The Pythagoras Tree as a Julia Set

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1. Introduction

Forty years ago in the dark days of the second world war the Dutch engineer A. Bosman constructed the so-called Pythagoras tree reproduced here in fig.1.1. It must have taken him many, many hours at the drawingboard. But now with a personal computer and a plotter a nice tree can be formed within an hour and generalizations can be made to order.

Our research on this tree actually started when we tried to determine the set of infinitesimally small squares formed in the limit if the construction is continued indefinitely. Let J denote the closure of this set then J, which we call the *blossom* of the Pythagoras tree, is a continuous curve which is invariant (i.e. mapped into itself) under two (!) similarity transformations A and B. Coordinates can be chosen in such a way that in complex notation

$$\begin{cases} A: \; z \mapsto 1 + (1+i)z \,/\, 2, \\ B: \; z \mapsto 1 + (1-i)z \,/\, 2. \end{cases} \tag{1.1}$$

We see that A has $1+i$ as its centre of rotation (or fixed point), the reduction factor $1 / \sqrt{2}$ and the rotation angle $\pi / 4$. For B the centre is at $1-i$ with reduction factor $1 / \sqrt{2}$ and rotation angle $-\pi / 4$. Both centres $1 \pm i$ are elements of J. More points of J (in fact a dense subset of it) can be obtained from them by subjecting them to a random sequence of operations of A and B. In this way fig.1.2 has been obtained as part of the blossom of the Pythagoras tree.

J is a continuous image of the unit interval $0 \leqslant r \leqslant 1$. Let r $(r<1)$ have the binary expansion

$$r = 0.r_1 r_2 r_3 \cdots \tag{1.2}$$

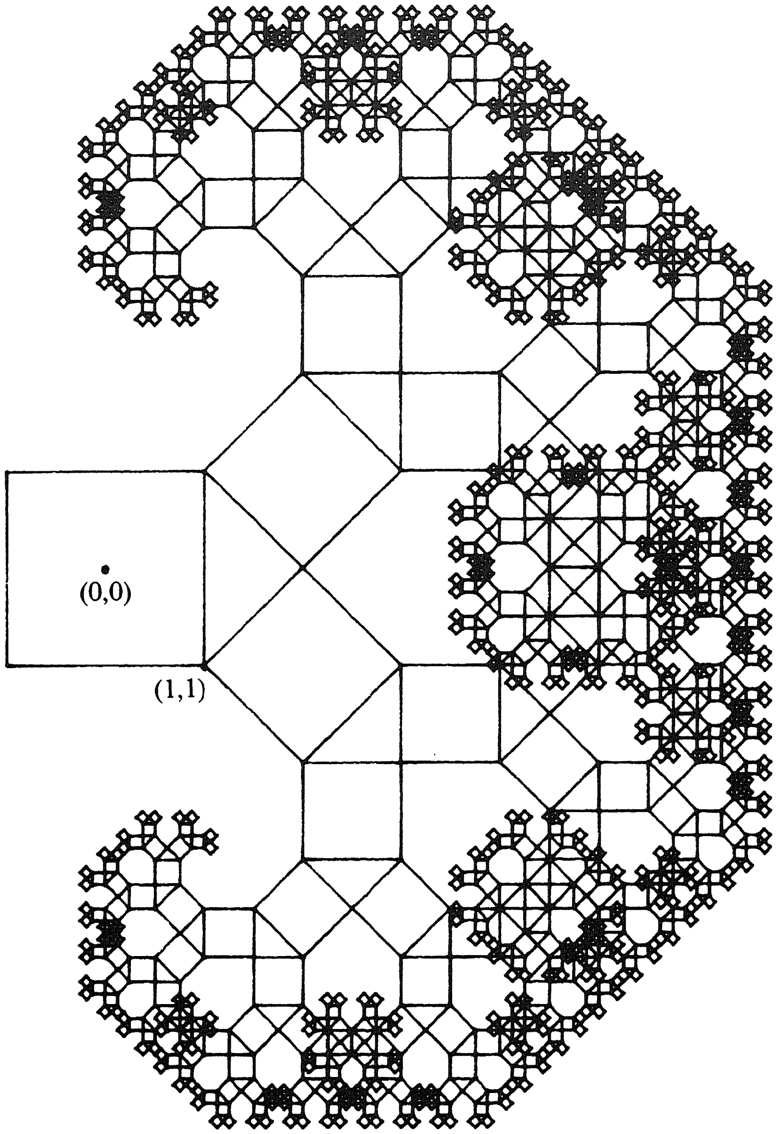

FIGURE 1.1. The Pythagoras tree

and define for $k \geqslant 1$

$$s_k = (1-r_k)a + r_k b, \tag{1.3}$$

with

$$a = (1+i)/2, \quad b = (1-i)/2. \tag{1.4}$$

This means that s_k is either a or b according to the value 0 or 1 of the kth binary digit of r. Then to r we may associate the following point of J

FIGURE 1.2. (Part of) the limit set of the Pythagoras tree

$$z = c_0 + c_1 + c_2 + c_3 + \cdots \tag{1.5}$$

where $c_0 = 1$ and for $k \geqslant 1$

$$c_k = s_k c_{k-1}. \tag{1.6}$$

Thus to $r = 0$ corresponds the point

$$z = 1 + a + a^2 + a^3 + \ldots = 1/(1-a) = 1+i,$$

the fixed point of A, and to $r = 1/3 = .010101\cdots$ corresponds

$$z = 1 + a + ab + a^2b + a^2b^2 + a^3b^2 + \cdots = 3 + i.$$

On J the actions of A and B are then translated into

$$\begin{cases} A: \ r \mapsto r/2, \\ B: \ r \mapsto (1+r)/2. \end{cases} \tag{1.7}$$

Thus a random sequence of transformations A and B corresponds to a uniform distribution of numbers in (0,1) and accordingly to what one could call a uniform distribution of points on J.

The obvious generalization is to give a and b arbitrary complex values with $|a| < 1$ and $|b| < 1$. The problem of determining the conditions under which the resulting curve can be considered as the limit set of some Pythagoras tree will be taken up in the next section 2.

There is a little problem about rational numbers with a terminating binary expansion. Sequences like $.1000\cdots$ and $.0111\cdots$ represent the same rational number (1/2). However, a simple calculation shows that for $a + b = 1$ the corresponding sequences of complex numbers $1 + b + ab + a^2b + a^3b + \cdots$ and $1 + a + ab + ab^2 + ab^3 + \cdots$ also represent the same point.

The overall situation is very reminiscent of the inverse logistic map in its complex form as studied by MANDELBROT [1]

$$z' = \pm \sqrt{z + \mu}. \tag{1.8}$$

We follow up this analogy in more detail in section 3. Its main features are as follows. For suitable values of μ this two-valued map (1.8) has a Julia set (cf.[1]) as the collection of limit points of random iterative sequences. The fixed points are $p/2$ and $1 - p/2$ where $\mu = (p^2 - 2p)/4$ in the usual notation of the logistic map as $x \rightarrow px(1-x)$. In fact, both the more general version of (1.1)

$$z' = 1 + az \quad \text{or} \quad z' = 1 + bz, \tag{1.9}$$

and (1.8) can be considered as the members of a family of quadratic (2,2)-maps described by a relation of the form

$$F(z', z) = 0, \tag{1.10}$$

where F is a quadratic polynomial of its arguments. In particular the blossom of the Pythagoras tree and the San Marco attractors (cf.[1]) can be interpreted as Julia sets of the map (1.10). However, the theory of iterated analytic maps (cf.[2]) is only fully developed for the case that $z'(z)$ (or its inverse) is a single-valued meromorphic analytic function. The examples given here may give rise to an extension of the theory to algebraic functions of the kind (1.10).

2. The Pythagoras tree

In the introduction we have seen that the construction of Bosman's Pythagoras tree can be based upon sequences of complex numbers (1.5), (1.6) with a and b given by (1.4). In fig.2.1 the initial part of what we call the skeleton of a tree is given. The endpoints $P_k(z_k)$ of the successive branches can be labelled in such a way that

$$z_0 = 0, \quad z_1 = 1, \quad z_2 = 1+a, \quad z_3 = 1+b,$$
$$z_4 = 1+a+a^2, \quad z_5 = 1+a+ab, \quad \text{etc.}$$

E.g. for $k = 50$, which is 110010 in binary notation, we have

$$z_{50} = 1+b+ab+a^2b+a^2b^2+a^3b^2.$$

What we have done in fig.2.1 with the special values of a and b can be done for any values of a and b. In this way we obtain a similar tree. The question arises whether such a tree can be interpreted as the skeleton of a generalized Pythagoras tree. Can we put squares or quadrilaterals onto the branches?

Before that question can be answered we need a little more analysis of the tree of fig.2.1 which we now interpret as an illustration of the general case. The tree is transformed into itself by either similarity transformation

$$\begin{cases} A: \; z \mapsto 1 + az, \\ B: \; z \mapsto 1 + bz. \end{cases} \tag{2.1}$$

The fixed points of these transformations, $1/(1-a)$, $1/(1-b)$ are indicated in fig.2.1 by A and B. An endpoint with index k is transformed into an endpoint with a higher index. In particular

$$Az_{50} = z_{82}, \quad Bz_{50} = z_{114}.$$

The general rule is as follows. Let

$$2^m \leqslant k < 2^{m+1}$$

then symbolically

$$A(k) = k+2^m, \quad B(k) = k+2^{m+1}.$$

We now consider the central question under what conditions for a and b the tree of fig.2.1 can be blown up into a generalized Pythagoras tree. By this we understand a tree like fig.1.1 where the basic pattern is a triangle with similar quadrilaterals on its sides. In fig.1.1 the quadrilaterals are squares and the triangle is half a square. If the triangle is rectangular but not isosceles the tree is called an *oblique Pythagoras tree.* In all other cases the tree is called a *generalized Pythagoras tree.*

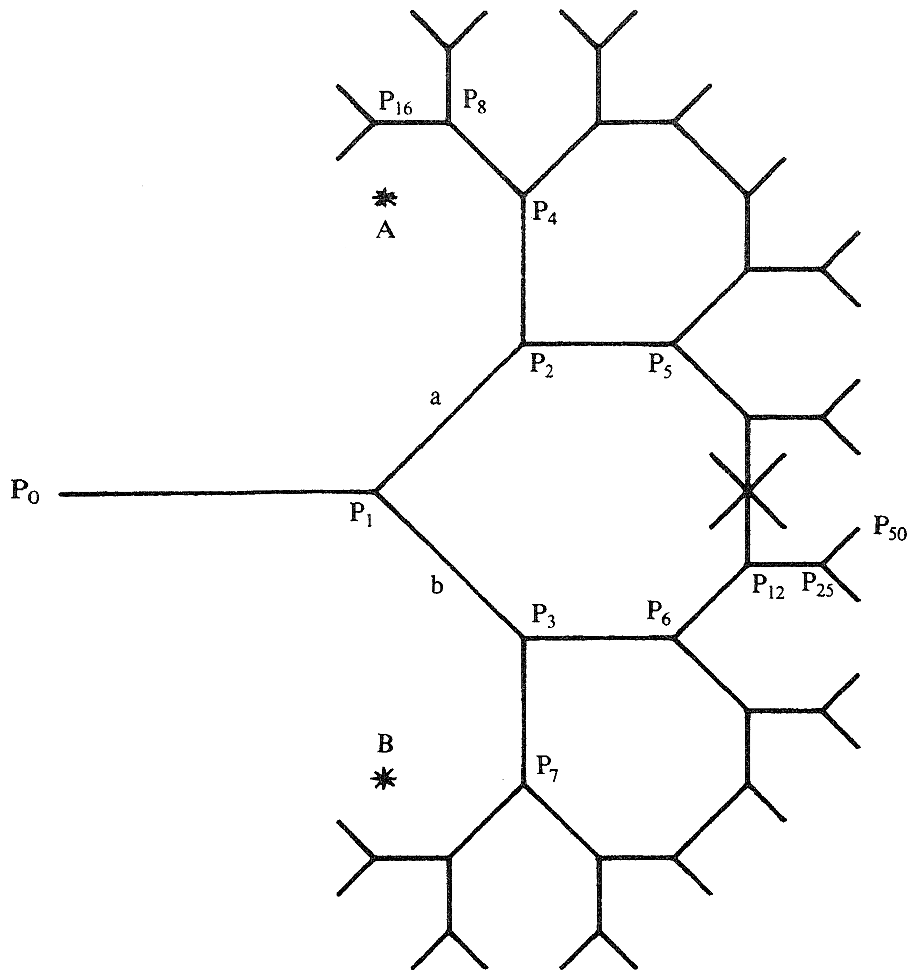

FIGURE 2.1. Initial part of the skeleton of the Pythagoras tree

In view of the similarity transformations (2.1) it is sufficient to consider the first three branches with the first three quadrilaterals as shown in fig.2.2. Let $UU'V'V$ be the first quadrilateral with $U' = A(U)$ and $V' = B(V)$ then there exists a point W which is both the A-image of V as well as the B-image of U. Labelling U and V by complex numbers u and v we obtain the condition

$$1+bu = 1+av$$

so that $bu = av$. This suggests the following construction. Let a and b be arbitrary complex numbers, of course with $|a| < 1$, $|b| < 1$ and a/b not real, then for any complex number λ a generalized Pythagoras tree can be constructed. The first quadrilateral is determined by the corners

$$\lambda a, \ \lambda b, \ 1+\lambda a^2, \ 1+\lambda b^2. \tag{2.2}$$

EXAMPLE. For $a = \frac{1}{2}(1+i)$, $b = \frac{1}{4}(3-2i)$ and $\lambda = 1$ we obtain a quadrilateral with the corner points $(1+i)/2$, $(3-2i)/4$, $(2+i)/2$, $(21-12i)/16$.

The situation is sketched in fig.2.2. The quadrilateral is a trapezium here. A simple calculation shows that always $U'V' /\!/ UV$ when $a+b$ is real. When the vectors UU' and VV' are equal, the quadrilaterals are parallelograms. In that case we should have $\lambda(b^2-a^2) = \lambda(b-a)$ which gives the condition

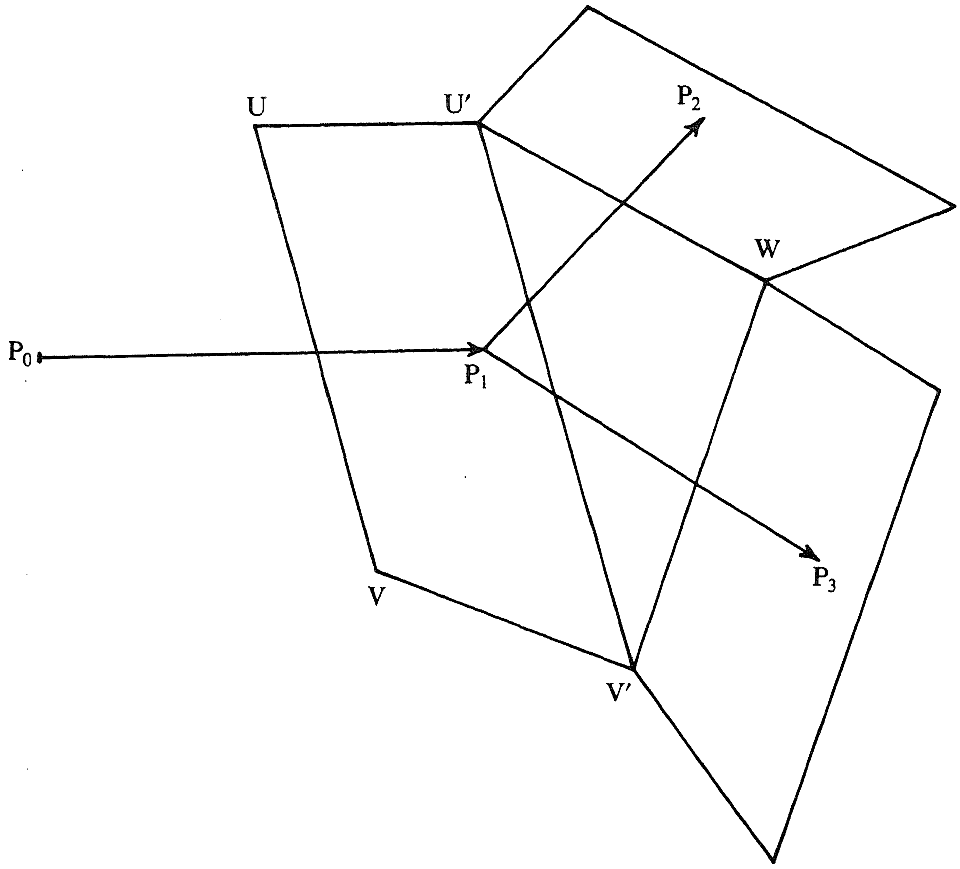

FIGURE 2.2. The beginning of a generalized Pythagoras tree

$$a+b = 1. \tag{2.3}$$

We may write

$$a = (1+ic)/2, \ b = (1-ic)/2 \tag{2.4}$$

where $c = i(b-a)$ is an arbitrary complex number for which $|a| < 1$ and $|b| < 1$. Thus c is restricted to a lens-shaped region bounded by the two circular arcs defined by $|c \pm i| < 2$.

The quadrilaterals are squares if

$$i(v-u) = u'-u$$

i.e. if

$$(1+\lambda a^2) - \lambda a = i\lambda(b-a).$$

Substitution of (2.4) gives the unique solution

$$\lambda = \frac{4}{c^2+4c+1}. \tag{2.5}$$

So given a and b satisfying (2.4) unless $c = -2 \pm \sqrt{3}$ a generalized Pythagoras tree with squares can be constructed. An oblique Pythagoras tree, i.e. a tree with squares and right-angled triangles, calls for a further specialization. A simple calculation shows that this requires that

$$c = -i\exp(2\alpha i), \ 0 < \alpha < \pi/4$$

and hence

$$\begin{cases} a = \cos^2\alpha + i\sin\alpha\cos\alpha, \\ b = \sin^2\alpha - i\sin\alpha\cos\alpha \end{cases} \tag{2.6}$$

Finally for $\alpha = \pi/4$ the original symmetric Pythagoras tree is obtained.

The corresponding geometric situation for an oblique tree is sketched in fig.2.3 (where $\alpha = 2\pi/9$).

A 'full' oblique Pythagoras tree with $\alpha = \pi/5$ is given in fig.2.4. The limit set of the infinitesimally small squares, its blossom, is given in fig.2.5. In the computer program it is obtained as the invariant set of the similarity transformations

$$z \mapsto 1+az, \quad z \mapsto 1+bz$$

with a and b given by (2.6). Each fixed point $1/(1-a)$ and $1/(1-b)$ is subjected to random sequences of similarity transformations.

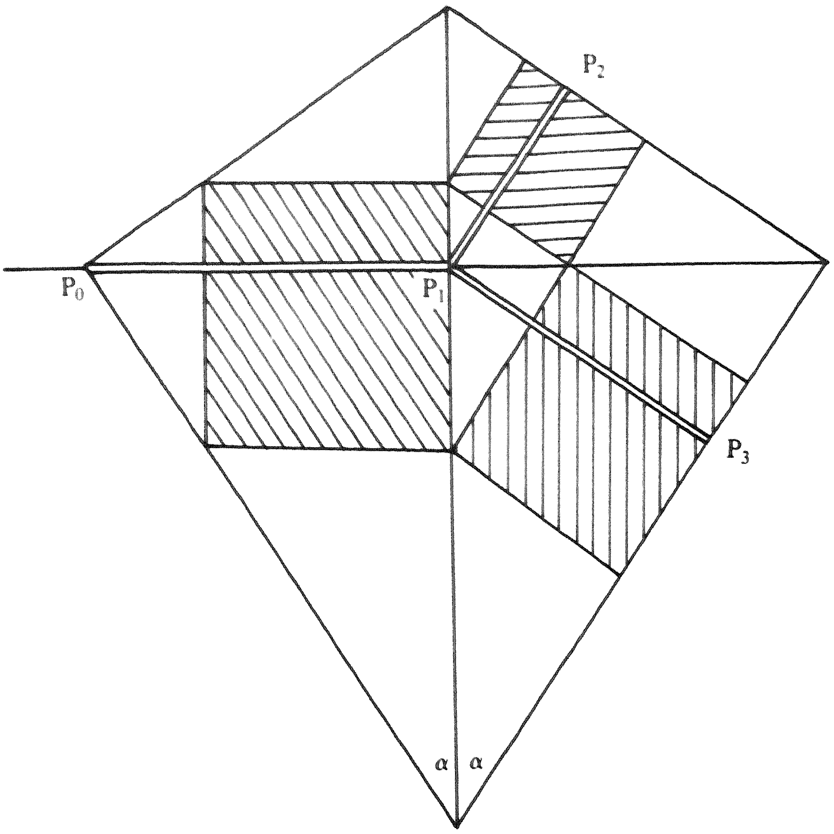

FIGURE 2.3. The basis of an oblique Pythagoras tree

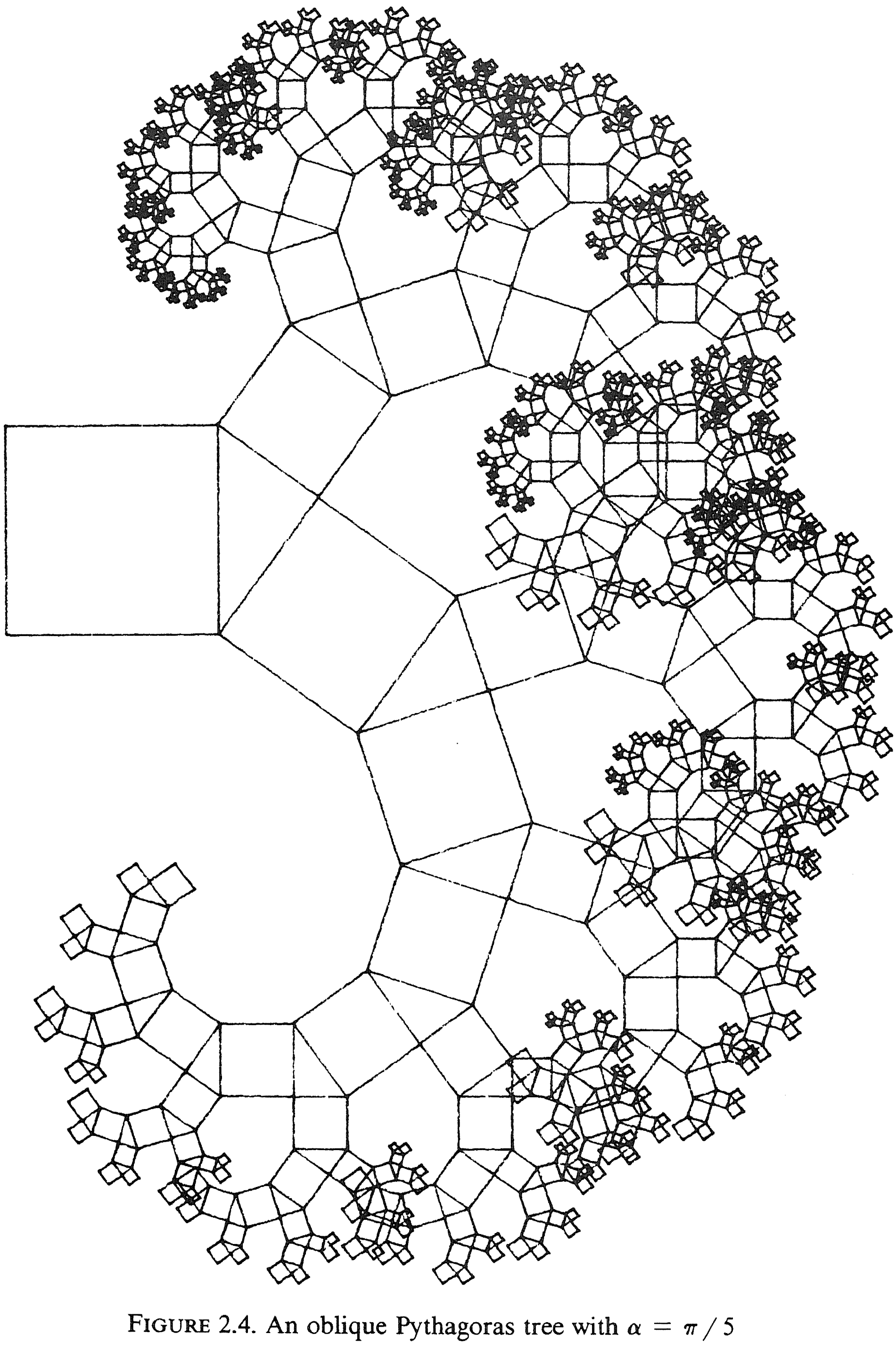

FIGURE 2.4. An oblique Pythagoras tree with $\alpha = \pi / 5$

FIGURE 2.5. The limit set of an oblique Pythagoras tree with $\alpha = \pi / 5$

3. Julia sets

A Julia set is a certain invariant set of an analytic map $z \mapsto f(z)$. It is obtained as the closure of the set of all unstable periodic points. Definitions, properties and many details can be found in the excellent survey paper by BLANCHARD [2]. In many cases the Julia set is a non-differentiable curve or a totally disconnected point set. The very special case

$$z \mapsto z^2$$

already shows many features of the general case. The Julia set is the unit circle here. It is densely covered by the pre-images of any of its points. It is a separatrix separating orbits converging to $z=0$ and orbits diverging to $z=\infty$. It is an attractor of the inverse map $z \mapsto \pm\sqrt{z}$.

Much attention has been paid to the properties of the quadratic map

$$z \mapsto z^2 - \mu \tag{3.1}$$

in the literature. Only for $\mu = 0$ and $\mu = 2$ do we have a Julia set in the form of a simple curve or arc. For $\mu = 3/4$ the Julia set has a nice shape called the 'San Marco attractor' by MANDELBROT. It is given in fig.3.1. The computer program is very similar to that for the blossom of the Pythagoras tree. Points of the Julia set are obtained from the iteration process

$$z_{k+1} = \sigma_k \sqrt{\mu + z_k}, \tag{3.2}$$

where σ_k, $k \in \mathbb{N}$ is a random sequence of $\pm$ 1's and where $z_0 = -1/2$, a critically stable fixed point which is an element of the Julia set.

The Julia set of (3.1) is invariant under the two transformations

$$\begin{cases} A: \ z \mapsto \sqrt{\mu+z}, \\ B: \ z \mapsto -\sqrt{\mu+z}. \end{cases} \tag{3.3}$$

the two inverses of (3.1). If this is compared with the corresponding transformations (2.1) of the generalized Pythagoras tree, we observe a striking similarity. The limit set of a Pythagoras tree and the Julia set of the quadratic transformation (3.3) appear to have common features. We have seen that the limit points of the Pythagoras tree formed from

$$z \mapsto 1 + az, \quad z \mapsto 1 + bz \tag{3.4}$$

with $|a| < 1$, $|b| < 1$ are explicitly given by

$$z = 1 + \sum_{k=0}^{\infty} a_0 a_1 a_2 \cdots a_k, \tag{3.5}$$

with

$$\begin{cases} a_k = a & \text{if } r_k = 0, \\ a_k = b & \text{if } r_k = 1, \end{cases} \tag{3.6}$$

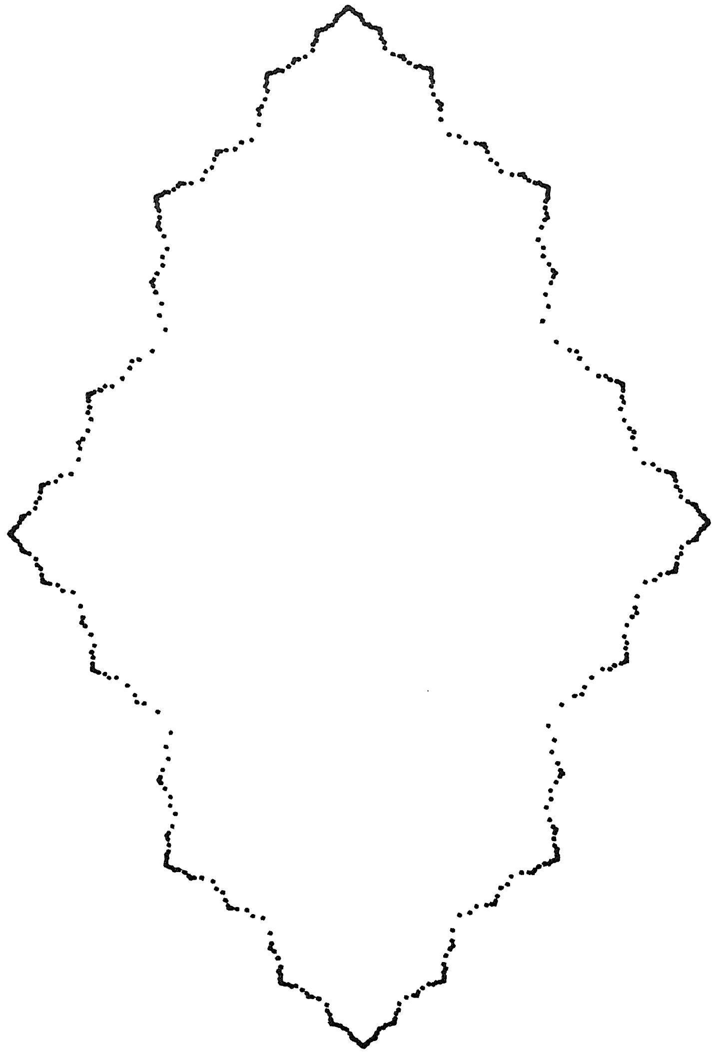

FIGURE 3.1. The San Marco attractor, the Julia set of $z \mapsto z^2 - 3/4$

where r_k is the kth binary digit of the binary expansion of a fraction r. Thus to each point of the limit set J corresponds a real number of [0,1]. (There is a little ambiguity for binary expansions terminating in an endless string of zeros or ones, but this concerns only a countable subset of J.)

The dynamics on the limit set J can be described by (see fig.3.2)

$$\begin{cases} Az: \ r \mapsto r/2, \\ Bz: \ r \mapsto (1+r)/2. \end{cases} \tag{3.7}$$

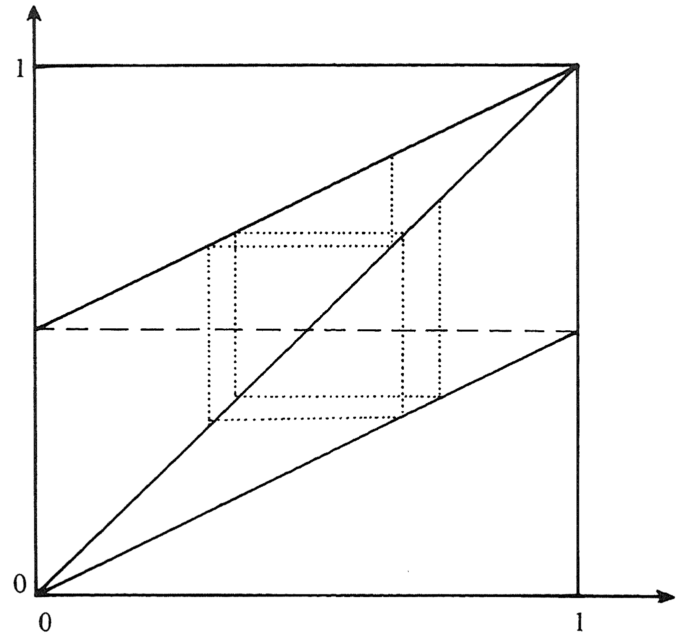

FIGURE 3.2.

This double-valued transformation has a unique inverse which is perhaps the simplest transformation showing chaotic behaviour.

Let us next consider the quadratic map (3.3) for the case $\mu = 2$. Then we may use the parametrization

$$z = 2\cos\pi r. \tag{3.8}$$

Substitution gives at once

$$\begin{cases} Az: \ r \mapsto r/2, \\ Bz: \ r \mapsto 1 - r/2, \end{cases} \tag{3.9}$$

the well-known tent map closely related to the map (3.7).

Thus there is every reason to extend the notion of the Julia set to non-unique analytic mappings. In both cases we have considered here the limit set J has the same chaotic behaviour. If z is an arbitrary point of J, then the sequences formed by subjecting z either to A or to B in some pseudo-random manner, e.g. prescribed by the binary digits of the binary expansion of a fraction, almost never converge. A generalized Julia set may then be defined as the limit set of all sequences found in this way from two or more analytic transformations, A, B etc. provided it exists. If the transformations are the branches of the inverse of a single-valued analytic function this coincides with the traditional definition. It would be tempting to sketch a general theory but, in our opinion, it is better to start with a number of interesting special cases.

We end this paper by considering the following (2-2)-complex map in which both the Pythagoras tree and the quadratic map are combined. We consider

$$F(w,z) = 0 \tag{3.10}$$

where F is a quadratic polynomial. It is assumed that $w = z = \pm 1$ are fixed points with multipliers dw/dz equal to a and b. Then F is determined by a further single complex parameter c and can be written as

$$(w - az - 1 + a)(w - bz + 1 - b) + c(w - z)^2 = 0. \tag{3.11}$$

For $c = 0$ this reduces to the Pythagoras map

$$\begin{cases} w = 1 + a(z-1), \\ w = -1 + b(z+1). \end{cases} \tag{3.12}$$

For $c = -ab = -\frac{1}{2}(a+b)$ the quadratic map is obtained in the form

$$\tfrac{1}{2}\sqrt{(1+c)/c}\,(w^2 - 1) + w - z = 0. \tag{3.13}$$

The maps (3.13) and (3.1) are equivalent with the following relation between the parameters

$$4\mu c = 1. \tag{3.14}$$

In particular the San Marco attractor is obtained for $a = -1$, $b = 1/3$, $c = 1/3$ as

$$w^2 + w - 1 = z. \tag{3.15}$$

As an illustration we consider the special case

$$a = (1+i)/2, \ b = (1-i)/2, \ c = -\tfrac{1}{4}.$$

The multipliers are those of the Pythagoras tree (1.1). The value $c = -\frac{1}{4}$ is chosen halfway the value $c = 0$ of the Pythagoras tree and $c = -\frac{1}{2}$ of the quadratic map. The result shown in fig.3.3 looks like the blossom of a generalized Pythagoras tree but has the cauliflower structure of known Julia sets of the quadratic map (see [2]). Shown are 1000 pre-images of the point $z = 1$ which is a fixed point of (3.13). It is a safe conjecture that this Julia set is entirely disconnected.

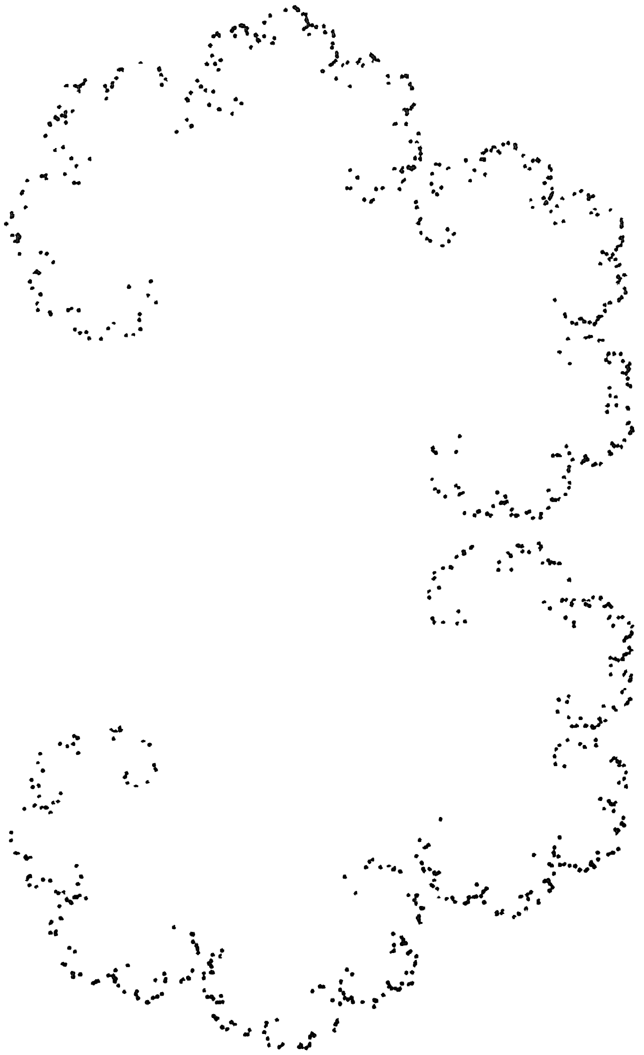

FIGURE 3.3. A generalized Julia set

e last illustration may give an idea of what to expect in a more general tion. We took

$$\begin{cases} A: \ z \mapsto 1 + iz/2, \\ B: \ z \mapsto a(1+z^2)/z, \end{cases} \tag{3.16}$$

$a = 4/5$. The fixed point of A is $0.8 + i0.4$. The fixed points of B are In fig.3.4 we have shown a representative part of the corresponding ralized Julia set with $z = 2$ as the starting-point of a random sequence.

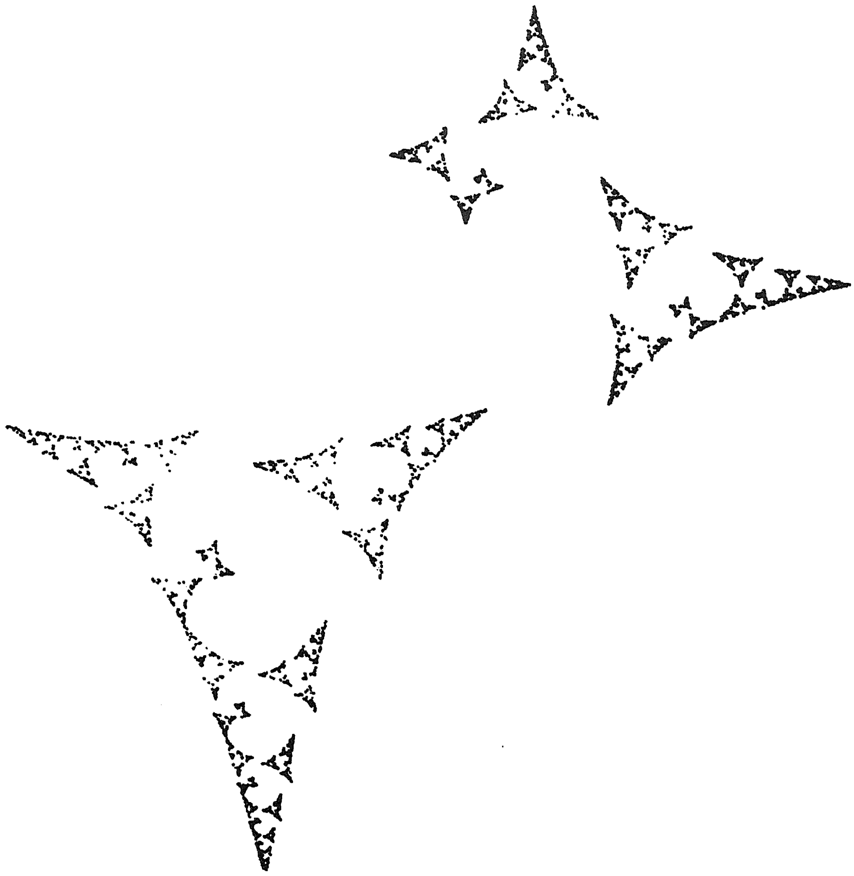

FIGURE 3.4. A generalized Julia set

REFERENCES

1. B. MANDELBROT (1980). Fractal aspects of the iteration of $z \to \lambda z(1-z)$. *Ann. New York Acad. Sci.* *357,* 249-259.
2. P. BLANCHARD (1984). Complex analytic dynamics. *Bull. Am. Math. Soc.* *11,* 85-141.
3. M.F. BARNSLEY, A.N. HARRINGTON (1985). A Mandelbrot set for pairs of linear maps. *Physica 15D,* 421-432.