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Action-Angle Maps and Scattering Theory for Some Finite-Dimensional Integrable Systems

I. The Pure Soliton Case

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Abstract. We construct an action-angle transformation for the Calogero-Moser systems with repulsive potentials, and for relativistic generalizations thereof. This map is shown to be closely related to the wave transformations for a large class $\mathscr{C}$ of Hamiltonians, and is shown to have remarkable duality properties. All dynamics in $\mathscr{C}$ lead to the same scattering transformation, which is obtained explicitly and exhibits a soliton structure. An auxiliary result concerns the spectral asymptotics of matrices of the form $M \exp(tD)$ as $t \to \infty$. It pertains to diagonal matrices D whose diagonal elements have pairwise different real parts and to matrices M for which certain principal minors are non-zero.

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1. Introduction

In this paper we study four classes of classical integrable N-particle systems on the line, which can be characterized by an $N \times N$ matrix-valued function L (referred to as the Lax matrix) on a $2N$-dimensional phase space Ω, cf. (2.1), (2.17), (2.31), (2.32), and (2.59) below. The symmetric functions $S_1, \ldots, S_N$ of L are in involution, so that the spectrum of L is conserved under the flow corresponding to any Hamiltonian in the maximal Abelian algebra generated by $S_1, \ldots, S_N$.

It follows from general principles (the Liouville-Arnold theorem) that there exists a canonical transformation $\Phi : \Omega \to \hat{\Omega}$ (the action-angle map) which diagonalizes the Abelian algebra in the sense that the functions $S_k \circ \Phi^{-1}$ depend only on the new generalized momenta (the action variables). However, this existence result yields neither an explicit picture of Φ nor a precise description of the action-angle phase space $\hat{\Omega}$. (This state of affairs is the classical analog of the quantum situation: There the spectral theorem ensures the existence of a unitary joint eigenfunction transformation $\mathscr{E}$ for the maximal Abelian quantum algebra, but does not provide detailed information concerning $\mathscr{E}$ and related matters such as existence of bound states, scattering, etc.)

Our main result is an explicit construction of an action-angle map Φ for the systems mentioned above. We also determine explicitly the wave and scattering maps for a large class of Hamiltonians, cf. Theorem 4.1. A crucial auxiliary result concerns the spectral asymptotics as $t \to \infty$ of two classes of t-dependent matrices. For the first class this amounts to a quite straightforward application of nondegenerate perturbation theory, since the t-dependence is linear, cf. Theorem A1. However, for the second class the dependence on t is exponential and the result (Theorem A2) is of independent interest.

The systems I_{nr} and II_{nr} studied below are commonly known as Calogero-Moser systems [1]. The subscript refers to the nonrelativistic Hamiltonians with pair potential $V(q) = 1/q^2$ for I_{nr} and $V(q) = 1/sh^2 q$ for II_{nr}. The relativistic generalizations presented and studied in [2] are denoted I_{rel} and II_{rel}. The results of this paper have a bearing on the relations of the latter systems to soliton solutions of various nonlinear PDE, which are detailed in [2, 3]. We intend to come back to this issue in a sequel to this paper, where we shall consider systems of particles that behave as solitons, antisolitons and their bound states [4].

For the case I_{nr} an explicit construction of Φ can already be found (in a somewhat different guise) in a paper by Airault et al. [5]. They observed that there exists a commutation relation of the Lax matrix L with an auxiliary matrix-valued function A on Ω, which can be used to infer crucial spectral properties of L. This state of affairs was further explained and elaborated on in a paper by Kazhdan et al. [6]. We have followed the lead of these papers and exploit a generalization of the commutation relation for the case I_{nr} to the other three cases. Further related work includes a paper by Adler [7], who obtains detailed information about the systems I_{nr} and II_{nr} with external potentials, and various references listed in the survey [1].

In all cases the action variables are simple functions of the eigenvalues of the Lax matrix L, whereas the matrix A is diagonal and depends only on the positions. Therefore, the similarity transformation turning L into a diagonal matrix $\hat{L}$ turns A into a matrix $\hat{A}$ whose symmetric functions are commuting Hamiltonians on the action-angle phase space $\hat{\Omega}$. It turns out that $\hat{A}$ is in essence equal to the Lax matrix of one of the four cases considered here. We shall express this state of affairs by saying that the two cases involved are dual to each other. Specifically, it turns out that I_{nr} is dual to itself, cf. (2.24), II_{nr} is dual to I_{rel}, cf. (2.49–50), whereas II_{rel} is again self-dual, cf. (2.73). The self-duality of I_{nr} (already pointed out in [5]) and II_{rel} can also be expressed by saying that Φ equals its inverse when Ω and $\hat{\Omega}$ are identified in the obvious way. (More precisely, for the case II_{rel} this holds after a scaling.)

It is known that the integrability of the systems I_{nr} and II_{nr} persists after quantization, cf. [8]. As proved in [9], the systems I_{rel} and II_{rel} can also be formally quantized in such a fashion that integrability is preserved. Elsewhere, we shall return to the quantum version of these systems and present evidence to the effect that the duality properties just described survive quantization, too; Moreover, there exist again intimate relations with various well-known integrable quantum systems [10].

We proceed by discussing the results and the organization of this paper in more detail. Though the cases I_{rel}, II_{nr}, I_{nr} may be viewed as special cases of II_{rel}, it turns out to be quite awkward to keep the action-angle map under control in the various parameter limits leading to the former systems. Therefore, we have opted for a case by case construction of Φ, which is presented in Sects. 2B–2D. As a bonus, this brings out the peculiarities of each case and leads to a clear picture of the duality properties. However, for conceptual and notational reasons we begin with Sect. 2A, which explains the construction and its consequences in general terms. The reader might skip this section on first reading and refer back to it when needed.

The construction performed in Chap. 2 only involves some simple linear algebra, including two versions of Cauchy's identity (listed at the end of this chapter). However, we have not found a way to avoid considerable analysis in proving that the map Φ is indeed a canonical transformation. We have relegated most of these analytic aspects to several appendices. Specifically, we prove in Appendix B that Φ and its inverse are real-analytic, whereas Appendix C is devoted to showing canonicity. In the latter appendix we make essential use of the results of Appendix A and Chap. 3. Appendix A contains the spectral asymptotics results already mentioned above, whereas Chap. 3 is devoted to a case by case study of special flows whose relevant features can be established without using the canonicity property of Φ.

Admittedly, our proof of this key property is not exactly straightforward. The main analytic difficulty in our approach (which hinges on exploiting scattering theory) is to justify a certain interchange of limits. Obviating this snag involves holomorphicity arguments and the uniform estimates of Appendix A, and is already nontrivial for the simplest case I_{nr}. (In the previous work on this case mentioned above this interchange is left unjustified.) Possibly, smoothness and canonicity of Φ can also be established by adopting a picture as presented in [6, 11], but from the information given there it is not obvious to us why the two different descriptions involved should be related by a canonical transformation.

In Chap. 4 we study the scattering for a certain class of Hamiltonians. We prove that all of these have the same wave and scattering maps as the special

Hamiltonian studied in Chap. 3. For the nonrelativistic systems this invariance principle had already been conjectured to hold in [12], where we presented and discussed a quantum analog (cf. also [13], where similar invariance principles are proved for several integrable field theories). Just as in [12], one may argue that due to this invariance principle the various dynamics involved are "equally important" from a mathematical point of view; in this picture the fundamental objects are L and Φ, and the special Hamiltonians of Chap. 3 are singled out solely by their simplicity and by their physical interpretation in terms of space-time symmetry groups.

Chapter 5 contains some further developments. Specifically, in Sect. 5A we prove an asymptotic property of the action-angle map, which may be viewed as a generalization of the invariance principle of Chap. 4. In Sect. 5B we show that one can get new integrable particle systems by restricting Φ to certain submanifolds. These restricted systems may be viewed as being associated with the root systems C_l and BC_l, in the same sense as the unrestricted systems are associated with A_{2l-1} and A_{2l}, respectively (cf. [1]). In the final Sect. 5C we collect some further observations of interest, including a striking property of matrices associated with the case I_{nr} (i), a one-parameter generalization of the Lax matrix whose symmetric functions still commute (ii), a symmetry property of Φ (iii), and last but not least, the relation between the four cases (iv).

This paper is in essence self-contained. In particular, we do not assume involutivity of the symmetric functions of the Lax matrix, since this information would not simplify our canonicity proof. Of course, once canonicity of Φ is proved, this commutativity property is an obvious corollary. Quite a few other previous results are simplified and subsumed, as well. For instance, the explicit description of the special flows studied in Chap. 3 can already be found in [1] for the cases I_{nr} and II_{nr}, and in [2] for the cases I_{rel} and II_{rel}, but its validity is proved here with a minimum of labor (avoiding e.g. the somewhat involved arguments of [2, Appendix B]).

We close this introduction by listing two versions of Cauchy's identity in a form which suits our later requirements:

$$\left|\left(\frac{\alpha}{\alpha+x_i-x_j}\right)\right| = \prod_{i<j}\left[1-\frac{\alpha^2}{\alpha^2-(x_i-x_j)^2}\right], \tag{1.1}$$

$$\left|\left(\frac{shz}{sh(z+y_i-y_j)}\right)\right| = \prod_{i<j}\left[1-\frac{sh^2 z}{sh(z+y_i-y_j)sh(z+y_j-y_i)}\right]. \tag{1.2}$$

Note that (1.1) follows from (1.2) by setting $z=\alpha\beta$, $y=\beta x$ and taking β to 0. For further discussion of these formulas we refer to [2].

2. The Construction of the Action-Angle Map

2A. Generalities

We begin by sketching the construction of the action-angle map Φ in general terms. The systems studied below all have a phase space and action-angle space given by

$$\Omega \equiv \{(q,\theta)\in \mathbb{R}^{2N} | q_N < \ldots < q_1\}, \quad \omega \equiv \sum_{i=1}^{N} dq_i \wedge d\theta_i, \tag{2.1}$$

$$\hat{\Omega} \equiv \{(\hat{q},\hat{\theta})\in \mathbb{R}^{2N} | \hat{\theta}_N < \ldots < \hat{\theta}_1\}, \quad \hat{\omega} \equiv \sum_{i=1}^{N} d\hat{q}_i \wedge d\hat{\theta}_i. \tag{2.2}$$

They are characterized by an $N \times N$ matrix L on Ω, which also depends on certain parameters, collectively denoted by g. These parameters take values in a region $G \subset \mathbb{C}^l$ which we shall not specify here, since G and l depend on the case at hand.

The key to the construction of Φ is a commutation relation of L with a diagonal matrix A that is a simple function of the matrix

$$Q \equiv \mathrm{diag}(q_1, \ldots, q_N). \tag{2.3}$$

This commutation relation is of the form

$$f(g)[A,L] = e \otimes e - F(A,L), \tag{2.4}$$

where the complex-valued and matrix-valued functions f and F and the vector $e(g;q,\theta)$ depend on the case. In all cases considered below L is diagonalizable and has real spectrum. Thus, an invertible matrix T exists satisfying

$$\hat{L} \equiv TLT^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N), \quad \lambda_i \in \mathbb{R}. \tag{2.5}$$

Then (2.4) implies

$$f(g)\hat{A}_{ij}(\lambda_j - \lambda_i) = \hat{e}_i \tilde{e}_j - F(\hat{A}, \hat{L})_{ij}. \tag{2.6}$$

Here, we have set

$$\hat{A} \equiv TAT^{-1}, \tag{2.7}$$

$$\hat{e} \equiv Te, \quad \tilde{e} \equiv T^{-1t}e \tag{2.8}$$

(where t denotes transpose).

We now render T unique by imposing several requirements. First, we exploit (2.6) to prove that the coordinates of $\hat{e}$ and $\tilde{e}$ are non-zero and that L has simple spectrum. Thus, we may and shall require

$$\lambda_N < \ldots < \lambda_1. \tag{2.9}$$

This determines T up to left multiplication by a diagonal matrix D with $D_{ii} \neq 0$. We then fix $D_{11}, \ldots, D_{NN}$ up to a sign by requiring

$$\tilde{e} = \hat{e}, \tag{2.10}$$

cf. (2.8). Finally, the sign is fixed by first proving that (2.10) entails $\hat{e}$ is real and then requiring

$$\hat{e}_i > 0, \quad i = 1, \ldots, N. \tag{2.11}$$

Next, we reparametrize $\hat{L}$, $\hat{e}$, and $\hat{A}$ with points in $\hat{\Omega}$. In particular, the eigenvalue λ_i is written as a simple function of $\hat{\theta}_i$. By virtue of the uniqueness of T, we obtain a well-defined transformation Φ from Ω into $\hat{\Omega}$ in this way.

To show that Φ is a bijection, one need only solve (2.6) for $\hat{A}$ and regard $\hat{A}$ and $\hat{L}$ as functions on $\hat{\Omega}$, after which the map can be "run backwards." That is, one can construct a map $\mathscr{E}:\hat{\Omega}\to\Omega$ such that

$$\mathscr{E}\circ\Phi=\mathrm{id}_{\Omega}, \qquad \Phi\circ\mathscr{E}=\mathrm{id}_{\hat{\Omega}}, \tag{2.12}$$

which entails that Φ is bijective.

Since the eigenvalue λ_i of L is reparametrized in terms of $\hat{\theta}_i$ only, the bijection Φ diagonalizes any Hamiltonian H which is defined in terms of L (e.g., $H=\mathrm{Tr}\,L^k$), in the sense that $H\circ\mathscr{E}$ depends only on $\hat{\theta}$. In particular, for any $h\in C^{\infty}_{\mathbb{R}}(\mathbb{R})$ one can define a Hamiltonian H_h such that

$$(H_h\circ\mathscr{E})(\hat{q},\hat{\theta})=\sum_{i=1}^{N}h(\hat{\theta}_i). \tag{2.13}$$

Viewed as a Hamiltonian on $\hat{\Omega}$, the right-hand side obviously generates the linear flow

$$(\hat{q}(t),\hat{\theta}(t))\equiv(\hat{q}_1+th'(\hat{\theta}_1),\dots,\hat{q}_N+th'(\hat{\theta}_N),\hat{\theta}). \tag{2.14}$$

However, it does *not* follow from this that the pullback to Ω,

$$(q(t),\theta(t))\equiv\mathscr{E}(g;\hat{q}(t),\hat{\theta}) \tag{2.15}$$

is the solution of Hamilton's equations for H_h, *unless* one can prove that $\mathscr{E}$ is a symplectic diffeomorphism.

We prove smoothness of Φ and $\mathscr{E}$ in Appendix B in the general context of this section. However, our proof of canonicity hinges on picking a *special h*, for which (2.15) can be shown to solve the Hamilton equations for H_h *without* assuming canonicity of $\mathscr{E}$. An important ingredient for showing this is a description of $q(t)$ in terms of eigenvalues of a matrix $A(t)$ defined below. This description (for general h) is given at the end of each of the following sections, and amounts to an explicit picture of the position part of the Hamiltonian flow generated by H_h, once we have proved that $\mathscr{E}$ is canonical.

The matrix $A(t)$ involves h and L in a simple fashion and reduces to A when $t=0$. The notation

$$\hat{A}(t)\equiv TA(t)T^{-1}, \tag{2.16}$$

which we shall use below is therefore consistent with (2.10). We shall denote evaluation of the matrices Q, L, and T in the point (2.15) by appending a subscript t. Finally, we shall use the symbol $\sim$ to denote similarity.

2B. The Case I_{nr}

The nonrelativistic rational case I_{nr} is characterized by the Lax matrix

$$L(\varrho;q,\theta)_{ij}\equiv\delta_{ij}\theta_j+\varrho(1-\delta_{ij})\frac{1}{q_i-q_j}, \qquad \varrho\in i\mathbb{R}^*. \tag{2.17}$$

Setting

$$A\equiv Q, \qquad e_i\equiv 1, \tag{2.18}$$

$$f\equiv 1/\varrho, \qquad F\equiv 1, \tag{2.19}$$

it is clear that L satisfies the commutation relation (2.4). Moreover, L is diagonalizable and has real spectrum, since (2.17) implies $L=L^*$. Thus, a matrix T exists satisfying (2.5). Then the transformed commutation relation (2.6) reads

$$\varrho^{-1}\hat{A}_{ij}(\lambda_j-\lambda_i)=\hat{e}_i\tilde{e}_j-\delta_{ij}. \tag{2.20}$$

Taking $i=j$, one sees that $\hat{e}_i\tilde{e}_i=1$, so that the coordinates of $\hat{e}$ and $\tilde{e}$ are nonzero. Taking then $i\neq j$, it follows that $\lambda_j-\lambda_i\neq 0$, so that $\sigma(L)$ is simple. Hence, we are now in the position to require (2.9) and (2.10). Doing so, we may reparametrize $\hat{L}$ and λ_i by setting

$$\hat{L}=\operatorname{diag}(\hat{\theta}_1,\ldots,\hat{\theta}_N), \qquad \hat{\theta}_i\equiv\lambda_i. \tag{2.21}$$

Since the requirement (2.10) entails $\hat{e}_i^2=1$, we can now fix T completely by requiring (2.11). Thus, in this case the requirements amount to imposing

$$\tilde{e}_i=\hat{e}_i=1, \qquad i=1,\ldots,N. \tag{2.22}$$

We claim that the matrix T, which has just been uniquely determined, is unitary. To prove this, we note that we could have started with a unitary T to diagonalize L, since L is self-adjoint. If we then require (2.9), the ambiguity left is a diagonal unitary. But unitarity entails $\tilde{e}=\bar{\hat{e}}$, cf. (2.8), so that the ambiguity can be removed by requiring (2.22). By uniqueness one then obtains the same T as before.

At this point the reader might wonder why we did not require that T be unitary to begin with. This would however lead to certain difficulties later on, which we wish to avoid. In fact, we only need the unitarity to conclude that the quantities $\hat{q}_i\equiv\hat{A}_{ii}$ are real. [Indeed, this is obvious from the fact that $\hat{A}=\hat{A}^*$ when T is unitary, cf. (2.18), (2.7).] Combining this definition with (2.20–22) and (2.17) we conclude

$$\hat{A}=L(-\varrho;\hat{\theta},\hat{q}). \tag{2.23}$$

Summarizing, we see that we have constructed a map $\Phi:\Omega\to\hat{\Omega},(q,\theta)\mapsto(\hat{q},\hat{\theta})$ by diagonalizing $L(\varrho;q,\theta)$ with a uniquely determined unitary T, $\hat{\theta}_i$ being the eigenvalues of L and $\hat{q}_i$ the diagonal elements of TQT^{-1}.

Theorem 2.1. *The map $\Phi(\varrho;q,\theta)$ is a smooth bijection from Ω onto $\hat{\Omega}$ whose inverse satisfies*

$$\mathscr{E}(\varrho;\hat{q},\hat{\theta})=P\circ\Phi(-\varrho;\hat{\theta},\hat{q}). \tag{2.24}$$

Here, P is the permutation

$$P(x,y)\equiv(y,x), \qquad (x,y)\in\mathbb{R}^{2N}. \tag{2.25}$$

Proof. Using the self-duality relation (2.23) it is obvious how to construct a map $\mathscr{E}:\hat{\Omega}\to\Omega$ satisfying (2.12), and bijectivity and (2.24) then follow.

Well-known facts concerning matrix-valued holomorphic functions entail real-analyticity, and hence smoothness, of $\hat{\theta}(\varrho;q,\theta)$. The less obvious fact that $\hat{q}(\varrho;q,\theta)$ is real-analytic, as well, is proved in Appendix B. □

As announced in Sect. 2A, we shall now conclude this section with a theorem which, when combined with the canonicity property proved later, yields an explicit description of the (position part of the) Hamiltonian flow generated by

$$H_h\equiv\operatorname{Tr}h(L), \qquad h\in C^\infty_{\mathbb{R}}(\mathbb{R}). \tag{2.26}$$

[Here, $h(L)$ is defined by the functional calculus.] Note that (2.26) entails that (2.13) holds true. The notation used in the following theorem and its proof is explained at the end of Sect. 2A.

Theorem 2.2. *Let*

$$A(t) \equiv Q + th'(L). \tag{2.27}$$

Then

$$A(t) \sim Q_t. \tag{2.28}$$

Proof. We have $\hat{A}(t) = \hat{A} + th'(\hat{L})$, from which it follows that

$$\hat{A}(t) = L(-\varrho; \hat{\theta}, \hat{q}) + t\,\mathrm{diag}(h'(\hat{\theta}_1), \ldots, h'(\hat{\theta}_N)), \tag{2.29}$$

cf. (2.23), (2.21). From this formula and the definition (2.15) of $(q(t), \theta(t))$ we can now conclude

$$T_t^{-1}\hat{A}(t)T_t = Q_t, \tag{2.30}$$

so that (2.28) follows. □

2C. *The Cases II_{nr} and I_{rel}*

We consider the nonrelativistic hyperbolic systems and the relativistic rational systems alongside, since they turn out to be dual to each other. We shall use subscripts to distinguish the two cases, unless the context prevents confusion. The Lax matrices are given by

$$L_{\mathrm{nr}}(\mu, \varrho; q, \theta)_{ij} \equiv \delta_{ij}\theta_j + \varrho(1-\delta_{ij})\frac{\mu}{2sh\frac{\mu}{2}(q_i - q_j)}, \quad \mu \in (0, \infty),\ \varrho \in i\mathbb{R}^*, \tag{2.31}$$

$$L_{\mathrm{rel}}(\beta, \varrho; q, \theta)_{ij} \equiv \exp\left[\frac{\beta}{2}(\theta_i + \theta_j)\right](V_i^{1/2}V_j^{1/2}C_{ij})(\beta\varrho; q), \qquad \beta \in (0, \infty),\ \varrho \in i\mathbb{R}^*, \tag{2.32}$$

where

$$V_i(\kappa; q) \equiv \prod_{l \neq i}\left[1 - \frac{\kappa^2}{(q_i - q_l)^2}\right]^{1/2}, \tag{2.33}$$

$$C(\kappa; q)_{ij} \equiv \frac{\kappa}{\kappa + q_i - q_j}. \tag{2.34}$$

(Here and henceforth, positive square roots are taken.)

The commutation relation (2.4), with $L = L_{\mathrm{nr}}$, is satisfied when one sets

$$A \equiv \exp[\mu Q], \quad e_i \equiv \exp\left[\frac{\mu}{2}q_i\right], \tag{2.35}$$

$$f \equiv 1/\mu\varrho, \quad F \equiv A. \tag{2.36}$$

Since L is self-adjoint, we can find an invertible T diagonalizing L, so that (2.6) reads here

$$\frac{1}{\mu\varrho}\hat{A}_{ij}(\lambda_j-\lambda_i)=\hat{e}_i\tilde{e}_j-\hat{A}_{ij}. \tag{2.37}$$

Solving for $\hat{A}$ we get

$$\hat{A}_{ij}=\hat{e}_i\tilde{e}_jC(-\mu\varrho;\lambda)_{ij}, \tag{2.38}$$

cf. (2.34). We now use Cauchy's identity (1.1) to conclude

$$|\hat{A}|=\prod_i\hat{e}_i\prod_j\tilde{e}_j\prod_{i<j}\left[1-\frac{\mu^2\varrho^2}{\mu^2\varrho^2-(\lambda_i-\lambda_j)^2}\right]. \tag{2.39}$$

Since $|\hat{A}|=|A|=\exp[\mu\Sigma_i q_i]\neq 0$, the coordinates of $\hat{e}$ and $\tilde{e}$ are non-zero and $\sigma(L)$ is nondegenerate. Hence we may and shall require (2.9) and (2.10).

Next, we claim that (2.10) entails reality of $\hat{e}$. Indeed, since $L^*=L$, we could have chosen a unitary T to diagonalize L. Then we would get $\tilde{e}=\bar{\hat{e}}$, so that (2.10) can be satisfied by multiplying this unitary T from the left with an appropriate diagonal unitary. Hence, the resulting unitary must be equal to the previous T up to a matrix D with $D_{ii}=\pm 1$, and reality follows. We can, therefore, require (2.11) in addition to (2.10), which yields a unique unitary T.

We are now in the position to reparametrize the relevant objects with points in $\hat{\Omega}$, as follows:

$$\hat{L}_{\mathrm{nr}}=\mathrm{diag}(\hat{\theta}_1,\ldots,\hat{\theta}_N), \quad \hat{\theta}_i\equiv\lambda_i, \tag{2.40}$$

$$(\hat{e}_{\mathrm{nr}})_i\equiv\exp\left[\frac{\mu}{2}\hat{q}_i\right]V_i(-\mu\varrho;\hat{\theta})^{1/2}, \tag{2.41}$$

which entails

$$\hat{A}_{\mathrm{nr}}=L_{\mathrm{rel}}(\mu,-\varrho;\hat{\theta},\hat{q}), \tag{2.42}$$

in view of (2.38) and (2.31–33). Thus, we obtain a map $\Phi_{\mathrm{nr}}:\Omega\to\hat{\Omega}$, $(q,\theta)\mapsto(\hat{q},\hat{\theta})$.

The fact that Φ_{nr} is a bijection will be obvious from (2.40–42) and the construction of Φ_{rel}, on which we now embark. When $L=L_{\mathrm{rel}}$, we can satisfy (2.4) by setting

$$A\equiv Q, \quad e_i\equiv\exp\left[\frac{\beta}{2}\theta_i\right]V_i(\beta\varrho;q)^{1/2}, \tag{2.43}$$

$$f\equiv 1/\beta\varrho, \quad F\equiv L. \tag{2.44}$$

Again, L is self-adjoint and hence there exists a matrix T satisfying (2.5). Now, (2.6) reads

$$\frac{1}{\beta\varrho}\hat{A}_{ij}(\lambda_j-\lambda_i)=\hat{e}_i\tilde{e}_j-\delta_{ij}\lambda_j. \tag{2.45}$$

Next, we invoke (1.1) once more, to conclude

$$\prod_i\lambda_i=|L|=\exp\left[\beta\sum_i\theta_i\right]\neq 0.$$

Hence, the eigenvalues λ_i are non-zero. Taking $i=j$ in (2.45) it follows that $\hat{e}_i$ and $\tilde{e}_i$ are non-zero. Taking then $i \neq j$, it follows that $\sigma(L)$ is simple. Thus we can require (2.9) and (2.10). Moreover, it follows in the same way as before that (2.10) entails unitarity of T and hence reality of $\hat{e}$. Thus, T can be rendered unique by imposing the extra condition (2.11).

We proceed by noting that (2.10–11), combined with (2.45), implies $\lambda_i > 0$. Hence we may set

$$\hat{L}_{\text{rel}} = \text{diag}(\exp[\beta\hat{\theta}_1], \ldots, \exp[\beta\hat{\theta}_N]), \quad \hat{\theta}_i \equiv \beta^{-1} \ln \lambda_i, \tag{2.46}$$

which implies

$$(\hat{e}_{\text{rel}})_i = \exp\left[\frac{\beta}{2}\hat{\theta}_i\right] \tag{2.47}$$

on account of (2.45) and (2.10–11). Finally, since T is unitary and A self-adjoint, the numbers $\hat{q}_i \equiv \hat{A}_{ii}$ are real and (2.45), (2.30) imply

$$\hat{A}_{\text{rel}} = L_{\text{nr}}(\beta, -\varrho; \hat{\theta}, \hat{q}). \tag{2.48}$$

This completes the construction of Φ_{rel}.

Theorem 2.3. *The maps $\Phi_{\text{nr}}(\mu, \varrho; q, \theta)$ and $\Phi_{\text{rel}}(\beta, \varrho; q, \theta)$ are smooth bijections from Ω onto $\hat{\Omega}$. Moreover, their inverses satisfy*

$$\mathscr{E}_{\text{nr}}(\mu, \varrho; \hat{q}, \hat{\theta}) = P \circ \Phi_{\text{rel}}(\mu, -\varrho; \hat{\theta}, \hat{q}), \tag{2.49}$$

$$\mathscr{E}_{\text{rel}}(\beta, \varrho; \hat{q}, \hat{\theta}) = P \circ \Phi_{\text{nr}}(\beta, -\varrho; \hat{\theta}, \hat{q}), \tag{2.50}$$

where P is the permutation (2.25).

Proof. Bijectivity and (2.49–50) are immediate from the above, cf. (2.40–42) and (2.46–48). Smoothness follows from Theorem B2. □

We continue by defining H_h for the case II_{nr} via (2.26) (with $L = L_{\text{nr}}$, of course). In view of (2.40) the relation (2.13) follows again. Hence, the following theorem gives information on the flow generated by H_h, in the sense explained in Sect. 2A.

Theorem 2.4. *Let*

$$A(t) \equiv \exp[\mu Q] \exp[t\mu h'(L_{\text{nr}})]. \tag{2.51}$$

Then

$$A(t) \sim \exp[\mu Q_t]. \tag{2.52}$$

Proof. The analog of (2.29) reads

$$\hat{A}(t) = L_{\text{rel}}(\mu, -\varrho; \hat{\theta}, \hat{q}) \exp[t\mu\, \text{diag}(h'(\hat{\theta}_1), \ldots, h'(\hat{\theta}_N))], \tag{2.53}$$

cf. (2.40), (2.42). Hence we conclude

$$T_t^{-1} \exp[\tfrac{1}{2}t\mu h'(\hat{L})]\hat{A}(t) \exp[-\tfrac{1}{2}t\mu h'(\hat{L})] T_t = \exp[\mu Q_t], \tag{2.54}$$

so (2.52) follows. □

Let us finally obtain the analogous theorem for the case I_{rel}. In this case we must replace (2.26) by

$$H_h \equiv \operatorname{Tr} h(\beta^{-1}\ln L), \qquad h \in C^{\infty}_{\mathbb{R}}(\mathbb{R}) \tag{2.55}$$

(with $L = L_{\mathrm{rel}}$) to ensure that (2.13) holds true. The significance of the following result for the H_h-flow is detailed in Sect. 2A.

Theorem 2.5. *Let*

$$A(t) \equiv Q + th'(\beta^{-1}\ln L_{\mathrm{rel}}). \tag{2.56}$$

Then

$$A(t) \sim Q_t. \tag{2.57}$$

Proof. By virtue of (2.46) and (2.48) we have

$$\hat{A}(t) = L_{\mathrm{nr}}(\beta, -\varrho; \hat{\theta}, \hat{q}) + t\,\mathrm{diag}(h'(\hat{\theta}_1), \ldots, h'(\hat{\theta}_N)). \tag{2.58}$$

Hence, (2.30) holds here, too, and (2.57) results. □

2D. The Case II_{rel}

The hyperbolic relativistic systems have Lax matrix

$$L(\beta, \mu, z; q, \theta)_{ij} \equiv \exp\left[\frac{\beta}{2}(\theta_i + \theta_j)\right](V_i^{1/2}V_j^{1/2}C_{ij})(\mu, z; q), \qquad \beta, \mu \in (0, \infty), \tag{2.59}$$

where

$$V_i(\mu, z; q) \equiv \prod_{l \neq i}\left[1 - \frac{sh^2 z}{sh^2\frac{\mu}{2}(q_i - q_l)}\right]^{1/2}, \tag{2.60}$$

$$C(\mu, z; q)_{ij} \equiv \frac{shz}{sh\left(z + \frac{\mu}{2}(q_i - q_j)\right)}. \tag{2.61}$$

Then (2.4) is satisfied with

$$A \equiv \exp[\mu Q], \qquad e_i \equiv \exp\left[\frac{\mu}{2}q_i + \frac{\beta}{2}\theta_i\right]V_i(\mu, z; q)^{1/2}, \tag{2.62}$$

$$f \equiv \frac{1}{2thz}, \qquad F \equiv \frac{1}{2}(AL + LA). \tag{2.63}$$

We first consider the choice

$$\pm z \in i(0, \pi). \tag{2.64}$$

Then C is self-adjoint, so L is self-adjoint, too. Choosing a unitary T diagonalizing L, we get for (2.6) in this case

$$\frac{1}{2thz}\hat{A}_{ij}(\lambda_j - \lambda_i) = \hat{e}_i\hat{e}_j - \tfrac{1}{2}\hat{A}_{ij}(\lambda_j + \lambda_i) \tag{2.65}$$

with $\tilde{e}=\bar{\hat{e}}$, since T is unitary and e real. Taking $i=j$ and noting $\hat{A}_{ii}>0$, we see that $\lambda_i>0$. Hence we may introduce real numbers $\hat{\theta}_i\equiv\beta^{-1}\ln\lambda_i$, in terms of which we can rewrite (2.65) as

$$\hat{A}_{ij}=\hat{e}_i\tilde{e}_j\exp\left[-\frac{\beta}{2}(\hat{\theta}_i+\hat{\theta}_j)\right]C(\beta,-z;\hat{\theta})_{ij}, \tag{2.66}$$

cf. (2.61). Hence, by Cauchy's identity (1.2)

$$|\hat{A}|=\prod_i|\hat{e}_i|^2\prod_{i<j}\left[1-\frac{sh^2z}{sh\left(z-\frac{\beta}{2}(\hat{\theta}_i-\hat{\theta}_j)\right)sh\left(z-\frac{\beta}{2}(\hat{\theta}_j-\hat{\theta}_i)\right)}\right]. \tag{2.67}$$

But we have $|\hat{A}|\neq 0$ [cf. (2.62)], so that $\hat{e}_i\neq 0$ and $\hat{\theta}_i\neq\hat{\theta}_j$. We can, therefore, render T unique by requiring (2.9–11).

In view of the above we have

$$\hat{L}=\mathrm{diag}(\exp[\beta\hat{\theta}_1],\dots,\exp[\beta\hat{\theta}_N]),\qquad \hat{\theta}_i\equiv\beta^{-1}\ln\lambda_i, \tag{2.68}$$

and we may introduce $\hat{q}\in\mathbb{R}^N$ by setting

$$\hat{e}_i\equiv\exp\left[\frac{\beta}{2}\hat{\theta}_i+\frac{\mu}{2}\hat{q}_i\right]V_i(\beta,-z;\hat{\theta})^{1/2}, \tag{2.69}$$

so that

$$\hat{A}=L(\mu,\beta,-z;\hat{\theta},\hat{q}) \tag{2.70}$$

by virtue of (2.66). This completes the definition of the map $\Phi:(q,\theta)\mapsto(\hat{q},\hat{\theta})$ for the case (2.64).

We shall now handle the case

$$z\pm\frac{i\pi}{2}\in\mathbb{R}^*. \tag{2.71}$$

For these values of z the Cauchy matrix C is not self-adjoint, but now C is real, cf. (2.61). Since one still has $V_i>0$ [cf. (2.60)], L is real, too. Thus the symmetric functions of L and traces of powers of L are real, and hence may be viewed as Hamiltonians on Ω. Apart from this, the regime (2.71) is of interest, since it connects the relativistic Calogero-Moser systems with the relativistic Toda systems: The latter arise in the limit $|\gamma|\to\infty$, where $\gamma\equiv z\pm\frac{i\pi}{2}$ [14].

In order to define Φ, let us fix $(q,\theta)\in\Omega$ and consider the spectrum of L as γ varies over $\mathbb{R}$. Recall we have proved already that $\sigma(L)$ is simple and positive when $\gamma=0$. Now let $c\in(0,\infty]$ be the largest number such that $\sigma(L)$ is simple and positive for $|\gamma|<c$. (The existence of c follows from the continuity in γ of the eigenvalues of L and from the reality of L for $\gamma\in\mathbb{R}$. In fact, we shall presently prove that $c=\infty$.) For these values of γ we can find a real and invertible T satisfying (2.5), which is moreover continuous in γ and reduces to the previous T when $\gamma=0$. (Indeed, the uniqueness of the matrix T constructed above is readily seen to imply its reality when $z=\pm i\pi/2$.) This leads again to (2.66), so that $\hat{e}$ and $\tilde{e}$ must have non-zero

coordinates. Now T is, so far, determined up to left multiplication by a matrix $\mathrm{diag}(d_1(\gamma), \ldots, d_N(\gamma))$, where $d_i(0)=1$ and $d_i(\gamma)$ continuous and positive. Hence we can and shall fix T by requiring $\hat{e}=\tilde{e}$, cf. (2.8). By virtue of continuity in γ and reality of T it then follows that $\hat{e}$ has positive coordinates. Thus we can define $\hat{q}\in\mathbb{R}^N$ via (2.69), as before, so that (2.68–70) are valid again.

A priori, c depends on the point (q, θ) we have fixed. However, we shall now prove that the assumption $c<\infty$ leads to a contradiction. In view of the above definition of c this amounts to the assumption that for $\gamma=\pm c<\infty$ the spectrum of L is not positive or not simple. First, assume $\sigma(L)$ is not positive. Since $\sigma(L)$ is positive for $|\gamma|<c$, this implies that L must be singular when $\gamma=\pm c$. But we have

$$|L|=\exp\left[\beta\sum_i \theta_i\right]\neq 0$$

for any γ, so that this possibility is ruled out. Hence, $\sigma(L)$ is not simple when $\gamma=\pm c$. But now consider the γ-independent function $\mathrm{Tr}\,A$: By virtue of (2.66) and the fact that $\hat{e}=\tilde{e}$, it can be written

$$\mathrm{Tr}\,A=\sum_{i=1}^{N}\hat{e}_i^2\exp[-\beta\hat{\theta}_i]\,. \tag{2.72}$$

The terms in this sum are positive, and the limits of the $\hat{\theta}_i$ when $|\gamma|\uparrow c$ are bounded. Therefore, the positive numbers $\hat{e}_1, \ldots, \hat{e}_N$ must remain bounded when $|\gamma|\uparrow c$. To exploit this, we consider (2.66) with $\tilde{e}=\hat{e}$: On one hand, one must have $|C|\to 0$ and hence $|\hat{A}|\to 0$ when $|\gamma|\uparrow c$, since two eigenvalues of L must collide for $|\gamma|\uparrow c$. On the other hand,

$$|\hat{A}|=|A|=\exp\left[\mu\sum_i q_i\right]$$

does not depend on γ. This is the contradiction announced above.

Summarizing, we have constructed a map Φ from Ω into $\hat{\Omega}$ for both z-regimes (2.64) and (2.71).

Theorem 2.6. *The map $\Phi(\beta, \mu, z; q, \theta)$ is a smooth bijection from Ω onto $\hat{\Omega}$, whose inverse is given by*

$$\mathscr{E}(\beta, \mu, z; \hat{q}, \hat{\theta})=P\circ\Phi(\mu, \beta, -z; \hat{\theta}, \hat{q})\,, \tag{2.73}$$

where P is the permutation (2.25).

Proof. Bijectivity and (2.73) are immediate from the duality relations (2.68–70). Smoothness is a consequence of Theorem B2. □

The relation between λ_i and $\hat{\theta}_i$ is the same as for the case I_{rel}. Hence we define H_h by (2.25) to ensure that (2.13) follows. At the end of Sect. 2A we have explained the relevance of the following result for the H_h-flow.

Theorem 2.7. *Set*

$$A(t)\equiv\exp[\mu Q]\exp[t\mu h'(\beta^{-1}\ln L)]\,. \tag{2.74}$$

Then

$$A(t)\sim\exp[\mu Q_t]\,. \tag{2.75}$$

Proof. In view of (2.68) and (2.70) we have

$$\hat{A}(t)=L(\mu,\beta,-z;\hat{\theta},\hat{q})\exp[t\mu\,\mathrm{diag}(h'(\hat{\theta}_1),\ldots,h'(\hat{\theta}_N))]. \tag{2.76}$$

Hence, (2.54) holds true, provided we replace $\hat{L}$ by $\beta^{-1}\ln\hat{L}$, and (2.75) follows. □

3. A Special Flow and Its Temporal Asymptotics

By virtue of the results established in the previous chapter, canonicity of $\mathscr{E}$ would imply that the trajectories $(q(t),\theta(t))$ [cf. (2.15)] are the integral curves of the H_h-vector field. In this chapter we shall prove that this is true for a function h whose choice depends on the case at hand, *without* assuming canonicity. The asymptotics of this special Hamiltonian flow is then determined by invoking the spectral asymptotics results of Appendix A. As we shall show in Appendix C, this information can then be exploited to obtain a proof that $\mathscr{E}$ is canonical. We now present the details for the four cases involved.

3A. The Case I_{nr}

Theorem 3.1. *Let*

$$H\equiv\tfrac{1}{2}\sum_{j=1}^{N}\theta_j^2-\varrho^2\sum_{1\le j<k\le N}\frac{1}{(q_j-q_k)^2},\qquad \varrho\in i\mathbb{R}^*. \tag{3.1}$$

Then the functions

$$(q(t),\theta(t))\equiv\mathscr{E}(\varrho;\hat{q}_1+t\hat{\theta}_1,\ldots,\hat{q}_N+t\hat{\theta}_N,\hat{\theta}) \tag{3.2}$$

(cf. (2.24)) solve Hamilton equations and satisfy

$$q_{\substack{j\\N-j+1}}(t)=\hat{q}_j+t\hat{\theta}_j+O(|t|^{-1}),\qquad t\to\pm\infty, \tag{3.3}$$

$$\theta_{\substack{j\\N-j+1}}(t)=\hat{\theta}_j+O(|t|^{-2}),\qquad t\to\pm\infty. \tag{3.4}$$

Proof. We begin by noting

$$H=\tfrac{1}{2}\mathrm{Tr}\,L^2=H_h,\qquad h(x)\equiv\tfrac{1}{2}x^2, \tag{3.5}$$

cf. (2.17), (2.26). Next, we claim that

$$Q_{t+\Delta t}\sim Q_t+\Delta tL_t. \tag{3.6}$$

(The notation used here and below is explained at the end of Sect. 2A.) Indeed, if we transform the left-hand side and the right-hand side with $T_{t+\Delta t}$ and T_t, then we obtain $\hat{A}(t+\Delta t)$ and $\hat{A}(t)+\Delta t\hat{L}$, respectively, and these matrices are equal, cf. Theorem 2.2 and its proof.

It now follows from (3.6) and nondegenerate perturbation theory that

$$\dot{q}_j=L_{jj},\qquad \ddot{q}_j=2\sum_{k\neq j}\frac{L_{jk}L_{kj}}{q_j-q_k}. \tag{3.7}$$

We have suppressed the argument and subscript t, since t is arbitrary. Using the definition (2.17) of L this can be rewritten as

$$\dot{q}_j=\theta_j, \quad \ddot{q}_j=-2\varrho^2\sum_{k\neq j}\frac{1}{(q_j-q_k)^3}. \tag{3.8}$$

But from the definition (3.1) of H one sees that the functions at the right-hand side are equal to $\{q_j, H\}$ and $\{\theta_j, H\}$, respectively, so that $(q(t), \theta(t))$ solves Hamilton's equations, as claimed.

It remains to prove the temporal asymptotics (3.3–4). To this end we observe that the right-hand side of (2.29), with $h'(\hat{\theta}_j)$ replaced by $\hat{\theta}_j$, is of the form (A11). Hence, $(3.3–4)_+$ are immediate consequences of (A12–15). To derive $(3.3–4)_-$ from (A12–15), we need only transform $\hat{A}(t)$ with the reversal matrix

$$\mathscr{R}\equiv\begin{pmatrix}0 & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0\end{pmatrix} \tag{3.9}$$

and replace t by $-s$. Indeed, (A12–15) then applies to the $s\to\infty$ limit, since we have

$$-\mathscr{R}\hat{L}\mathscr{R}=\mathrm{diag}(-\hat{\theta}_N, \dots, -\hat{\theta}_1)\in\mathscr{D}. \tag{3.10}$$

Thus the proof is complete. □

3B. *The Case* II_{nr}

Theorem 3.2. *Let*

$$H\equiv\tfrac{1}{2}\sum_{j=1}^{N}\theta_j^2-\varrho^2\sum_{1\leq j<k\leq N}\frac{\mu^2}{4sh^2\frac{\mu}{2}(q_j-q_k)}, \quad \mu\in(0,\infty),\ \varrho\in i\mathbb{R}^* \tag{3.11}$$

Then the functions

$$(q(t), \theta(t))\equiv\mathscr{E}(\mu, \varrho; \hat{q}_1+t\hat{\theta}_1, \dots, \hat{q}_N+t\hat{\theta}_N, \hat{\theta}) \tag{3.12}$$

(cf. (2.49)*) solve Hamilton's equations and satisfy*

$$\underset{N-j+1}{q_j}(t)=\hat{q}_j\mp\tfrac{1}{2}\Delta_j(\hat{\theta})+t\hat{\theta}_j+O(\exp[\mp tR]), \quad t\to\pm\infty, \tag{3.13}$$

$$\underset{N-j+1}{\theta_j}(t)=\hat{\theta}_j+O(\exp[\mp tR]), \quad t\to\pm\infty, \tag{3.14}$$

where

$$R\equiv\mu\min\{\hat{\theta}_1-\hat{\theta}_2, \dots, \hat{\theta}_{N-1}-\hat{\theta}_N\}, \tag{3.15}$$

$$\Delta_j(\theta)\equiv\sum_{k<j}\delta(\theta_j-\theta_k)-\sum_{k>j}\delta(\theta_j-\theta_k), \tag{3.16}$$

and

$$\delta(\theta)\equiv\mu^{-1}\ln\left[1-\frac{\mu^2\varrho^2}{\theta^2}\right]. \tag{3.17}$$

Proof. From (2.31) it follows that (3.5) holds here, too. The analog of (3.6) reads

$$\exp[\mu Q_{t+\Delta t}] \sim \exp[\mu Q_t] \exp[\Delta t \mu L_t]. \tag{3.18}$$

Indeed, this relation follows from Theorem 2.4 and its proof in the same way as (3.6) follows from Theorem 2.2 and its proof.

From (3.18) and nondegenerate perturbation theory it now follows by a long, but straightforward, calculation that

$$\dot{q}_j = L_{jj}, \quad \ddot{q}_j = \mu \sum_{k \neq j} L_{jk} L_{kj} cth \frac{\mu}{2}(q_j - q_k). \tag{3.19}$$

Using (2.31) this reads

$$\dot{q}_j = \theta_j, \quad \ddot{q}_j = -\tfrac{1}{4}\varrho^2 \mu^3 \sum_{k \neq j} ch \frac{\mu}{2}(q_j - q_k)/sh^3 \frac{\mu}{2}(q_j - q_k), \tag{3.20}$$

which equals $\{q_j, H\}$ and $\{\theta_j, H\}$, respectively, by virtue of (3.11). Hence it remains to prove (3.13–14).

To this end we observe that Theorem A2 applies to the matrix (2.53) with $h'(x) = x$. Indeed, the matrices

$$M \equiv \hat{A}(0), \quad D \equiv \mu \operatorname{diag}(\hat{\theta}_1, \ldots, \hat{\theta}_N) \tag{3.21}$$

belong to $\mathcal{M}$ and $\mathcal{D}$, respectively. Therefore, we may conclude from (A31) that

$$\exp[\mu q_j(t)] = \exp[t d_j](1 + \varrho_j(t)) \tag{3.22}$$

for t large. In the case at hand the numbers d_j and m_j [cf. (A29)] read

$$d_j = \mu \hat{\theta}_j, \tag{3.23}$$

$$m_j = \exp[\mu \hat{q}_j] \exp\left[-\frac{\mu}{2} \Delta_j(\hat{\theta})\right], \tag{3.24}$$

by virtue of (3.21) and Cauchy's identity (1.1). Thus, $(3.13\text{–}14)_+$ follow from (A31–34) and the relation $\theta_j = \dot{q}_j$, cf. (3.20). Finally, $(3.13\text{–}14)_-$ can be reduced to (A31–34) by using the reversal permutation $\mathcal{R}$, just as in the proof of Theorem 3.1, cf. (3.9–10). □

3C. The Case I_{rel}

Theorem 3.3. *Let*

$$H \equiv \beta^{-1} \sum_{j=1}^{N} \exp[\beta\theta_j] \prod_{k \neq j} \left[1 - \frac{\beta^2 \varrho^2}{(q_j - q_k)^2}\right]^{1/2}, \quad \beta \in (0, \infty), \ \varrho \in i\mathbb{R}^*. \tag{3.25}$$

Then the functions

$$(q(t), \theta(t)) \equiv \mathcal{E}(\beta, \varrho; \hat{q}_1 + t \exp[\beta\hat{\theta}_1], \ldots, \hat{q}_N + t \exp[\beta\hat{\theta}_N], \hat{\theta}) \tag{3.26}$$

(cf. (2.50)) solve Hamilton's equations and satisfy

$$q_{\substack{j \\ N-j+1}}(t) = \hat{q}_j + t \exp[\beta\hat{\theta}_j] + O(|t|^{-1}), \quad t \to \pm\infty, \tag{3.27}$$

$$\theta_{\substack{j \\ N-j+1}}(t) = \hat{\theta}_j + O(|t|^{-2}), \quad t \to \pm\infty. \tag{3.28}$$

Proof. In this case we have

$$H=\beta^{-1}\operatorname{Tr}L=H_h, \quad h(x)\equiv\beta^{-1}\exp[\beta x], \tag{3.29}$$

cf. (2.32), (2.55). Moreover, (3.6) holds true by virtue of Theorem 2.5 and its proof. Hence, (3.7) follows, as before. In this case we should use (2.32) to rewrite it. This yields

$$\dot{q}_j=\exp[\beta\theta_j]V_j, \tag{3.30}$$

$$\ddot{q}_j=2\dot{q}_j\sum_{k\neq j}\dot{q}_k\partial_k\ln V_j. \tag{3.31}$$

Here, we used the relation

$$\frac{1}{q_j-q_k}C_{jk}C_{kj}=\partial_k\ln V_j, \quad k\neq j, \tag{3.32}$$

which readily follows from (2.33–34). If we now equate the time derivative of the right-hand side of (3.30) to the right-hand side of (3.31) and solve for $\dot{\theta}_j$, we obtain

$$\dot{\theta}_j=-\beta^{-1}\sum_k\exp[\beta\theta_k]\partial_jV_k. \tag{3.33}$$

Here, we also used the equality

$$\partial_k\ln V_j=-\partial_j\ln V_k, \quad k\neq j, \tag{3.34}$$

which follows from (3.32). Since the right-hand sides of (3.30) and (3.33) are equal to $\{q_j,H\}$ and $\{\theta_j,H\}$, respectively [cf. (3.25)], we have now proved that the function $(q(t),\theta(t))$ solves Hamilton's equations.

To prove (3.27–28), we first note that the right-hand side of (2.58), with $h'(\hat{\theta}_j)$ replaced by $\exp[\beta\hat{\theta}_j]$, is of the form (A11). Thus, $(3.27)_+$ follows from (A12) and (A14). Moreover, (3.30) implies

$$\theta_j=\beta^{-1}\ln[\dot{q}_j/V_j], \tag{3.35}$$

so that $(3.28)_+$ follows from (A12–15) by using (2.33). Finally, $(3.27\text{–}28)_-$ follow from (A12–15) by using $\mathscr{R}$ in the same way as before, with the relation

$$-\mathscr{R}\hat{L}\mathscr{R}=\operatorname{diag}(-\exp[\beta\hat{\theta}_N],\ldots,-\exp[\beta\hat{\theta}_1])\in\mathscr{D} \tag{3.36}$$

playing the role of (3.10). □

3D. The Case II_{rel}

Theorem 3.4. *Let*

$$H\equiv\beta^{-1}\sum_{j=1}^{N}\exp[\beta\theta_j]\prod_{k\neq j}\left[1-\frac{sh^2 z}{sh^2\frac{\mu}{2}(q_j-q_k)}\right]^{1/2},$$

$$\beta,\mu\in(0,\infty), \quad \pm z\in i(0,\pi)\cup\left(\frac{i\pi}{2}+\mathbb{R}\right). \tag{3.37}$$

Then the functions

$$(q(t),\theta(t))\equiv\mathscr{E}(\beta,\mu,z;\hat{q}_1+t\exp[\beta\hat{\theta}_1],\dots,\hat{q}_N+t\exp[\beta\hat{\theta}_N],\hat{\theta}) \tag{3.38}$$

(cf. (2.73)*) solve Hamilton's equations and satisfy*

$$\underset{N-j+1}{q_j}(t)=\hat{q}_j\mp\tfrac{1}{2}\Delta_j(\hat{\theta})+t\exp[\beta\hat{\theta}_j]+O(\exp[\mp tR]),\quad t\to\pm\infty, \tag{3.39}$$

$$\underset{N-j+1}{\theta_j}(t)=\hat{\theta}_j+O(\exp[\mp tR]),\quad t\to\pm\infty, \tag{3.40}$$

where

$$R\equiv\mu\min\{\exp[\beta\hat{\theta}_1]-\exp[\beta\hat{\theta}_2],\dots,\exp[\beta\hat{\theta}_{N-1}]-\exp[\beta\hat{\theta}_N]\}, \tag{3.41}$$

and where $\Delta_j(\theta)$ *is given by* (3.16), *with*

$$\delta(\theta)\equiv\mu^{-1}\ln\left[1-\frac{sh^2 z}{sh^2\frac{\beta}{2}\theta}\right] \tag{3.42}$$

Proof. Due to (2.59) and (2.55) the relation (3.29) holds in this case, too. Furthermore, (3.18) is satisfied by virtue of Theorem 2.7 and its proof. Thus, (3.19) follows. The definition (2.59) of L then implies that (3.30–31) hold true, the relation

$$\frac{\mu}{2}cth\frac{\mu}{2}(q_j-q_k)C_{jk}C_{kj}=\partial_k\ln V_j,\quad k\neq j \tag{3.43}$$

[which follows from (2.60–61)] playing the role of (3.32). Thus (3.33) follows again, so that $(q(t),\theta(t))$ solves Hamilton's equations.

To prove (3.39–40) we note that Theorem A2 applies to the matrix (2.76) with $h'(x)=\exp[\beta x]$, since one has

$$M\equiv\hat{A}(0)\in\mathscr{M},\qquad D\equiv\mu\,\mathrm{diag}(\exp[\beta\hat{\theta}_1],\dots,\exp[\beta\hat{\theta}_N])\in\mathscr{D}. \tag{3.44}$$

Thus, for t large (3.22) holds true again, with

$$d_j=\mu\exp[\beta\hat{\theta}_j], \tag{3.45}$$

and m_j given by (3.24), (3.16), and (3.42). Indeed, this follows from the definition (A29) of m_j and Cauchy's identity (1.2). The proof is now reduced to (A31–34) by arguing in the same way as in the proof of Theorem 3.2, using the relations (3.35–36). □

4. The Invariance Principle for the Wave and Scattering Maps

In Appendix C we have proved that the diffeomorphisms Φ constructed in Chap. 2 are canonical by using the results of Chap. 3, which pertain to a special Hamiltonian flow on Ω. We shall now determine the temporal asymptotics for a large class of flows on Ω, which are defined as pullbacks under Φ of linear flows on $\hat{\Omega}$. Since Φ is a canonical transformation and the $\hat{\Omega}$-flows are complete and Hamiltonian, the Ω-flows are complete and Hamiltonian, too.

We shall formulate the asymptotics in terms of notions from time-dependent classical scattering theory (cf. [15, 16]). To this end we first introduce the incoming and outgoing phase spaces

$$\Omega^{\pm} \equiv \{(q^{\pm}, \theta^{\pm}) \in \mathbb{R}^{2N} | \theta_N^{\pm} \lessgtr \ldots \lessgtr \theta_1^{\pm}\}, \quad \omega^{\pm} \equiv \sum_{i=1}^{N} dq_i^{\pm} \wedge d\theta_i^{\pm}. \tag{4.1}$$

Thus, Ω^+ can and will be identified with $\hat{\Omega}$ [cf. (2.2)], whereas the reversal map

$$R(q_1^-, \ldots, q_N^-, \theta_1^-, \ldots, \theta_N^-) \equiv (q_N^-, \ldots, q_1^-, \theta_N^-, \ldots, \theta_1^-) \tag{4.2}$$

yields a canonical transformation from Ω^- onto $\Omega^+ \simeq \hat{\Omega}$. It is convenient to regard $\Omega^{\pm}$ as subsets of the auxiliary phase space

$$\Omega_0 \equiv \{(x, y) \in \mathbb{R}^{2N}\}, \quad \omega_0 \equiv \sum_{i=1}^{N} dx_i \wedge dy_i. \tag{4.3}$$

We shall consider a class $\mathscr{C}_0$ of Hamiltonians H_0 on Ω_0 which depend only on y. Hence, the corresponding flows are linear and leave $\Omega^{\pm}$ invariant. The class $\mathscr{C}_0$ consists of all functions

$$H_0 \equiv F(y) \in C^{\infty}_{\mathbb{R}}(\mathbb{R}^N) \tag{4.4}$$

which are invariant under permutations of $y_1, \ldots, y_N$ and satisfy

$$(\partial_N F)(y) < \ldots < (\partial_1 F)(y) \quad \text{when} \quad y_N < \ldots < y_1. \tag{4.5}$$

Due to the symmetry of F this is equivalent to

$$(\partial_N F)(y) > \ldots > (\partial_1)(y) \quad \text{when} \quad y_N > \ldots > y_1. \tag{4.6}$$

We are now prepared to introduce the class $\mathscr{C}$ of Hamiltonians H on Ω for which we shall determine the scattering. This class is defined as the pullback of the class $\mathscr{C}_0$ under the canonical transformation $\Phi: \Omega \to \hat{\Omega} \simeq \Omega^+ \subset \Omega_0$. Thus we have

$$H(q, \theta) \equiv H_0 \circ \Phi(q, \theta) = F(\hat{\theta}), \tag{4.7}$$

cf. (4.4). We note that for the Hamiltonians H_h defined by (2.26) and (2.55) in the nonrelativistic and relativistic cases, respectively, we get

$$F(\hat{\theta}) = \sum_{i=1}^{N} h(\hat{\theta}_i) \tag{4.8}$$

cf. (2.13). Thus we have

$$H_h \in \mathscr{C} \Leftrightarrow h'(y) \text{ is strictly increasing} \tag{4.9}$$

cf. (4.5). In particular, the Hamiltonians $\mathrm{Tr}\, L^n$ belong to $\mathscr{C}$ when $n = 2, 4, 6, \ldots$ for I_{nr} and II_{nr}, and when $n = 1, 2, 3, \ldots$ for I_{rel} and II_{rel}. In the latter two cases the symmetric functions $S_1, \ldots, S_{N-1}$ of L also belong to $\mathscr{C}$, as a moment of reflection shows.

We continue by defining the canonical transformations in terms of which the temporal asymptotics of the flows generated by Hamiltonians in $\mathscr{C}$ can be described. First, we introduce

$$T: \hat{\Omega} \to \hat{\Omega}, \quad (\hat{q}, \hat{\theta}) \mapsto (\hat{q}_1 - \tfrac{1}{2}\Delta_1(\hat{\theta}), \ldots, \hat{q}_N - \tfrac{1}{2}\Delta_N(\hat{\theta}), \hat{\theta}), \tag{4.10}$$

where

$$\Delta_j(\theta) \equiv \sum_{k<j} \delta(\theta_j - \theta_k) - \sum_{k>j} \delta(\theta_j - \theta_k), \tag{4.11}$$

and

$$\delta(\theta) \equiv \begin{cases} 0, & I_{\mathrm{nr}}, I_{\mathrm{rel}} \\ \mu^{-1} \ln[1 - \mu^2 \varrho^2/\theta^2], & II_{\mathrm{nr}} \\ \mu^{-1} \ln\left[1 - sh^2 z / sh^2 \frac{\beta}{2}\theta\right], & II_{\mathrm{rel}} \end{cases}. \tag{4.12}$$

It is easily seen that T is a canonical transformation with inverse

$$T^{-1}(\hat{q}, \hat{\theta}) = (\hat{q}_1 + \tfrac{1}{2}\Delta_1(\hat{\theta}), \ldots, \hat{q}_N + \tfrac{1}{2}\Delta_N(\hat{\theta}), \hat{\theta}). \tag{4.13}$$

Second, we set

$$U_- \equiv \mathscr{E} T R, \quad U_+ \equiv \mathscr{E} T^{-1}, \tag{4.14}$$

where $R: \Omega^- \to \Omega^+$ is defined by (4.2). Third, we define

$$S \equiv U_+^{-1} U_- = T^2 R, \tag{4.15}$$

which amounts to

$$S(q_1^-, \ldots, q_N^-, \theta_1^-, \ldots, \theta_N^-) \equiv (q_N^- + \Delta_N(\theta^-), \ldots, q_1^- + \Delta_1(\theta^-), \theta_N^-, \ldots, \theta_1^-) \tag{4.16}$$

in view of (4.10–11).

Theorem 4.1. *The symplectic diffeomorphisms U_-, U_+, and S from Ω^- onto Ω, Ω^+ onto Ω and Ω^- onto Ω^+, respectively, are the wave maps and scattering map for any $H \in \mathscr{C}$ with comparison dynamics $H_0 \in \mathscr{C}_0$. That is, one has*

$$\lim_{t \to \pm\infty} [e^{tH}(q, \theta) - e^{tH_0}(q^\pm, \theta^\pm)] = 0 \tag{4.17}$$

uniformly on compacts of Ω, where

$$(q^\pm, \theta^\pm) \equiv U_\pm^{-1}(q, \theta). \tag{4.18}$$

Proof. We begin by setting

$$(q(t), \theta(t)) \equiv e^{tH}(q, \theta), \tag{4.19}$$

and noting that the above relation between H and H_0 amounts to

$$(q(t), \theta(t)) = \mathscr{E}(g; \hat{q} + t(\nabla F)(\hat{\theta}), \hat{\theta}). \tag{4.20}$$

Then (4.17) can be rewritten as

$$\underset{N-j+1}{q_j}(t) - \hat{q}_j \pm \tfrac{1}{2}\Delta_j(\hat{\theta}) - t(\partial_j F)(\hat{\theta}) \to 0, \quad t \to \pm\infty, \tag{4.21}$$

$$\underset{N-j+1}{\theta_j}(t) - \hat{\theta}_j \to 0, \quad t \to \pm\infty. \tag{4.22}$$

To prove (4.21) we set

$$D \equiv \mathrm{diag}(d_1, \ldots, d_N), \quad d_j \equiv \begin{cases} (\partial_j F)(\hat{\theta}), & I_{\mathrm{nr}}, I_{\mathrm{rel}} \\ \mu(\partial_j F)(\hat{\theta}), & II_{\mathrm{nr}}, II_{\mathrm{rel}} \end{cases}, \tag{4.23}$$

$$M \equiv L(\hat{g}; \hat{\theta}, \hat{q}), \tag{4.24}$$

where $\hat{g}$ denotes the dual coupling constants, cf. (2.23), (2.42), (2.48), and (2.70). Then (4.21) follows in the same way as (3.3), (3.13), (3.27), and (3.39), since $D \in \mathscr{D}$ due to (4.5).

To prove (4.22) we cannot proceed as in Chap. 3, since for a general F there exists no sufficiently explicit expression for $\theta_j(t)$ in terms of $q(t)$ and $\dot{q}(t)$. Instead, we exploit the relation (4.21) and various other results already obtained. We shall only prove (4.22) for the case $II_{\rm rel}$, the proof for the remaining cases being similar, but simpler.

We begin by showing that the quantities $|\theta_j(t)|$ are uniformly bounded in t. To this end we consider the Hamiltonian

$$P_0 \equiv \mathrm{Tr}(L+L^{-1})/2 = H_{h_0}, \quad h_0(x) \equiv ch\beta x, \tag{4.25}$$

which is explicitly given by

$$P_0(q,\theta) = \sum_{j=1}^{N} ch\beta\theta_j V_j(\mu, z; q). \tag{4.26}$$

[To verify this, use (2.59) and Cauchy's identity (1.2) to obtain

$$(L^{-1})_{jj} = \exp[-\beta\theta_j] V_j. \tag{4.27}$$

From this (4.26) is obvious.] The desired a priori bound on the $|\theta_j(t)|$ then follows by recalling that $V_j \geqq 1$ and noting that the quantity $P_0(q(t), \theta(t))$ does not depend on t. [Indeed, it equals $\sum ch\beta\hat{\theta}_j$ on account of (4.20).]

Next, we set

$$L_t = B_t + S_t, \tag{4.28}$$

where

$$B_t \equiv \mathrm{diag}(\exp[\beta\theta_1(t)], \ldots, \exp[\beta\theta_N(t)]). \tag{4.29}$$

We claim that there exists a constant $C>0$ such that

$$\|S_t\| \leqq C\exp[-|t|R/2], \quad \forall t \in \mathbb{R}, \tag{4.30}$$

where R is the minimal distance between the quantities $d_1, \ldots, d_N$, cf. (4.23) and (A4–5). Indeed, the error term in (4.21) is $O(\exp[-|t|R])$ due to (A33), so (4.30) follows from the definition (2.59) of L and the boundedness of the $|\theta_j(t)|$. (Of course, no a priori bound is needed in the nonrelativistic cases.)

We now assert that there exist permutations $\tau_\pm \in S_N$ and a number $T>0$ such that

$$|\exp[\beta\hat{\theta}_j] - \exp[\beta\theta_{\tau_\pm(j)}(t)]| \leqq \|S_t\| \leqq \tilde{R}/4, \quad \forall j \in \{1, \ldots, N\}, \quad \forall t \gtrless \pm T, \tag{4.31}$$

where $\tilde{R}$ is defined by

$$\tilde{R} \equiv \min_j \{\exp[\beta\hat{\theta}_j] - \exp[\beta\hat{\theta}_{j+1}]\}. \tag{4.32}$$

To prove this, we first observe that points in $\mathbb{C}$ whose distance to $\sigma(B_t)$ is larger than $\|S_t\|$ belong to the resolvent set of $B_t + S_t = L_t$. Indeed, this is clear from the second resolvent formula and the self-adjointness of B_t. Since L_t has spectrum

$$\{\exp[\beta\hat{\theta}_1], \ldots, \exp[\beta\hat{\theta}_N]\},$$

this is equivalent to the distance of $\exp[\beta\hat{\theta}_j]$ to $\sigma(B_t)$ being $\leqq \|S_t\|$ for all $j\in\{1,\ldots,N\}$. But in view of the bound (4.30) we can ensure that $\|S_t\|\leqq\tilde{R}/4$ by picking $|t|$ large enough, from which the above assertion readily follows.

It is now clear that (4.22) holds true, provided we can prove that τ_+ equals the identity and τ_- the reversal permutation τ_r. We shall prove $\tau_+=\mathrm{id}$, the proof that τ_- equals τ_r being analogous. To this end we introduce the eigenprojection

$$P_j\equiv\frac{1}{2\pi i}\int_{\Gamma_j}R(z,L_t)dz\,, \tag{4.33}$$

where Γ_j is a circle around $\exp[\beta\hat{\theta}_j]$ with radius $\tilde{R}/2$, and where $t>T$. Then the distance of $\sigma(B_t)$ to Γ_j is $\geqq\tilde{R}/4$ due to (4.31–32). Iterating the second resolvent formula for $R(z,L_t)$ and using the bound (4.30), one now concludes that L_t has eigenvectors given by

$$u_j\equiv P_jb_{\tau_+(j)}=b_{\tau_+(j)}+x_j\,,\qquad |x_j|=O(\exp[-tR/2])\,, \tag{4.34}$$

where $\{b_1,\ldots,b_N\}$ is the standard basis of $\mathbb{C}^N$. Hence, the matrix T_t^{-1} diagonalizing L_t is of the form

$$T_t^{-1}=\mathrm{Col}(u_1,\ldots,u_N)\,\mathrm{diag}(c_1,\ldots,c_N)\,, \tag{4.35}$$

where $c_1,\ldots,c_N$ are non-zero normalizing functions, cf. Chap. 2. But then T_t is given by

$$T_t=\mathrm{diag}(c_1^{-1},\ldots,c_N^{-1})\,\mathrm{Col}(\tilde{u}_1,\ldots,\tilde{u}_N)\equiv\mathrm{Col}(v_1,\ldots,v_N)\,, \tag{4.36}$$

where

$$\tilde{u}_j=b_{\sigma(j)}+\tilde{x}_j\,,\qquad \sigma\equiv\tau_+^{-1}\,,\qquad |\tilde{x}_j|=O(\exp[-tR/2])\,. \tag{4.37}$$

Let us now specialize to the regime $\pm iz\in(0,\infty)$, the point being that then T_t is unitary. This entails that the normalizing functions satisfy

$$\|c_j|-1|=O(\exp[-tR/2]) \tag{4.38}$$

in view of (4.34–35). Thus, setting

$$w_j\equiv c_jv_j\,, \tag{4.39}$$

it follows from (4.36–37) that

$$w_j=b_{\sigma(j)}+y_j\,,\qquad |y_j|=O(\exp[-tR/2])\,. \tag{4.40}$$

We now recall from Chap. 2 that T_t diagonalizes the matrix

$$\exp[tD/2]M\exp[tD/2]\,,$$

where D and M are given by (4.23) and (4.24) with $\hat{g}=(\mu,\beta,-z)$. Hence we may conclude that

$$\exp[tD/2]M\exp[tD/2]w_j=\lambda_jw_j\,, \tag{4.41}$$

where

$$\lambda_j\equiv\exp[\mu q_j(t)]\,. \tag{4.42}$$

We are now prepared to derive a contradiction from the assumption that $\sigma \neq id$. Indeed, if this holds true, then there exists $j \in \{1, \ldots, N\}$ such that

$$\sigma(i) = i, \quad i = 1, \ldots, j-1, \quad \sigma(j) = k > j. \tag{4.43}$$

Using (A 31) and setting $\delta_k \equiv d_k - d_j$, it then follows from (4.40) that the upper $j-1$ components of (4.41) can be rewritten as

$$M_{j-1} \begin{pmatrix} y_{j1} \exp[t\delta_1/2] \\ \vdots \\ y_{jj-1} \exp[t\delta_{j-1}/2] \end{pmatrix} + O(\exp[-tR/2])$$
$$= m_j(1+\varrho_j) \begin{pmatrix} y_{j1} \exp[-t\delta_1/2] \\ \vdots \\ y_{jj-1} \exp[-t\delta_{j-1}/2] \end{pmatrix}, \quad t \to \infty. \tag{4.44}$$

[Note that the second vector at the left-hand side would be $O(1)$ when $\sigma(j) = j$.] Let us now multiply this by M_{j-1}^{-1} and take $t \to \infty$. Then it follows that

$$\lim_{t \to \infty} y_{jl} \exp[t\delta_l/2] = 0, \quad l = 1, \ldots, j-1, \tag{4.45}$$

cf. also (A 33). Therefore, if we multiply (4.41) by the matrix

$$\operatorname{diag}(\exp[-t\delta_1/2], \ldots, \exp[-t\delta_{j-1}/2], \exp[-td_j], \ldots, \exp[-td_j]) \tag{4.46}$$

and take t to ∞, then the left-hand side converges to 0, whereas the right-hand side converges to $m_j b_k \neq 0$. Thus, we have arrived at the contradiction announced above, so that $(4.22)_+$ follows when $\pm iz \in (0, \pi)$.

We shall now handle the second z-regime (2.71). Then T_t is not unitary, so that we have no control over the normalizing functions c_j, and the above argument cannot be used. Instead, we reduce this case to the previous one, as follows. We fix $(\hat{q}, \hat{\theta}) \in \hat{\Omega}$ and then consider the point $(q(t), \theta(t))$, defined by (4.20), in its dependence on $\gamma \equiv z \pm i\pi/2 \in \mathbb{R}$. Inspection of the bound (A 33) and the z-dependence of M then shows that the error term in (4.21) (understood to refer to the fixed point in $\hat{\Omega}$ instead of a given point in Ω) can be chosen uniformly for $\gamma \in K$, where $K \equiv [-l, l]$ with $l > 0$. Likewise, the a priori bound on the $|\theta_j(t)|$ can be chosen uniform on K. From this it readily follows that we can choose the constant C in (4.30) uniformly on K. But then we can find a $T > 0$ such that the estimate (4.31) holds true for any $\gamma \in K$. A priori, the permutations $\tau_\pm$ occurring there could depend on γ. However, it follows from Theorem B2 that $\mathscr{E}$ is continuous in γ, so that the quantities $|\theta_j(t)|$ are continuous in γ in view of (4.20). Hence, the permutations $\tau_\pm$ in (4.31) must be constant on K for a fixed t with $|t| > T$. Since we know already that $\tau_+ = id$ and $\tau_- = \tau_r$ when $\gamma = 0$, it follows that this holds true on K, too. Since l is arbitrary, we have now proved (4.22) for both z-regimes.

To complete the proof of the theorem, we claim that the error terms in (4.21) and (4.22) (viewed again as corresponding to given points in Ω) are locally uniform on Ω. Indeed, for (4.21) this is an easy consequence of the estimates (A 14) and (A 33). But this implies that one can get a locally uniform bound on $\|S_t\|$, so that our claim for (4.22) follows from the estimate (4.31) with $\tau_+ = id$, $\tau_- = \tau_r$, and its obvious analogs for the three remaining cases. Uniformity on compacts then follows from a standard argument. □

5. Further Developments

5A. Asymptotic Constancy

The following result amounts of a reformulation and generalization of Theorem 4.1. It is included so as to make clear that the asymptotics of the point $\exp[tH](q,\theta)$, $H\in\mathscr{C}$, may be viewed as a special case of an "asymptotic constancy" property of $\mathscr{E}$.

Theorem 5.1. *Let*

$$(q(t),\theta(t))\equiv\mathscr{E}(g;\hat{q}(t),\hat{\theta}), \tag{5.1}$$

where

$$\hat{q}_j(t)\equiv\hat{q}_j+ta_{\tau(j)}, \quad a_N<\ldots<a_1, \quad \tau\in S_N. \tag{5.2}$$

Then one has

$$q_{\tau(j)}(t)-\hat{q}_j(t)\to-\tfrac{1}{2}\Delta_j(\hat{\theta}), \quad t\to\infty, \tag{5.3}$$

$$\theta_{\tau(j)}(t)-\hat{\theta}_j\to 0, \quad t\to\infty \tag{5.4}$$

uniformly on compacts of $\hat{\Omega}$, *where* Δ_j *is given by* (4.11–12).

Proof. The proof of (5.3) proceeds in the same way as for the special case considered in Chap. 3: One need only invoke Theorems A1 and A2 for the pair

$$D\equiv\operatorname{diag}(d_1,\ldots,d_N), \quad d_j\equiv\begin{cases}a_j & I_{\rm nr}, I_{\rm rel}\\ \mu a_j, & II_{\rm nr}, II_{\rm rel}\end{cases}, \tag{5.5}$$

$$M\equiv P_\tau^{-1}L(\hat{g};\hat{\theta},\hat{q})P_\tau, \tag{5.6}$$

and recall the relation of the quantities $q_N(t)<\ldots<q_1(t)$ to the eigenvalues of the matrix $E(t)$. Here, P_τ is the permutation matrix

$$(P_\tau)_{jk}\equiv\delta_{\tau(j),k}, \tag{5.7}$$

and $\hat{g}$ denotes the dual coupling constants.

To prove (5.4), we observe that (5.3) leads to $(4.31)_+$ via the same arguments as for the special case $\tau=id$. Thus, we need only show that the permutation τ_+ in $(4.31)_+$ is equal to τ. This can be proved along the same lines as before; Here, the matrix

$$\exp\left[\frac{t}{2}D\right]M\exp\left[\frac{t}{2}D\right]$$

is diagonalized by $P_\tau^{-1}T_tP_v$, so that one should replace w_j by $w_{\tau(j)}$ in (4.41). Then the assumption that $\sigma\tau=\tau_+^{-1}\tau\neq id$ leads again to a contradiction, so that (5.4) results. Finally, the uniformity assertion is easily seen to follow from (A14) and (A33). □

5B. Integrable Systems Associated with C_l and BC_l

The integrable systems considered so far may be viewed as being associated with the root system A_{N-1}, cf. the review [1]. We shall now show that one can obtain new integrable systems on the phase space

$$\Omega_r \equiv \{(q,\theta)\in \mathbb{R}^{2l} | 0<q_l<\ldots<q_1\}, \qquad \omega_r \equiv \sum_{i=1}^{l} dq_i \wedge d\theta_i, \tag{5.8}$$

associated with the root systems C_l and BC_l by restricting the pair (A, L) with $N=2l$ and $N=2l+1$ to the submanifolds of Ω given by

$$\begin{aligned}\Omega^e \equiv \{(q,\theta)\in\Omega | (q_1,\ldots,q_l,\theta_l,\ldots,\theta_l)\in\Omega_r \\ q_{l+1}=-q_l,\ldots,q_{2l}=-q_1,\theta_{l+1}=-\theta_l,\ldots,\theta_{2l}=-\theta_1\},\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\Omega^0 \equiv \{q,\theta)\in\Omega | (q_1,\ldots,q_l,\theta_1,\ldots,\theta_l)\in\Omega_r, q_{l+1}=\theta_{l+1}=0, \\ q_{l+2}=-q_l,\ldots,q_{2l+1}=-q_1,\theta_{l+2}=-\theta_l,\ldots,\theta_{2l+1}=-\theta_1\}\end{aligned} \tag{5.10}$$

respectively. To this end we introduce the phase space

$$\hat{\Omega}_r \equiv \{(\hat{q},\hat{\theta})\in\mathbb{R}^{2l} | 0<\hat{\theta}_l<\ldots<\hat{\theta}_1\}, \qquad \hat{\omega}_r \equiv \sum_{i=1}^{l} d\hat{q}_i \wedge d\hat{\theta}_i, \tag{5.11}$$

and submanifolds $\hat{\Omega}^e$ and $\hat{\Omega}^0$ of $\hat{\Omega}$ via (5.9) and (5.10) (with carets added, of course), and identify Ω^e and Ω^0 with Ω_r, and $\hat{\Omega}^e$ and $\hat{\Omega}^0$ with $\hat{\Omega}_r$, in the obvious way. The following theorem has various consequences that parallel results obtained above for the root system A_{N-1}, so we refrain from spelling them out.

Theorem 5.2. *The map Φ restricts to a symplectic diffeomorphism Φ_r from Ω_r onto $\hat{\Omega}_r$.*

Proof. From the construction of Φ in Chap. 2 it is far from obvious that $\Phi(\Omega^s)\subset\hat{\Omega}_s$, $s=e, 0$. Therefore, we proceed in another way, exploiting results already obtained. We only consider the case $II_{\rm rel}$, the other cases having a similar, but simpler, proof. First, we introduce the Hamiltonians

$$H_r^e(q,\theta)\equiv\sum_{j=1}^{l} ch\beta\theta_j f(2q_j)\prod_{k\neq j} f(q_j-q_k)f(q_j+q_k) \tag{5.12}$$

and

$$H_r^0(q,\theta)\equiv\sum_{j=1}^{l} ch\beta\theta_j f(q_j)f(2q_j)\prod_{k\neq j} f(q_j-q_k)f(q_j+q_k)+\tfrac{1}{2}\prod_{j=1}^{l} f^2(q_j) \tag{5.13}$$

on Ω_r, where

$$f(q)\equiv\left[1-sh^2z/sh^2\frac{\mu q}{2}\right]^{1/2}. \tag{5.14}$$

These are obtained from the Hamiltonian $\frac{1}{2}P_0$, given by (4.25–26) with $N=2l$ and $N=2l+1$, upon restriction to Ω^e and Ω^0, respectively.

Next, we introduce the (a priori local) flows

$$(q(t),\theta(t))\equiv\exp[tH_r^s](q,\theta), \qquad (q,\theta)\in\Omega_r, \qquad s=e,0, \tag{5.15}$$

and define corresponding trajectories $(q^s(t),\theta^s(t))$ in $\Omega^s\subset\Omega$. Then a long, but straightforward calculation shows that the trajectory $(q^s(t),\theta^s(t))$ is an integral curve for the Hamiltonian P_0 on Ω.

Now we have $P_0 \in \mathscr{C}$ in view of (4.25) and (4.9), so that

$$q_j^s(t) - \hat{q}_j + \tfrac{1}{2}\Delta_j(\hat{\theta}) - t\beta sh\beta\hat{\theta}_j \to 0, \quad t \to \infty, \tag{5.16}$$

$$\theta_j^s(t) - \hat{\theta}_j \to 0, \quad t \to \infty, \tag{5.17}$$

on account of (4.21) and (4.22). On the other hand, one has, e.g.,

$$\theta_1^s(t) + \theta_N^s(t) = 0, \quad \forall t \in \mathbb{R}, \tag{5.18}$$

since the trajectory belongs to Ω^s. Hence, (5.17) implies $\hat{\theta}_1 + \hat{\theta}_N = 0$, with a similar conclusion for the other $\hat{\theta}_i$. Using now (5.16) in the same way, the desired conclusion $\Phi(\Omega^s) \subset \hat{\Omega}^s$ readily follows.

By duality it is clear that $\subset$ may be replaced by $=$, and real-analyticity of the restriction Φ_r and its inverse $\mathscr{E}_r$ is evident from the real-analyticity of Φ and $\mathscr{E}$ established in Appendix B. Canonicity of Φ_r can be seen from canonicity of Φ, but also follows by using Appendix C for the Hamiltonians H_r^s on Ω_r: The explicit description of the associated flows which we have just obtained plays the role of Theorem 3.4 for H, so that we are reduced again to justifying an interchange of limits. To prove that this is legitimate, it clearly suffices to show that the function $(\tilde{q}(t, q, \theta), \tilde{\theta}(t, q, \theta))$ associated with the Hamiltonian P_0 on Ω has a holomorphic extension converging uniformly to the holomorphic extension of the function $(q^+(q, \theta), \theta^+(q, \theta))$. But this follows in the same way as for the Hamiltonian H: One need only replace the function $\exp(\cdot)$ by the function $\beta sh(\cdot)$ in (C10–12), and reinterpret ϱ_j and $\dot{\varrho}_j$ accordingly. □

5C. *Miscellanea*

(i) *(Functional equations for I_{nr})*. The following result concerns functions of the Lax matrix for the case I_{nr}. Note that its proof only involves the properties of the matrix T constructed in Sect. 2B.

Proposition 5.3. *Let $f: \mathbb{R} \to \mathbb{C}$ be an arbitrary function. Then one has*

$$\sum_{i \neq j} f(L)_{ij} = 0, \tag{5.19}$$

where L is given by (2.17).

Proof. We have shown in Sect. 2B that a matrix T exists such that

$$TLT^{-1} = \hat{L}, \quad Te = T^{-1t}e = e, \quad e \equiv (1, \dots, 1), \tag{5.20}$$

where $\hat{L}$ is diagonal. Hence we infer

$$\begin{aligned} \left(\sum_i f(L)_{ii}\right) e \otimes e &= (\mathrm{Tr}\, f(\hat{L})) e \otimes e = e \otimes e f(\hat{L}) e \otimes e \\ &= e \otimes e T^{-1} f(L) T e \otimes e = e \otimes e f(L) e \otimes e \\ &= \left(\sum_{i,j} f(L)_{ij}\right) e \otimes e . \end{aligned} \tag{5.21}$$

From this (5.19) is evident. □

(ii) *(A generalized Lax matrix).* Let us denote by L_α the matrix obtained when one replaces the Cauchy matrix C in (2.59) by

$$(C_\alpha)_{ij} \equiv C(\mu, z; q)_{ij} + \alpha \exp\left[-\frac{\mu}{2}(q_i - q_j)\right], \quad \alpha \in \mathbb{C}. \tag{5.22}$$

Proposition 5.4. *The symmetric functions of L_α commute.*

Proof. From (2.62) it follows that

$$L_\alpha = L + \alpha A^{-1} e \otimes e. \tag{5.23}$$

Transforming this with T yields

$$\hat{L}_\alpha = \operatorname{diag}(\hat{\theta}_1, \ldots, \hat{\theta}_N) + \alpha \hat{A}^{-1} \hat{e} \otimes \hat{e}. \tag{5.24}$$

Recalling (2.69–70) we see that $\hat{L}_\alpha$ is of the form

$$(\hat{L}_\alpha)_{ij} = \exp\left[-\frac{\mu}{2}\hat{q}_i\right] M_\alpha(\hat{\theta})_{ij} \exp\left[\frac{\mu}{2}\hat{q}_j\right]. \tag{5.25}$$

This clearly implies that the symmetric functions of L_α, transformed to $\hat{\Omega}$, depend only on $\hat{\theta}$. Since Φ is a canonical transformation, the proposition follows. □

In fact, the symmetric functions of L_α are proportional to those of L, the proportionality factor depending only on α and z. This is a consequence of a generalized Cauchy identity established in [9]: If one sets

$$\alpha \equiv e^{-\tau} shz / sh(\tau - z), \tag{5.26}$$

then the right-hand side of (5.22) can be written

$$e^{-x}\left(\frac{e^{-z} sh\tau}{sh(\tau - z)}\right)\left(\frac{sh(x+\tau)}{sh\tau} \frac{shz}{sh(x+z)}\right), \quad x \equiv \frac{\mu}{2}(q_i - q_j). \tag{5.27}$$

The assertion now follows by setting

$$v\mu \to z, \quad v\lambda \to \tau, \quad v \to \mu/2 \tag{5.28}$$

in Eqs. (3.19–20) of [9].

(iii) *(Evenness in ϱ and z).* The following result implies that the minus signs in the duality relations (2.24), (2.49), (2.50), and (2.73) may be omitted.

Proposition 5.5. *The map Φ is even in ϱ (cases $I_{\rm nr}$, $II_{\rm nr}$, $I_{\rm rel}$) and z (case $II_{\rm rel}$).*

Proof. The substitution $\varrho \to -\varrho$ in the Lax matrices (2.17), (2.31), and (2.32), and $z \to -z$ in the Lax matrix (2.59), is equivalent to transposing $L(g; q, \theta)$. Hence, the corresponding vector $\hat{\theta}$ is invariant, whereas

$$T(-\sigma) = T(\sigma)^{-1t}, \quad \sigma = \varrho, z \tag{5.29}$$

in view of (2.5), (2.8), (2.10) and the evenness of e. Now for the cases $II_{\rm nr}$ and $II_{\rm rel}$ the vector $\hat{q}$ is determined by $\hat{e}$, cf. (2.41), (2.69). Hence, (5.29) implies $\hat{q}$ is unchanged. For the cases $I_{\rm nr}$ and $I_{\rm rel}$ one has $\hat{q}_i \equiv A_{ii}$, cf. (2.23), (2.48). Since A is real and does not depend on ϱ, and since T is unitary in the latter cases, (5.29) again implies $\hat{q}$ is even. □

(iv) *(The relation between the four cases).* We conclude this final chapter by specifying the parameter limits needed to reach the cases $I_{\rm rel}$, $II_{\rm nr}$, and $I_{\rm nr}$ from the case $II_{\rm rel}$. To this end we substitute

$$z \equiv \beta\mu\varrho/2 \tag{5.30}$$

in (2.59–63). If we then take μ to 0, the matrices L and $\mu^{-1}(A-\mathbf{1})$ converge to the matrices $L_{\rm rel}$ and $A_{\rm rel}$ of Sect. 2C. If, instead, we take β to 0, then the matrices $\beta^{-1}(L-\mathbf{1})$ and A converge to the matrices $L_{\rm nr}$ and $A_{\rm nr}$ of Sect. 2C. Finally, the matrices L and A of $I_{\rm nr}$ result by taking either μ to 0 in the matrices L and $\mu^{-1}(A-\mathbf{1})$ of $II_{\rm nr}$ or by taking β to 0 in the matrices $\beta^{-1}(L-\mathbf{1})$ and A of $I_{\rm rel}$.

Appendix A. Spectral Asymptotics

In this appendix we determine the $t\to\infty$ asymptotics of the spectrum of $N\times N$ matrices of the form

$$E(t) \equiv M + tD\,, \tag{A1}$$

and of the form

$$E(t) \equiv M \exp(tD)\,. \tag{A2}$$

The first type of t-dependence arises for the rational systems $I_{\rm nr}$ and $I_{\rm rel}$, the second one for the hyperbolic systems $II_{\rm nr}$ and $II_{\rm rel}$. Throughout this appendix the matrices D are assumed to belong to the set

$$\mathscr{D} \equiv \{\mathrm{diag}(d_1, \ldots, d_N) | d \in \mathbb{C}^N, \mathrm{Re}\, d_N < \ldots < \mathrm{Re}\, d_1\}\,. \tag{A3}$$

We also use the notation

$$\begin{aligned} r_1 &\equiv \mathrm{Re}(d_1 - d_2)\,, \qquad r_N \equiv \mathrm{Re}(d_{N-1} - d_N)\,, \\ r_j &\equiv \min\{\mathrm{Re}(d_{j-1} - d_j), \mathrm{Re}(d_j - d_{j+1})\}\,, \qquad j = 2, \ldots, N-1\,, \end{aligned} \tag{A4}$$

and we set

$$R \equiv \min\{r_1, \ldots, r_N\}\,. \tag{A5}$$

The matrix M in (A1) is arbitrary, whereas in (A2) it has properties to be specified below.

We shall need information on $\sigma(E(t))$ for pairs (M, D) with M self-adjoint and D real to determine the pointwise asymptotics of the Hamiltonian flows occurring above. However, we also need information that is uniform on complex neighborhoods of a given initial point $(q, \theta) \in \Omega$, in order to obtain a rigorous proof that the bijection Φ of Chap. 2 is a canonical transformation. Therefore, we consider pairs (M, D) of a more general type and obtain bounds on error terms that are expressed in terms of appropriate norms. The uniform information we need involves a ball B_0 in $\mathscr{D}$ around a fixed D_0, which is given by

$$B_0 \equiv \{D \in \mathscr{D} | \; \|D - D_0\| < R_0/4\}\,. \tag{A6}$$

Here and below, $\|\cdot\|$ denotes the operator norm derived from the standard scalar product on $\mathbb{C}^N$. Note that one has

$$r_j \geqq r_{0j} - R_0/2 \geqq R_0/2 \tag{A7}$$

for any $D \in \bar{B}_0$.

To obtain explicit formulas for eigenvalues we use some standard techniques from finite-dimensional perturbation theory (cf. [17, Chap. II]). Specifically, setting

$$R(\zeta, A) \equiv (\zeta - A)^{-1}, \tag{A8}$$

where A is an $N \times N$ matrix and $\zeta \notin \sigma(A)$, we shall employ the formula

$$\lambda \equiv \operatorname{Tr} A P_\lambda, \tag{A9}$$

valid when λ is a simple eigenvalue of A. The eigenprojection P_λ is given by

$$P_\lambda = \frac{1}{2\pi i} \int_\Gamma R(\zeta, A) d\zeta, \tag{A10}$$

where Γ is a circle around λ whose radius is smaller than the distance of λ to the remaining spectrum of A. Also, here and below contours are oriented counterclockwise.

We are now prepared to deal with matrices of the form (A1). From now on the symbol C denotes positive numbers that do not depend on the relevant variables, and whose magnitude is of no importance.

Theorem A1. *Let*

$$E(t) = M + tD, \qquad M \in M_N(\mathbb{C}), \qquad D \in \mathscr{D}. \tag{A11}$$

Then there exists $T_E \geqq 1$ such that $E(t)$ has simple spectrum for $t \geqq T_E$. The (suitably ordered) eigenvalues $\lambda_1(t), \ldots, \lambda_N(t)$ satisfy

$$\lambda_j(t) = M_{jj} + t d_j + \varrho_j(t), \tag{A12}$$

$$\dot{\lambda}_j(t) = d_j + \dot{\varrho}_j(t), \tag{A13}$$

where the remainder functions obey

$$|\varrho_j(t)| \leqq C t^{-1} \|M\|^2 (\|D\| r_j^{-2} + r_j^{-1}), \tag{A14}$$

$$|\dot{\varrho}_j(t)| \leqq C t^{-2} \|M\|^2 (\|D\| r_j^{-2} + r_j^{-1}) \tag{A15}$$

for any $t \geqq T_E$. Now fix $D_0 \in \mathscr{D}$ and let B_0 be defined by (A6). *Then T_E can be chosen uniformly for (M, D) in the closure of*

$$\mathscr{U}_0(K) \equiv \{M \mid \|M\| < K\} \times B_0. \tag{A16}$$

Proof. Let us introduce the auxiliary matrix

$$A(t) \equiv D + t^{-1} M = t^{-1} E(t). \tag{A17}$$

We denote the circle with radius $r_j/2$ around d_j by Γ_j. Then

$$\|R(z, D)\| = 2 r_j^{-1}, \qquad \forall z \in \Gamma_j, \tag{A18}$$

so picking T_j with

$$T_j \geqq 4r_j^{-1}\|M\|+1 \tag{A19}$$

ensures

$$\|t^{-1}MR(z,D)\| \leqq \tfrac{1}{2}, \quad \forall t \geqq T_j, \quad \forall z \in \Gamma_j. \tag{A20}$$

Hence, the iteration of the second resolvent formula

$$R(z,A(t)) = R(z,D) \sum_{n=0}^{\infty} [t^{-1}MR(z,D)]^n \tag{A21}$$

converges uniformly on Γ_j for any $t \geqq T_j$, so

$$P_j(t) \equiv \frac{1}{2\pi i} \int_{\Gamma_j} R(z,A(t))dz \tag{A22}$$

is well defined. Moreover, $P_j(t)$ is one-dimensional, since $P_j(\infty)$ is. The eigenvalue inside Γ_j is then given by [cf. (A9)]

$$a_j(t) = d_j + t^{-1}M_{jj} + t^{-1}\varrho_j(t), \tag{A23}$$

where

$$\varrho_j(t) \equiv \frac{1}{2\pi i} \sum_{n=1}^{\infty} \mathrm{Tr}(tD+M) \int_{\Gamma_j} R(z,D)[t^{-1}MR(z,D)]^n dz. \tag{A24}$$

But we have

$$\mathrm{Tr}D \int_{\Gamma_j} R(z,D)MR(z,D)dz = 0, \tag{A25}$$

since D is diagonal. Hence, using (A18) and (A20) the bounds (A14–15) easily follow. Moreover, putting

$$T_E \equiv \max\{T_1, \ldots, T_N\} \tag{A26}$$

it follows that for any $t \geqq T_E$ the matrix $E(t)$ has one and only one simple eigenvalue $\lambda_j(t)$ inside $t\Gamma_j$, which is such that (A12–15) hold true, cf. (A17).

It remains to prove the uniformity claim. To this end we note that the above T_j is restricted only by (A19). Recalling (A7), we conclude that it suffices to choose

$$T_E \geqq 8K/R_0 + 1 \tag{A27}$$

to handle all (M,D) in $\bar{\mathcal{U}}_0(K)$ simultaneously. □

To control the spectral asymptotics for matrices of the form (A2) is a lot more arduous. The main problem is to bypass two related difficulties: There is no formula for the norm of the resolvent $R(\zeta, A)$ in terms of the distance of ζ to $\sigma(A)$ when A is not normal, and, secondly, the presence of diverging matrix elements (as $t \to \infty$) can a priori cause drastic spectral changes under small perturbations. (For instance, the matrix

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

has spectrum $\{0\}$ for any b, whereas

$$\begin{pmatrix} 0 & b \\ s & 0 \end{pmatrix}$$

has spectrum $\{\pm(bs)^{1/2}\}$; if b is big, this may amount to a sizable change even when s is small.) However, it turns out to be possible to obtain explicit formulas for the relevant resolvents in terms of matrices whose elements do not diverge as $t \to \infty$, and this is how we shall be able to avoid these snags in the case at hand.

We proceed to define the set $\mathcal{M}$ of matrices M for which we shall study (A2). To this end we denote by M_j the $j \times j$ matrix obtained from M by deleting the rows and columns $j+1, \ldots, N$. Then we set

$$\mathcal{M} \equiv \{M \in M_N(\mathbb{C}) \mid |M_j| \neq 0, j=1, \ldots, N\}. \tag{A28}$$

Note that $\mathcal{M}$ is an open set containing the positive matrices and the regular diagonal ones. For $M \in \mathcal{M}$ we also set

$$m_1 \equiv M_{11}, \quad m_j \equiv |M_j|/|M_{j-1}|, \quad j=2, \ldots, N. \tag{A29}$$

Theorem A2. *Let*

$$E(t) = M\exp(tD), \quad M \in \mathcal{M}, \quad D \in \mathcal{D}. \tag{A30}$$

Then there exists T_E such that $E(t)$ has simple spectrum for $t \geqq T_E$. The (suitably ordered) eigenvalues $\lambda_1(t), \ldots, \lambda_N(t)$ satisfy

$$\lambda_j(t) = m_j \exp(td_j)[1 + \varrho_j(t)], \tag{A31}$$

$$\dot{\lambda}_j(t) = m_j \exp(td_j)[d_j + d_j\varrho_j(t) + \dot{\varrho}_j(t)], \tag{A32}$$

where

$$|\varrho_j(t)| \leqq \exp(-tr_j) P(|m_j|, |m_j|^{-1}, \|M_j^{-1}\|, \|M\|), \tag{A33}$$

$$|\dot{\varrho}_j(t)| \leqq \exp(-tr_j)(|d_j - d_1| + \ldots + |d_j - d_N|) Q(|m_j|, |m_j|^{-1}, \|M_j^{-1}\|, \|M\|) \tag{A34}$$

for any $t \geqq T_E$, with P and Q polynomials. Now fix $D_0 \in \mathcal{D}$ and let B_0 be defined by (A6). *Also, fix $M_0 \in \mathcal{M}$ and choose ε so small that the closure of*

$$B_{M_0}(\varepsilon) \equiv \{M \in M_N(\mathbb{C}) \mid \|M - M_0\| < \varepsilon\} \tag{A35}$$

belongs to $\mathcal{M}$. Then T_E can be chosen uniformly for (M, D) in the closure of

$$\mathcal{U}_0(\varepsilon) \equiv B_{M_0}(\varepsilon) \times B_0. \tag{A36}$$

To prove this theorem we need the following lemma, which concerns a $j \times j$ matrix of the form

$$F(t) \equiv \Delta(t)^{-1} G, \tag{A37}$$

where G is a $j \times j$ matrix and

$$\Delta(t) \equiv \mathrm{diag}(\exp(t\delta_1), \ldots, \exp(t\delta_{j-1}), 1), \quad 0 < \mathrm{Re}\,\delta_{j-1} < \ldots < \mathrm{Re}\,\delta_1. \tag{A38}$$

Thus we can write

$$F(t) = H + V(t), \tag{A39}$$

where

$$H \equiv \begin{pmatrix} & 0 & \\ G_{j1} & \dots & G_{jj} \end{pmatrix}, \tag{A40}$$

$$V(t) \equiv \operatorname{diag}(\exp(-t\delta_1), \dots, \exp(-t\delta_{j-1}), 0)G. \tag{A41}$$

Lemma A3. *Suppose $G_{jj} \neq 0$ and let Γ be the circle around G_{jj} with radius $\frac{1}{2}|G_{jj}|$. Then one has*

$$\|R(z,H)\| \leqq \alpha(G), \qquad \forall z \in \Gamma, \tag{A42}$$

where

$$\alpha(G) \equiv 12|G_{jj}|^{-2}\|G\|_2, \tag{A43}$$

and $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. Now fix T such that

$$\exp(-T\operatorname{Re}\delta_{j-1})\|G\|\alpha(G) \leqq \tfrac{1}{2}. \tag{A44}$$

Then one has for any $t \geqq T$ and $z \in \Gamma$,

$$\|R(z,F(t))\| \leqq 2\alpha(G), \tag{A45}$$

$$\|\dot{R}(z,F(t))\| \leqq 4\exp(-t\operatorname{Re}\delta_{j-1})(|\delta_1| + \dots + |\delta_{j-1}|)\|G\|\alpha(G)^2. \tag{A46}$$

Moreover, the matrix $F(t)$ has one and only one simple eigenvalue $e(t)$ inside Γ, given by

$$e(t) = G_{jj} + R(t), \tag{A47}$$

where

$$|R(t)| \leqq \exp(-t\operatorname{Re}\delta_{j-1})P_1(|G_{jj}|^{-1}, \|G\|), \tag{A48}$$

$$|\dot{R}(t)| \leqq \exp(-t\operatorname{Re}\delta_{j-1})(|\delta_1| + \dots + |\delta_{j-1}|)P_2(|G_{jj}|^{-1}, \|G\|) \tag{A49}$$

for any $t \geqq T$, with P_1, P_2 polynomials.

Proof. It is easily verified that the H-resolvent is explicitly given by

$$R(z,H) = \frac{1}{z-G_{jj}}\,\frac{1}{z}\begin{pmatrix} z-G_{jj} & & 0 & \\ & \ddots & & \\ 0 & & z-G_{jj} & \\ G_{j1} & \dots & G_{j,j-1} & z \end{pmatrix}. \tag{A50}$$

Hence,

$$\|R(z,H)\| \leqq \frac{1}{|z-G_{jj}|}\,\frac{1}{|z|}(|z-G_{jj}| + |z| + \|G\|_2) \leqq 12|G_{jj}|^{-2}\|G\|_2 \qquad \forall z \in \Gamma, \tag{A51}$$

proving (A42).

Next, we note that

$$\|V(t)R(z,H)\| \leqq \exp(-t\operatorname{Re}\delta_{j-1})\|G\|\alpha(G) \leqq \tfrac{1}{2}, \quad \forall t \geqq T, \quad \forall z \in \Gamma, \tag{A52}$$

cf. (A 44). Thus

$$R(z,F(t))=R(z,H)\sum_{n=0}^{\infty}[V(t)R(z,H)]^n \tag{A53}$$

converges uniformly for $z\in\Gamma$ and $t\geqq T$, and the bounds (A 45–46) readily follow. Moreover, the projection

$$P(t)\equiv\frac{1}{2\pi i}\int_{\Gamma}R(z,F(t))dz \tag{A54}$$

is well defined for $t\geqq T$, and since the projection

$$P(\infty)=\frac{1}{2\pi i}\int_{\Gamma}R(z,H)dz=\begin{pmatrix} & 0 & \\ G_{j1}/G_{jj}\dots 1 \end{pmatrix} \tag{A55}$$

[cf. (A 50)] has rank one, $P(t)$ is one-dimensional, too.

We conclude that $F(t)$ has one and only one eigenvalue $e(t)$ inside Γ given by (A 47), where

$$R(t)\equiv\mathrm{Tr}\left[V(t)P(\infty)+\frac{1}{2\pi i}\sum_{n=1}^{\infty}(H+V(t))\int_{\Gamma}R(z,H)[V(t)R(z,H)]^n dz\right]. \tag{A56}$$

From this it is easy to verify (A 48–49) by using (A 41) and the bounds (A 51–52). □

Proof of Theorem A2. We are going to study the auxiliary matrix

$$A(t)\equiv\exp(-td_j)E(t),\quad j\in\{1,\dots,N\}. \tag{A57}$$

To this end we introduce the complex numbers

$$\delta_k\equiv d_k-d_j,\quad k=1,\dots,N \tag{A58}$$

and the $j\times j$ matrix $\Delta(t)$ given by (A 38). (Here and below we have suppressed dependence on j to ease the notation.) Now we split up M in 2×2 form and write $A(t)$ as the sum of a "big" and a "small" matrix, as follows:

$$B(t)\equiv\begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--}\end{pmatrix}\begin{pmatrix}\Delta(t) & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}M_{++}\Delta(t) & 0\\ M_{-+}\Delta(t) & 0\end{pmatrix}, \tag{A59}$$

$$S(t)\equiv A(t)-B(t)=M\,\mathrm{diag}(0,\dots,0,\exp(t\delta_{j+1}),\dots,\exp(t\delta_N)). \tag{A60}$$

Thus, $B(t)$ has the same spectrum as the $j\times j$ matrix

$$Z(t)\equiv M_{++}\Delta(t) \tag{A61}$$

(up to an eigenvalue 0 when $j<N$).

Next, we note that $M_{++}=M_j$, so that $Z(t)$ is regular due to our assumption $M\in\mathcal{M}$. Moreover, if we set

$$F(t)\equiv Z(t)^{-1},\quad G\equiv M_j^{-1}, \tag{A62}$$

then the assumptions of Lemma A3 are satisfied, with G_{jj} being given by

$$G_{jj}=(M_j^{-1})_{jj}=|M_{j-1}|/|M_j|=m_j^{-1} \tag{A63}$$

[cf. (A29)]. Hence, it follows that for $t \geqq T$ [where T satisfies (A44)] $B(t)$ has one and only one eigenvalue $b(t)$ inside the contour

$$\tilde{\Gamma} \equiv \{\zeta \in \mathbb{C} | \zeta^{-1} \in \Gamma\}, \tag{A64}$$

which is given by

$$b(t) = m_j/[1 + m_j R(t)], \tag{A65}$$

cf. (A47), (A63).

We now make the key observation that the $B(t)$-resolvent can be simply expressed in terms of the $F(t)$-resolvent, as follows:

$$R(\zeta, B(t)) = \begin{pmatrix} -\zeta^{-1} F(t) R(\zeta^{-1}, F(t)) & 0 \\ -\zeta^{-2} M_{-+} M_{++}{}^{-1} R(\zeta^{-1}, F(t)) & \zeta^{-1} \end{pmatrix}. \tag{A66}$$

Indeed, the validity of this formula can be readily verified by using the above relations between B and F. Consequently, we are able to estimate the norms of the $B(t)$-resolvent and its time derivative on the contour $\tilde{\Gamma}$ by using the bounds (A45–46). This yields

$$\|R(\zeta, B(t))\| \leqq Q_1(|m_j|, |m_j|^{-1}, \|M_j^{-1}\|, \|M\|), \tag{A67}$$

$$\|\dot{R}(\zeta, B(t))\| \leqq \exp(-t \operatorname{Re}\delta_{j-1})(|\delta_1| + \ldots + |\delta_{j-1}|) Q_2(|m_j|, |m_j|^{-1}, \|M_j^{-1}\|, \|M\|)$$

for any $t \geqq T$ and $\zeta \in \tilde{\Gamma}$, with Q_1, Q_2 polynomials. (A68)

We proceed by concluding from (A67) that one has

$$\|S(t) R(\zeta, B(t))\| \leqq \tfrac{1}{2}, \quad \forall t \geqq T_j \geqq T, \quad \forall \zeta \in \tilde{\Gamma}, \tag{A69}$$

provided T_j is chosen such that

$$\exp(T_j \operatorname{Re}\delta_{j+1}) \|M\| Q_1(|m_j|, |m_j|^{-1}, \|M_j^{-1}\|, \|M\|) \leqq \tfrac{1}{2} \tag{A70}$$

[cf. the definition (A60) of $S(t)$]. Hence,

$$R(\zeta, A(t)) = R(\zeta, B(t)) \sum_{n=0}^{\infty} [S(t) R(\zeta, B(t))]^n \tag{A71}$$

converges uniformly when $t \geqq T_j$ and $\zeta \in \tilde{\Gamma}$. Also, the projection

$$\tilde{P}(t) \equiv \frac{1}{2\pi i} \int_{\tilde{\Gamma}} R(\zeta, A(t)) d\zeta \tag{A72}$$

is well defined for any $t \geqq T_j$, and using (A66) and (A37–38) one infers that $\lim_{t \to \infty} \tilde{P}(t)$ exists and is one-dimensional, so that $\tilde{P}(t)$ is one-dimensional, too.

As a result we have now shown that both $B(t)$ and $A(t)$ have one and only one eigenvalue $b(t)$ and $a(t)$, respectively, inside $\tilde{\Gamma}$ for any $t \geqq T_j$. Moreover, these eigenvalues are related by

$$a(t) = \operatorname{Tr} A(t) \tilde{P}(t) = b(t) + \frac{1}{2\pi i} \operatorname{Tr} \int_{\tilde{\Gamma}} S(t) R(\zeta, B(t)) d\zeta$$
$$+ \frac{1}{2\pi i} \sum_{n=1}^{\infty} \operatorname{Tr} \int_{\tilde{\Gamma}} [-\mathbf{1} + (\zeta + S(t)) R(\zeta, B(t))] [S(t) R(\zeta, B(t))]^n d\zeta. \tag{A73}$$

Indeed, this follows by using (A 71–72); we have gotten rid of the "big" matrix $B(t)$ by using

$$BR(\zeta,B) = -\mathbf{1} + \zeta R(\zeta,B).$$

Defining now $\varrho_j(t)$ by setting

$$a(t) \equiv m_j[1+\varrho_j(t)], \tag{A 74}$$

and combining this with (A 73) and (A 65), it is not hard to obtain the bounds (A 33–34) on ϱ_j and $\dot{\varrho}_j$: One need only use the estimates (A 48–49) and (A 67–69), recall the definitions (A 4) and (A 60) of r_j and $S(t)$, and observe that the modulus of the denominator in (A 65) is bounded below by $\frac{1}{2}$, since $b(t)$ lies inside $\tilde{\Gamma}$ for any $t \geqq T_j$. It is then obvious from the definition (A 57) of $A(t)$ that $E(t)$ has one and only one eigenvalue $\lambda_j(t)$ inside $\exp(td_j)\tilde{\Gamma}$, which satisfies (A 31–34). Finally, defining T_E by (A 26) and noting that all bounds are decreasing in t, it follows that $E(t)$ has simple spectrum for any $t \geqq T_E$, with eigenvalues satisfying (A 31–34).

It remains to prove the uniformity statement. To this end we recall that the above T_j is restricted only by (A 70) and the requirement $T_j \leqq T$, where T is solely restricted by (A 44). These requirements are expressed in terms of functions of M that are continuous on $\mathscr{M}$ and in terms of exponential functions involving D. The latter functions can be uniformly majorized on $\bar{B}_0$ by using the lower bound (A 7). Also, since $\bar{B}_{M_0}(\varepsilon)$ is a compact set which belongs to $\mathscr{M}$, the former functions are uniformly bounded on it. Hence we can choose T_j uniformly on $\mathscr{U}_0(\varepsilon)$, and defining T_E by (A 26) the proof of Theorem A2 is complete. □

Appendix B. Real-Analyticity

In this appendix we show that the bijections Φ and $\mathscr{E}$ constructed in Chap. 2 are real-analytic functions of (q,θ) and, therefore, diffeomorphisms from Ω onto $\hat{\Omega}$ and $\hat{\Omega}$ onto Ω, respectively. In fact, we shall prove more, namely, that these maps are real-analytic in the coupling constants, too. Just as in Sect. 2A we denote these parameters collectively by g and their definition domain by $G \subset \mathbb{C}^l$; this enables us to handle all cases simultaneously.

We have occasion to use the following lemma, which summarizes some results from nondegenerate perturbation theory. In essence, these facts can be found in [17, Chap. II], but since this may not be visible to the unaided eye, we sketch a proof.

Lemma B1. *Suppose $M \equiv M(z)$ is an $N \times N$ matrix that is holomorphic in a polydisc around $z_0 \in \mathbb{C}^k$ and suppose that $M_0 \equiv M(z_0)$ has simple spectrum. Then there exists a (possibly smaller) polydisc D around z_0 with the following properties: The spectrum of M is simple in D, the eigenvalues $\lambda_1, \ldots, \lambda_N$ of M are holomorphic in D, and there exist corresponding eigenvectors $u_1, \ldots, u_N$ that are holomorphic in D.*

Proof. Let λ_0 be an eigenvalue of M_0 with corresponding eigenvector u_0 and consider the series

$$R(\zeta,M) = R(\zeta,M_0) \sum_{n=0}^{\infty} [(M-M_0)R(\zeta,M_0)]^n \tag{B1}$$

[cf. (A8)] where ζ belongs to a circle Γ around λ_0 such that all other eigenvalues lie outside Γ. Picking z close enough to z_0 ensures that the series converges uniformly on Γ, so that we can define P_λ via (A10). Then P_λ is a one-dimensional eigenprojection of M, since $P_\lambda(z_0)=\lim_{z\to z_0} P_\lambda(z)$ is one-dimensional. Using Hartog's theorem one now infers that P_λ and the corresponding eigenvalue λ [given by (A9)] are holomorphic near z_0. Hence, the function $u\equiv P_\lambda u_0$ is holomorphic near z_0. Moreover, since $u(z_0)=P_\lambda(z_0)u_0=u_0\neq 0$, one has $u\neq 0$ near z_0. It is now clear how to complete the proof. □

Theorem B2. *The bijections Φ and $\mathscr{E}\equiv\Phi^{-1}$ constructed in Chap. 2 are real-analytic functions in $G\times\Omega$ and $G\times\hat{\Omega}$, respectively.*

Proof. We shall only prove this for Φ, since the assertion for $\mathscr{E}$ is then obvious from duality. Let us fix a point P in $G\times\Omega$. Inspection of the definitions of $L(g;q,\theta)$ and $e(g;q,\theta)$ shows that there exists a polydisc in $\mathbb{C}^l\times\mathbb{C}^{2N}$ around P in which L and e are holomorphic. Moreover, we have proved in Chap. 2 that L has simple spectrum on $G\times\Omega$. Hence, Lemma B1 applies, with $M=L$ and $z_0=P$. Using the notation introduced there, we can now define a regular matrix

$$H\equiv\mathrm{Col}(u_1,\dots,u_N), \tag{B2}$$

which is holomorphic in D. Furthermore, eventually performing a permutation and multiplying H from the right by a constant, diagonal and invertible matrix, we can achieve that $H(P)$ equals the matrix $T^{-1}(P)$ of Chap. 2.

Next, we introduce the vectors

$$\hat{a}\equiv H^{-1}e, \quad \tilde{a}\equiv H^t e, \tag{B3}$$

which are holomorphic in D, as well. Now consider the functions $h_i\equiv\hat{a}_i/\tilde{a}_i$. Since we have

$$\hat{a}_i(P)=\tilde{a}_i(P)=\hat{e}_i(P)>0 \tag{B4}$$

(cf. Chap. 2), there exists a polydisc $U\subset D$ around P such that $h_1,\dots,h_N$ are non-zero and holomorphic in U. Hence, the functions $r_i\equiv h_i^{1/2}$, with $r_i(P)\equiv 1$, are holomorphic and non-zero in U, too. Multiplying the eigenvector u_i by r_i, we get a regular matrix, denoted again by H, for which the vectors $\hat{a}$ and $\tilde{a}$ [cf. (B3)] are equal, and which is holomorphic in U. Moreover, the coordinates $\hat{a}_1,\dots,\hat{a}_N$ are holomorphic in U and positive in P.

We are now in the position to invoke the uniqueness of the matrix T, established in Chap. 2, to conclude that we must have

$$H^{-1}=T, \quad \hat{a}=\hat{e}, \quad \forall(g;q,\theta)\in(U\cap G\times\Omega). \tag{B5}$$

As a consequence, T and $\hat{e}$ are real-analytic in $G\times\Omega$. Real-analyticity of the functions $\hat{q}_j$ and $\hat{\theta}_j$ is then clear from their definitions, and the proof is complete. □

Appendix C. Canonicity

The purpose of this appendix is to state and prove the following theorem which justifies our interpretation of the maps Φ of Chap. 2 as action-angle transformations.

Theorem C1. *The diffeomorphisms Φ from Ω onto $\hat{\Omega}$ constructed in Chap. 2 are symplectic.*

Proof. We shall prove this for the case II_{rel}. The proof for the remaining three cases proceeds along the same lines, with simplifications occurring at various points. First, let us introduce

$$q_j^+(q,\theta)\equiv \hat{q}_j-\tfrac{1}{2}\Delta_j(\hat{\theta}), \tag{C1}$$

$$\theta_j^+(q,\theta)\equiv\hat{\theta}_j, \tag{C2}$$

and observe [cf. (3.16) and (3.42)] that Δ_j can be written

$$\Delta_j(\hat{\theta})=2\mu^{-1}\ln\Big(\prod_{k<j}\hat{f}(\hat{\theta}_j-\hat{\theta}_k)\Big/\prod_{k>j}\hat{f}(\hat{\theta}_j-\hat{\theta}_k)\Big), \tag{C3}$$

$$\hat{f}(\theta)\equiv\left[1-\frac{sh^2 z}{sh^2\frac{\beta}{2}\theta}\right]^{1/2}. \tag{C4}$$

From these relations it follows that it suffices to prove that the map $\Phi^+:\Omega\to\hat{\Omega}$, $(q,\theta)\mapsto(q^+,\theta^+)$ is canonical. To this end we use (3.39–40) to infer

$$q^+(q,\theta)=\lim_{t\to\infty}\tilde{q}(t,q,\theta), \tag{C5}$$

$$\theta^+(q,\theta)=\lim_{t\to\infty}\tilde{\theta}(t,q,\theta), \tag{C6}$$

where

$$\tilde{q}_j(t,q,\theta)\equiv q_j(t)-t\exp[\beta\theta_j(t)], \tag{C7}$$

$$\tilde{\theta}_j(t,q,\theta)\equiv\theta_j(t). \tag{C8}$$

Now we have

$$(q(t),\theta(t))=\exp(tH)(q,\theta)$$

by virtue of Theorem 3.4. Since Hamiltonian flows are canonical, it follows that the functions $\tilde{q}_j,\tilde{\theta}_j$ have Poisson brackets

$$\begin{aligned}&\{\tilde{q}_j,\tilde{q}_k\}=\{\tilde{\theta}_j,\tilde{\theta}_k\}=0,\\&\{\tilde{q}_j,\tilde{\theta}_k\}=\delta_{jk}\end{aligned} \tag{C9}$$

for any $t\in\mathbb{R}$.

In view of (C5–6) it remains to prove that one may interchange the $t\to\infty$ limit and the differentiations with respect to q_i and θ_i implied in (C9). To this end, let us fix $(q_0,\theta_0)\in\Omega$ with image $(\hat{q}_0,\hat{\theta}_0)\in\hat{\Omega}$ under Φ. From Theorem B2 and its proof we then conclude that there exists a polydisc $X\subset\mathbb{C}^{2N}$ around (q_0,θ_0) such that $(\hat{q},\hat{\theta})$ depends holomorphically on $(q,\theta)\in X$ and such that the "pair potentials" $\hat{f}(\hat{\theta}_j-\hat{\theta}_k)$ do not vanish on X. Hence, (q^+,θ^+) extends to a holomorphic function in X by virtue of the monodromy theorem and Hartog's theorem, cf. (C1–4).

Next, we invoke Theorem A2 to infer that there exists a polydisc $\hat{Y}$ around $(\hat{q}_0,\hat{\theta}_0)$, whose closure belongs to $\hat{X}\equiv\Phi(X)$, and a number $T\in\mathbb{R}$ such that the matrix

$$E(t) \equiv L(\mu, \beta, -z; \hat{\theta}, \hat{q}) \exp[t\mu \operatorname{diag}(\exp[\beta\hat{\theta}_1], \ldots, \exp[\beta\hat{\theta}_N])] \tag{C10}$$

has simple spectrum for any $(\hat{q}, \hat{\theta}) \in \bar{\hat{Y}}$ and $t \geqq T$. Indeed, if we denote the matrices in (C10) corresponding to $(\hat{q}_0, \hat{\theta}_0)$ by M_0 and D_0, then it is clear that choosing $\hat{Y}$ small enough ensures that the corresponding pairs (M, D) belong to $\mathscr{U}_0(\varepsilon)$, cf. (A 35–36) and (A6). Moreover, it follows that on $\hat{Y} \cap \hat{\Omega}$ we have

$$q_j(t) = \hat{q}_j - \tfrac{1}{2}\Delta_j(\hat{\theta}) + t\exp[\beta\hat{\theta}_j] + \mu^{-1}\ln[1 + \varrho_j(t)], \tag{C11}$$

$$\exp[\beta\theta_j(t)] = (\exp[\beta\hat{\theta}_j] + \mu^{-1}\dot{\varrho}_j(t)[1 + \varrho_j(t)]^{-1}) \times \prod_{k \neq j} \left[1 - \frac{sh^2 z}{sh^2 \frac{\mu}{2}(q_j(t) - q_k(t))}\right]^{-1/2}, \tag{C12}$$

cf. the proof of Theorem 3.4 and (3.30). Now $\varrho_j(t)$ and $\dot{\varrho}_j(t)$ are expressed in terms of series that converge uniformly, and the terms of the series are clearly holomorphic in $\hat{Y}$, cf. the proof of Theorem A2. Hence, $\varrho_j(t)$ and $\dot{\varrho}_j(t)$ are holomorphic in $\hat{Y}$ for any $t \geqq T$. Also, the function $1 + \varrho_j(t)$ in (C11) is non-zero on $\hat{Y}$, since $E(t)$ is regular on $\hat{Y}$. Thus, for any $t \geqq T$ the function $q_j(t)$ has a holomorphic extension to $\hat{Y}$. Moreover, we may view $q_j(t)$ as a holomorphic function of $(q, \theta) \in Y \equiv \mathscr{E}\hat{Y}$, since $(\hat{q}, \hat{\theta})$ is holomorphic in $Y \subset X$.

At first sight the same assertion for $\theta_j(t)$ may seem to follow from (C12), but in fact it is not clear from the above that choosing $t \in [T, \infty)$ ensures that the terms $[\ldots]^{-1/2}$ at the right-hand side do not diverge on $\hat{Y}$ and that the first term does not vanish on $\hat{Y}$. However, we shall now prove that these snags can be avoided by eventually increasing T. To this end we first note that (C11) and the estimates (A7), (A33) imply

$$|\mathrm{Re}(q_j(t) - q_k(t))| \geqq Ct, \quad \forall (\hat{q}, \hat{\theta}) \in \hat{Y}, \quad \forall t \geqq T. \tag{C13}$$

Thus we can achieve that the radicands in (C12) are non-zero on $\hat{Y}$ by picking T large enough. Also, eventually increasing T once more, we can ensure that the first term at the right-hand side of (C12) is non-zero on $\hat{Y}$ for any $t \geqq T$. [Indeed, this is clear from the estimates (A33–34).] Thus, $\theta_j(t)$ has a holomorphic extension to $\hat{Y}$, and hence may be regarded as a holomorphic function in Y, for t large enough.

We are now in the position to conclude that $(\tilde{q}, \tilde{\theta})$ has a holomorphic extension to Y for t large enough, cf. (C7–8). Moreover, due to the bounds (A33–34) the convergence of this holomorphic function to the holomorphic function (q^+, θ^+) as $t \to \infty$ is uniform on $\hat{Y}$. But then all derivatives of $(\tilde{q}, \tilde{\theta})$ converge to those of (q^+, θ^+), so that (C9) holds for (q^+, θ^+), too. □

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