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Residue Formulas for Meromorphic Matrices

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In the analysis of the vibrations of mechanical systems, it is not only important to compute the resonance frequencies, but also to find the so-called "participation matrices" which govern the distribution of the energy over the various resonance modes. These matrices appear as residue matrices for certain meromorphic matrix-valued functions (transfer matrices from forces to displacements), the poles of which correspond to the resonance frequencies. Also, these poles are simple as a consequence of the law of conservation of energy. So the problem comes down to the computation of the residue at a simple pole of a meromorphic matrix. This matrix is in general not given through its entries, but rather as the inverse of another matrix or as a fraction of holomorphic matrices. Extending earlier results of Lancaster and of Gohberg and Sigal, we work out a convenient residue formula for matrices in fractional form. Several variants will be discussed as well. In all versions, one constructs a "normalizing matrix" which is invertible if and only if the pole one considers is simple, and one writes down a formula for the residue which features the inverse of the normalizing matrix. Proofs are based on the "local Smith form" for meromorphic matrices. The normalizing matrix can also be used in stability tests, and we show an application of this.

1. Introduction

Let $Y(\lambda)$ be a matrix whose entries are meromorphic functions of λ. By expanding each entry in a Laurent series around a given point α, we get a Laurent series development for $Y(\lambda)$:

$$Y(\lambda) = Y_{-r}(\lambda-\alpha)^{-r} + \cdots + Y_{-1}(\lambda-\alpha)^{-1} + \\ + Y_0 + Y_1(\lambda-\alpha) + \cdots . \tag{1.1}$$

The matrix Y_{-1} is called the *residue* of $Y(\lambda)$ at α. We say that Y has a *simple pole* at α if $(\lambda-\alpha)Y(\lambda)$ is analytic in a neighborhood of α. The purpose of the present paper is to obtain convenient formulas for the computation of the residue of Y at a simple pole under certain assumptions on the way that this matrix function is given.

This problem is directly motivated by engineering applications. For a very simple example of this, consider the equations of a vibrating string with forces and displacements at both ends being of interest. (The electrical analog of this would be the lossless transmission line.) The equations are as follows:

$$\frac{\partial^2}{\partial t^2}w(x,t) = \frac{\partial^2}{\partial x^2}w(x,t) \tag{1.2}$$

$$-\frac{\partial}{\partial x}w(0,t) = F_1(t) \tag{1.3}$$

$$\frac{\partial}{\partial x} w(1,t) = F_2(t) \tag{1.4}$$

$$w(0,t) = y_1(t) \tag{1.5}$$

$$w(1,t) = y_2(t) \tag{1.6}$$

For this system, we determine a matrix that relates the amplitudes of the forces to the amplitudes of the displacements under the assumption that the system is in harmonic motion at frequency ω. We set

$$w(x,t) = w_0(x)e^{i\omega t} \tag{1.7}$$

$$\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} e^{i\omega t}, \quad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} e^{i\omega t}. \tag{1.8}$$

The equations (1.2-6) then lead to

$$w_0''(x) = -\omega^2 w_0(x) \tag{1.9}$$

$$\begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} = \begin{bmatrix} -w_0'(0) \\ w_0'(1) \end{bmatrix} \tag{1.10}$$

$$\begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} = \begin{bmatrix} w_0(0) \\ w_0(1) \end{bmatrix} \tag{1.11}$$

From (1.9), we get

$$w_0(x) = a\,\omega^{-1} \sin \omega x + b \cos \omega x \,, \tag{1.12}$$

where $\omega^{-1} \sin \omega x$ is taken as an analytic function of ω for every x, so that its value at $\omega = 0$ is simply x. Using (1.10) and (1.11), one now expresses the force and displacement amplitudes in terms of the parameters a and b:

$$\begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\cos \omega & \omega \sin \omega \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{1.13}$$

$$\begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^{-1} \sin \omega & \cos \omega \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{1.14}$$

Eliminating the parameters a and b, one obtains the "admittance matrix" $Y(\omega)$:

$$Y(\omega) = \begin{bmatrix} 0 & 1 \\ \omega^{-1} \sin \omega & \cos \omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\cos \omega & \omega \sin \omega \end{bmatrix}^{-1}. \tag{1.15}$$

Notice that the admittance matrix appears as a fraction of two analytic matrices. This representation of the relation between forces and displacements at both ends of a string is known among mechanical engineers as the "dynamic direct-stiffness method" (cf. [4], Ch.20). Of course, in system-theoretic terms the matrix $Y(\omega)$ is just the transfer matrix from the forces to the displacements. The fact that it is a symmetric matrix which is even as a function of ω is no surprise since the considered system (1.2-6) is time-reversible Hamiltonian (cf. [17]). The symmetry with respect to the transverse diagonal reflects the left-right symmetry of the system. The poles of the admittance matrix correspond to the "natural frequencies" of the system (no force input). As a consequence of the law of conservation of energy, all poles are real and simple.

In this very simple example, one could compute the residues by computing the entries of $Y(\omega)$ separately and using the standard rules for the computation of residues for scalar functions. However, for large structures this becomes hardly an attractive way of doing the computation. One would like to use a method which is adapted to the form in which the admittance matrix appears.

In [13], Lancaster has given a residue formula for the case in which $Y(\omega) = Z(\omega)^{-1}$, and $Z(\omega)$ is a polynomial matrix (cf. also the more recent work [6], p.64). In the engineering literature, Lancaster's formula has been used also in situations where $Z(\omega)$ is not polynomial but rational or even meromorphic (cf. [8,12,15]). The techniques of [13] and [6] are not readily adapted to these more general situations; moreover, as we shall see, the use of Lancaster's formula in the more general context is not always possible. In this paper, we use methods similar to those of [5] in order to prove residue formulas for meromorphic matrices appearing in various forms (and that are not necessarily square). We may note that, if $Y(s)$ is a strictly proper rational matrix function having only simple poles, then knowledge of the poles and the corresponding residues means that one can write down the partial fraction expansion of $Y(s)$, and this is practically equivalent to finding a state-space realization for Y. The use of the partial fraction expansion for computing realizations has been suggested in [16], and was recommended as a numerically robust procedure in [18]. Of course, it is a classical observation that the inverse Laplace transform can be computed in a convenient way by using the partial fraction expansion.

The organization of the paper is as follows. We start with some algebraic preliminaries in Section 2. Next, we discuss what can be said about the residue of Y at α under the assumption that Y is available through its inverse. Although we de obtain a residue formula, it will appear that this formula is not quite satisfactory. Another direction in which Lancaster's work may be generalized is given by coprime factorization. This is considered in section 4, and it turns out that it is possible (as in Lancaster's formula) to determine the residue by calculating the derivative at α of a matrix that is analytic in a neighborhood of α, plus some operations on constant matrices. We also get criteria for the pole at α to simple. Such criteria can be used in stability tests, and we show an application of this in Section 5.

2. Preliminaries

Let Ω be a region of the complex plane, which will be fixed throughout the discussion below. In most applications, one will have $\Omega = \mathbf{C}$. Let α be a point in Ω. We let F denote the field of meromorphic functions on Ω, and we write R_α for the subset of F consisting of functions that are analytic in a neighborhood af α. In other words, $f \in F$ belongs to R_α if and only if f does not have a pole at α. It is straightforward to verify that R_α is a ring, and it is also easily seen that R_α is in fact a principal ideal domain, the ideals being of the form $(\lambda-\alpha)^k R_\alpha$, $k \geqslant 0$. The set of $p \times m$-matrices with elements in F (resp. R_α) will be denoted by $F^{p \times m}$ (resp. $R_\alpha^{p \times m}$).

The situation we have here is a particular instance of the following set-up. Let R be a commutative ring, and let D and E be multiplicative subsets of R such that $D \subset E$. A matrix U with elements in the factor ring $R\,/\,D$ is said to be *D-unimodular* if it has an inverse with elements in $R\,/\,D$. Matrices N and M with elements in the factor ring $R\,/\,E$ are said to be *D-equivalent* if there exist D-unimodular matrices U and V such that $M = UNV$. The key result in this context is the following.

THEOREM 2.1 *In the situation described above, suppose that R / D is a principal ideal domain. Then every matrix over R / E is D-equivalent to a matrix of the form*

$$N = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \tag{2.1}$$

where $\Delta = diag(h_1, \cdots, h_r)$ and the elements h_j may be chosen such that $h_{j+1} / h_j \in R$.

The proof of this is essentially standard (cf., for instance, [3], [9], [20]). In our setting, R is the ring of analytic functions on Ω, D is the set of all elements of R that are nonzero at α, and E is the set of all elements that are not identically zero on Ω. Instead of *D-unimodular,* we shall say *locally unimodular (at α)* and we will leave out the reference to the specific point α on most occasions, since the point will be fixed throughout the discussion. Because of the simple ideal structure of $R / D = R_\alpha$, the special form (2.1), which will be called the *local Smith form* (at α), can be taken such that

$$h_j = (\lambda - \alpha)^{d_j} \quad (j = 1, \cdots, r) \tag{2.3}$$

and $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$. This particular appearance of the local Smith form will be used extensively below. The form was used earlier, for instance in [5] (p.607) and in [19]; if one deals with rational matrices, it can also be used with $\alpha = \infty$, replacing $\lambda - \alpha$ by λ^{-1} [7]. The local Smith form of a meromorphic matrix is relatively easy to compute (cf. [7] and [10], p.139); nevertheless, it contains much more information than the residue does, and so it should be easier to compute the latter. We shall only use the local Smith form for proofs, for which it is, in fact, a most convenient tool.

Often, "local" properties of meromorphic matrices can be expressed in terms of constant matrices. In fact, the situation could be interpreted as a special case of reduction to the space of maximal ideals, as explained in [20], Section 8.1. Of course, the ring R_α has just a single maximal ideal, generated by the function $\lambda - \alpha$. Whatever approach one takes, it is easy to prove results such as the following.

LEMMA 2.2 *A matrix $U \in R_\alpha^{m \times m}$ is unimodular if and only if $U(\alpha) \in \mathbb{C}^{m \times m}$ is invertible.*

We now consider coprimeness in the sense of the local ring R_α. Given $N \in R_\alpha^{p \times m}$, a matrix $P \in R_\alpha^{m \times m}$ is said to be a *right factor* of N if there exists $\hat{N} \in R_\alpha^{p \times m}$ such that $N = \hat{N}P$. Two matrices $N_1 \in R_\alpha^{p \times m}$ and $N_2 \in R_\alpha^{q \times m}$ are said to be *locally right coprime* (at α) if all their common right factors are locally unimodular. The following characterization of this concept is classical (see [14], p.35).

PROPOSITION 2.3 *Two matrices $N_1 \in R_\alpha^{p \times m}$ and $N_2 \in R_\alpha^{q \times m}$ are right coprime if and only if there exist matrices $G \in R_\alpha^{m \times p}$ and $H \in R_\alpha^{m \times q}$ such that $GN_1 + HN_2 = I_m$.*

The proof in [14] shows that, in fact, the following is true.

PROPOSITION 2.4 *Two matrices $N_1 \in R_\alpha^{p \times m}$ and $N_2 \in R_\alpha^{q \times m}$ are right coprime if and only if there exist unimodular matrices $S \in R_\alpha^{(p+q) \times (p+q)}$ and $T \in R_\alpha^{m \times m}$ such that*

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = S \begin{bmatrix} I_m \\ 0 \end{bmatrix} T. \tag{2.4}$$

The characterization of Prop. 2.3 can be read in terms of left invertibility, and hence there is a translation in terms of constant matrices ([20, Thm. 8.1.12]).

COROLLARY 2.5 *Two matrices $N_1 \in R_\alpha^{p\times m}$ and $N_2 \in R_\alpha^{q\times m}$ are right coprime if and only if the matrix*

$$\begin{bmatrix} N_1(\alpha) \\ N_2(\alpha) \end{bmatrix} \in \mathbf{C}^{(p+q)\times m} \tag{2.5}$$

is full column rank.

Of course, one can also define *left factors* and *left coprimeness* for pairs of matrices having an equal number of rows, and the above results can be duplicated; we won't spell this out.

A *locally right coprime factorization* (at α) of a matrix $Y \in F^{p\times m}$ is a representation of Y in the form

$$Y = ND^{-1} \tag{2.6}$$

where $N \in R_\alpha^{p\times m}$, $D \in R_\alpha^{m\times m}$, the matrices N and D are right coprime, and D is invertible as an element of $F^{m\times m}$. We will now display a particular locally coprime factorization that will turn out to be useful. Using Thm.2.1, we can write

$$Y \Rightarrow S \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} T \tag{2.7}$$

where S and T are locally unimodular matrices, and Δ is as described in (2.3). Now, we can split the negative powers of $(\lambda-\alpha)$ off from the nonnegative powers:

$$\Delta = \begin{bmatrix} \Delta_-(\lambda) & 0 \\ 0 & \Delta_+(\lambda) \end{bmatrix} \tag{2.8}$$

$$\Delta_-(\lambda) = diag\,[(\lambda-\alpha)^{d_1}, \cdots ,(\lambda-\alpha)^{d_k}],\ d_1 \leqslant \cdots \leqslant d_k < 0 \tag{2.9}$$

$$\Delta_+(\lambda) = diag\,[(\lambda-\alpha)^{d_{k+1}}, \cdots ,(\lambda-\alpha)^{d_r}],\ 0 \leqslant d_{k+1} \leqslant \cdots \leqslant d_r. \tag{2.10}$$

Using the notation of (2.7-10), define

$$N = S \begin{bmatrix} I & 0 & 0 \\ 0 & \Delta_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.11}$$

$$D = T^{-1} \begin{bmatrix} (\Delta_-)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \cdot \tag{2.12}$$

It is easily verified that $Y = ND^{-1}$ is a right coprime factorization of Y at α. The dimension of the matrix Δ_- in (2.7) will be called the *total pole multiplicity* of Y at α.

One also defines *left coprime factorizations* $Y = D^{-1}N$. A left coprime factorization for Y can be obtained by forming a right coprime factorization for Y^t and taking transposes. Of course, every statement about right coprime matrices has an analog for left coprime matrices.

The next lemma shows to what extent coprime factorizations and corresponding "Bézout factors" (as in Prop. 2.3) are unique.

LEMMA 2.6 *Let $Y = ND^{-1}$ and $Y = \tilde{D}^{-1}\tilde{N}$ be a right and a left coprime* *factorization, respectively, of* $Y \in F^{p\times m}$. *Let* $G \in R_\alpha^{m\times m}$ *and* $H \in R_\alpha^{m\times p}$ *be such that*

$$GD + HN = I_m. \tag{2.13}$$

Suppose now that $Y = \bar{D}^{-1}\bar{N}$ *is also a left coprime factorization, and that* $\hat{G} \in R_\alpha^{m\times m}$ *and* $\hat{H} \in R_\alpha^{m\times p}$ *are matrices that satisfy*

$$\hat{G}D + \hat{H}N = I_m. \tag{2.14}$$

Then there exist matrices $E \in R_\alpha^{p\times p}$ *and* $F \in R_\alpha^{m\times p}$, *with E unimodular, such that*

$$\bar{D} = E\tilde{D},\ \bar{N} = E\tilde{N} \tag{2.15}$$

$$\hat{G} = G - F\tilde{N},\ \hat{H} = H + F\tilde{D}. \tag{2.16}$$

Proof It follows from the 'left' version of Prop.2.3 that there exist matrices $\tilde{G} \in R_\alpha^{p\times p}$ and $\tilde{H} \in R_\alpha^{m\times p}$ such that

$$\tilde{D}\tilde{G} + \tilde{N}\tilde{H} = I_p. \tag{2.17}$$

Using (2.13), (2.17), and the equality $ND^{-1} = \tilde{D}^{-1}\tilde{N}$, we get

$$\begin{bmatrix} G & H \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{H} \\ N & \tilde{G} \end{bmatrix} = \begin{bmatrix} I_m & -G\tilde{H} + H\tilde{G} \\ 0 & I_p \end{bmatrix}. \tag{2.18}$$

It is clear that the matrix on the right hand side in this equation is unimodular, and it follows that the two square matrices on the left must also be unimodular. Now, let $\hat{G}$, $\hat{H}$, $\bar{N}$ and $\bar{D}$ be as in the statement of the lemma. From (2.18), we then have

$$\begin{bmatrix} \hat{G} & \hat{H} \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} G & H \\ -\tilde{N} & \tilde{D} \end{bmatrix}^{-1} \begin{bmatrix} I_m \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{G} & \hat{H} \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}. \tag{2.19}$$

This means that there exist matrices $F \in R_\alpha^{m\times p}$ and $E \in R_\alpha^{p\times p}$ such that

$$\begin{bmatrix} \hat{G} & \hat{H} \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} G & H \\ -\tilde{N} & \tilde{D} \end{bmatrix}^{-1} = \begin{bmatrix} I_m & F \\ 0 & E \end{bmatrix}. \tag{2.20}$$

Moreover, E must be unimodular because the left hand side of the equation is unimodular. Multiplying out the inverse, we find (2.15) and (2.16).

The result in (2.15) is standard (see, for instance, [9] (p.441) or [3] (p.60)). An alternative version of the uniqueness result on the Bézout factors can be found in [20] (Lemma 4.1.32).

3. Formula based on the inverse

In this section, we suppose that $Y(\lambda) \in F^{m\times m}$ is invertible, and we want to find a formula which expresses the residue at a simple pole in terms of the inverse. First, let us introduce some notation. We let R_α^m denote the free module of rank m over R_α, and we define $\phi_\alpha : R_\alpha^m \to \mathbf{C}^m$ to be the evaluation map at α:

$$\phi_\alpha(f) = f(\alpha) \qquad (f \in R_\alpha^m). \tag{3.1}$$

The inverse of $Y(\lambda)$ will be denoted by $Z(\lambda)$. We define

$$N_\alpha(Z) = \{f \in R_\alpha^m \mid Zf \in \ker \phi_\alpha\}. \tag{3.2}$$

It is easily seen that $N_\alpha(Z)$ is a submodule of R_α^m, and consequently $\phi_\alpha N_\alpha(Z)$ is a subspace of $\mathbf{C}^m$. We are now ready to formulate the main results of this section:

THEOREM 3.1 *Assume that $Y(\lambda) \in F^{m\times m}$ is invertible, with inverse $Z(\lambda)$. Let C_R be a matrix over R_α such that $\phi_\alpha C_R$ is a basis matrix for $\phi_\alpha N_\alpha(Z)$. Let C_L be a matrix over R_α such that $\phi_\alpha C_L^t$ is a basis matrix for $\phi_\alpha N_\alpha(Z^t)$. Under these conditions, the following holds.*

1. *The constant matrix $M(C_L, C_R)$ defined by*

$$M(C_L,C_R) = \lim_{\lambda\to\alpha} \frac{d}{d\lambda}[C_L(\lambda)Z(\lambda)C_R(\lambda)] \tag{3.3}$$

is square. Its dimension is equal to the total pole multiplicity of Y at α, and its rank equals the multiplicity of the first order pole of Y at α.

2. *The pole of Y at α is simple if and only if the matrix $M(C_L,C_R)$ is invertible, and in this case the residue is given by*

$$\mathrm{Res}(Y;\alpha) = C_R(\alpha)M(C_L,C_R)^{-1}C_L(\alpha). \tag{3.4}$$

REMARK. It follows from the theorem that, in the case of a simple pole, there exist matrices C_R and C_L, satisfying the conditions of the theorem, such that $\mathrm{Res}(Y;\alpha) = C_R(\alpha)C_L(\alpha)$. This has been shown earlier in [5] (Thm.7.1) (extending still earlier results in [10]), where, in fact, a much more general situation was considered, involving operator-valued (rather than matrix-valued) functions of λ, and dealing with the complete principal part at an arbitrary pole rather than just at a simple pole. However, the normalizing matrix $M(C_L,C_R)$ was not given in [5].

PROOF Of course, the matrices C_R and C_L are not determined uniquely by the requirements of the theorem. First of all, we note that if we add to C_R a matrix H_R with columns in $N_\alpha(Z) \cap \ker\phi_\alpha$, then the result still satisfies the requirements. The same is true if we right multiply C_R by an invertible constant matrix G_R. On the other hand, suppose that both C_R and $\hat{C}_R$ satisfy the requirements of the theorem. Then there must exist an invertible constant matrix G_R such that $\hat{C}_R(\alpha) = C_R(\alpha)G_R$, because $\hat{C}_R(\alpha)$ and $C_R(\alpha)$ are basis matrices for the same subspace. The columns of the matrix $H_R \overset{\mathrm{def}}{=} \hat{C}_R - C_R G_R$ will then belong to $N_\alpha(Z) \cap \ker\phi_\alpha$. So we can conclude that the nonuniqueness in C_R and C_L is described by a transformation group which involves two invertible constant matrices G_R and G_L and two matrices H_R and H_L over R_α, such that the columns of both H_R and H_L^t belong to $N_\alpha(Z) \cap \ker\phi_\alpha$, and which acts as follows:

$$C_R \to \hat{C}_R = C_R G_R + H_R \tag{3.5}$$

$$C_L \to \hat{C}_L = C_L G_L + H_L. \tag{3.6}$$

Note that

$$\lim_{\lambda\to\alpha} \frac{d}{d\lambda}[C_L(\lambda)Z(\lambda)H_R(\lambda)] = 0 \tag{3.7}$$

by the product rule of differentiation, since both $C_L(\lambda)Z(\lambda)$ and $H_R(\lambda)$ vanish at α. Repeated use of the rule shows that the effect of the transformation group on

$M(C_L,C_R)$ is

$$M(\hat{C}_L,\hat{C}_R) = G_L M(C_L,C_R) G_R. \tag{3.8}$$

This shows that the dimension and the rank of $M(C_L,C_R)$ are invariants under the transformation group (3.5-6). We can therefore evaluate these two numbers for any particular value of C_R and C_L. We select suitable values in the following way. Because $Y(\lambda)$ is invertible, the local Smith form of $Y(\lambda)$ reduces to (cf. (2.7-10)):

$$Y(\lambda) = S(\lambda)\begin{bmatrix}\Delta_-(\lambda) & 0 \\ 0 & \Delta_+(\lambda)\end{bmatrix} T(\lambda). \tag{3.9}$$

It is easily verified that we can take

$$C_R(\lambda) = S(\lambda)\begin{bmatrix} I_k \\ 0 \end{bmatrix}, C_L(\lambda) = [I_k \;\; 0]\, T(\lambda) \tag{3.10}$$

where k is the dimension of $\Delta_-(\lambda)$, i.e., the total pole multiplicity of Y at α. For this selection of C_R and C_L, we get (cf. (2.9)):

$$M(C_L,C_R) = \frac{d}{d\lambda}(\Delta_-(\lambda)^{-1})\Big]_{\lambda=\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix} \tag{3.11}$$

where q is the multiplicity of the first-order pole of Y at α. The statements under 1. of the theorem now follow immediately. It is also a direct consequence that Y has a simple pole at α if and only if $M(C_L,C_R)$ is invertible, and so it remains to verify the residue formula (3.4).

From (3.5), (3.6) and (3.8), we see that the right hand side of (3.4) is an invariant under the transformation group described above. Therefore, it suffices to verify (3.4) for the particular selection (3.10) of C_R and C_L. In the case of a simple pole, (3.11) gives $M(C_L,C_R) = I$ and so

$$C_R(\alpha) M(C_L,C_R)^{-1} C_L(\alpha) = S(\alpha)\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} T(\alpha) = \text{Res}(Y;a). \tag{3.12}$$

The final equality follows because, in the case of a simple pole, the matrix $\Delta_-(\lambda)$ in (3.9) equals $(\lambda-\alpha)^{-1} I_k$. The proof is complete.

Remark. The problem that Thm.3.1 leaves us with is, how to compute C_R and C_L (without doing something that is equivalent to already computing the residue). It should be noted that it is not allowed, in general, to simplify the formula (3.4) by replacing C_R and C_L in the expression for $M(C_L,C_R)$ by their limit values. To see this, consider the following example:

$$Y(\lambda) = \frac{1}{\lambda}\begin{bmatrix} 1 & -\lambda \\ -\lambda & 2\lambda^2 \end{bmatrix}. \tag{3.13}$$

This corresponds to

$$Z(\lambda) = \frac{1}{\lambda}\begin{bmatrix} 2\lambda^2 & \lambda \\ \lambda & 1 \end{bmatrix}. \tag{3.14}$$

We are interested in the pole at $\lambda = 0$. It is easily verified that one can take

$$C_R(\lambda) = \begin{bmatrix} 1 \\ -\lambda \end{bmatrix}, C_L(\lambda) = [1 \quad -\lambda]. \tag{3.15}$$

Calculation shows that $M(C_L,C_R) = 1$. On the other hand, one has

$$\lim_{\lambda\to 0} \frac{d}{d\lambda}\left\{[1 \quad 0]\frac{1}{\lambda}\begin{bmatrix} 2\lambda^2 & \lambda \\ \lambda & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = 2. \tag{3.16}$$

One easily sees that this phenomenon is caused by the fact that $Z(\lambda)$ has a pole at $\lambda = 0$. In general, if $Z(\lambda)$ is analytic in a neighborhood of α, then it may be shown that it is allowed to replace C_L and C_R by their limit values in (3.3), and, moreover, in this case the constant matrices $C_R(\alpha)$ and $C_L(\alpha)$ may be constructed directly as basis matrices for the right and left null spaces, respectively, of $Z(\alpha)$. This will follow as a special case of the result of the next section.

4. Formula based on coprime factorization

In this section, we develop formulas for the residue at a simple pole, based on the availability of coprime factorizations. Several versions will be presented, each of which has its own merits.

Theorem 4.1 *Suppose that* $Y(\lambda) = N(\lambda)D(\lambda)^{-1}$ *and* $Y(\lambda) = \tilde{D}(\lambda)^{-1}\tilde{N}(\lambda)$ *are right and left coprime factorizations, respectively, of* $Y \in F^{p\times m}$ *at* α. *Define*

$$P(\lambda) = \tilde{D}(\lambda)N(\lambda) = \tilde{N}(\lambda)D(\lambda). \tag{4.1}$$

Also, let T_R *be a full column rank matrix such that*

$$\operatorname{im} T_R = \ker D(\alpha) \tag{4.2}$$

and let $\tilde{T}_L$ *be a full row rank matrix such that*

$$\ker \tilde{T}_L = \operatorname{im} \tilde{D}(\alpha). \tag{4.3}$$

Under these conditions, the following holds.

1. *The matrix* $\tilde{T}_L P'(\alpha) T_R$ *is square. Its dimension is equal to the total pole multiplicity of* Y *at* α, *and its rank equals the multiplicity of the first-order pole of* Y *at* α.
2. *The pole of* Y *at* α *is simple if and only oif the matrix* $\tilde{T}_L P'(\alpha) T_R$ *is invertible, and in this case the residue is given by*

$$\operatorname{Res}(Y;\alpha) = N(\alpha)T_R[\tilde{T}_L P'(\alpha)T_R]^{-1}\tilde{T}_L\tilde{N}(\alpha). \tag{4.4}$$

3. *Moreover, for a simple pole at* α *one has*

$$\ker \operatorname{Res}(Y;\alpha) = \operatorname{im} D(\alpha) \tag{4.5}$$

$$\operatorname{im} \operatorname{Res}(Y;\alpha) = \ker \tilde{D}(\alpha). \tag{4.6}$$

Proof We divide the proof in three parts corresponding to the three claims in the theorem.

Claim 1. It is clear that the matrix $P \in R_\alpha^{p\times m}$ and the constant matrices T_R and $\tilde{T}_L$ are not determined uniquely by Y. This is due to the nonuniqueness of left and right coprime factorizations, and to the fact that a matrix is only determined by (4.2) up to nonsingular transformations from the right, and by (4.3) up to nonsingular transformations from the left. One can define a transformation group by which all triples $(P, T_R, \tilde{T}_L)$ are related to eachother. As is seen from Lemma 2.6, the action of the transformation group is specified by two unimodular matrices U and $\tilde{U}$ and two invertible constant matrices M and $\tilde{M}$, in the following way:

$$P \to \hat{P} = \tilde{U}PU \tag{4.7}$$

$$T_R \to \hat{T}_R = U(\alpha)^{-1}T_R M \tag{4.8}$$

$$\tilde{T}_L \to \hat{\tilde{T}}_L = \tilde{M}\tilde{T}L\tilde{U}(\alpha)^{-1}. \tag{4.9}$$

To determine the behavior of the matrix $\tilde{T}_L P'(\alpha) T_R$ under the action of the transformation group, we first note that

$$\hat{P}'(\alpha) = \tilde{U}(\alpha)P'(\alpha)U(\alpha) + \tilde{U}'(\alpha)P(\alpha)U(\alpha) + \tilde{U}(\alpha)P(\alpha)U'(\alpha). \tag{4.10}$$

Multiplying this from the left by $\hat{\tilde{T}}_L = \tilde{M}\tilde{T}_L\tilde{U}(\alpha)^{-1}$ and from the right by $\hat{T}_R = U(\alpha)^{-1}T_R M$, and using the fact that $\tilde{T}_L P(\alpha) = \tilde{T}_L \tilde{D}(\alpha)\tilde{N}(\alpha) = 0$ and $P(\alpha)T_R = N(\alpha)D(\alpha)T_R = 0$, we find

$$\hat{\tilde{T}}_L \hat{P}'(\alpha)\hat{T}_R = \tilde{M}\tilde{T}_L P'(\alpha) T_R M. \tag{4.11}$$

This shows that the size and the rank of the matrix $\tilde{T}_L P'(\alpha) T_R$ are invariants under the transformation group defined above. So it suffices to compute these quantities for a particular selection of $P(\lambda)$, T_R, and $\tilde{T}_L$. We take the right coprime factorization given in (2.11-12), and the left factorization that can, in an obvious way, be defined similarly. This leads to

$$P(\lambda) = \begin{bmatrix} (\Delta_-(\lambda))^{-1} & 0 & 0 \\ 0 & \Delta_+(\lambda) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.12}$$

For T_R and $\tilde{T}_L$, we can take

$$T_R = \begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{T}_L = \begin{bmatrix} I_k & 0 & 0 \end{bmatrix} \tag{4.13}$$

where k is the dimension of the matrix $\Delta_-(\lambda)$, i.e., the total pole multiplicity of Y at α. Then the matrix $\tilde{T}_L P'(\alpha) T_R$ is a $k \times k$-matrix, and an easy computation shows that, in fact,

$$\tilde{T}_L P'(\alpha) T_R = \frac{d}{d\lambda}(\Delta_-(\lambda)^{-1})\Big|_{\lambda=\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix} \tag{4.14}$$

where q is the multiplicity of the first-order pole of Y at α. This completes the first part of the proof.

Claim 2. It is immediate from the above that Y has a simple pole at α if and only if the matrix $\tilde{T}_L P'(\alpha) T_R$ is invertible. To verify the residue formula, we extend the transformation group defined above with its action on $N(\alpha)$ and $\tilde{N}(\alpha)$:

$$N(\alpha) \to \hat{N}(\alpha) = N(\alpha)U(\alpha) \tag{4.15}$$

$$\tilde{N}(\alpha) \to \bar{N}(\alpha) = \tilde{U}(\alpha)\tilde{N}(\alpha)\cdot \tag{4.16}$$

It is now a straightforward matter to see that the right hand side of (4.4) is an invariant under the transformation group. To prove that it does indeed represent the residue of Y at α, we compute its value for the particular selection of matrices that was also used above. In the case of a simple pole, (4.14) gives $\tilde{T}_L P'(\alpha) T_R = I$ and so we find

$$N(\alpha)T_R[\tilde{T}_L P'(\alpha)T_R]^{-1}\tilde{T}_L\tilde{N}(\alpha) =$$

$$= S(\alpha)\begin{bmatrix} I_k & 0 & 0 \\ 0 & \Delta_+(\alpha) & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix}[I_k \ \ 0 \ \ 0]\begin{bmatrix} I_k & 0 & 0 \\ 0 & \Delta_+(\alpha) & 0 \\ 0 & 0 & 0 \end{bmatrix}T(\alpha) =$$

$$= \mathrm{Res}(Y;\alpha). \tag{4.17}$$

Claim 3. Again, the formula (4.5) is proved by noting that both sides do not depend on the selection of a particular coprime factorization, and that equality holds (as is seen by inspection) for the factorization displayed in (2.11-12). For (4.6), it's the same story.

Remark 1. It is true in general (whether α is a simple pole or not) that the subspaces $\mathrm{im}\, D(\alpha)$ and $\ker \tilde{D}(\alpha)$ are uniquely determined by Y. We could call $\ker \tilde{D}(\alpha)$ the *right modal subspace* of Y at α, and the row space of left null vectors of $D(\alpha)$ could be termed the *left modal subspace* of Y at α. If $Y(\lambda) = (\lambda I - A)^{-1}$, where $A \in \mathbb{C}^{m \times m}$, then α is a pole of Y if and only if α is an eigenvalue of A, and the right and left modal subspaces of Y at α are equal to the right and left eigenspaces of A corresponding to the eigenvalue α.

Remark 2. Suppose that $Y(\lambda) = (Z(\lambda))^{-1}$ and $Z(\lambda)$ does not have a pole at α. In this case, right and left locally coprime factorizations are given by $N(\lambda) = \tilde{N}(\lambda) = I$, $D(\lambda) = \tilde{D}(\lambda) = Z(\lambda)$. We get $P(\lambda) = Z(\lambda)$, and T_R and $\tilde{T}_L$ are determined as basis matrices for the right and left null space of $Z(\alpha)$, respectively. The residue formula becomes

$$\mathrm{Res}(Y;\alpha) = T_R[\tilde{T}_L Z'(\alpha) T_R]^{-1}\tilde{T}_L. \tag{4.18}$$

This formula is applicable in particular when $Z(\lambda)$ is a polynomial matrix, and for this case the result was given by Lancaster ([13], pp.60-65; see also [6], p.64).

Remark 3. Suppose that $Y(\lambda) \in F^{m \times m}$ is symmetric. If in this case $Y = ND^{-1}$ is a right coprime factorization, then a left coprime factorization is obtained simply by taking $\tilde{D} = D'$, $\tilde{N} = N'$. The matrix function P is equal to $D'N = N'D$, so we see that P is symmetric. If T_R satisfies (4.2) then it is clear that (4.3) is satisfied by $\tilde{T}_L = T_R'$. The residue formula (4.4) becomes

$$\mathrm{Res}(Y;\alpha) = N(\alpha)T_R[T_R' P'(\alpha) T_R]^{-1} T_R' N'(\alpha). \tag{4.19}$$

We see that the residue is symmetric, as, of course, it should be.

One may ask whether it is possible to give a residue formula which more clearly reflects the properties (4.5) and (4.6). In other words, suppose that we define matrices $\tilde{T}_R$ and T_L, having full column rank and full row rank respectively, such that

$$\mathrm{im}\, \tilde{T}_R = \ker \tilde{D}(\alpha) \tag{4.20}$$

$$\ker T_L = \operatorname{im} D(\alpha). \tag{4.21}$$

Can we then find a residue formula which has $\tilde{T}_R$ on the left and T_L on the right? It turns out that this is possible, but of course the normalizing matrix has to be adjusted.

THEOREM 4.2 *Suppose that $Y = ND^{-1}$ and $Y = \tilde{D}^{-1}\tilde{N}$ are right and left locally coprime factorizations, respectively, of $Y \in F^{p\times m}$. Define $\tilde{T}_R$ and T_L as in (4.21-22). Let $G \in \mathbf{C}^{m\times m}$ and $H \in \mathbf{C}^{m\times p}$ be such that*

$$GD(\alpha) + HN(\alpha) = I_m. \tag{4.22}$$

(Such matrices exist by Cor.2.5). Under these conditions, the matrix $T_L D'(\alpha) H\tilde{T}_R$ is square, with its dimension being equal to the total pole multiplicity of Y at α, and its rank to the multiplicity of the first order pole of Y at α. The pole of Y at α is simple if and only if the matrix $T_L D'(\alpha) H\tilde{T}_R$ is invertible, and in this case the residue is given by

$$\operatorname{Res}(Y;\alpha) = \tilde{T}_R[T_L D'(\alpha) H\tilde{T}_R]^{-1} T_L. \tag{4.23}$$

PROOF The invariance of the proposed formula follows in the same way as in the previous proof. The correctness of our claims is then again established by looking at the special factorization (2.11-12), and using the following selections for T_L, $\tilde{T}_R$, G and H:

$$T_L = [I_k \;\; 0 \;\; 0]\, T(\alpha), \quad \tilde{T}_R = S(\alpha)\begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix} \tag{4.24}$$

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} T(\alpha), \quad H = \begin{bmatrix} I_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1}(\alpha) \tag{4.25}$$

where k is the total pole multiplicity of Y at α. Our conclusions now follow by straightforward computation.

The formula (4.24) is 'right-handed'; one could apply it to Y' and take the transpose of the resulting formula to get a corresponding 'left-handed' version. The only point where the left coprime factorization enters in the proposition is through the definition (4.21). But even this could be eliminated, because what is actually needed is only $\ker \tilde{D}(\alpha)$, and it is seen from Lemma 2.6 that if $[P \;\; Q]$ is a basis matrix for the row space of left null vectors of $[N'(\alpha) \;\; D'(\alpha)]'$, then $\ker \tilde{D}(\alpha)$ is determined as $\ker Q$. In this way, one obtains a residue formula that is based only on a right coprime factorization. This may be an advantage in terms of computation. Note that what is actually needed to determine the "H" and "Q" matrices is the reduction of $[D(\alpha)' \;\; N(\alpha)']'$ to $[I_m \;\; 0]'$ by elementary row operations:

$$\begin{bmatrix} G & H \\ P & Q \end{bmatrix}\begin{bmatrix} D(\alpha) \\ N(\alpha) \end{bmatrix} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}. \tag{4.26}$$

One can avoid doing row operations, however, by using the formula of the following corollary, which will close this section. This formula is probably in general the most convenient for computational purposes.

COROLLARY 4.3 *Suppose that $Y = ND^{-1}$ is a right coprime factorization of $Y \in F^{p\times m}$. Write $k = \dim\ker D(\alpha)$. Let $T_R \in \mathbf{C}^{m\times k}$ be a full column rank matrix satisfying*

$D(\alpha)T_R = 0$, and let $T_L \in \mathbf{C}^{k\times m}$ be a full row rank matrix such that $T_L D(\alpha) = 0$. Under these conditions, the following conclusions hold:

1. *The total pole multiplicity of Y at α is equal to k. The multiplicity of the first order pole of Y at α is equal to the rank of the matrix $T_L D'(\alpha)T_R \in \mathbf{C}^{k\times k}$.*
2. *The pole of Y at α is simple if and only if the matrix $T_L D'(\alpha)T_R$ is invertible. In this case, the residue is given by*

$$\mathrm{Res}(Y;\alpha) = N(\alpha)T_R[T_L D'(\alpha)T_R]^{-1}T_L. \tag{4.27}$$

PROOF The proof can be given along the same lines that have been used above. Alternatively, one may apply Proposition 4.2 by showing that $N(\alpha)T_R$ qualifies as a "$\tilde{T}_R$" matrix.

REMARK 4. Another way to derive (4.35) would be the following: first apply (4.19) to compute the residue of the matrix function D^{-1} at α, and then multiply by $N(\alpha)$ to obtain the residue of $Y=ND^{-1}$. This is correct, provided that one shows that Y has a simple pole at α if and only if D^{-1} has a simple pole at α. This property is indeed a consequence of coprimeness and has, in fact, been shown in the corollary, since the criterion given under 2. depends only on D.

5. APPLICATION AS A STABILITY TEST

It has already been noted that our results can also be used in stability tests. Indeed, for stability (in the sense of Lyapunov) one should have that all poles are in the open left half plane or on the imaginary axis, and in the latter case they should be simple - which is what can be tested by looking at the normalizing matrix. To show an application of this idea, let us re-derive a classical result on stability.

Suppose we have matrices $A \in \mathbf{C}^{n\times n}$ and $C \in \mathbf{C}^{p\times n}$. Then it is a standard result that the matrix A is stable in the Lyapunov sense if there exists a self-adjoint, strictly positive definite matrix $P \in \mathbf{C}^{n\times n}$ such that

$$A^*P + PA = -C^*C. \tag{5.1}$$

Let us now see how to prove this from the theory presented above. Let μ be an eigenvalue of A and let $x \in \mathbf{C}^n$ be a corresponding eigenvector. From (5.1), we have

$$x^*A^*Px + x^*PAx = -x^*C^*Cx \tag{5.2}$$

which leads to

$$(2\,\mathrm{Re}\,\mu)(x^*Px) = -x^*C^*Cx \leqslant 0 \tag{5.3}$$

and hence $\mathrm{Re}\,\mu \leqslant 0$, because $x^*Px > 0$. It remains to show that μ is a simple eigenvalue if its real part is zero.

So, suppose $\mathrm{Re}\,\mu = 0$, and let T_R be a basis matrix for $\ker(\mu I - A)$. It follows from any of the criteria derived above that μ will be a simple pole of the matrix function $Y(\lambda) = (\lambda I - A)^{-1}$ (and hence a simple eigenvalue of A) if and only if we can find a basis matrix T_L for the row space of left null vectors of $\mu I - A$ such that $T_L T_R$ is invertible. As in (5.2-3), we find

$$-T_R^*C^*CT_R = (2\,\mathrm{Re}\,\mu)T_R^*PT_R = 0 \tag{5.4}$$

which implies, of course,

$$CT_R = 0. \tag{5.5}$$

Therefore, multiplying (5.1) from the right by T_R, we get

$$(A^* + \mu I)PT_R = 0. \tag{5.6}$$

Since $\bar{\mu} = -\mu$, taking adjoints leads to

$$T_R^* P(\mu I - A) = 0. \tag{5.7}$$

This shows that we can take $T_L = T_R^* P$. Obviously, $T_L T_R = T_R^* P T_R$ is nonsingular and so our test shows that we have, indeed, stability.

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