modulus formed by the functions

$$m(t) = n(t) + \sum_{\sigma=1}^{s} w_\sigma (t) \quad , \tag{3.10}$$

where $n(t)$ is an arbitrary function belonging to the modulus $N(\infty)$ defined in the preceding theorem, where s is an arbitrary integer $\geqq 0$ and where $w_1(t), w_2(t), \cdots, w_s(t)$ denote arbitrary functions belonging to $W(\infty)$.

Then infinity is an ordinary point with respect to the modulus $M(\infty)$.

Proof. We must show that, if a function $m(t)$ of the form (3.10) tends to a finite limit as t in T tends to infinity, then this limit is equal to zero. The special case $s = 0$ is treated in the preceding theorem, since in that case $m(t) = n(t)$ belongs to $N(\infty)$. We may therefore assume that $s \geqq 1$ and that we have already obtained the required result in the cases in which s is replaced by a smaller integer $\geqq 0$. We shall deduce a contradiction from the assumption that $m(t)$ tends to a finite limit $\neq 0$, as t in T approaches infinity.

We know that the function $n(t)$ belonging to $N(\infty)$ can be written as a linear combination

(3.11) $$n(t) = \sum_{\lambda=1}^{\gamma} c_\lambda \; v_\lambda \; (t)$$

of different functions belonging to $V(\infty)$. We know also that $w_\sigma(t)$ $(\sigma = 1, \ldots, s)$ is the sum of a convergent series

(3.12) $$w_\sigma(t) = \sum_{h=0}^{\infty} c_{\sigma h} \, v_{\sigma h} \, (t) \quad ,$$

where the functions $v_{\sigma h}(t)$ belong to $V(\infty)$, such that for each fixed integer $h \geqq 0$

(3.13) $$\frac{v_{\sigma, h+1}(t)}{v_{\sigma, h} \; (t)} \to 0 \quad \text{as } t \to \infty$$

and that

$$(3.14) \quad w_{\sigma}(t) - c_{\sigma 0}\, v_{\sigma 0}(t) - c_{01}\, v_{\sigma 1}(t) \cdots - c_{0,h-1}\, v_{\sigma,h-1}(t)$$

is at most of the same order of magnitude as $v_{\sigma h}(t)$. Let $z_1(t)$ be the function of highest order occurring among the functions denoted by

$$v_\lambda(t) \ (1 \leq \lambda \leq \ell) \quad ; \qquad v_{\sigma 0}(t) \ (1 \leq \sigma \leq s) \quad .$$

The right hand side of (3.10) contains at least one term which is equal to the function $z_1(t)$ multiplied by a constant. If $\gamma\, z_1(t)$ denotes the total contribution of all these terms to the right hand side of (3.10), then

$$(3.15) \qquad m(t) = \gamma\, z_1(t) + n_1(t) + \sum_{\sigma=1}^{s} w_{\sigma 1}(t) \quad ;$$

here $n_1(t)$ is the linear combination occurring in (3.11) apart from the possible term which involves $z_1(t)$; in the same way $w_{\sigma 1}(t)$ is the sum of the convergent series occurring in (3.12) apart from the possible term which involves $z_1(t)$.

Because we have cancelled the term of highest order, we know that

$$\frac{n_1(t)}{z_1(t)} \to 0 \quad \text{and} \quad \frac{w_{\sigma 1}(t)}{z_1(t)} \to 0 \quad ,$$

as t in T approaches infinity. The constant γ occurring in (3.15) is equal to zero, for otherwise we would have

$$m(t) = \gamma z_1(t) \left\{ 1 + \frac{n_1(t)}{\gamma z_1(t)} + \sum_{\sigma=1}^{s} \frac{w_{\sigma 1}(t)}{\gamma z_1(t)} \right\},$$

where the expression between braces tends to 1; from the fact that $m(t)$ tends to a finite limit $\neq 0$, it would follow that the function $z_1(t)$ belonging to $V(\infty)$ would also tend to a finite limit $\neq 0$, as t in T approaches infinity, contrary to the hypothesis that $V(\infty)$ does not contain a function which tends to a finite limit $\neq 0$. Consequently $\gamma = 0$, so that

$$m(t) = n_1(t) + \sum_{\sigma=1}^{s} w_{\sigma 1}(t) .$$

In other words: formula (3.10) remains true if we cancel on the right hand side all terms which contain $z_1(t)$ as factor.

We can repeat this argument by introducing the function $z_2(t)$ which is the function $\neq z_1(t)$ of highest order occurring among the functions denoted by

$$v_\lambda(t) \ (1 \leq \lambda \leq \chi) \quad ; \quad v_{\sigma 0}(t) \quad ; \quad v_{\sigma 1}(t) \ (1 \leq \sigma \leq s) .$$

Precisely as above we obtain

$$m(t) = n_2(t) + \sum_{\sigma=1}^{s} w_{\sigma 2}(t) ;$$

$n_2(t)$ is the linear combination occurring in (3.11) apart from the possible terms which involve $z_1(t)$ or $z_2(t)$; furthermore $w_{\sigma 2}(t)$ is the sum of the convergent series occurring in (3.12) apart from the possible terms which involve $z_1(t)$ or $z_2(t)$. Thus we have cancelled the terms which involve $z_1(t)$ or $z_2(t)$.

Continuing in this way we define $z_k(t)$ for each positive integer k as the function $\neq z_h(t)$ $(h = 1,2, \cdots, k-1)$ of highest order occurring among the functions denoted by

$$v_\lambda(t) \ (1 \leqq \lambda \leqq \chi) \ ; \qquad v_{\sigma h}(t) \ (1 \leqq \sigma \leqq s \ ; \ 0 \leqq h < k) \ .$$

Then

$$m(t) = n_k(t) + \sum_{\sigma=1}^{s} w_{\sigma k}(t) \ ; \tag{3.16}$$

$n_k(t)$ is the linear combination occurring in (3.11) apart from the possible terms which involve one of the functions $z_h(t)(h = 1,2, \cdots, k)$, whereas $w_{\sigma k}(t)$ is the sum of the convergent series occurring in (3.12) apart from the possible terms which involve one of the functions $z_h(t)$ $(h = 1,2, \cdots k)$.

Let us now examine the behavior as $k \to \infty$ of the terms on the right hand side of formula (3.16). To that end we write

$$\sum_{\lambda=1}^{\chi} c_\lambda v_\lambda(t) = \sum_{\lambda=1}^{\chi}{}' c_\lambda v_\lambda(t) + \sum_{\lambda=1}^{\chi}{}'' c_\lambda v_\lambda(t) \ ,$$

where the last sum contains the terms such that $v_\lambda(t)$ is of the same or higher order than at least one of the functions $z_k(t)$ $(k \geqq 1)$; the sum Σ' contains the functions $v_\lambda(t)$ which are of smaller order of magnitude than each function $z_k(t)$ $(k \geqq 1)$. Then we have for sufficiently large k

$$n_k(t) = \sum_{\lambda=1}^{\chi}{}' c_\lambda v_\lambda(t) \ ,$$

since all of the terms of Σ'' (and only these terms) have been cancelled.

If each given term in the series $\Sigma\ c_{\sigma h}\ v_{\sigma h}$ has the property that $v_{\sigma h}$ is of the same or higher order than at least one of the functions $z_k(t)$ $(k \geqq 1)$, then

$$(3.17) \qquad \lim_{k \to \infty} w_{\sigma k}(t) = 0 \quad ,$$

since all the terms are cancelled. Otherwise there exists an integer $q_\sigma \geqq 0$ such that $v_{\sigma h}(t)$ is for $h = 0, 1, \cdots, q_\sigma - 1$ of the same or higher order than at least one of the functions $z_k(t)$ $(k \geqq 1)$, whereas $v_{\sigma h}(t)$ is for $h \geqq q_\sigma$ of smaller order of magnitude than each function $z_k(t)$ $(k \geqq 1)$. In that case we find

$$(3.18) \qquad \lim_{k \to \infty} w_{\sigma k}(t) = \sum_{h=q_\sigma}^{\infty} c_\sigma\ v_{\sigma h}(t) \quad ,$$

since the terms (and only the terms) with $k < q_\sigma$ have been cancelled.

Let Σ^* denote the sum extended over the positive integers $\sigma \leqq s$ for which formula (3.18) holds; the other positive integers $\sigma \leqq s$ satisfy formula (3.17). If

$$(3.19) \qquad k > \chi + \sum_{\sigma=1}^{s}{}^{*}\ q_\sigma \quad ,$$

the number of cancelled terms is greater than the right hand side of (3.19), so that it is impossible that the sum Σ^* is extended over all positive integers $\sigma \leqq s$. This sum contains therefore at most $s - 1$ terms.

Taking in (3.16) the limit as $k \to \infty$ we obtain

$$m(t) = \sum_{\lambda=1}^{\gamma}{}' c_\lambda v_\lambda(t) + \sum_{\sigma=1}^{s}{}^{*} w_\sigma^*(t) \quad ,$$

where

$$w_\sigma^*(t) = \sum_{h=q_\sigma}^{\infty} c_\sigma v_{\sigma h}(t) \quad .$$

Thus we have written $m(t)$ in a form similar to the original form (3.10), but in such a way that s is replaced by the number of terms of the sum Σ^* and therefore by an integer $\geqq 0$ which is less than s. According to our induction hypothesis it is impossible that this function $m(t)$ tends to a finite limit $\neq 0$. This gives the required contradiction.

We will proceed to give three applications of the last theorem.

THEOREM 5. *Let* $V(\infty)$ *be the set formed by the functions*

$$v(t) = t_0^{\alpha_0} t_1^{\alpha_1} \dots t_k^{\alpha_k} \quad , \tag{3.20}$$

where k *is a given integer* $\geqq 0$ *and where* $t_0 = t$ *and* $t_{h+1} = \log t_h$.

Furthermore we assume

$$\text{(3.21)} \quad \begin{cases} \alpha_0 \geqq 0; \textit{ if } \alpha_0 = 0, \textit{ then } \alpha_1 \geqq 0; \textit{ if } \alpha_0 = \alpha_1 = 0, \textit{ then } \alpha_2 \geqq 0; \\ \dots, \textit{ finally, if } \alpha_0 = \alpha_1 = \dots = \alpha_{k-1} = 0, \textit{ then } \alpha_k > 0 \quad . \end{cases}$$

Choose the positive number p *so large that* t_k *is positive for* $t \geqq p$. *Let* $W(\infty)$ *be the set formed by the functions* $w(t)$ *which can be written for* $t \geqq p$ *as a sum of a convergent series*

$$w(t) = \sum_{h=0}^{\infty} c_h \, v_h(t) \quad ,$$

where for each integer $h \geqq 0$ *the coefficient* c_h *is a constant, and where the functions* $v_h(t)$ $(h = 0, 1, \cdots)$ *belong to* $V(\infty)$. *We assume that for each fixed integer* $h \geqq 0$

$$\frac{v_{h+1}(t)}{v_h(t)} \to 0 \qquad \textit{as } t \to \infty$$

and that

$$w(t) - c_0 \, v_0(t) - c_1 \, v_1(t) - \cdots - c_{h-1} \, v_{h-1}(t)$$

is for large t *at most of the same order of magnitude as* $v_h(t)$.

Then the functions

$$(3.22) \qquad m(t) = \chi(t) + \sum_{\sigma=1}^{s} w_{\sigma}(t) \qquad (t \geqq p) \quad ,$$

where $\chi(t)$ *is an arbitrary function belonging to the logarithmic modulus* $L(\infty)$, *where* s *is an arbitrary integer* $\geqq 0$ *and where* $w_1(t)$, $w_2(t)$, $\cdots$, $w_s(t)$ *denote arbitrary functions belonging to* $W(\infty)$, *form a modulus for which infinity is an ordinary point.*

Proof. We must show that, if a function of the form (3.22) tends to a finite limit λ as $t \to \infty$, this limit is equal to zero.

We can write $\chi(t) = n(t) + r(t)$, where $n(t)$ is a linear combination of a finite number of terms each belonging to $V(\infty)$ and where $r(t) \to 0$ as $t \to \infty$. We know therefore that

$$n(t) + \sum_{\sigma=1}^{s} w_\sigma(t) \longrightarrow \lambda \text{ as } t \longrightarrow \infty \quad .$$

The absolute value of the quotient of two different functions belonging to $V(\infty)$ tends to infinity or zero as $t \to \infty$. Also, since each function in $V(\infty)$ tends to infinity as $t \to \infty$, we may now apply the Theorem 4, which gives $\lambda = 0$. Therefore the functions of the form $m(t)$ form a modulus for which infinity is an ordinary point.

<u>Example:</u> Suppose α and b are real and α is not an integer $\geqq 0$. Then the functions

$$m(t) = \chi(t) + c_1 t^{\frac{1}{2}}(\log\log t) \sin\frac{1}{\sqrt{\log t}} + c_2(t+b)^\alpha \quad ,$$

where $\chi(t)$ is an arbitrary function of $L(\infty)$ and where c_1 and c_2 are arbitrary constants, form a modulus, for which infinity is an ordinary point. For we have

$$t^{\frac{1}{2}}(\log\log t) \sin\frac{1}{\sqrt{\log t}} = \sum_{h=o}^{\infty} (-)^h \frac{t^{\frac{1}{2}} \log\log t}{(2h+1)!(\log t)^{h+1/2}}$$

and

$$(t+b)^\alpha = \sum_{o \leqq h < \alpha} \binom{\alpha}{h} t^{\alpha-h} + r(t) \quad ,$$

where $r(t) \to 0$ as $t \to \infty$.

The function

$$\chi_1(t) = \chi(t)+C_2 \sum_{0 \leq h < \alpha} \binom{\alpha}{h} t^{\alpha - h}$$

belongs to $L(\infty)$ and we have

$$m(t) = \chi_1(t) + w_1(t) + r(t) \quad , \tag{3.23}$$

where

$$w_1(t) = c_1 \, t^{\frac{1}{2}}(\log \log t) \, \sin \frac{1}{\sqrt{\log t}}$$

$$= c_1 \sum_{h=0}^{\infty} \frac{(-)^h}{(2h+1)!} \, \frac{t^{\frac{1}{2}} \log \log t}{(\log t)^{h+1/2}} \quad .$$

The set $V(\infty)$ mentioned in the preceding theorem contains the functions

$$t^{\frac{1}{2}}(\log t)^{-h-\frac{1}{2}} \log \log t \quad (h = 0,1,2, \dots)$$

and for each fixed integer $h \geq 0$

$$w_1(t) - c_1 \sum_{q=0}^{h-1} \frac{(-)^q}{(2q+1)!} \, \frac{t^{\frac{1}{2}} \log \log t}{(\log t)^{q+1/2}}$$

is at most of the same order of magnitude as $\dfrac{t^{\frac{1}{2}} \log \log t}{(\log t)^{h+1/2}}$, so that the conditions of the preceding theorems are satisfied.

If $m(t)$ tends to a finite limit λ as $t \to \infty$, it follows from (3.23) and $r(t) \to 0$ that $\chi_1(t) + w_1(t)$ tends also to λ and accordint to the preceding theorems this limit is equal to zero. Consequently infinity is an ordinary point with respect to the modulus formed by the functions $m(t)$.

THEOREM 6. Let k be a given integer $\geqq 0$ and let p be a positive number such that $x_k > 0$ for $x \geqq p$; here $x_0 = x$ and $x_{h+1} = \log x_h$.

Let $V(\infty)$ be a set formed by functions of the form

$$(3.24) \qquad v(t) = \gamma + \int_p^t x_0^{\beta_0} x_1^{\beta_1} \cdots x_k^{\beta_k} \, dx \quad (t \geqq p)$$

where γ is an arbitrary constant and where $\beta_0, \beta_1, \ldots, \beta_k$ are arbitrary real numbers satisfying the following inequalities:

$$(3.25) \qquad \begin{aligned} &\beta_0 \geqq -1; \text{ if } \beta_0 = -1, \text{ then } \beta_1 \geqq -1; \text{ if } \beta_0 = \beta_1 = -1, \text{ then } \beta_2 \geqq -1, \\ &\ldots, \text{ finally, if } \beta_0 = \beta_1 = \cdots = \beta_{k-1} = -1, \text{ then } \beta_k \geqq -1 \ . \end{aligned}$$

We assume that $V(\infty)$ does not contain two different functions (3.24) and

$$(3.26) \qquad v^*(t) = \gamma^* + \int_p^t x_0^{\beta_0^*} \cdots x_k^{\beta_k^*} \, dx \ ,$$

such that

$$\beta_0 = \beta_0^* \ , \quad \beta_1 = \beta_1^* \ , \ \ldots \ , \ \beta_k = \beta_k^* \ .$$

Let $W(\infty)$ be the set formed by all functions $w(t)$ which for $t \geqq p$ can be written as the sum of a convergent series

$$w(t) = \sum_{h=0}^{\infty} c_h \, v_h(t) \ ,$$

where for each integer $h \geqq 0$ the coefficient c_h is a constant and where the functions $v_h(t)$ $(h = 0,1 \cdots)$ belong to $V(\infty)$; we assume that for each fixed integer $h \geqq 0$

$$\frac{v_{h+1}(t)}{v_h(t)} \to 0 \qquad \text{as } t \to \infty$$

and that

$$w(t) - c_o\, v_o(t) - c_1\, v_1(t) - \cdots - c_{h-1}\, v_{h-1}(t)$$

is for large t at most of the same order of magnitude as $v_h(t)$.

Let $M(\infty)$ be the modulus formed by the functions of the form

$$m(t) = n(t) + w_1(t) + w_2(t) + \cdots + w_s(t) \quad ;$$

here $n(t)$ is an arbitrary linear combination of a finite number of functions belonging to $V(\infty)$; furthermore s is an arbitrary integer $\geqq 0$ and $w_\sigma(t)$ $(\sigma = 1, \cdots s)$ denote arbitrary functions belonging to $W(\infty)$.

Under these conditions infinity is an ordinary point with respect to $M(\infty)$.

Proof: From the inequalities (3.25) it follows that for $x \geqq p$

$$x_o^{\beta_o}\, x_1^{\beta_1} \cdots x_k^{\beta_k} \geqq x_o^{-1}\, x_1^{-1} \cdots x_k^{-1} \quad ,$$

so that

$$\int_p^t x_o^{\beta_o}\, x_1^{\beta_1} \cdots x_k^{\beta_k}\, dx \geqq \int_p^t x_o^{-1}\, x_1^{-1} \cdots x_k^{-1}\, dx$$

$$= \Big|_p^t \; x_{k+1}$$

and the last side increases indefinitely as $t \to \infty$. Consequently each function $v(t)$ in $V(\infty)$ approaches infinity as $t \to \infty$. Any pair of different functions $v(t)$ and $v^*(t)$ belonging to $V(\infty)$ has the property that their quotient tends to infinity or to zero as $t \to \infty$. This follows from the fact that the functions $v(t)$ and $v^*(t)$, defined in (3.24) and (3.26) have the property that $\frac{v(t)}{v^*(t)}$ tends to infinity as $t \to \infty$ in each of the cases:

$$\beta_0 > \beta_0^* \; ; \; \beta_0 = \beta_0^* \; , \; \beta_1 > \beta_1^* \; ; \; \beta_0 = \beta_0^* \; , \; \beta_1 = \beta_1^* \; , \; \beta_2 > \beta_2^* \; ;$$

$$\cdots \; ; \text{ finally } \beta_0 = \beta_0^* \; , \; \cdots \; , \; \beta_{k-1} = \beta_{k-1}^* \; ; \; \beta_k > \beta_k^* \; ;$$

interchanging $v(t)$ and $v^*(t)$ we find that in each remaining case $\frac{v^*(t)}{v(t)}$ tends to infinity.

It may now be observed that Theorem 6 is a corrollary of Theorem 4.

Example: For $q > 1$ the integral

$$I = \int_q^\infty x^{3/2} (\log x) \, e^{1/\log x} \, dx$$

exists with respect to this modulus $M(\infty)$, if we choose $V(\infty)$ in such a way that it contains the functions

$$v_h(t) = \int_p^t x^{3/2} (\log x)^{1-h} \, dx \qquad (h = 0, 1, \cdots) \quad .$$

Then $M(\infty)$ contains the integral

$$\int_p^t x^{3/2}(\log x)e^{1/\log x}\,dx = \sum_{h=o}^{\infty} \frac{1}{h!} \int_p^t x^{3/2}(\log x)^{1-h}dx = \sum_{h=o}^{\infty} \frac{1}{h!} v_h(t)$$

so that this integral tends to zero modulo $M(\infty)$, as $t \to \infty$. In this way we find

$$I = \int_q^p x^{3/2} (\log x)e^{1/\log x}\,dx \ (M(\infty)) \quad .$$

Applying the same argument with summation instead of integrations we find

THEOREM 7. Let k be a given integer $\geqq 0$ and let p be a positive integer such that $n_k > 0$ for each integer $n \geqq p$; here $n_o = n$ and $n_{h+1} = \log n_h$.

Let $V(\infty)$ be a set formed by functions of the form

$$(3.27) \qquad v(t) = \gamma + \sum_{n=p}^{t} n_o^{\beta_o} n_1^{\beta_1} \cdots n_k^{\beta_k} \quad (t \text{ integer} \geqq p) \quad ,$$

where γ is an arbitrary constant and where $\beta_o, \beta_1, \cdots, \beta_k$ are arbitrary real numbers satisfying the inequalities (3.25). We assume that $V(\infty)$ does not contain two different functions (3.27) and

$$(3.28) \qquad v^*(t) = \gamma^* + \sum_{n=p}^{t} n_o^{\beta_o^*} n_1^{\beta_o^*} \cdots n_k^{\beta_k^*}$$

such that

$$\beta_o = \beta_o^*, \ \beta_1 = \beta_1^* \ , \cdots, \beta_k = \beta_k^* \quad .$$

Let $W(\infty)$ be the set formed by all functions $w(t)$ which can be written for each integer $t \geqq p$ as the sum of a convergent series

$$w(t) = \sum_{h=o}^{\infty} c_h v_h(t) \quad ,$$

where each c_h denotes a constant and where the functions $v_h(t)$ $(h = 0, 1 \cdots)$ belong to $V(\infty)$; we assume that for each fixed integer $h \geqq 0$

$$\frac{v_{h+1}(t)}{v_h(t)} \longrightarrow 0 \qquad \text{as the integer } t \longrightarrow \infty$$

and that

$$w(t) - c_o v_o(t) - c_1 v_1(t) - \cdots - c_{h-1} v_{h-1}(t)$$

is for large integers t at most of the same order of magnitude as $v_h(t)$.

Let $M(\infty)$ be the modulus formed by the functions of the form

$$m(t) = g(t) + \sum_{\sigma=1}^{s} w_\sigma(t) \qquad (t \text{ integer} \geqq p) \; ;$$

here $g(t)$ is an arbitrary linear combination of a finite number of functions belonging to $V(\infty)$; furthermore s is an arbitrary integer $\geqq 0$ and $w_\sigma(t)$ $(\sigma = 1, \cdots, s)$ denote arbitrary functions belonging to $W(\infty)$.

Under these conditions infinity is an ordinary point with respect to $M(\infty)$.

Example: The series

$$\sum_{n=2}^{\infty} n^{3/2} (\log n) e^{1/\log n}$$

is convergent with respect to this modulus $M(\infty)$, if we choose the set $V(\infty)$ in such a way that it contains the functions

$$\sum_{n=p}^{t} n^{3/2} (\log n)^{1-h} \qquad (h = 0, 1, \cdots) \; .$$

Section 4. INTEGRATION BY PARTS

In the theory of integration the most important rules of calculation are those of the integration by parts and of the substitution of a new integration variable. In this section we show that the method of integrating by parts can be applied also in the theory of integration with respect to given moduli. For the integrals occurring in ordinary analysis the method of integrating by parts can be formulated as follows:

If f(x) and g(x) are continuously differentiable in the open interval $a < x < b$ (a may be $-\infty$ and b may be ∞), then

$$(4.1)\int_a^b f(x)g'(x)dx = \lim_{\substack{v<b\\ v\to b}} f(v)g(v) - \lim_{\substack{u>a\\ u\to a}} f(u)g(u) - \int_a^b f'(x)g(x)\,dx \quad ,$$

provided that the three terms on the right hand side exist; then the integral on the left exists and is equal to the right hand side.

In the theory of the integrals with respect to given moduli we obtain the following similar theorem.

THEOREM 8. Suppose that f(x) and g(x) are continuously differentiable in the open interval $a < x < b$ (a may be $-\infty$ and b may be $+\infty$). Assume that

$$\lim_{\substack{v<b\\ v\to b}} f(v)\ g(v)$$

exists with respect to a given modulus M(b−) for which b− is an ordinary point and that

$$\lim_{\substack{u > a \\ u \to a}} f(u)\ g(u)$$

exists with respect to a given modulus M(a+) for which a+ is an ordinary point. Finally we assume that the integral

$$\int_a^b f'(x)\ g(x)\ dx$$

exists modulis M(a+) and M(b−). Under these conditions the integral

$$\int_a^b f(x)\ g'(x)\ dx$$

exists modulis M(a+) and M(b−) and satisfies formula (4.1) modulis M(a+) and M(b−).

Proof. The proof is simple. We have for $a < u < v < b$

$$\int_u^v f(x)\ g'(x)\ dx = f(v)\ g(v) - f(u)\ g(u) - \int_u^v f'(x)\ g(x)\ dx \quad .$$

Taking the limits modulis M(b−) and M(a+) of the three terms on the right as v tends to b and u tends to a, we find the required result.

The application of the limits modulo a given class of functions enables us to generalize considerably some well-known formulae occurring in calculus.

THEOREM 9. Suppose

$$-\frac{\pi}{2} \leqq \arg \epsilon \leqq \frac{\pi}{2} \quad , \qquad \alpha > 0 \quad ;$$

if $-\frac{\pi}{2} < \arg \epsilon < \frac{\pi}{2}$, let β be any real number, but if $\arg \epsilon = \pm \frac{\pi}{2}$, we assume $\beta < \alpha - 1$. Let m be an integer $\geqq 0$.

Then the integral

$$I = \int_0^\infty x^\beta (\log x)^m \, e^{-\epsilon x^\alpha} \, dx$$

exists modulo $L(0+)$. Furthermore, if $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$, then we have modulo $L(0+)$

$$I = \frac{\partial^m}{\partial \beta^m} \left\{ \frac{1}{\alpha} \, \Gamma\left(\frac{\beta+1}{\alpha}\right) \epsilon^{-\frac{\beta+1}{\alpha}} \right\} . \tag{4.2}$$

In the case that $-\frac{\beta+1}{\alpha}$ is equal to an integer $k \geqq 0$, we define I_{km} as follows:

$$I = \int_0^\infty x^{-1-\alpha k} \, (\log x)^m \, e^{-\epsilon x^\alpha} \, dx = \frac{m! \, \epsilon^k}{\alpha^{m+1}} I_{km} . \tag{4.3}$$

Then for $k \geqq 0$

$$I_{ko} = \frac{(-)^k}{k!} \left(-\log \epsilon - \gamma + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}\right) , \tag{4.4}$$

where γ denotes the constant of Euler, and for $m \geqq 0$

$$\left\{ \begin{aligned} I_{om} &= \frac{\epsilon}{(m+1)!} \left(\frac{d^{m+1} \, \epsilon^{-s} \, \Gamma(s)}{ds^{m+1}} \right)_{s=1} = \\ &= \frac{1}{(m+1)!} \sum_{h=o}^{m+1} (-)^h \binom{m+1}{h} (\log \epsilon)^h \, \Gamma^{(m+1-h)}(1) , \end{aligned} \right. \tag{4.5}$$

<u>and finally</u> I_{km} <u>for</u> $k \geqq 1$ <u>and</u> $m \geqq 1$ <u>is a linear combination of</u> I_{00}, $I_{01}, I_{02}, \ldots, I_{om}, I_{10}, I_{20}, \ldots, I_{ko}$. <u>The coefficients in this linear combination are determined by the recurrence relation</u>

$$(4.6) \qquad k\, I_{km} = I_{k,m-1} - I_{k-1,m} \quad (k \geqq 1,\ m \geqq 1) \quad .$$

<u>Remark.</u> As we can see from (4.4) and (4.5) the two values obtained for I_{oo} are the same, namely

$$-\log \epsilon - \gamma \quad ,$$

since $\Gamma'(1) = -\gamma$.

We will divide the proof into 5 parts, as follows:

I. Investigation of the behavior of I at the upper limit, ∞.

II. Behavior of the integral at 0, for $-\frac{\beta+1}{\alpha} < 0$, that is, $\beta > -1$.

III. Behavior of the integral at 0 for $\beta \leqq -1$, that is, $-\frac{\beta+1}{\alpha} \geqq 0$, and $-\frac{\beta+1}{\alpha}$ not an integer. There are two subcases:

A| $m = 0$ B| $m \geqq 1$.

IV. Behavior of the integral at 0 for $\beta = -1$, that is $-\frac{\beta+1}{\alpha} = 0$.

V. Behavior of the integral at 0 for $\beta < -1$, that is $-\frac{\beta+1}{\alpha} > 0$, and $-\frac{\beta+1}{\alpha}$ an integer. We have again two subcases:

A| $m = 0$ B| $m \geqq 1$.

<u>I.|</u> The integral I converges in the usual sense at infinity, since the substitution $x^{\alpha} = y$ gives

$$(4.7)\int_1^\infty x^\beta (\log x)^m e^{-\epsilon x^\alpha} dx = \frac{1}{\alpha^{m+1}} \int_1^\infty y^{\frac{\beta+1}{\alpha}-1} (\log y)^m e^{-\epsilon y} dy \quad ;$$

the last integral converges, since we have assumed either

$$-\frac{\pi}{2} < \arg \epsilon < \frac{\pi}{2} \quad \text{thus Re } \epsilon < 0 \quad ,$$

in which case because of the $e^{-(\mathrm{Re}\,\epsilon)y}$ factor the integral converges, or

$$\arg \epsilon = \pm \frac{\pi}{2} \qquad \text{and} \qquad \frac{\beta+1}{\alpha} - 1 < 0$$

in which case we use the fact that the integrals

$$\int_1^\infty \chi(y) \cos \lambda y \, dy \quad \text{and} \quad \int_1^\infty \chi(y) \sin \lambda y \, dy$$

(λ real $\neq 0$) exist in the usual sense, if $\chi(y)$ is continuous and tends monotonically to zero for sufficiently large y as $y \to \infty$.

II. If $\beta > -1$, the integral I converges in the usual sense at the origin also, so that in that case the integral I exists in the usual sense. Thus the convergent integral

$$\int_0^\infty x^\beta e^{-\epsilon x^\alpha} dx$$

is equal to

$$\frac{1}{\alpha}\int_0^\infty y^{\frac{\beta+1}{\alpha}-1} e^{-\epsilon y} dy = \frac{1}{\alpha} \Gamma\left(\frac{\beta+1}{\alpha}\right) \epsilon^{-\frac{\beta+1}{\alpha}}$$

and taking the partial derivative m times with respect to β we find

$$\int_0^\infty x^{\beta}(\log x)^m e^{-\epsilon x^\alpha}\, dx = \frac{\partial^m}{\partial\beta^m}\int_0^\infty x^{\beta} e^{-\epsilon x^\alpha}\, dx =$$

$$= \frac{\partial^m}{\partial\beta^m}\left\{\frac{1}{\alpha}\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+1}{\alpha}}\right\} .$$

This gives the required result in the case that $\beta > -1$.

III.| Suppose that $\beta \leqq 1$ and that $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$. In this part III we shall prove that the assertion holds for β under the assumption that we have already proved the assertion for $\beta + \alpha$ instead of β, so that

$$(4.8)\quad \int_0^\infty x^{\beta+\alpha}(\log x)^m e^{-\epsilon x^\alpha}\, dx = \frac{\partial^m}{\partial\beta^m}\left\{\frac{1}{\alpha}\Gamma\left(\frac{\beta+\alpha+1}{\alpha}\right)\epsilon^{-\frac{\beta+\alpha+1}{\alpha}}\right\}$$

modulo L(0+) if $\beta \leqq -1$ and $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$. Then the assertion will hold for each real $\beta \leqq -1$ for which $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$.

To show this we introduce the smallest positive integer h such that $\beta + \alpha h > -1$, and since the assertion has been proven for $\beta > -1$, the assertion holds if β is replaced by $\beta + \alpha h$. Using (4.8) with $\beta + \alpha$ replaced by $\beta + \alpha h$, we see that the assertion holds if β is replaced by $(\beta + \alpha h) - \alpha = \beta + \alpha(h-1)$ (notice that $\beta + \alpha(h-1) \leqq -1$). Using (4.8) again, but now with $\beta + \alpha(h-1)$ instead of $\beta + \alpha$, we see that the assertion holds if β is replaced by $\beta + \alpha(h-2)$. Continuing in this way we notice that the assertion holds if β is replaced by $\beta + \alpha h$ or

$\beta + \alpha(h-1)$ or $\beta + \alpha(h-2)$, $\cdots$, or $\beta + \alpha$ and therefore finally for $\beta - \alpha(h-h) = \beta$.

Now we must prove the assertion under the assumption (4.8).

Since $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$, the number $\beta + 1$ is different from zero. Integrating by parts we obtain that

$$I = \frac{1}{\beta+1} \int_0^\infty (\log x)^m e^{-\epsilon x^\alpha} dx^{\beta+1}$$

is modulo L(0+) equal to

$$(4.9) \qquad \left\{ \begin{aligned} I = \frac{1}{\beta+1} \Big/_0^\infty & (\log x)^m e^{-\epsilon x^\alpha} x^{\beta+1} + \\ + \frac{\epsilon\alpha}{\beta+1} \int_0^\infty & x^{\beta+\alpha} (\log x)^m e^{-\epsilon x^\alpha} dx \\ - \frac{m}{\beta+1} \int_0^\infty & x^\beta (\log x)^{m-1} e^{-\epsilon x^\alpha} dx , \end{aligned} \right.$$

provided the terms on the right hand side exist modulo L(0+), which we shall prove presently. If m = 0 the last term does not occur.

<u>A)</u> Let us consider first the special case in which m = 0. Then by (4.9) I assumes the form

$$I = \frac{1}{\beta+1} \Big/_0^\infty e^{-\epsilon x^\alpha} x^{\beta+1} + \frac{\epsilon\alpha}{\beta+1} \cdot \frac{1}{\alpha} \Gamma\left(\frac{\beta+\alpha+1}{\alpha}\right) \epsilon^{-\frac{\beta+\alpha+1}{\alpha}} ,$$

where the last term is equal to

$$\frac{1}{\alpha}\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+1}{\alpha}} \quad .$$

Now we shall show that the integrated part is equal to zero and to show that, we write

$$e^{-\epsilon x^{\alpha}} x^{\beta+1}$$

as

$$(4.10) \qquad e^{-\epsilon x^{\alpha}} x^{\beta+1} = \sum_{0 \leqq h \leqq -\frac{\beta+1}{\alpha}} \frac{(-\epsilon)^h}{h!} x^{\beta+1+\alpha h} + r(x) \quad ,$$

where $r(x) \to 0$ as $x \to 0$. Since each exponent $\beta + 1 + \alpha h$ is different from zero, the right hand side of (4.10) tends to zero modulo $L(0+)$ as $x \to 0$. Moreover

$$e^{-\epsilon x^{\alpha}} x^{\beta+1}$$

tends to zero in the ordinary sense as $x \to \infty$; that is obvious if $-\frac{\pi}{2} < \arg \epsilon < \frac{\pi}{2}$ but it is also true if $\arg \epsilon = \pm \frac{\pi}{2}$, since we have assumed $\beta \leqq -1$ and $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$, so that β is different from -1 and therefore less than -1. Consequently the integrated part is equal to zero, so that modulo $L(0+)$

$$\int_0^{\infty} x^{\beta} e^{-\epsilon x^{\alpha}} dx = \frac{\epsilon\alpha}{\beta+1} \cdot \frac{1}{\alpha} \Gamma\left(\frac{\beta+1}{\alpha} + 1\right) \epsilon^{-\frac{\beta+\alpha+1}{\alpha}} =$$

$$= \frac{1}{\alpha}\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+1}{\alpha}} \quad .$$

This gives the required result for $m = 0$.

B| Now we treat the case m $\geqq$ 1 under the assumption that we have already proved the required result with m − 1 instead of m, so that modulo L(0+)

$$(4.11)\int_0^\infty x^\beta (\log x)^{m-1}\, e^{-\epsilon x^\alpha}\, dx = \frac{\partial^{m-1}}{\partial\beta^{m-1}}\left\{\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+1}{\alpha}}\right\} .$$

We apply again formula (4.9) and observe that the integrated part is again equal to zero. Using (4.8) and (4.11) we obtain

$$I = \frac{\epsilon\alpha}{\beta+1}\,\frac{\partial^m}{\partial\beta^m}\left\{\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+\alpha+1}{\alpha}\right)\epsilon^{-\frac{\beta+\alpha+1}{\alpha}}\right\} -$$

$$-\frac{m}{\beta+1}\,\frac{\partial^{m-1}}{\partial\beta^{m-1}}\left\{\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+1}{\alpha}}\right\} .$$

We have

$$\frac{1}{\alpha}\left(\frac{\beta+\alpha+1}{\alpha}\right)\epsilon^{-\frac{\beta+\alpha+1}{\alpha}} = \frac{\beta+1}{\alpha}\cdot\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+\alpha+1}{\alpha}}$$

and therefore, according to the rule of Leibniz's

$$\frac{\partial^m}{\partial\beta^m}\left(\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+\alpha+1}{\alpha}\right)\epsilon^{-\frac{\beta+\alpha+1}{\alpha}}\right) = \frac{\beta+1}{\alpha}\,\frac{\partial^m}{\partial\beta^m}\left(\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+\alpha+1}{\alpha}}\right) +$$

$$+\frac{m}{\alpha}\,\frac{\partial^{m-1}}{\partial\beta^{m-1}}\left(\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+\alpha+1}{\alpha}}\right) .$$

Substituting this value we obtain

$$I = \frac{\partial^m}{\partial\beta^m}\left\{\frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+1}{\alpha}}\right\} ,$$

which completes the proof in the case that $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$.

IV.| Now we pass to the proof in the case that $\beta = -1$, therefore $-\frac{\beta+1}{\alpha} = 0$. Then we have modulo L(0+)

$$I = \lim_{\delta \to 0} \int_{\delta}^{\infty} x^{-1} (\log x)^{m} e^{-\epsilon x^{\alpha}} dx =$$

$$= \frac{1}{\alpha^{m+1}} \lim_{\delta \to 0} \int_{\delta^{\alpha}}^{\infty} y^{-1} (\log y)^{m} e^{-\epsilon y} dy =$$

$$= \frac{1}{\alpha^{m+1}} \lim_{\delta \to 0} \left\{ -\frac{(\alpha \log \delta)^{m+1}}{m+1} e^{-\epsilon \delta^{\alpha}} + \frac{\epsilon}{m+1} \int_{\delta^{\alpha}}^{\infty} (\log y)^{m+1} e^{-\epsilon y} dy \right\}.$$

Here

$$(\alpha \log \delta)^{m+1} e^{-\epsilon \delta^{\alpha}} = (\alpha \log \delta)^{m+1} + (\alpha \log \delta)^{m+1} \left(e^{-\epsilon \delta^{\alpha}} - 1\right) ;$$

the last term tends, in the usual sense, to zero as $\delta \to 0$ and the first term on the right hand side tends modulo L(0+) to zero, so that, modulo L(0+),

$$I = \frac{\epsilon}{(m+1)\alpha^{m+1}} \int_{0}^{\infty} (\log y)^{m+1} e^{-\epsilon y} dy .$$

For each point s with Re s > 0 we have

$$\epsilon^{-s} \Gamma(s) = \int_{0}^{\infty} y^{s-1} e^{-\epsilon y} dy ,$$

and therefore

$$\frac{d^{m+1}(\epsilon^{-s} \Gamma(s))}{ds^{m+1}} = \int_{0}^{\infty} y^{s-1} (\log y)^{m+1} e^{-\epsilon y} dy ,$$

so that

$$I = \frac{\epsilon}{(m+1)\alpha^{m+1}} \left(\frac{d^{m+1}(\epsilon^{-s}\,\Gamma(s))}{ds^{m+1}} \right)_{s=1} .$$

Thus we have found the required result in the case $\beta = -1$, making use of the remark added to the theorem in the case $\beta = -1$, $m = 0$.

V.| A| Now we treat the case that $m = 0$ and $\beta = -1 - \alpha k$, where k is a positive integer. Since the case $k = 0$ has already been treated in part III we may assume that the assertion has already been proved in the case $m = 0$ for $k - 1$ instead of k. If k is replaced by $k - 1$, then $\beta = -1-\alpha k$ is replaced by

$$-1 - \alpha(k - 1) = \beta + \alpha ,$$

so that we may use the formula

$$(4.12) \int_0^\infty x^{\beta+\alpha} e^{-\epsilon x^\alpha} dx = \frac{(-)^{k-1}}{(k-1)!\,\alpha} \epsilon^{k-1} \left(- \log \epsilon - \gamma + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k-1} \right)$$

modulo $L(0+)$. Applying (4.9) with $m = 0$ we obtain

$$(4.13) \qquad I = \frac{1}{\beta+1} \int_0^\infty e^{-\epsilon x^\alpha} x^{\beta+1} dx + \frac{\epsilon\alpha}{\beta+1} \int_0^\infty x^{\beta+\alpha} e^{-\epsilon x^\alpha} dx .$$

In this case the last term has the value

$$-\frac{\epsilon}{k} \cdot \frac{(-)^{k-1}\epsilon^{k-1}}{(k-1)!\,\alpha} \left(- \log \epsilon - \gamma + \frac{1}{1} + \cdots + \frac{1}{k-1} \right) =$$

$$= \frac{(-)^k \epsilon^k}{k!\,\alpha} \left(- \log \epsilon - \gamma + \frac{1}{1} + \cdots + \frac{1}{k-1} \right) .$$

since $k = \frac{\beta+1}{\alpha}$. We apply (4.10) to the integrated part of (4.13), but now the sum, occurring in that formula, contains a term with h = k, for which the exponent $\beta + 1 + \alpha h$ is equal to zero, so that modulo L(0+)

$$e^{-\epsilon x^{\alpha}} x^{\beta+1} \rightarrow \frac{(-\epsilon)^k}{k!} \qquad \text{as } x \rightarrow 0 \quad .$$

The integrated part in (4.13) is therefore equal to

$$-\frac{1}{\beta+1} \cdot \frac{(-\epsilon)^k}{k!} = \frac{1}{\alpha} \frac{(-\epsilon)^k}{k!k} \quad ,$$

so that

$$I = \frac{1}{\alpha} \frac{(-\epsilon)^k}{k!k} + \frac{(-)^k \epsilon^k}{k!\alpha} \left(-\log \epsilon - \gamma + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k-1} \right) \quad ,$$

which gives the required result for $k \geqq 1$, m = 0.

<u>B</u>| Finally we consider the case in which both $-\frac{\beta+1}{\alpha} = k$ and m are positive integers. We again use formula (4.9). The integrated part is again equal to zero, for applying formula (4.10) we obtain

$$(\log x)^m e^{-\epsilon x^{\alpha}} x^{\beta+1} = \sum_{0 \leqq h \leqq -\frac{\beta+1}{\alpha}} \frac{(-\epsilon)^h}{h!} x^{\beta+1+\alpha h} (\log x)^m + r(x)(\log x)^m \quad ,$$

where $r(x)(\log x)^m \rightarrow 0$ as $x \rightarrow 0$ and where the sum does not contain a term which is constant so that the sum tends to zero modulo L(0+) as $x \rightarrow 0$. Formula (4.9) tells us therefore that according to the definition of I_{km} given in the statement of the theorem

$$\frac{m!\epsilon^k}{\alpha^{m+1}} I_{km} = \frac{\epsilon\alpha}{\beta+1} \cdot \frac{m!\epsilon^{k-1}}{\alpha^{m+1}} I_{k-1,m} - \frac{m}{\beta+1} \cdot \frac{(m-1)!\epsilon^k}{\alpha^m} I_{k,m-1} \quad ,$$

hence

$$I_{km} = -\frac{1}{k} I_{k-1,m} + \frac{1}{k} I_{k,m-1}$$

This shows that $I_{k,m}$ for $k \geq 1$ and $m \geq 1$ is a linear combination of I_{00}, $I_{01}, I_{02}, \ldots, I_{om}, I_{10}, I_{20}, \ldots, I_{ko}$. This completes the proof.

Another case where we apply the generalized limit is in establishing (modulis $L(0+)$ and $L(1-)$) the formula

$$\int_0^1 x^{p-1} (1-x)^{q-1}\, dx = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \quad ,$$

which is valid for each pair of complex numbers p and q, provided that neither p nor q is an integer ≤ 0. The formula is well known for $\operatorname{Re} p > 0$ and $\operatorname{Re} q > 0$. We may suppose that we have proved the formula already in the case that $p + q$ is replaced by $p + q + 1$, so that

$$\int_0^1 x^{p-1} (1-x)^{q-1}\, dx = \int_0^1 x^{p} (1-x)^{q-1}\, dx + \int_0^1 x^{p-1} (1-x)^{q}\, dx$$

$$= \frac{\Gamma(p+1)\,\Gamma(q)}{\Gamma(p+q+1)} + \frac{\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$= \frac{\Gamma(p)\,\Gamma(q)\,(p+q)}{\Gamma(p+q+1)} = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \quad .$$

Thus if the formula holds for $p + 1$ and q it holds for p and q, and if it holds for p and $q + 1$ it also holds for p and q. By repeating this process we see that the formula holds for each p and q which are not integers $\leqq 0$, since it holds in the case where Re p > 0 and Re 1 > 0.

We have calculated (III,III,2,page 3) the integral

$$I = \int_1^{\infty} \sqrt{x+1} \log (x + 2)\, dx \quad .$$

We can also evaluate this integral as follows. Since

$$(x + 1)^{3/2} \log (x + 2) = x^{3/2} \log x + \frac{7}{2} x^{1/2} + \epsilon (x) \quad ,$$

where $\epsilon(x)$ tends to zero as x approaches infinity, we have modulo $L(\infty)$

$$(x + 1)^{3/2} \log (x + 2) \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad ,$$

so that integrating by parts we find

$$I = -\frac{2}{3} \cdot 2^{3/2} \log 3 - \frac{2}{3} \int_1^{\infty} \frac{(x+1)^{3/2}}{x+2}\, dx \quad .$$

From

$$\frac{(x+1)^{3/2}}{x+2} = (x + 1)^{1/2} - (x + 1)^{-1/2} + \frac{1}{(x+2)\sqrt{x+1}}$$

and

$$(x + 1)^{3/2} \rightarrow 0 \;\; ; \quad (x + 1)^{1/2} \rightarrow 0 \quad \text{as } x \rightarrow \infty \; (L(\infty))$$

it follows therefore that

$$I = \frac{2}{3} \cdot 2^{3/2} \log 3 + \frac{4}{9} \cdot 2^{3/2} - \frac{4}{3} \cdot 2^{1/2} - \frac{2}{3} \int_1^\infty \frac{dx}{(x+2)\sqrt{x+1}}$$

$$= -\frac{4}{3}\sqrt{2}\left(\frac{1}{3} + \log 3\right) - \frac{2}{3} \int_1^\infty \frac{dx}{(x+2)\sqrt{x+1}} \quad .$$

Let us now give some examples involving the modulus $P(\infty)$ defined on (III,III,2,12).

THEOREM 10. _Suppose the hyperpolynomial_ $g(x)$ _is_ k _times_ $(k \geqq 0)$ _continuously differentiable for_ $x \geqq a$ _such that_

$$\int_a^\infty \left| g^{(k)}(x) \right| dx < \infty \quad .$$

Let $\chi(x)$ _be a bounded integrable function with period_ 1. _Then the integral_

$$\int_a^\infty \chi(x)\ g(x)\ dx$$

modulo $P(\infty)$ _exists_.

Proof. The assertion is obvious for $k = 0$, so that I may suppose that $k \geqq 1$. Integrating k times by parts we obtain for $t > a$

$$\int_a^t \chi(x)\ g(x)\ dx = \sum_{h=0}^{k-1} (-)^h \left\{ \chi_{h+1}(t)\ g^{(h)}(t) - \chi_{h+1}(a)\ g^{(h)}(a) \right\}$$

$$+ (-)^k \int_a^t \chi_k(x)\ g^{(k)}(x)\ dx \quad .$$

Here $\chi_1(x)$, $_2(x)_1$ ... are defined by

$$\chi_1(x) = \int_0^x \chi(u)\,du + c_1$$

and

$$\chi_h(x) = \int_0^x \chi_{h-1}(u)\,du + c_h \qquad (h \geqq 2) \quad ,$$

where we determine the constants c_1, c_2 ... such that

$$\int_0^1 \chi_h(x)\,dx = 0 \qquad (h \geqq 1) \quad .$$

Then $\chi_1(x), \chi_2(x), \cdots, \chi_k(x)$ satisfy the conditions occurring in the definition of $P(\infty)$, so that the integral

$$\int_a^\infty \chi(x)\,g(x)\,dx$$

modulo $P(\infty)$ exists and is equal to

$$-\sum_{h=0}^{k-1} (-)^h \chi_{h+1}(a)\,g^{(h)}(a) + (-1)^k \int_a^\infty \chi_k(x)\,g^{(k)}(x)\,dx \quad .$$

THEOREM 11. Let m be an integer $\geqq 0$. If $s \neq 0$ and $\neq 1$, then

$$(-)^m \zeta^{(m)}(s) = \int_0^\infty \psi_1(x)\,(x^{-s}\log^m x)'\,dx \quad ,$$

where the integral is taken modulis $P(\infty)$ and $L(0+)$. This formula holds also in the case $s = 0$, $m \geqq 1$. In order for the formula to hold in the case $s = m = 0$ the right hand side (which is then equal to zero) must be replaced by $-\frac{1}{2}$. The formula holds also for $s = 1$ if $(-)^m \zeta^{(m)}(s)$ is replaced by $\gamma - 1$ when $m = 0$ (where γ is the constant of Euler) and $(-1)^m \zeta^{(m)}(s)$ is replaced by

$$\lim_{s \to 1} \left\{ (-)^m \zeta^{(m)}(s) - \frac{m!}{(s-1)^{m+1}} \right\} \quad \text{when } m \geqq 1 \quad .$$

Remark. Since the zeta function of Riemann has a simple pole at the point $s = 1$ with residue 1, this function possesses in the neighborhood of that point an expansion of the form

$$\zeta(s) = \frac{1}{s-1} + c_o + \frac{c_1}{1!}(s-1) + \frac{c_2}{2!}(s-1)^2 + \cdots \quad .$$

In this expansion c_o is the constant of Euler. We have

$$\lim_{s \to 1} \left((-)^m \zeta^{(m)}(s) - \frac{m!}{(s-1)^{m+1}} \right) = (-)^m c_m \quad ,$$

in particular

$$\lim_{s \to 1} \left(\zeta(s) - \frac{1}{s-1} \right) = c_o = \gamma \quad .$$

For the proof we first consider the half plane $\operatorname{Re} s > 1$. Then according to the first fundamental identity in the theory of the sum formula of Euler

$$(-)^m \zeta^{(m)}(s) = \sum_{n=1}^{\infty} n^{-s} \log^m n$$

$$= \Theta + \int_1^{\infty} x^{-s} \log^m x \, dx + \int_1^{\infty} \Psi_1(x)(x^{-s} \log^m x)' \, dx \quad ,$$

where $\Theta = \frac{1}{2}$ if $m = 0$ and $\Theta = 0$ if $m > 0$. Moreover for $0 < \delta < 1$

$$\int_\delta^1 x^{-s} \log^m x \, dx + \int_\delta^1 \Psi_1(x)(x^{-s} \log^m x)' \, dx$$

$$= \int_\delta^1 d(\Psi_1(x) \, x^{-s} \log^m x) = \Theta - (\delta - \tfrac{1}{2}) \, \delta^{-s} \log^m \delta \quad .$$

Subtracting we obtain

$$(-)^m \zeta^{(m)}(s) = \int_\delta^{\infty} x^{-s} \log^m x \, dx + \int_\delta^{\infty} \Psi_1(x)(x^{-s} \log^m x)' \, dx$$

$$+ (\delta - \tfrac{1}{2}) \, \delta^{-s} \log^m \delta \quad .$$

The first term on the right hand side is equal to

$$- \delta^{1-s} \left\{ \frac{\log^m \delta}{1-s} - m \frac{\log^{m-1} \delta}{(1-s)^2} + \cdots + (-)^m m! \frac{1}{(1-s)^{m+1}} \right\} \quad ,$$

hence

$$(-)^m \zeta^{(m)}(s) = - \delta^{1-s} \left\{ \frac{\log^m \delta}{1-s} - m \frac{\log^{m-1} \delta}{(1-s)^2} + \cdots + (-)^m m! \frac{1}{(1-s)^{m+1}} \right\}$$

(4.15)

$$+ \int_\delta^{\infty} \Psi_1(x) \, (x^{-s} \log^m x)' \, dx + (\delta - \tfrac{1}{2}) \, \delta^{-s} \log^m \delta \quad .$$

Let us now prove that the integral

$$I_1 = \int_\delta^\infty \psi_1(x)(x^{-s}\log^m x)'\,dx$$

represents a function which is analytic in the whole complex s-plane. It is evident that I_1 denotes a function of s which is analytic in the half plane Re s > 1. Integrating by parts we obtain

$$I_1 = -\psi_2(\delta)(\delta^{-s}\log^m \delta)' - I_2 \quad , \quad \text{where } I_2 = \int_\delta^\infty \psi_2(x)(x^{-s}\log^m x)''\,dx ,$$

where I_2 represents a function of s which is analytic in the half plane Re s > 0, so that also the function denoted by I_1 is analytic in that half plane. Furthermore

$$I_2 = -\psi_3(\delta)(\delta^{-s}\log^m \delta)'' - I_3 \quad , \quad \text{where } I_3 = \int_\delta^\infty \psi_3(x)(x^{-s}\log^m x)'''\,dx \quad .$$

The function denoted by I_3 and therefore also those denoted by I_2 and I_1 are analytic in the half plane Re s > -1. Continuing in this way we see that the function represented by I_1 is everywhere analytic.

The other terms in (4.15) represent functions of s, which are analytic for each $s \neq 1$, so that that formula holds, not only in the half plane Re s > 1, but in the whole complex s-plane, the point s = 1 excepted.

Taking now the limit modulo L(0+) as $\delta \to 0$ in (4.15), we find, for $s \neq 1$,

$$(-)^m \zeta^{(m)}(s) = \int_0^\infty \Psi_1(x)(x^{-s}\log^m x)'\,dx \quad ,$$

provided that in the special case s = m = 0 the right hand side is augmented with

$$\lim_{\delta \to 0} (\delta - \tfrac{1}{2}) = -\tfrac{1}{2} \quad .$$

To find the required result for s = 1 we notice that the first term on the right hand side of (4.15) is equal to

$$\frac{(-)^{m+1}m!}{(1-s)^{m+1}} \sum_{h=0}^{m} \delta^{1-s} \frac{(-)^h}{h!}(1-s)^h \log^h \delta \quad .$$

Here

$$\delta^{1-s} = e^{(1-s)\log\delta} = \sum_{k=0}^{m+1-h} \frac{1}{k!}(1-s)^k \log^k \delta + (1-s)^{m+1-h} r_h \quad ,$$

where $r_h \to 0$ as $s \to 1$. The first term on the right hand side of (4.15) can therefore be written as

$$\frac{(-)^{m+1}m!}{(1-s)^{m+1}} \sum_{h=0}^{m} \frac{(-)^h}{h!}(1-s)^h \log^h \delta \sum_{k=0}^{m+1-h} \frac{(1-s)^k \log^k \delta}{k!} + r \quad ,$$

(where $r \to 0$ as $s \to 1$)

$$= \frac{(-)^{m+1}m!}{(1-s)^{m+1}} \sum_{n=0}^{m+1} \frac{1}{n!}(1-s)^n \log^n \delta \sum_{h=0}^{n}{}' (-)^h \binom{n}{h} + r \quad ,$$

the dash indicating that the term with h = n = m + 1 does not occur.

Here

$$\sum_{h=o}^{n}{}' (-)^h \binom{n}{h} = 1 \qquad \text{for } n = 0$$

$$= (1-1)^n = 0 \qquad \text{for } 1 \leqq n \leqq m$$

$$= (1-1)^{m+1} - (-)^{m+1} = (-)^m \quad \text{for } n = m+1 \quad .$$

The first term on the right hand side of (4.15) is therefore equal to

$$\frac{(-)^{m+1} m!}{(1-s)^{m+1}} - \frac{1}{m+1} \log^{m+1} \delta + r \quad .$$

Thus formula (4.15) assumes for Re s > 1 the form

$$(-)^m \zeta^{(m)}(s) - \frac{m!}{(s-1)^{m+1}} = -\frac{1}{m+1} \log^{m+1} \delta + r$$

$$+ \int_{\delta}^{\infty} \psi_1(x)(x^{-s} \log^m x)' \, dx + (\delta - \tfrac{1}{2}) \delta^{-s} \log^m \delta \quad .$$

If the number s, of which the real part is greater than 1, tends to 1, then $r \to 0$ and we find for each fixed integer $m \geqq 0$

$$\lim_{s \to 1} \left((-)^m \zeta^{(m)}(s) - \frac{m!}{(s-1)^{m+1}} \right) = -\frac{1}{m+1} \log^{m+1} \delta$$

$$+ \int_{\delta}^{\infty} \psi_1(x)(x^{-1} \log^m x)' \, dx + (\delta - \tfrac{1}{2}) \delta^{-1} \log^m \delta \quad .$$

If $\delta \to 0$, the last term tends, modulo L(0+), to 1 if m = 0 and to zero if $m \geqq 1$. Passing to the limit modulo L(0+) we find therefore

$$\lim_{s\to 1}\left\{(-)^m \zeta^{(m)}(s) - \frac{m!}{(s-1)^{m+1}}\right\} = \int_0^\infty \psi_1(x)(x^{-1}\log^m x)'\,dx \quad ,$$

provided that in the special case m = 0 the right hand side is augmented by 1. This completes the proof.

Remarks. For $1 < s < 2$ we find therefore by integration by parts

$$\frac{\zeta(-s)}{s} = \int_0^\infty \psi_1(x)\, x^{s-1}\,dx = -(s-1)\int_0^\infty \psi_2(x)\, x^{s-2}\,dx \quad .$$

The last integral converges. According to formula (4.8) in part III, Chapter I, Section 4,

$$\psi_2(x) = 2\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{(2\pi n)^2} \quad ,$$

where the series converges uniformly. Therefore

$$\frac{\zeta(-s)}{s} = -2(s-1)\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^2}\int_0^\infty (\cos 2\pi n x)\, x^{s-2}\,dx$$

$$= -2(s-1)\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^{1+s}}\int_0^\infty (\cos y)\, y^{s-2}\,dy$$

$$= -2(s-1)\,(2\pi)^{-1-s}\,\zeta(1+s)\,\Gamma(s-1)\sin\frac{\pi s}{2} \quad .$$

In this way we find for $1 < s < 2$ the functional equation of the zeta function

$$(4.16) \qquad \zeta(-s) = -2(2\pi)^{-1-s}\,\zeta(1+s)\,\Gamma(s+1)\sin\frac{\pi s}{2} \quad .$$

Of course this formula holds for each complex $s \neq -1$, since both sides represent analytic functions of s.

$$\lim_{s\to 1}\left\{(-)^m \zeta^{(m)}(s) - \frac{m!}{(s-1)^{m+1}}\right\} = \int_0^\infty \psi_1(x)(x^{-1}\log^m x)'\,dx \quad ,$$

provided that in the special case $m = 0$ the right hand side is augmented by 1. This completes the proof.

Remarks. For $1 < s < 2$ we find therefore by integration by parts

$$\frac{\zeta(-s)}{s} = \int_0^\infty \psi_1(x)\, x^{s-1}\,dx = -(s-1)\int_0^\infty \psi_2(x)\, x^{s-2}\,dx \quad .$$

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where the series converges uniformly. Therefore

$$\frac{\zeta(-s)}{s} = -2(s-1)\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^2}\int_0^\infty (\cos 2\pi n x)\, x^{s-2}\,dx$$

$$= -2(s-1)\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^{1+s}}\int_0^\infty (\cos y)\, y^{s-2}\,dy$$

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In this way we find for $1 < s < 2$ the functional equation of the zeta function

(4.16) $$\zeta(-s) = -2(2\pi)^{-1-s}\,\zeta(1+s)\,\Gamma(s+1)\sin\frac{\pi s}{2} \quad .$$

Of course this formula holds for each complex $s \neq -1$, since both sides represent analytic functions of s.

Section 5. ON THE SUBSTITUTION OF A NEW INTEGRATION VARIABLE

In this section we make some remarks about the substitution of a new integration variable.

Let $\phi(x)$ be a monotonic, non-decreasing function with continuous derivative in the interval $a \leqq x < b$. Let $\phi(a) = \alpha$ and let β denote the limit to which $\phi(x)$ tends as $x \to b$. Then, as is well known,

$$\int_a^b f(\phi(x))\, \phi'(x)\, dx = \int_\alpha^\beta f(y)\, dy \quad ,$$

provided that the last integral exists; in that case the first integral exists also and has the same value.

To generalize this result, it is necessary to introduce a modulus $M(b-)$ for which $b-$ is an ordinary point and a modulus $M(\beta-)$ for which $\beta-$ is an ordinary point. We suppose that these two moduli are equivalent. In section 2, page 2, we have already given the definition of two equivalent moduli, which implies that, if $M(b-)$ is formed by the functions $g(t)$, then $M(\beta-)$ is formed by the functions $\chi(\tau)$, where

$$\begin{aligned} \chi(\tau) &= g(\tau + b - \beta) && \text{if } b \text{ and } \beta \text{ are finite} \\ &= g(\tau) && \text{if } b = \beta = \infty \\ &= g(\tfrac{1}{\beta - \tau}) && \text{if } b = \infty \text{ and } \beta \text{ is finite} \\ &= g(b - \tfrac{1}{\tau}) && \text{if } b \text{ is finite and } \beta = \infty. \end{aligned}$$

For instance, if b and β are finite and the corresponding equivalent modulus $M(\infty)$ is formed by the functions $G(t')$, then according to the definition given in Section 2 the modulus $M(b-)$ is formed by the functions $G\left(\frac{1}{b-t}\right)$ and $M(\beta-)$ is formed by the functions $G\left(\frac{1}{\beta-\tau}\right)$, which implies the relation $\frac{1}{b-t} = \frac{1}{\beta-\tau}$, hence $t = \tau + b - \beta$.

THEOREM 12. Let $\phi(x)$ be a monotonic non-decreasing function with continuous derivative in a given interval $a \leq x < b$. Put $\alpha = \phi(a)$ and let β denote the limit to which $\phi(x)$ tends as $x \to b$. Let

$$(5.1)\quad \begin{cases} t = \tau + b - \beta & \text{if } b \text{ and } \beta \text{ are finite,} \\ t = \tau & \text{if } b = \beta = \infty, \\ t = \frac{1}{\beta-\tau} & \text{if } b = \infty \text{ and } \beta \text{ is finite,} \\ t = b - \frac{1}{\tau} & \text{if } b \text{ is finite and } \beta = \infty \end{cases}$$

Therefore, if $\tau < \beta$ tends to β, then $t < b$ tends to b, so that $\phi(t)$ tends to β.

Under these conditions

$$(5.2)\quad \int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_\alpha^\beta f(y)\,dy + \lim_{\substack{\tau < \beta \\ \tau \to \beta}} \int_\tau^{\phi(t)} f(y)\,dy ,$$

provided that the integral and the limit, occurring on the right hand side, exist modulo $M(\beta-)$; then the integral on the left hand side exists with respect to the equivalent modulus $M(b-)$ and is equal to the right hand side.

Proof. We have for $a < t < b$

$$(5.3) \quad \begin{cases} \displaystyle\int_a^t f(\phi(x))\,\phi'(x)\,dx = \int_\alpha^{\phi(t)} f(y)\,dy \\ \displaystyle\qquad\qquad = \int_\alpha^\tau f(y)\,dy + \int_\tau^{\phi(t)} f(y)\,dy \;. \end{cases}$$

It follows from the hypothesis that the last side tends to λ, modulo $M(\beta-)$, as $\tau < \beta$ tends to β, where λ denotes the right hand side of (5.2). Consequently $M(\beta-)$ contains a function $\chi(\tau)$ such that

$$(5.4) \quad \int_\alpha^\tau f(y)\,dy + \int_\tau^{\phi(t)} f(y)\,dy - \chi(\tau) \to \lambda$$

(in the usual sense) as $\tau < \beta$ tends to β. From (5.1) it follows that $\chi(\tau)$ is a function $g(t)$ of t and since $M(\beta-)$ and $M(b-)$ are equivalent, $g(t)$ belongs to the modulus $M(b-)$.

From (5.3) and (5.4) it follows that

$$\int_a^t f(\phi(x))\,\phi'(x)\,dx - g(t) \to \lambda$$

(in the usual sense) as $t < b$ tends to b, so that the integral

$$\int_a^b f(\phi(x))\,\phi'(x)\,dx$$

exists modulus $M(b-)$ and is equal to λ. This completes the proof.

In a similar way we find:

THEOREM 13. _Let $\phi(x)$ be a monotonic non-decreasing function with a continuous derivative in a given interval_ $a < x \leqq b$. _Let_ $\beta = \phi(b)$ _and let α denote the limit to which $\phi(x)$ tends as_ $x \to a$.

Let

$$t = \tau + a - \alpha \qquad \text{if } a \text{ and } \alpha \text{ are finite,}$$

$$t = \tau \qquad \text{if } a = \alpha = -\infty;$$

$$t = \frac{1}{\alpha - \tau} \qquad \text{if } a = -\infty \text{ and } \alpha \text{ is finite}$$

$$t = a - \frac{1}{\tau} \qquad \text{if } a \text{ is finite and } \alpha = -\infty$$

Under these conditions

$$\int_a^b f(\phi(x))\, \phi'(x)\, dx = \int_\alpha^\beta f(y)\, dy - \lim_{\substack{\tau > \alpha \\ \tau \to \alpha}} \int_\tau^{\phi(t)} f(y)\, dy \quad ,$$

provided that the integral and the limit, occurring on the right hand side, exist modulo $M(\alpha+)$; then the integral on the left hand side exists with respect to the equivalent modulus $M(a+)$ and is equal to the right hand side.

THEOREM 14. Let $\phi(x)$ be a monotonic non-increasing function with a continuous derivative in a given interval $a \leq x < b$. Put $\alpha = \phi(a)$ and let β denote the limit to which $\phi(x)$ tends as $x \to b$.

Put

$$t = -\tau + b + \beta \qquad \text{if } b \text{ and } \beta \text{ are finite}$$

$$t = -\tau \qquad \text{if } b = -\beta = \infty$$

$$t = \frac{1}{\tau - \beta} \qquad \text{if } b = \infty \text{ and } \beta \text{ is finite}$$

$$t = b + \frac{1}{\tau} \qquad \text{if } b \text{ is finite and } \beta = -\infty \quad .$$

Under these conditions

$$\int_a^b f(\phi(x))\, \phi'(x)\, dx = \int_\alpha^\beta f(y)\, dy + \lim_{\substack{\tau > \beta \\ \tau \to \beta}} \int_\tau^{\phi(t)} f(y)\, dy \quad ,$$

provided that the integral and the limit, occurring on the right hand side, exist modulo $M(\beta+)$; then the integral on the left hand side exists with respect to the equivalent modulus $M(b-)$ and is equal to the right hand side.

THEOREM 15'. Let $\phi(x)$ be in a given interval $a < x \leq b$ a monotonic non-increasing function with continuous derivative. Put $\beta = \phi(b)$ and let α denote the limit to which $\phi(x)$ tends as $x \to a$. Put

$t = -\tau + a + \alpha$ if a and α are finite,

$t = -\tau$ if $\alpha = -a = \infty$

$t = \frac{1}{\tau - \alpha}$ if $a = -\infty$ and α is finite

$t = a + \frac{1}{\tau}$ if a is finite and $\alpha = \infty$

Under these conditions

$$\int_a^b f(\phi(x))\, \phi'(x)\, dx = \int_\alpha^\beta f(y)\, dy - \lim_{\substack{\tau < \alpha \\ \tau \to \alpha}} \int_\tau^{\phi(t)} f(y)\, dy \quad ,$$

provided that the integral and the limit, occurring on the right hand side, exist modulo $M(\alpha-)$; then the integral on the left hand side exists with respect to the equivalent modulus $M(a+)$ and is equal to the right hand side.

In the following examples of Theorems 12-15 we use the logarithmic moduli.

$$\int_1^{\infty} \frac{dx}{x} = \int_1^{\infty} \frac{2dx}{2x} = \int_2^{\infty} \frac{dy}{y} + \lim_{\tau \to \infty} \int_{\tau}^{2\tau} \frac{dy}{y}$$

$$= \int_2^{\infty} \frac{dy}{y} + \log 2, \text{ modulo } L(\infty)$$

which is obvious anyway since

$$\int_1^{\infty} \frac{dx}{x} - \int_2^{\infty} \frac{dx}{x} = \int_1^{2} \frac{dx}{x} = \log 2 \quad .$$

Moreover

$$\int_0^{1} \frac{dx}{x} = \int_0^{1} \frac{2dx}{2x} = \int_0^{2} \frac{dy}{y} - \lim_{\substack{\tau > 0 \\ \tau \to 0}} \int_{\tau}^{2\tau} \frac{dy}{y}$$

$$= \int_0^{2} \frac{dy}{y} - \log 2, \text{ modulo } L(0+) \quad .$$

Also

$$\int_{-1}^{0} \frac{dx}{x} = \int_{-1}^{0} \frac{-2dx}{-2x} = -\int_0^{2} \frac{dy}{y} + \lim_{\substack{\tau > 0 \\ \tau \to 0}} \int_{\tau}^{2\tau} \frac{dy}{y}$$

$$= -\int_0^{2} \frac{dy}{y} + \log 2, \text{ modulis } L(0-) \text{ and } L(0+)$$

and finally

$$\int_{-\infty}^{-1} \frac{dx}{x} = \int_{-\infty}^{-1} \frac{-2dx}{-2x} = -\int_{2}^{\infty} \frac{dy}{y} - \lim_{\tau \to \infty} \int_{\tau}^{2\tau} \frac{dy}{y}$$

$$= -\int_{2}^{\infty} \frac{dy}{y} - \log 2, \text{ modulis } L(-\infty) \text{ and } L(\infty) .$$

For the calculation of the integral

$$\int_{0}^{\infty} x^{p-1} (1+x)^{-p-q} \, dx \quad ,$$

where p and q denote arbitrary complex numbers such that neither of them is equal to an integer ≤ 0, we divide the path of integration by a positive number a into two parts and we apply the transformation

$$y = \emptyset(x) = \frac{x}{1+x} \quad , \quad \text{so that } x = \frac{y}{1-y} \quad .$$

According to Theorem 12 we have

$$\int_{a}^{\infty} x^{p-1}(1+x)^{-p-q} \, dx = \int_{\alpha}^{1} y^{p-1}(1-y)^{q-1} \, dy + \lim_{\tau \to 1} \int_{\tau}^{\emptyset(t)} y^{p-1}(1-y)^{q-1} \, dy \quad ,$$

where

$$\alpha = \emptyset(a) \quad ; \quad t = \frac{1}{1-\tau} \text{ and } \emptyset(t) = \frac{t}{1+t} = \frac{1}{2-\tau} \quad ,$$

provided the integral and limit exist with respect to some modulus $M(1-)$ for which $1-$ is an ordinary point. We will find that $L(1-)$ possesses this property.

We have

$$(1-v)^{p-1}\,v^{q-1} = v^{q-1} - \binom{p-1}{1} v^{q} + \cdots \pm \binom{p-1}{h} v^{q-1+h} + r(1-v) \quad .$$

If the positive integer h is large enough, the remainder $r(1-v)$ tends to zero as $v \to 0$. Consequently, we have

$$y^{p-1}(1-y)^{q-1} = (1-y)^{q-1} - \binom{p-1}{1}(1-y)^{q} + \cdots \pm \binom{p-1}{h}(1-y)^{q-1+h} + r(y) \; ,$$

where $r(y) \to 0$ as $y \to 1$ for sufficiently large h. Then the integral

$$(5.5) \qquad \int_{\tau}^{(2-\tau)^{-1}} y^{p-1}(1-y)^{q-1}\,dy$$

can be written as

$$\begin{aligned}
&-\frac{1}{q}(1-\tau)^{q}\left\{(2-\tau)^{-q}-1\right\} \\
&+\binom{p-1}{1}\frac{1}{q+1}(1-\tau)^{q+1}\left\{(2-\tau)^{q+1}-1\right\} + \cdots \\
&\pm\binom{p-1}{h}\frac{1}{q+h}(1-\tau)^{q+h}\left\{(2-\tau)^{q+h}-1\right\} + \int_{\tau}^{(2-\tau)^{-1}} r(y)\,dy \\
&= a_0(1-\tau)^{q} + a_1(1-\tau)^{q+1} + \cdots + a_h(1-\tau)^{q+h} \\
&\quad + \rho(\tau) + \int_{\tau}^{(2-t)^{-1}} r(y)\,dy \quad ,
\end{aligned}$$

where $\rho(\tau) \to 0$ as $\tau \to 1$. Since none of the exponents q, $q+1$, $\cdots$ $q+h$ is equal to zero, integral (5.5) tends to zero, modulo L(1−), as $\tau \to 1$. Consequently $\int_a^{\infty} x^{p-1}(1+x)^{-p-q}\,dx$ exists modulo $L(\infty)$ and

$$\int_a^{\infty} x^{p-1}\,(1+x)^{-p-q}\,dx = \int_{\alpha}^{1} y^{p-1}\,(1-y)^{q-1}\,dy \quad .$$

In a similar way we find modulo L(0+)

$$\int_0^{a} x^{p-1}\,(1+x)^{-p-q}\,dx = \int_0^{\alpha} y^{p-1}\,(1-y)^{q-1}\,dy \quad ,$$

therefore modulis L(0+) and $L(\infty)$

$$\int_0^{\infty} x^{p-1}\,(1+x)^{-p-q}\,dx = \int_0^{1} y^{p-1}\,(1-y)^{q-1}\,dy = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

according to formula (4.4) in the preceding section.

As a last example we consider the integral

$$I = \int_0^{\infty} x^{2p-1}\,(1+x^2)^{-p-q}\,dx \quad ,$$

where neither p nor q is an integer $\leqq 0$. Applying Theorem 12 with the substitution $y = x^2$ we get for positive a

$$I_1 = \int_{a^2}^{\infty} x^{2p-1}\,(1+x^2)^{-p-q}\,dx$$

(5.6)

$$= \tfrac{1}{2} \int^{\infty} y^{p-1}\,(1+y)^{-p-q}\,dy + \tfrac{1}{2} \lim_{\tau\to\infty} \int_{\tau}^{\tau^2} y^{p-1}\,(1+y)^{-p-q}\,dy$$

for $\emptyset(t)$, the upper limit of the last integral is equal to $t^2 = \tau^2$ since $\tau = t$.

The integrand in the last integral can be written as

$$a_0 y^{-q-1} + a_1 y^{-q-2} + \cdots + a_h y^{-q-h-1} + r(y) \quad .$$

If h is sufficiently large

$$\int_{\tau}^{\tau^2} r(y)\, dy \to 0 \qquad \text{as } \tau \to \infty \quad .$$

Since all the integrated terms $\int_{\tau}^{\tau^2} a_k\, y^{-q-(k+1)}\, dy$ $(k = 0,1 \cdots h)$ belong to the logarithmic modulus $L(\infty)$, the last term in (5.6) is equal to zero and

$$I_1 = \tfrac{1}{2} \int_{a^2}^{\infty} y^{p-1}\, (1 + y)^{-p-q}\, dy \quad .$$

In a similar way we find modulo $L(0+)$

$$\int_0^{a^2} x^{2p-1}\, (1 + x^2)^{-p-q}\, dx = \tfrac{1}{2} \int_0^{a} y^{p-1}\, (1 + y)^{-p-q}\, dy \quad .$$

Therefore modulis $L(0+)$ and $L(\infty)$

$$I = \tfrac{1}{2} \int_0^{\infty} y^{p-1}\, (1 + y)^{-p-q}\, dy = \frac{\Gamma(p)\Gamma(q)}{2\,\Gamma(p+q)} \quad ,$$

where the last equality follows from the previous result.

The generalized limits can be also used in the theory of series. For instance let us evaluate modulo $L(\infty)$

$$\sum_{h=1}^{\infty} \frac{1}{(h-1+w)^s} \quad \text{where } 0 < w \leqq 1 \quad .$$

That means that we must calculate

$$\lim_{t \to \infty} \sum_{h=1}^{t} \frac{1}{(h-1+w)^s} \quad (L(\infty)) \quad .$$

If s is equal to an integer $\leqq 0$, then according to Part III, Chapter I, Section 7, Theorem 13, the sum

$$\sum_{h=1}^{t} (h-1+w)^{-s}$$

is equal to a polynomial in t, in which the constant term is equal to zero. This polynomial, therefore, belongs to $L(\infty)$, so that

$$\sum_{h=1}^{\infty} \frac{1}{(h-1+w)^s} = 0 \qquad (L(\infty))$$

for each integer $s \leqq 0$. Let us now consider the case in which s is not an integer $\leqq 1$. According to Part III, Chapter I, Section 11, formula (11.1), we have

$$\sum_{h=1}^{t} \frac{1}{(h-1+w)^s} = \zeta(s,w) + g(t) + R$$

where g(t) has the form

$$g(t) = a_0\, t^{-s+1} + a_1\, t^{-s} + \cdots + a_{m+1}\, t^{-s-m} ,$$

If m is sufficiently large, the remainder R tends to zero as $t \to \infty$. Since none of the exponents $-s+1, -s, \cdots, -s-m$ is equal to zero, g(t) belongs to $L(\infty)$. Consequently

$$\sum_{h=1}^{\infty} \frac{1}{(h-1+w)^s} = \zeta(s,w) \qquad (L(\infty))$$

for $0 < w \leq 1$, if s is not an integer ≤ 1.

Section 6. ON THE SUM FORMULA OF EULER IN THE THEORY OF THE MODULO LIMITS

Let us start with

$$\sum_{n=1}^{N} f(n) - \frac{1}{2} f(1) - \frac{1}{2} f(N) - \int_1^N f(x)\, dx = \int_1^N \Psi_1(x)\, f'(x)\, dx \quad ,$$

where N is an integer > 1 and where f(x) denotes a function which is continuously differentiable for x > 0. Here $\Psi_1(x)$ is the function with period 1 which is equal to zero at x = 0 and which is equal to $x - \frac{1}{2}$ in the interval $0 < x < 1$. For each number δ between 0 and 1 we find

$$\int_\delta^1 f(x)\, dx + \int_\delta^1 \Psi_1(x)\, f'(x)\, dx = \int_\delta^1 d\left(\Psi_1(x)\, f(x)\right) =$$

$$= \Psi_1(1-)\, f(1) - \Psi_1(\delta)\, f(\delta) = \frac{1}{2} f(1) - \left(\delta - \frac{1}{2}\right) f(\delta) \quad .$$

Subtracting we obtain

$$\sum_{n=1}^{N} f(n) - \frac{1}{2} f(N) - \int_\delta^N f(x)\, dx = \left(\delta - \frac{1}{2}\right) f(\delta) + \int_\delta^N \Psi_1(x)\, f'(x)\, dx \quad .$$

Let us now assume that we have for f'(x) a formal expansion

$$f'(x) = \sum_{h=0}^{\infty} g_h(x) \quad .$$

Then we write

$$f'(x) = \sum_{h=0}^{k-1} g_h(x) + G_k(x) \quad ,$$

so that

$$\sum_{n=1}^{N} f(n) - \frac{1}{2} f(N) - \int_{\delta}^{N} f(x)\,dx - (\delta - \tfrac{1}{2})\, f(\delta) =$$

$$= \sum_{h=0}^{k-1} \int_{\delta}^{N} \Psi_1(x)\, g_h(x)\,dx + \int_{\delta}^{N} \Psi_1(x)\, G_k(x)\,dx \quad .$$

Taking the limit modulo $P(\infty)$ as $N \to \infty$ and the limit modulo $L(0+)$ as $\delta \to 0$, we obtain

$$(6.1) \quad \begin{cases} \displaystyle\sum_{n=1}^{\infty} f(n) - \frac{1}{2} \lim_{N\to\infty} f(N) - \int_{0}^{\infty} f(x)\,dx = \\ \displaystyle= \lim_{\delta\to\frac{1}{2}} (\delta - \tfrac{1}{2})\, f(\delta) + \sum_{h=0}^{k-1} \int_{0}^{\infty} \Psi_1(x)\, g_h(x)\,dx + R_k \quad , \end{cases}$$

where

$$R_k = \int_{0}^{\infty} \Psi_1(x)\, G_k(x)\,dx \quad ,$$

provided of course that these limits exist, modulis $P(\infty)$ and $L(0+)$.

In this way we obtain the following theorem:

THEOREM 16. Suppose that $f(x)$ is continuously differentiable for $x > 0$. Let

$$f'(x) = \sum_{h=0}^{k-1} g_h(x) + G_k(x) \quad .$$

Then formula (6.1) holds, if the terms in that relation exist with respect to the moduli $P(\infty)$ and $L(0+)$.

Remarks: 1. If $f(x)$ depends not only on x but also on an unbounded variable ω and the series

$$\sum_{k=0}^{\infty} \int_0^{\infty} \Psi_1(x)\, G_k(x)\, dx$$

is asymptotic, then the function

$$(6.2) \quad \sum_{n=1}^{\infty} f(n) - \frac{1}{2} \lim_{N \to \infty} f(N) - \int_0^{\infty} f(x)\, dx - \lim_{\delta \to 0} (\delta - \tfrac{1}{2})\, f(\delta)$$

is asymptotically equal to

$$(6.3) \qquad \sum_{h=0}^{\infty} \int_0^{\infty} \Psi_1(x)\, g_h(x)\, dx$$

under the conditions of Theorem 9.

2. If under the conditions of Theorem 9 the remainder R_k tends to zero as $k \to \infty$, then expression (6.2) is equal to the sum of the convergent series (6.3).

Section 7. NEUTRALIZERS

The application of the sum formula of Euler, given in the preceding section, leads to integrals of the form

$$\int_0^\infty \Psi_1(x)\ g(x)\ dx \quad . \tag{7.1}$$

Sometimes it is necessary to divide such an integral into two parts

$$\int_0^t \Psi_1(x)\ g(x)\ dx + \int_t^\infty \Psi_1(x)\ g(x)\ dx \quad ,$$

where $t > 0$. If the integral (7.1) converges at infinity in the usual sense, the integral

$$\int_t^\infty \Psi_1(x)\ g(x)\ dx \tag{7.2}$$

tends to zero as $t \to \infty$, but possibly very slowly and perhaps too slowly for our purpose. If the integral (7.1) does not converge at infinity in the ordinary sense, but only with respect to a certain given modulus $M(\infty)$, then the integral (7.2) does not even tend to zero as $t \to \infty$; it may happen that small changes in the large number t produce very large changes in the value of the integral (7.2). To overcome the difficulties which are consequences of this phenomenon we introduce a neutralizer. Let n be a given integer $\geqq 0$. We call $N(u)$ the neutralizer of the nth order, if

$$N(u) = 0 \qquad \text{for } u \leqq 0 \ ,$$

(7.3) $$\qquad = 1 \qquad \text{for } u \geqq 1$$

$$\qquad = c \int_0^u v^n(1 - v)^n \, dv \text{ in the interval } 0 \leqq u \leqq 1 \ ,$$

where the constant c is chosen such that $N(1) = 1$, hence

$$c = \frac{(2n+1)!}{n!n!} \quad .$$

This function $N(u)$ is everywhere, therefore also at the points $u = 1$ and at the origin, n times differentiable and satisfies the relations

(7.4) $$N^{(h)}(0) = N^{(h)}(1) = 0 \qquad (1 \leqq h \leqq n) \quad .$$

The identity

(7.5) $$N(u) + N(1 - u) = 1$$

is evident for $u \geqq 1$ and also for $u \leqq 0$, and in the interval $0 \leqq u \leqq 1$ we have

$$N(1 - u) = c \int_0^{1-u} v^n(1 - v)^n \, dv = c \int_u^1 v^n(1 - v)^n \, dv \quad ,$$

so that

$$N(u) + N(1 - u) = c \int_0^1 v^n \, (1 - v)^n \, dv = N(1) = 1 \quad .$$

This identity (7.5) enables us to divide the integral (7.1) into two parts as follows

$$(7.6)\quad \int_0^{\infty} \Psi_1(x)g(x)dx = \int_0^{2t} \Psi_1(x)g(x)\, N\left(2 - \frac{x}{t}\right) dx + \int_t^{\infty} \Psi_1(x)g(x)\, N\left(\frac{x}{t} - 1\right) dx ;$$

notice that $N\left(2 - \frac{x}{t}\right) = 0$ for $x \geqq 2t$ and that $N\left(\frac{x}{t} - 1\right) = 0$ for $x \leqq t$.

In this way we obtain instead of the integral (7.2), which is difficult to handle, the integral

$$(7.7)\qquad \int_t^{\infty} \Psi_1(x)\, g(x)\, N\left(\frac{x}{t} - 1\right) dx \quad ,$$

whose absolute value is, as we shall see, under general conditions, small for large t and for suitably chosen n. In other words, the function $N\left(\frac{x}{t} - 1\right)$ neutralizes almost completely the influence of t, provided that t is large enough and that n is conveniently chosen. Let us begin with a simple example.

THEOREM 17. *Let t be a positive number and let N(u) be the neutralizer of positive order n. Let g(x) be n + 1 times continuously differentiable for $x \geqq t$ such that for $h = 0, 1, \cdots , n$*

$$(7.8)\qquad |g^{(h)}(x)| \leqq K\, \mu^h \quad (t \leqq x \leqq 2t) \quad ,$$

where K and μ denote numbers $\geqq 0$ which are independent of x. Assume moreover that as $x \to \infty$

$$(7.9)\qquad \Psi_{h+2}(x)g^{(h)}(x) \to 0 \qquad (h = 0,1, \cdots n)$$

with respect to a certain modulus $M(\infty)$, for which infinity is an ordinary point. Finally we suppose that the integral

$$J = \int_t^{\infty} |g^{(n+1)}(x)|\,dx \tag{7.10}$$

<u>converges in the ordinary sense.</u> <u>Then the integral</u>

$$I = \int_t^{\infty} \Psi_1(x)\; g(x)\; N\left(\frac{x}{t} - 1\right) dx \tag{7.11}$$

<u>exists modulo</u> $M(\infty)$ <u>and satisfies the inequality</u>

$$|I| \leqq c_n \left\{ J + Kt\,(\mu + \frac{1}{t})^{n+1} \right\} \quad , \tag{7.12}$$

<u>where</u> c_n <u>denotes a suitably chosen number which depends only on</u> n.

<u>Proof:</u> Integrating by parts n + 1 times we find modulo $M(\infty)$

$$I = \pm \int_t^{\infty} \Psi_{n+2}(x) \left\{ \frac{d^{n+1}}{dx^{n+1}} \left(N\left(\frac{x}{t} - 1\right) g(x)\right)\right\} dx \quad ;$$

notice that the integrated parts are equal to zero, since the contribution of infinity is equal to zero according to (7.9) and the contribution of t is equal to zero according to (7.4).

In this formula

$$\frac{d^{n+1}}{dx^{n+1}} \left(N\left(\frac{x}{t} - 1\right) g(x)\right) = \sum_{h=o}^{n+1} \binom{n+1}{h}\, t^{-h}\, N^{(h)}\left(\frac{x}{t} - 1\right) f^{(n+1-h)}(x) \quad .$$

If $x \geqq 2t$ all terms on the right hand side, the first term excepted, are equal to zero, so that we obtain modulo $M(\infty)$

$$\pm I = \int_t^{\infty} \Psi_{n+2}(x)\, N\left(\frac{x}{t} - 1\right)\, g^{(n+1)}(x)\, dx$$

(7.13)

$$+ \sum_{h=1}^{n+1} \binom{n+1}{h} t^{-h} \int_t^{2t} \Psi_{n+2}(x)\, N^{(h)}\left(\frac{x}{t} - 1\right)\, g^{(n+1-h)}(x)\, dx \quad .$$

The periodic function $\Psi_{n+2}(x)$ is in absolute value less than a convenient number which depends only on n. Moreover $0 \leq N\left(\frac{x}{t} - 1\right) \leq 1$, so that the first term on the right hand side of (7.13) is at most equal to J multiplied by a coefficient depending only on n. The absolute value of $N^{(h)}\left(\frac{x}{t} - 1\right)$ is also less than a suitably chosen number which depends only on n, so that the sum Σ occurring on the right hand side of (7.13) is in absolute value

$$\leq c \sum_{h=0}^{n+1} \binom{n+1}{h} t^{1-h} K\mu^{n+1-h} = cKt\left(\mu + \frac{1}{t}\right)^{n+1} \quad ,$$

where c depends only on n. This gives the required result.

The preceding theorem is very useful in the examination of asymptotic expansions, since in that theory we generally can choose n fixed. But in the theory of convergent expansions it is often necessary to choose for n a number which increases indefinitely; in such a case we must know a convenient upper bound for the coefficient c_n occurring in the assertion of the preceding theorem. We shall give this upper bound in theorem 18 but first we formulate a lemma which shall be applied in the proof of theorem 18.

LEMMA. We have in the interval $0 \leqq u \leqq 1$ for each integer h which is $\geqq 0$ and $\leqq n$

$$\left|\frac{d^h u^n(1-u)^n}{du^h}\right| \leqq n^h\, u^{n-h}\,(1-u)^{n-h} \leqq n^h\, 2^{-2n+2h} \quad .$$

Proof. We know that

$$\frac{d^h u^n(1-u)^n}{du^h} = \Sigma_1 \pm \binom{h}{k} \sigma_k\, u^{n-k}\,(1-u)^{n-h+k} \quad ,$$

where

$$\sigma_k = n(n-1)\cdots(n+1-k)\, n(n-1)\cdots(n+1-h+k) \quad ;$$

the sum Σ_1 is extended over the integers $k \geqq 0$ such that

$$k \leqq h \quad ; \qquad k \leqq n \quad ; \qquad k \geqq n-h \quad .$$

The coefficient σ_k is a product of h positive factors, each $\leqq n$, so that $0 \leqq \sigma_k \leqq n^h$ and

$$|\Sigma_1| \leqq \sum_{k=0}^{h} \binom{h}{k} n^h\, u^{n-k}\,(1-u)^{n-h+k}$$

$$= n^h\, u^{n-h}\,(1-u)^{n-h}\,(u+(1-u))^h$$

$$= n^h\, u^{n-h}\,(1-u)^{n-h} \quad .$$

This completes the proof.

THEOREM 18. Under the conditions of Theorem 17

$$|I| < \frac{1}{12(2\pi)^n} J + \frac{Kt}{16(2\pi)^n}\left(\mu + \frac{4n}{t}\right)^{n+1} \quad .$$

Proof. From part III, Chapter I, Section 4, Page 3, formulas (4.7) and (4.8) it follows that

$$|\Psi_{n+2}(x)| \leqq \sum_{h=1}^{\infty} \frac{2}{(2\pi h)^{n+2}} \leqq \frac{2}{(2\pi)^{n+2}} \sum_{h=1}^{\infty} \frac{1}{h^2}$$

$$= \frac{2}{(2\pi)^{n+2}} \cdot \frac{\pi^2}{6} = \frac{1}{12(2\pi)^n} \quad .$$

Consequently the first term on the right hand side of (7.13) is in absolute value at most equal to

$$\frac{1}{12(2\pi)^n} \int_t^{\infty} \left| g^{(n+1)}(x)\right| dx \quad ,$$

since $0 \leqq N\left(\frac{x}{t} - 1\right) \leqq 1$.

According to the preceding lemma, applied with $h - 1$ instead of h, we find in the interval $0 \leqq u \leqq 1$ for $h = 1, 2, \cdots, n + 1$

$$|N^{(h)}(u)| \leqq cn^{h-1}\, 2^{-2n+2h-2} \quad .$$

Here

$$c = \frac{(2n+1)!}{n!n!} = (2n+1) \cdot \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{n!n!}$$

$$\leqq (2n+1) \cdot \frac{2^n n! \cdot 2^n n!}{n!n!} < 3n\, 2^{2n} \quad ,$$

therefore

$$|N^{(h)}(u)| < \frac{3}{4} n^h 2^{2h} .$$

Consequently the sum Σ occurring in (7.13) is by (7.8) in absolute value at most equal to

$$\frac{1}{16(2\pi)^n} \sum_{h=0}^{n+1} \binom{n+1}{h} t^{1-h} n^h 2^{2h} K \mu^{n+1-h}$$

$$= \frac{Kt}{16(2\pi)^n} \left(\frac{4n}{t} + \mu\right)^{n+1} .$$

This result gives the required result.

An important condition in the two preceding theorems is the inequality

$$\left|g^{(h)}(x)\right| \leqq K \mu^h .$$

If a function g(x), the number h and a point x are given, how can we find two numbers K and μ, such that this inequality holds? To that end the following theorems may be useful.

THEOREM 19 (Product Theorem). _If_

$$|g^{(h)}(x)| \leqq K_h \mu_h^h \qquad (h = 0,1, \cdots , n)$$

and

(7.14) $$|\chi^{(h)}(x)| \leqq L_h \nu_h^h \qquad (h = 0,1, \cdots , n)$$

hold, when K_h, L_h, μ_h and ν_h are monotonic non-decreasing functions $\geqq 0$ of h, then the product

$$p(x) = g(x)\,\chi(x)$$

satisfies the inequalities

$$(7.15) \qquad |p^{(h)}(x)| \leqq K_h\, L_h(\mu_h + \nu_h)^h \qquad (h = 0,1, \cdots , n) \quad .$$

Proof. We have

$$|p^{(h)}(x)| = \left| \sum_{k=0}^{h} \binom{h}{k} g^{(k)}(x)\, \chi^{(h-k)}(x) \right|$$

$$\leqq \sum_{k=0}^{h} \binom{h}{k} K_h\, \mu_h^{k}\, L_h\, \nu_h^{h-k}$$

$$= K_h L_h (\mu_h + \nu_h)^h \quad .$$

THEOREM 20 (On a function of a function). Consider

$$f(x) = \chi(g(x)) \quad .$$

Suppose

$$(7.16) \qquad \frac{1}{h!}\left|\frac{d^h g(x)}{dx^h}\right| \leqq L_h\, v_h^{h} \qquad (h = 1,2, \cdots , n) \quad .$$

and

$$(7.17) \qquad \frac{1}{h!}\left|\frac{d^h \chi(y)}{dy^h}\right| \leqq a_h \qquad (h = 0,1, \cdots , n) \quad ,$$

where $y = g(x)$. Let us assume that L_h and $v_h (0 \leq h \leq n)$ and $\frac{a_{h-1}}{a_h}$ $(1 \leq h \leq n)$ are monotonic non-decreasing functions of h.

Then the inequality

$$\frac{1}{(h+1)!} \left| f^{(h)}(x) \right| \leq (2\, L_h\, v_h)^h\, a_h \tag{7.18}$$

holds for $h = 0$ and also for the positive integers $h \leq n$ with

$$a_h\, L_h \geq a_{h-1} \quad . \tag{7.19}$$

For the other positive integers $h \leq n$ there exists a smallest positive integer q (this integer is $\leq h$) such that

$$a_q\, L_h < a_{q-1} \quad ; \tag{7.20}$$

for these integers h we have

$$\frac{1}{(h+1)!} \left| f^{(h)}(x) \right| \leq 2^h\, {v_h}^h\, {L_h}^q\, a_q \quad . \tag{7.21}$$

Proof. We must prove the inequalities (7.18) and (7.21) for $x = x_o$ under the assumption that the inequalities (7.16) and (7.17) hold at $x = x_o$. If we replace $g(x)$ and $\chi(y)$ by the polynomials

$$\sum_{k=o}^{h} \frac{g^{(k)}(x_o)}{k!} (x - x_o)^k \quad \text{and} \quad \sum_{k=o}^{h} \frac{\chi^{(k)}(y)}{k!} (y - y_o)^k \quad ,$$

where $y_o = g(x_o)$, the function $f(x) = \chi(g(x))$ is replaced by a polynomial whose derivatives of order $\leq h$ at $x = x_o$ are the same as those of the functions f(x) itself. Without loss of generality we may therefore suppose that g(x) and $\chi(y)$ are polynomials of degree $\leqq h$.

Proof. In this way we find by (7.16)

$$|g(x) - g(x_o)| = \left| \sum_{k=1}^{h} \frac{g^{(k)}(x_o)}{k!} (x - x_o)^k \right|$$

$$\leqq \sum_{k=1}^{h} L_h v_h^{\,h} |x - x_o|^k < L_h$$

for the points x with $|x - x_o| v_h = \frac{1}{2}$. For these points x we have

$$f(x) = \chi(g(x)) = \sum_{k=o}^{h} \frac{\chi^{(k)}(g(x_o))}{k!} (g(x) - g(x_o))^k ,$$

so that, according to (7.17),

$$|f(x)| \leqq \sum_{k=o}^{h} a_k L_h^{\,k} . \tag{7.22}$$

Now we distinguish two cases.

1. Consider first the integer h = 0 and the positive integers $h \leqq n$ which satisfy inequality (7.19). For h = 0 the left hand side of (7.18) is equal to

$$|f(x_o)| = |\chi(g(x_o)| \leqq a_o$$

according to (7.17), so that formula (7.18) holds for h = 0 at $x = x_o$. Consider now an integer $h \leqq n$ subject to (7.19). Since $\frac{a_k}{a_{k-1}}$ $(1 \leqq k \leqq n)$ is a monotonic non-increasing function of k, we have

$$\frac{a_k}{a_{k-1}} L_h \geqq 1 \qquad (k = 1,2, \cdots , h) \quad .$$

Multiplying we obtain for $k = 0,1, \cdots , h$

$$\frac{a_h}{a_k} L_h \geqq 1 \quad ,$$

so that

$$a_k L_h^k \leqq a_h L_h^h \quad .$$

It follows therefore from (7.22) that

$$|f(x)| \leqq (h + 1) \; a_h L_h^h$$

for the points x lying on the circle Γ with center x_o and radius $\frac{1}{2v}$.

The polynomial $f(x) = \chi(g(x))$ satisfies the identity

$$\frac{1}{h!} f^{(h)}(x_o) = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{(x-x_o)^{h+1}} dx \quad , \tag{7.23}$$

hence

$$\frac{1}{h!} |f^{(h)}(x_o)| \leqq (h + 1) \; a_h L_h^h \; (2v_h)^h \quad ,$$

which gives the required result (7.18) at $x = x_o$.

2. Let us now consider a positive integer $h \leqq n$ which does not satisfy (7.19), so that

$$\frac{a_h}{a_{h-1}} < L_h \quad .$$

The smallest positive integer q with (7.20) is therefore $\leqq$ h. Then

$$\frac{a_k}{a_{k-1}} L_h \geqq 1 \qquad \text{for } k = 1,2, \cdots , q-1$$

$$< 1 \qquad \text{for } k = q, q+1, \cdots , h \ ,$$

therefore

$$a_k \, L_h^{\ k} \leqq a_q \, L_h^{\ q} \qquad \text{for } k = 0,1, \cdots , n \ .$$

For the points x on the circle Γ we find therefore by (7.22)

$$|f(x)| \leqq (h+1) \; a_q \, L_h^{\ q} \ ,$$

so that it follows from (7.23) that

$$\frac{1}{h!} \, |f^{(h)}(x_o)| \leqq (h+1) \; a_q \, L_h^{\ q} \, (2v_h) \ .$$

This completes the proof.

Examples. To find an upper bound for the absolute value of the derivatives of

$$f(x) = e^{-\epsilon x^{\alpha}}$$

where $x > 0$ and $\alpha > 0$, we let

$$g(x) = \epsilon x^{\alpha} \qquad \text{and } \chi(y) = e^{-y} \ .$$

Then we have for $h \geqq 1$

$$\frac{1}{h!} \, |g^{(h)}(x)| = \frac{1}{h!} \, |\epsilon \alpha (\alpha - 1) \cdots (\alpha + 1 - h)| x^{\alpha - h} \ ,$$

hence

$$\frac{1}{h!} \, |g^{(h)}(x)| \leqq |\epsilon| \; \alpha \; x^{\alpha - h}$$

if $0 < \alpha \leqq 1$. If $\alpha \geqq 1$, we have

$$|k - 1 - \alpha| \leqq k\,\alpha \qquad \text{for } k \geqq 1 \quad ,$$

hence

$$\left| \frac{\alpha}{1} \cdot \frac{\alpha - 1}{2} \cdot \ldots \cdot \frac{\alpha + 1 - h}{h} \right| \leqq \alpha^{h} \quad ,$$

so that in that case

$$\frac{1}{h!} \left| g^{(h)}(x) \right| \leqq |\epsilon| \alpha^{h} x^{\alpha - h} \quad .$$

Let $L = |\epsilon| \alpha\, x^{\alpha}$. We can apply the preceding theorem with

$$L_h = L \qquad \text{and } v_h = x^{-1} \qquad \text{if } 0 < \alpha \leqq 1$$

and with

$$L_h = |\epsilon| \; x^{\alpha} \qquad \text{and } v_h = \alpha\, x^{-1} \qquad \text{if } \alpha \geqq 1 \quad .$$

Furthermore

$$\frac{1}{h!} \left| \chi^{(h)}(y) \right| = \frac{|e^{-y}|}{h!} \quad ,$$

so that we may choose in the preceding theorem

$$a_h = \frac{|e^{-y}|}{h!} = \frac{|e^{-\epsilon x^{\alpha}}|}{h!} \quad .$$

The inequality (7.19) assumes here the form $h \leqq L^{*}$, where

$$L^* = L = |\epsilon| \alpha x^\alpha \qquad \text{if} \qquad 0 < \alpha \leqq 1 \ ,$$

$$L^* = \alpha^{-1} L = |\epsilon| x^\alpha \qquad \text{if} \qquad \alpha \geqq 1 \ .$$

Consequently we find for the integers h which are $\geqq 0$ and $\leqq L^*$

$$(7.24) \qquad |f^{(h)}(x)| \leqq (h+1) \left(\frac{2L}{x}\right)^h |e^{-\epsilon x^\alpha}| \ .$$

This result enables us to find an upper bound for the absolute values of the derivatives of the more general function

$$p(x) = x^\beta e^{-\epsilon x^\alpha} \ ,$$

where

$$x > 0 \ , \qquad \alpha > 0 \qquad \text{and } \beta \text{ real} \ .$$

We notice that

$$(7.25) \qquad \begin{aligned} \left|\frac{d^h x^\beta}{dx^h}\right| &= |\beta(\beta-1) \cdots (\beta+1-h)| \, x^{\beta-h} \\ &\leqq (h+|\beta|)^h x^{\beta-h} \ . \end{aligned}$$

Combining this result with (7.24) and applying the product theorem (theorem 18), we obtain for the integers h which are $\geqq 0$ and $\leqq L^*$

$$(7.26) \qquad |p^{(h)}(x)| \leqq (h+1) \left(\frac{h+|\beta|+2L}{x}\right)^h |p(x)| \ .$$

Also the more general function

$$q(x) = x^\beta (\log x)^m e^{-\epsilon x^\alpha} \ ,$$

where

$$x \geqq e \quad , \qquad \alpha > 0 \quad , \qquad \beta \text{ real} \quad , \qquad m \text{ integer} \geqq 0$$

can be treated in this way. If h denotes a positive integer, we have

$$\frac{d^h \log x}{dx^h} = \pm (h-1)!\, x^{-h}$$

and therefore for $h \geqq 0$

$$\left|\frac{d^h \log x}{dx^h}\right| \leqq h^h\, x^{-h} \log x \quad . \tag{7.27}$$

This gives for each positive integer m the inequality

$$\left|\frac{d^h (\log x)^m}{dx^h}\right| \leqq h^h\, m^h\, x^{-h} (\log x)^m \quad . \tag{7.28}$$

To prove that, we may assume that $m \geqq 2$ and that we know already

$$\left|\frac{d^h (\log x)^{m-1}}{dx^h}\right| \leqq h^h\, (m-1)^h\, x^{-h} (\log x)^{m-1} \quad ; \tag{7.29}$$

the product theorem (Theorem 18) tells us that (7.28) follows from (7.27) and (7.29).

Applying the product theorem and using the inequalities (7.25) and (7.28) we find for each integer h which is $\geqq 0$ and $\leqq L^*$

$$|q^{(h)}(x)| \leqq (h+1) \left(\frac{mh+h+|\beta|+2L}{x}\right)^h |q(x)| \quad . \tag{7.30}$$

Applying the product theorem and using the inequalities (7.25) and (7.28) we find for $x \geqq e$ and for each integer $h \geqq 0$

$$(7.31)\qquad \left|\frac{d^h(x^{\beta}(\log x)^m)}{dx^h}\right| \leqq (hm + h + |\beta|)^h\, x^{\beta-h}(\log x)^m \quad .$$

Let us apply these results to find upper bounds for the absolute values of some integrals of the form

$$\int_t^{\infty} \Psi_1(x)\ g(x)\ N\left(\frac{x}{t} - 1\right)\ dx \quad .$$

We consider first the case in which $g(x) = x^{\beta}(\log x)^m$ and $t \geqq e$. According to inequality (7.31) the condition (7.8) occurring in Theorem 17 is satisfied, if we choose

$$K = (2^{\beta} + 1)t^{\beta}(\log 2t)^m \quad \text{and} \quad \mu = \frac{nm+n+|\beta|}{t} \quad .$$

Formula (7.9) holds with respect to the modulus $P(\infty)$ defined in Section 2. Applying Theorem 18 we obtain

$$(7.32)\begin{cases} \left|\int_t^{\infty} \Psi_1(x)\ x^{\beta}(\log x)^m\ N\left(\frac{x}{t} - 1\right) dx\right| \\ < \frac{1}{12(2\pi)^n}\ J + \frac{(2^{\beta}+1)t^{\beta+1}(\log 2t)^m}{16(2\pi)^n}\left(\frac{nm+5n+|\beta|}{t}\right)^{n+1} \quad , \end{cases}$$

where

$$J = \int_t^{\infty}\left|\frac{d^{n+1}x^{\beta}(\log x)^m}{dx^{n+1}}\right| dx \quad .$$

According to (7.31), applied with $h = n + 1$,

$$|g^{(n+1)}(x)| \leqq \left((n+1)(m+1) + |\beta|\right)^{n+1} x^{\beta-n-1}(\log x)^m ,$$

so that

$$|J| \leqq \left((n+1)(m+1) + |\beta|\right)^{n+1} J_m ;$$

here

$$J_m = \int_t^{\infty} x^{\beta-n-1}(\log x)^m \, dx .$$

Choosing $n \geqq \beta + 2m + 1$ and integrating by parts we obtain

$$J_m = \frac{t^{\beta-n}}{n-\beta}(\log t)^m + \frac{m}{n-\beta} J_{m-1}$$

$$\leqq t^{\beta-n}(\log t)^m \left\{\frac{1}{n-\beta} + \frac{m}{(n-\beta)^2} + \frac{m(m-1)}{(n-\beta)^3} + \cdots + \frac{m!}{(n-\beta)^{m+1}}\right\}$$

$$(7.33) \qquad < t^{\beta-n}(\log t)^m \left\{\frac{1}{n-\beta} + \frac{1}{2(n-\beta)} + \frac{1}{2^2(n-\beta)^2} + \cdots\right\}$$

$$= 2\, t^{\beta-n}(\log t)^m .$$

THEOREM 21. *If* $t \geqq e$ *and* $n \geqq \beta + 2m + 1$, *then*

$$(7.34) \quad \left\{ \begin{array}{l} \left| \int_t^{\infty} \Psi_1(x)\, x^{\beta} (\log x)^m \, N\left(\frac{x}{t} - 1\right) dx \right| \\ < \dfrac{(2^{\beta}+1)\, t^{\beta+1}(\log 2t)^m}{4\,(2\pi)^n} \left(\dfrac{(n+1)(m+1)+5n+|\beta|}{t}\right)^{n+1} . \end{array} \right.$$

This inequality is very sharp, if n is large and t is very large.

Section 8. ON SUMS OF THE FORM $\sum_{n=1}^{\infty} a(n)\, b(\epsilon\, n^{\alpha})$.

The generalized limits, introduced in this chapter, are convenient for the determination of the behavior of sums of the form

$$\sum_{n=1}^{\infty} a(n)\, b(\epsilon\, n^{\alpha}) \quad .$$

For the sake of simplicity we restrict ourselves here to the sums

$$(8.1) \qquad S_m = \sum_{n=1}^{\infty} f(n) \quad , \quad \text{where } f(x) = x^{\beta} (\log x)^m e^{-\epsilon x^{\alpha}} \quad .$$

We suppose

$$(8.2) \quad \alpha > 0 \; ; \; \beta \text{ real } ; \; m \text{ is an integer} \geqq 0 \; , \; -\frac{\pi}{2} + p \leqq \arg \epsilon \leqq \frac{\pi}{2} - p \; ,$$

where p denotes a fixed positive number $\leqq \frac{\pi}{2}$.

THEOREM 22. If the condition (8.2) is satisfied, then S_o is for small values of $|\epsilon|$ asymptotically equal to

$$\frac{1}{\alpha}\, \Gamma\left(\frac{\beta+1}{\alpha}\right) \epsilon^{-\frac{\beta+1}{\alpha}} + \sum_{h=o}^{\infty} \frac{(-\epsilon)^h}{h!}\, \zeta(-\beta - \alpha\, h)$$

if $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$, and S_o is asymptotically equal to

$$\frac{(-)^k}{k!\,\alpha}\, \epsilon^k \left(-\log \epsilon - \gamma + \alpha\, \gamma + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}\right)$$

$$+ \sum_{\substack{h=o \\ h \neq k}}^{\infty} \frac{(-)^h}{h!}\, \epsilon^h\, \zeta(-\beta - \alpha\, h)$$

if $-\frac{\beta+1}{\alpha}$ is equal to an integer $k \geqq 0$; here γ denotes the constant of Euler.

At the same time we give a proof of the following theorem.

THEOREM 23. If the condition (8.2) is satisfied and m is a fixed positive integer, then S_m is for small values of $|\epsilon|$ asymptotically equal to

$$(8.5)\quad \int_0^\infty x^\beta (\log x)^m e^{-\epsilon x^\alpha} dx + \sum_{h=0}^\infty \frac{(-\epsilon)^h}{h!} \int_0^\infty \psi_1(x) \left(x^{\beta+\alpha h} (\log x)^m\right)' dx ,$$

where the integrals, taken modulo L(0+), are calculated in the theorems 9 and 11 in Section 4.

Proof. If Re $w \leqq 0$, then

$$\left| e^w - \sum_{h=0}^{q-1} \frac{w^h}{h!} \right| \leqq \frac{|w|^q}{q!} \qquad (q = 0,1, \cdots) ,$$

so that

$$f'(x) = (x^\beta (\log x)^m e^{-\epsilon x^\alpha})'$$

$$= \beta x^{\beta-1} (\log x)^m e^{-\epsilon x^\alpha} - \epsilon \alpha x^{\alpha+\beta-1} (\log x)^m e^{-\epsilon x^\alpha}$$

$$+ m x^{\beta-1} (\log x)^{m-1} e^{-\epsilon x^\alpha}$$

is equal to

$$f'(x) = \sum_{h=0}^{q-1} \frac{(-\epsilon)^h}{h!} (x^{\beta+\alpha h} (\log x)^m)' + r_q (x) ,$$

where

$$(8.6)\quad \begin{cases} |r_q(x)| \leqq |\beta| \dfrac{|\epsilon|^q}{q!} x^{\beta+\alpha q-1} |\log x|^m \\ \qquad + \alpha \dfrac{|\epsilon|^q}{(q-1)!} x^{\beta+\alpha q-1} |\log x|^m + m \dfrac{|\epsilon|^q}{q!} x^{\beta+\alpha q-1} |\log x|^{m-1} \\ \qquad = \dfrac{|\epsilon|^q}{q!} x^{\beta+\alpha q-1} \left\{ (|\beta| + \alpha q) |\log x|^m + m \, |\log x|^{m-1} \right\} \quad . \end{cases}$$

Applying Theorem 16 in Section 6 we obtain modulo L(0+)

$$(8.7)\quad \begin{aligned} S_m &= \int_0^\infty f(x)\,dx + \lim_{\delta\to 0} (\delta - \tfrac{1}{2}) f(\delta) \\ &+ \sum_{h=0}^{q-1} \frac{(-\epsilon)^h}{h!} \int_0^\infty \psi_1(x) \left(x^{\beta+\alpha h} (\log x)^m \right)' dx + R_q \quad , \end{aligned}$$

where

$$(8.8)\qquad R_q = \int_0^\infty \psi_1(x)\, r_q(x)\, dx \quad .$$

The expression

$$(\delta - \tfrac{1}{2}) f(\delta) - (\delta - \tfrac{1}{2}) \sum_{0 \leqq h \leqq \frac{\beta+1}{\alpha}} \frac{(-\epsilon)^h}{h!} \delta^{\beta+\alpha h} (\log \delta)^m \to 0$$

and consequently

$$(8.9)\qquad (\delta - \tfrac{1}{2}) f(\delta) \to 0 \ (\text{mod. } L(0+)) \quad , \quad \text{as } \delta \to 0 \quad ,$$

except in the case in which $m = 0$ and $\beta + \alpha h = 0$ or -1 for suitable integer $h \geqq 0$. In Theorem 23 the integer m is different from zero, so that $(\delta - \frac{1}{2}) f(\delta)$ tends to zero modulo L(0+), hence

$$S_m = \int_0^\infty f(x)\,dx + \sum_{h=0}^{q-1} \frac{(-\epsilon)^h}{h!} \int_0^\infty \psi_1(x) \left(x^{\beta+\alpha h} (\log x)^m \right)' dx + R_q \quad .$$

The proof of Theorem 23 is therefore established as soon as we have shown that the remainder $|R_q|$ is for each fixed integer $q \geqq 0$ for small values of $|\epsilon|$ at most of the same order of magnitude as $|\epsilon|^q$.

In Theorem 22 we have $m = 0$. Then formula (8.9) holds if $\beta + \alpha h \neq 0$ and $\neq -1$ for each integer $h \geqq 0$. If there exists an integer $\chi \geqq 0$ such that $\beta + \alpha \chi = 0$, then we have modulo L(0+)

$$(8.10) \qquad (\delta - \tfrac{1}{2})\, f(\delta) \to -\frac{1}{2}\,\frac{(-\epsilon)^{\chi}}{\chi!} = \zeta(0)\,\frac{(-\epsilon)^{\chi}}{\chi!}$$

as $\delta \to 0$. If there exists an integer $k \geqq 0$ such that $\beta + \alpha\, k = -1$, then we have modulo L(0+)

$$(8.11) \qquad (\delta - \tfrac{1}{2})\, f(\delta) \to \frac{(-\epsilon)^{k}}{k!} \qquad \text{as } \delta \to 0 .$$

Finally, if there exist two integers $\chi \geqq 0$ and $k \geqq 0$ such that $\beta + \alpha\,\chi = 0$ and $\beta + \alpha\, k = -1$, then we have modulo L(0+)

$$(8.12) \qquad (\delta - \tfrac{1}{2})\, f(\delta) \to \zeta(0)\,\frac{(-\epsilon)^{\chi}}{\chi!} + \frac{(-\epsilon)^{k}}{k!} \quad \text{as } \delta \to 0 .$$

The special case $m = 0$ of Theorem 11 in Section 4 gives

$$\begin{aligned} - s\int_0^{\infty} \psi_1(x)\, x^{-s-1}\, dx &= \zeta(s) && \text{for } s \neq 0 \text{ and } s \neq 1 \\ &= 0 && \text{for } s = 0 \\ &= \gamma - 1 && \text{for } s = 1 . \end{aligned}$$

Therefore

$$\sum_{h=0}^{q-1} \frac{(-\epsilon)^h}{h!} (\beta + \alpha h) \int_0^\infty \psi_1(x)\, x^{\beta+\alpha h-1}\, dx \tag{8.13}$$

$$= \sum_{h=0}^{q-1} \frac{(-\epsilon)^h}{h!} \zeta(-\beta - \alpha h) \quad ,$$

if there does not exist an integer $h \geqq 0$ for which $\beta + \alpha h$ is equal to zero or -1. If there exists an integer $\chi \geqq 0$ such that $\beta + \alpha \chi = 0$, then we can choose q such that $\chi < q$ and then the term with $h = \chi$ on the right hand side of (8.13) must be cancelled. Finally, if there exists an integer $k \geqq 0$ and $< q$ such that $\beta + \alpha k = -1$, then the term with $h = k$ on the right hand side of (8.13) must be replaced by $\frac{(-\epsilon)^k}{k!}(\gamma - 1)$.

Thus we find modulo $L(0+)$ that

$$\lim_{\delta \to 0} (\delta - \tfrac{1}{2}) f(\delta) + \sum_{h=0}^{q-1} \frac{(-\epsilon)^h}{h!} (\beta + \alpha h) \int_0^\infty \psi_1(x)\, x^{\beta+\alpha h-1}\, dx \tag{8.14}$$

$$= \sum_{h=0}^{q-1} \frac{(-\epsilon)^h}{h!} \zeta(-\beta - \alpha h)$$

if there does not exist an integer $h \geqq 0$ such that $\beta + \alpha h = -1$; for if there is an integer $\chi \geqq 0$ and $< q$ with $\beta + \alpha \chi = 0$, then the term

$$\frac{(-\epsilon)^\chi}{\chi!} \zeta(-\beta - \alpha \chi) = \frac{(-\epsilon)^\chi}{\chi!} \zeta(0)$$

is given by $\lim_{\delta \to 0} (\delta - \frac{1}{2}) f(\delta)$. Formula (8.14) holds also if $-\frac{\beta+1}{\alpha}$ is equal to an integer $k \geqq 0$, but in that case the term with $h = k$ on the right hand side of (8.14) must be replaced by $\frac{(-\epsilon)^k}{k!}\gamma$. Consequently, if $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$, then

$$(8.15)\quad S_o = \frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\epsilon^{-\frac{\beta+1}{\alpha}} + \sum_{h=o}^{q-1}\frac{(-\epsilon)^h}{h!}\,\zeta(-\beta-\alpha h) + R_q$$

and if $-\frac{\beta+1}{\alpha}$ is equal to an integer $k \geqq 0$, and $< q$, then

$$(8.16)\quad \begin{aligned} S_o &= \frac{(-\epsilon)^k}{k!}\left(-\log\epsilon - \gamma + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}\right) \\ &+ \sum_{\substack{h=o\\ h\neq k}}^{q-1}\frac{(-\epsilon)^h}{h!}\,\zeta(-\beta-\alpha h) + \gamma\,\frac{(-\epsilon)^k}{k!} + R_q \quad . \end{aligned}$$

Consequently, not only the proof of Theorem 23, but also that of Theorem 22 is established, as soon as we have shown that the remainder $|R_q|$ is for each fixed integer $q \geqq 0$ and for small values of $|\epsilon|$ at most of the same order of magnitude as $|\epsilon|^q$. It is even sufficient to show that for sufficiently large fixed integer $s \geqq q$ (fixed means here: independent of ϵ) the remainder $|R_s|$ is for small values of $|\epsilon|$ at most of the same order of magnitude as $|\epsilon|^q$, for it follows from (8.7) that

$$R_q = \sum_{h=q}^{s-1}\frac{(-\epsilon)^h}{h!}\int_o^\infty \psi_1(x)\left(x^{\beta+\alpha h}(\log x)^m\right)'\,dx + R_s \quad .$$

We choose s so large that $\beta + \alpha\, s > 0$. Since we are only interested in the behavior of the sum S_m for small values of $|\epsilon|$ we may suppose that $|\epsilon| < \frac{1}{2}$.

We divide R_s into two parts $U_s + V_s$, where

$$(8.17)\quad U_s = \int_o^{2t}\psi_1(x)\,r_s(x)\,N(2-\tfrac{x}{t})\,dx$$

and

$$V_s = \int_t^{\infty} \psi_1(x)\; r_s(x)\; N(\tfrac{x}{t} - 1)\; dx \quad , \tag{8.18}$$

where

$$t = |\epsilon|^{-\frac{1}{\alpha+1}} \tag{8.19}$$

and where the order n of the neutralizer N(u) is a sufficiently large fixed integer. For sufficiently small $|\epsilon|$ we have $t \geqq e$.

Applying (8.6) with s instead of q we obtain

$$|U_s| \leqq \frac{|\epsilon|^s}{s!}\,(|\beta| + \alpha\; s + m) \int_t^{\infty} x^{\beta+\alpha s-1}\,(\log x)^m\, dx \quad . \tag{8.20}$$

We have chosen s so large that $\beta + \alpha\; s > 0$, so that

$$|U_s| \leqq c_1\; |\epsilon|^s\; t^{\beta+\alpha s}\; (\log t)^m \quad .$$

In this proof $c_1, c_2, \cdots c_9$ denote suitable numbers which are independent of ϵ, t and x. In this way we have found

$$|U_s| \leqq c_1\; |\epsilon|^{\frac{s-\beta}{\alpha+1}} \left(\frac{1}{\alpha+1}\log\frac{1}{|\epsilon|}\right)^{m+1} \leqq c_1\; |\epsilon|^q \quad .$$

for sufficiently large fixed number s. It is therefore sufficient to show that V_s is for given integer s and for suitably chosen integer n at most of the same order of magnitude as $|\epsilon|^q$.

From the definition of $r_q(x)$ it follows that

$$(8.21)\qquad r_s(x) = f'(x) - \sum_{h=o}^{s-1} \frac{(-\epsilon)^h}{h!} \left(x^{\beta+\alpha h} (\log x)^m\right)' ,$$

therefore

$$(8.22)\quad V_s(x) = W - \sum_{h=o}^{s-1} \frac{(-\epsilon)^h}{h!} \int_t^{\infty} \psi_1(x) \left(x^{\beta+\alpha h}(\log x)^m\right)' N(\tfrac{x}{t} - 1)\, dx ,$$

where

$$W = \int_t^{\infty} \psi_1(x)\, f'(x)\, N(\tfrac{x}{t} - 1)\, dx .$$

Applying (7.34) with $\beta + \alpha h - 1$ instead of β and with m and m - 1 instead of m, we find

$$\left| \int_t^{\infty} \psi_1(x)\, (x^{\beta+\alpha h} (\log x)^m)'\, N(\tfrac{x}{t} - 1)\, dx \right|$$

$$< c_2\, t^{\beta+\alpha h-n-1} (\log t)^m ,$$

if we choose the fixed integer $n \geqq \beta + \alpha(s - 1) + 2m$. Then the sum Σ, occurring in (8.22), is in absolute value at most equal to

$$c_2 \sum_{h=o}^{s-1} \frac{|\epsilon|^h}{h!} t^{\beta+\alpha h-n-1} (\log t)^m \leqq c_2\, |\epsilon|^q$$

if the fixed integer n is large enough. The only thing we have to do now is to show that also W is for sufficiently large fixed integer n at most of the same order of magnitude as $|\epsilon|^q$.

We have

$$f^{(h)}(x) = x^{\beta-h} P_h(x)\, e^{-\epsilon x^{\alpha}} ,$$

where $P_h(x)$ is a polynomial in ϵx^{α} and $\log x$; the degree in ϵx^{α} is $\leqq h$ and the degree in $\log x$ is $\leqq m$; this assertion is obvious for $h = 0$ and can be proved for $h \geqq 1$ by means of the principle of mathematical induction. Thus we find for each fixed integer $h \geqq 0$ and for $x \geqq e$

$$|f^{(h)}(x)| \leqq c_3\, x^{\beta-h} (1 + (\epsilon\, x^{\alpha})^h)(\log x)^m\, e^{-\eta x^{\alpha}} ,$$

where $\eta = \operatorname{Re} \epsilon \geqq |\epsilon| \sin p > 0$. From

$$|\epsilon|\, 2^{\alpha}\, t^{\alpha} = 2^{\alpha}\, |\epsilon|^{\frac{1}{\alpha+1}} < 2^{\alpha}$$

it follows that in the interval $t \leqq x \leqq 2t$

$$|f^{(h)}(x)| \leqq c_4\, t^{\beta-h} (\log t)^m .$$

Applying (7.13) with $I = W$ and $g(x) = f(x)$ we obtain

$$|W| < c_5 \int_t^{\infty} x^{\beta-n-1} (1 + (|\epsilon| x^{\alpha})^{n+1})(\log x)^m\, e^{-\eta x^{\alpha}}\, dx$$

$$+ c_6\, t^{\beta-n} (\log t)^m$$

$$< c_5 |\epsilon|^{n+1} \int_t^{\infty} x^{\beta+(\alpha-1)(n+1)} (\log x)^m\, e^{-\eta x^{\alpha}}\, dx$$

$$+ c_7\, t^{\beta-n} (\log t)^m ,$$

if n is sufficiently large. This inequality implies, if $0 < \alpha < 1$,

$$|W| < c_8 \ |\epsilon|^q \quad ,$$

if t is sufficiently large. If $\alpha \geqq 1$ we write

$$|\epsilon|^{n+1} \int_t^\infty x^{\beta+(\alpha-1)(n+1)} (\log x)^m e^{-\eta x^\alpha} dx$$

$$= |\epsilon|^{n+1} \eta^{-\frac{\beta+1+(\alpha-1)(n+1)}{\alpha}} \alpha^{-m-1} \int_{\eta t^\alpha}^\infty y^{\frac{\beta+1+(\alpha-1)(n+1)}{\alpha}} (\log \frac{y}{\eta})^m e^{-y} dy$$

$$\leqq |\epsilon|^{\frac{n-\beta}{\alpha}} |\sin p|^{-\frac{\beta+1+(\alpha-1)(n+1)}{\alpha}} \alpha^{-m-1} \int_{\eta t^\alpha}^\infty y^{\frac{\beta+1+(\alpha-1)(n+1)}{\alpha}} (\log \frac{n}{\eta})^m e^{-y} dy$$

$$< c_9 \ |\epsilon|^q$$

for sufficiently large n. This completes the proof.

In the preceding theorem we have found for S_m an asymptotic expansion, which is valid for small values of $|\epsilon|$. This expansion is, as we shall prove now, convergent (1) if $0 < \alpha < 1$ (2) if $\alpha = 1$ and $|\epsilon| < 2\pi$. To that end we examine the behavior for large values of h of the integral

$$I_h = \int_0^\infty \psi_1(x) \ (x^{\beta+\alpha h} (\log x)^m)' \ dx \quad .$$

We have found in Theorem 11 in Section 4 that this integral is equal to

$$I_h = (-)^m \ \zeta^{(m)} (-\beta - \alpha h) \quad ,$$

if $\beta + \alpha h$ is different from zero and 1.

The zeta function of Riemann satisfies, as we have seen in formula (4.16), the functional equation

$$\zeta(-s) = -2\,(2\pi)^{-1-s}\,\zeta(1+s)\,\Gamma(s+1)\,\sin\frac{\pi s}{2}\,.$$

For $s \geqq 1$ we have

$$0 < \zeta(s+1) \leqq \zeta(2) \leqq \frac{\pi^2}{6} \quad\text{and}\quad \left|\sin\frac{\pi s}{2}\right| \leqq 1\,,$$

so that

$$|\zeta(-s)| \leqq \frac{1}{12}\,(2\pi)^{1-s}\,\Gamma(s+1)\,.$$

The series

$$\sum_{\substack{h=0\\ \beta+\alpha h \geqq 1}}^{\infty} \frac{(-\epsilon)^h}{h!}\,\zeta(-\beta-\alpha h)$$

has therefore the majorant

$$\frac{1}{12}\sum_{\substack{h=0\\ \beta+\alpha h \geqq 1}}^{\infty} (2\pi)^{1-\beta-\alpha h}\,\Gamma(\beta+\alpha h+1)\,\frac{|\epsilon|^h}{h!}$$

and this majorant converges (1) if $0 < \alpha < 1$ (2) if $\alpha = 1$ and $|\epsilon| < 2\pi$. This gives the required result in the particular case $m = 0$. In the general case $m \geqq 0$ we write $\zeta^{(m)}(-s)$ by means of the functional equation of the zeta function as a linear combination with constant coefficients of terms of the form

$$(2\pi)^{-s}\,\zeta^{(g)}(1+s)\,\Gamma^{(j)}(1+s)\binom{\sin}{\cos}\frac{s\pi}{2}\,,$$

where $g + j \leq m$. Consequently for $s \geq 1$

$$|\zeta^{(m)}(-s)| \leq \gamma\,(2\pi)^{-s} \sum_{j=0}^{m} |\Gamma^{(j)}(1+s)| \;,$$

where γ depends only on m. The series

$$\sum_{\substack{h=0\\ \beta+\alpha h\geq 1}}^{\infty} \zeta^{(m)}(-\beta-\alpha h)\,\frac{(-\epsilon)^h}{h!}$$

has therefore the majorant

$$\gamma \sum_{\substack{h=0\\ \beta+\alpha h\geq 1}}^{\infty} (2\pi)^{-\beta-\alpha h}\,\frac{|\epsilon|^h}{h!} \sum_{j=0}^{m} |\Gamma^{(j)}(\beta+\alpha h+1)|$$

Since also this majorant converges (1) if $0 < \alpha < 1$ (2) if $\alpha = 1$ and $|\epsilon| < 2\pi$, we find in this way the required result for each integer $m \geq 0$.

We see even: if $|\epsilon| < 2\pi$ and $0 < \alpha \leq 1$, then the expansion, obtained above for the sum S_m, converges uniformly in α.

The question arises whether S_m is in these two cases the sum of this convergent series. If we assume that $-\frac{\pi}{2} < \arg\epsilon < \frac{\pi}{2}$, the reader finds the affirmative answer in

THEOREM 24. _Assume_ $-\frac{\pi}{2} < \arg\epsilon < \frac{\pi}{2}$. _Suppose either_ $0 < \alpha < 1$ _or_ $\alpha = 1$ _and_ $|\epsilon| < 2\pi$. _Let_ β _be real and let_ m _be an integer_ ≥ 0. _Then the sum_

(8.23) $$S_m = \sum_{n=0}^{\infty} f(n) \quad \underline{\text{where}} \quad f(x) = x^{\beta}(\log x)^m e^{-\epsilon x^{\alpha}}$$

can be written as a sum of a convergent series, namely

$$(8.24)\quad S_o = \frac{1}{\alpha}\,\Gamma\left(\frac{\beta+1}{\alpha}\right)\,\epsilon^{-\frac{\beta+1}{\alpha}} + \sum_{h=o}^{\infty} \frac{(-\epsilon)^h}{h!}\,\zeta(-\beta-\alpha h)$$

if $-\frac{\beta+1}{\alpha}$ is not an integer $\geqq 0$,

$$(8.25)\quad \begin{aligned} S_o &= \frac{(-)^k}{k!}\,\epsilon^k\left(-\log\epsilon - \gamma + \alpha\gamma + \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{k}\right) \\ &\quad + \sum_{\substack{h=o \\ h\neq k}}^{\infty} \frac{(-\epsilon)^h}{h!}\,\zeta(-\beta-\alpha h) \end{aligned}$$

if $-\frac{\beta+1}{\alpha}$ is equal to an integer $k \geqq 0$, finally for $m \geqq 1$

$$(8.26)\quad S_m = \int_0^{\infty} x^{\beta}(\log x)^m e^{-\epsilon x^{\alpha}}\,dx + \sum_{h=o}^{\infty} \frac{(-\epsilon)^h}{h!}\int_0^{\infty} \psi_1(x)\left(x^{\beta+\alpha h}(\log x)^m\right)'\,dx,$$

where the integrals are taken modulo $L(0+)$.

Proof. We begin in the same way as in the proof of the preceding theorem, but now we must show that the remainder R_q tends to zero as $q \to \infty$; in this argument ϵ, α, β and m are supposed to be fixed.

Let us treat first the case that $0 < \alpha < 1$. We choose a number t depending on q such that

$$(8.27)\quad t \to \infty \;;\; \frac{t^{\alpha}}{q} \to 0 \;;\; q\,\frac{\log t}{t} \to 0 \quad \text{as} \quad q \to \infty\ .$$

Furthermore we define the order n of the neutralizer $N(u)$ as the largest integer $\leqq \frac{t}{(m+7)e} - 1$, so that n depends on q and tends to infinity as $q \to \infty$, even so rapidly that

(8.28) $$\frac{q}{n}\log t \to 0 \quad \text{as} \quad q \to \infty \ .$$

For sufficiently large q we have

$$t \geqq e \quad \text{and} \quad \beta + \alpha q > 0 \ .$$

We divide again R_q into the two parts $U_q + V_q$, where U_q is according to (8.6), in absolute value at most equal to

$$\frac{|\epsilon|^q}{q!}\int_0^{2t} x^{\beta+\alpha q-1}\left\{(|\beta| + \alpha\ q)\ |\log x|^m + m|\log x|^{m-1}\right\}\ dx$$

$$\leqq \gamma\frac{|\epsilon|^q}{(q-1)!}\ (2t)^{\beta+\alpha q}\ (\log t)^m \ ,$$

where γ is fixed. From $\frac{t^\alpha}{q} \to 0$ it follows that $U_q \to 0$ as $q \to \infty$. Furthermore

(8.29) $$\left|\int_t^\infty \psi_1(x)\ f'(x)\ N(\tfrac{x}{t} - 1)\ dx\right| \leqq \int_t^\infty |f'(x)|\ dx \to 0$$

as $t \to \infty$. It is therefore sufficient to show that

$$\sigma_q = \sum_{h=0}^{q-1}\frac{(-\epsilon)^h}{h!}I_h \ , \quad \text{where } I_h = \int_t^\infty \psi_1(x)\ f'(x)\ N(\tfrac{x}{t} - 1)\ dx \ ,$$

tends to zero as $q \to \infty$, for then it follows from (8.28) that V_q and therefore also $R_q = U_q + V_q$ tends to zero as $q \to \infty$. To that end we apply inequality (7.34) with $\beta + \alpha h - 1$ instead of β. The sufficient condition

$$n \geqq \beta + \alpha h + 2m$$

is satisfied for $0 \leqq h < q$ and for sufficiently large q, since it follows from (8.28) that then

$$n \geqq \beta + \alpha q + 2m > \beta + \alpha h + 2m \quad .$$

We find therefore for $0 \leqq h < q$

$$(8.30) \quad |I_h| < \tfrac{1}{4}\left(2^{\beta+\alpha h-1}+1\right) t^{\beta+\alpha h-1}(\log 2t)^m \left(\frac{(m+1)(n+1)+5n+|\beta|}{t}\right)^{n+1} .$$

For sufficiently large q

$$n + 1 \geqq |\beta| \quad ,$$

hence

$$\frac{(n+1)(m+1)+5n+|\beta|}{t} \leqq \frac{(n+1)(m+7)}{t} \leqq \frac{1}{e}$$

by the definition of n. Consequently it follows from (8.30) and (8.28)

$$\log |I_h| < (\beta + \alpha q - 1) \log 2t + m \log \log 2t - n - 1 < -\tfrac{1}{2} n$$

for sufficiently large q. In this way we find

$$|\sigma_q| \leqq e^{-\frac{1}{2}n} \sum_{k=0}^{\infty} \frac{|\epsilon|^h}{h!} = e^{|\epsilon|-\frac{1}{2}n} \to 0$$

as $q \to \infty$. This completes the proof for $0 < \alpha < 1$.

The sum S_m is a continuous function of α in the interval $0 < \alpha \leq 1$ and we have seen that the expansion, obtained for S_m, converges uniformly in α if $|\epsilon| < 2\pi$, so that the required formulas (8.23), (8.24) and (8.25) hold also if $\alpha = 1$ and $|\epsilon| < 2\pi$.

It is easy to see that the series S_m remains convergent if the condition $-\frac{\pi}{2} < \arg \epsilon < \frac{\pi}{2}$ is replaced by

$$(8.31) \quad \epsilon \neq 0 \;;\; \arg \epsilon = \pm \frac{\pi}{2} \;;\; 0 < \alpha \leq 1 \text{ and } \beta < \alpha - 1 \;,$$

and that the sum S_m is a continuous function of ϵ. Since the expansions, obtained for S_m, converge uniformly in α (1) if $0 < \alpha < 1$ (2) if $\alpha = 1$ and $|\epsilon| < 2\pi$, we obtain finally

THEOREM 25. Assume (8.31); let m be an integer ≥ 0; if $\alpha = 1$, we suppose that $|\epsilon| < 2\pi$. Under these conditions the formulas (8.24), (8.25) and (8.26), obtained in the preceding theorem, remain true.

In the chapters I, II and II we have not exhausted the theory of the sum formula of Euler. Still many other applications can be given, even in the domain of the real variables, and the sum formula of Euler in the complex plane has not been treated at all in these chapters. The reader can find an excellent exposition of this sum formula in:
W. B. Ford, Studies on Divergent Series and Summability, Michigan Science Series, vol. II, New York, The Macmillan Company, 1916, XI + 194 pages (compare in particular p. 1 - 63.)

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