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THE GL(n)-INVARIANT IDEALS OF THE COORDINATE RING OF PAIRS OF SYMMETRIC MATRICES WITH PRODUCT ZERO

G.C.M. Ruitenburg
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Introduction.

Let W be the variety of pairs (s,t) of symmetric $n\times n$-matrices over k with $st = 0$, where n a non-negative integer and k a field of characteristic zero. We define a $G = GL(n,k)$-action on W by

$$g\cdot(s,t) = \left((g^{-1})^t s g^{-1}, g t g^t\right) \text{ , } g \in G \text{ and } (s,t) \in W \text{ .}$$

There is an induced action on the coordinate ring $R = k[X_{ij},Y_{ij}]/I$ of W, $1 \leq i,j \leq n$, where I is the ideal generated by the elements

$$X_{ij}-X_{ji} \text{ , } Y_{ij}-Y_{ji} \text{ and } \sum_{\ell=1}^{n} X_{i\ell}Y_{\ell j} \text{ , } 1 \leq i,j \leq n \text{ ,}$$

given by

$$g\cdot X_{ij} = (g^t X g)_{ij} \text{ and } g\cdot Y_{ij} = (g^{-1}Y(g^{-1})^t)_{ij}$$

such that

$$(gf)(g\omega) = f(\omega) \qquad \text{for all } g \in G \text{ , } f \in R \text{ and } w \in W \text{ .}$$

Our purpose is to study R as G-module and to give a description of the G-invariant (prime, primary, radical) ideals. Moreover we give an algorithm for forming a primary decomposition for any G-invariant ideal, describe the symbolic powers of prime ideals and describe for any G-invariant ideal the integral closure.

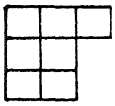
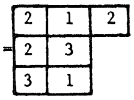
Other problems of this kind are studied in [1],[2] and [3]. They have in common that with help of certain generators and relations for R as k-module a multiplicity free decomposition in irreducible components for R as G-module can be obtained.

The proof in our case goes along the same line as in the case of determinantal varieties [2]. We will use pairs of bitableaux to indicate products of minors of X and Y and by combining straightening formulas given in [4] and [3] we will prove that a certain subset of standard pairs of bitableaux form a k-free basis of R . (Here is is sufficient to assume that k is a commutative ring.) Next we will give a multiplicity free decomposition in irreducible components $R = \oplus M_{[\sigma,\tau]}$ as G-module, where the sum is over all pairs of diagrams $[\sigma,\tau]$ with $\sigma_1+\tau_1 \leq n$. After that we will use a lemma and the results in [1] to describe all sets D of pairs of diagrams, the D-ideals, such that $\underset{D}{\oplus} M_{[\sigma,\tau]}$ is an ideal and thus find all G-invariant ideals. In order to get remaining results on G-invariant ideals we will translate our questions in terms of D-ideals and then answer them in a combinatorial way. Now for a more geometric point of view, let k be an algebraic closed field of characteristic zero. The orbits of GL(n) in W are the sets $V_{p,q} = \{(s,t)\in W \mid \text{rank } s = p, \text{rank } t = q\}$ with $0 \leq p,q$ and $p+q \leq n$. Their closures $W_{p,q} = \bar{V}_{p,q} = \{(s,t)\in W \mid \text{rank } s\leq p, \text{rank } t\leq q\}$ are the only G-invariant irreducible subvarieties of W . From our results it follows that the G-invariant prime ideal $J_{p+1,q+1}$ of functions vanishing on $W_{p,q}$ is generated by the p+1-order minors of X and q+1-order minors of Y . In Proposition 2.6 we describe the ideals $J^{(m)}_{p+1,q+1}$ (the m-th symbolic power of $J_{p+1,q+1}$) of functions vanishing to order $\geq m$ along $W_{p,q}$.

1. Combinatorics and R as G-module.

A (Young)-diagram σ is a finite subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that if $(i,j) \in \sigma$ and $i' \leq i$, $j' \leq j$ then $(i',j') \in \sigma$. Each diagram can be respresented by a sequence $(\sigma_1,\sigma_2,\dots)$ or even $(\sigma_1,\sigma_2,\dots,\sigma_\ell)$ if $\sigma_{\ell+1} = 0$, again denoted by σ, where $\sigma_i = \max\{j \mid (i,j)\in\sigma\}$ (max (empty set) = 0), $\sigma_1 \geq \sigma_2 \geq \dots$ and $\sigma_i = 0$ for i large enough. By interchanging the factors in $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ we get the dual diagram $\check{\sigma}$. The degree of σ is $|\sigma| = \sum_{i\geq 1} \sigma_i$, its length $\ell(\sigma) = \check{\sigma}_1 = \max\{i \mid \sigma_i \neq 0\}$ and for $k \in \mathbb{N}$ $\gamma_k(\sigma) = \sum_{i\geq k} \check{\sigma}_i$. If $\check{\sigma} + \check{\tau} = (\check{\sigma}_1+\check{\tau}_1, \check{\sigma}_2+\check{\tau}_2,\dots) = \check{\mu}$ then we write $\sigma\cdot\tau = \mu$. The diagrams are partially ordered as subsets of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by the inclusion $\subseteq$. A second finer partial order is given by $\sigma \leq \tau$ iff $\gamma_k(\sigma) \leq \gamma_k(\tau)$ for all $k \geq 1$. These orders provide product orders on the set of pairs of diagrams $[\sigma,\tau]$, again denoted by $\subseteq$ and $\leq$. For pairs of diagrams $[\sigma,\tau]$ and $[\sigma',\tau']$ we define degree $[\sigma,\tau] = (|\sigma|,|\tau|) \in \mathbb{N}\times\mathbb{N}$ and $[\sigma,\tau]\cdot[\sigma',\tau'] = [\sigma\sigma',\tau\tau']$.

A tableau A on $\{1,\dots n\}$ with shape σ is a map $A\colon \sigma \to \{1,\dots,n\}$. The content of A is the sequence of numbers $C_A = (\omega_1,\dots,\omega_n)$ where $\omega_i = |A^{-1}(i)|$. We think of σ as a set of boxes and A as a way of filling it with numbers between 1 and n. We will often denote A by a (in general not rectangular) matrix (a_{ij}) with $a_{ij} = A((i,j))$ $(i,j) \in \sigma$. Example:

$\sigma = (3,2,2) =$  $A =$  $= \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & \\ 3 & 1 & \end{pmatrix}$

A bitableau is a pair $(A|B)$ of tableau on $\{1,\dots,n\}$ of the same shape, in matrix notation $(a_{ij}|b_{ij})$ with $a_{ij},b_{ij} \in \{1,..,n\}$ $(i,j) \in \sigma$. For fixed i the bitableau $(a_{i1}\dots a_{i\sigma_i}|b_{i1}\dots b_{i\sigma_i})$ of shape (σ_i) is named the i-th row of $(A|B)$. We use bitableaux to indicate products of minors of a matrix $X = (X_{ij})$ in $R_X = k[X_{ij}]$, where k is a commutative ring and the X_{ij} are indeterminates, $1 \leq i,j \leq n$. First to a bitableau $(a_1\dots a_p|b_1\dots b_p)$ of shape (p) we associate the minor involving to rows $a_1,..,a_p$ and columns $b_1,..,b_p$. For an arbitrary bitableau we take the product of the minors associated to its rows. Up to a sign the element in R_X associated to a bitableau $(A|B)$ does not depend on the order of the rows of A and B. Since the monomials in the X_{ij} span R_X and X_{ij} is associated to the bitableau $(i|j)$, it is clear that the bitableau $(A|B)$ of which A and B

have strictly increasing rows indicate a set of generators for R_X as k-module. The set of tableaux is partially ordered by the relation

$$A = (a_{ij})_{(i,j)\in\sigma} < B = (b_{ij})_{(i,j)\in\tau} \text{ iff for all } k \text{ and } \ell$$

$$\left|\{(i,j)\in\sigma \mid i<k,\ a_{ij}<\ell\}\right| < \left|\{(i',j')\in\tau \mid i'<k,\ b_{i'j'}<\ell\}\right| .$$

A tableau is called standard if its rows are strictly increasing and its columns non-decreasing.

From a bitableau $(A|B) = (a_{ij}|b_{ij})_{(i,j)\in\sigma}$ we form the single tableau

$$s(A|B) = \begin{pmatrix} a_{11} & \cdots & a_{1\sigma_1} \\ b_{11} & \cdots & b_{1\sigma_1} \\ \vdots & & \vdots \\ a_{\ell 1} & \cdots & a_{\ell\sigma_\ell} \\ b_{\ell 1} & \cdots & b_{\ell\sigma} \end{pmatrix} \qquad , \ \ell = \ell(\sigma) .$$

Now the order on single tableaux provides via s an order on bitableaux. A bitableau $(A|B)$ is called standard if $s(A|B)$ is standard. The content of a bitableau is defined as $c_{(A|B)} = c_{s(A|B)}$. Let $R_{SX} = R_X/I_{SX}$, where I_{SX} is the ideal generated by the elements $X_{ij}-X_{ji}$, $1 < i,j < n$. The following proposition holds [4,section 5]:

PROPOSITION 1.1. <u>The standard bitableaux form a k-free basis for R_{SX} and each bitableau can be written as a linear combination of standard bitableaux with the same content that are later in the order.</u> □

We will need this proposition in the proof of a similar result for our ring R . Before we can state this result we have to define generators and an order on them. Define a $\mathbb{N}^2$-grading on R by $\deg(X_{ij}) = (1,0)$ and $\deg(Y_{ij}) = (0,1)$. Then $R_{SX} = \bigoplus_{p\in\mathbb{N}} R_{(p,0)}$, $R_{SY} = \bigoplus_{q\in\mathbb{N}} R_{(0,q)}$ and $R = R_{SX}.R_{SY}$.
For $(A|B) \in R_{SX}$ of shape σ and $(C|D) \in R_{SY}$ of shape τ we associate to the pair of bitableaux $[(A|B),(C|D)]$ of shape $[\sigma,\tau]$ their product in $R_{(|\sigma|,|\tau|)} \subseteq R$. From [3,prop. 1.3 i] follows that this product equals zero for $\sigma_1+\tau_1 > n$, so we restrict our attention to the case $\sigma_1+\tau_1 < n$. From each pair of bitableaux

$[(A|B),(C|D)]$ of shape $[\sigma,\tau]$ (with $\sigma_1+\tau_1 \leq n$), where A,B,C and D has strictly increasing rows, we can form the single tableau:

$$s[(A|B),(C|D)] = \begin{pmatrix} \hat{d}_{q1} & \cdots & \hat{d}_{q\mu_q} \\ \hat{c}_{q1} & \cdots & \hat{c}_{q\mu_q} \\ \vdots & & \vdots \\ \hat{d}_{11} & \cdots & \hat{d}_{1\mu_1} \\ \hat{c}_{11} & \cdots & \hat{c}_{1\mu_1} \\ a_{11} & \cdots & a_{1\sigma_1} \\ b_{11} & \cdots & b_{1\sigma_1} \\ \vdots & & \vdots \\ a_{\ell 1} & \cdots & a_{\ell\sigma_\ell} \\ b_{\ell 1} & \cdots & b_{\ell\sigma_\ell} \end{pmatrix}$$

with $\ell = \ell(\sigma)$, $q = |\tau|$, $\mu_i = n-\tau_i$ $1 \leq i \leq q$ and strictly increasing rows, and such that $\{c_{i1},\ldots,c_{i\tau_i}\} \cup \{\hat{c}_{i1},\ldots,\hat{c}_{i\mu_i}\} = \{1,\ldots,n\} =$
$= \{d_{i1},\ldots,d_{i\tau_i}\} \cup \{\hat{d}_{i1},\ldots,\hat{d}_{i\mu_i}\}$ for all $1 \leq i \leq q$.

The order on single tableaux provides via s an order on each set of pairs of bitableaux with fixed degree (and all rows strictly increasing). $[(A|B),(C|D)]$ is called standard if $s[(A|B),(C|D)]$ is standard.

PROPOSITION 1.2. <u>The standard pairs of bitableaux form a k-free basis for R and each pair of bitableaux can be written as a linear combination of standard bitableaux that are later in the order.</u>

<u>Proof.</u> First we prove the second part of the proposition. It is an easy calculation to prove:

LEMMA 1.3. <u>Let</u> $(A|B)$, $(A'|B')$ <u>and</u> $(C|D)$ <u>three bitableaux of shape</u> σ,σ' <u>and</u> τ <u>respectively. Assume that</u> $\sigma_1+\tau_1 \leq n$ <u>and</u> $\sigma'_1+\tau_1 \leq n$, <u>and</u> $C_{(A|B)} = C_{(A'|B')}$. <u>Then</u>

$$[(A|B),(C|D)] \leq [(A'|B'),(C|D)] \text{ and } [(C|D),(A|B)] \leq [(C|D),(A'|B')]. \qquad \square$$

Fix $(p,q) \in \mathbb{N}^2$. The set of pairs of bitableaux $[(A|B),(C|D)]$, with shape $[\sigma,\tau]$ such that $\deg[\sigma,\tau] = (|\sigma|,|\tau|) = (p,q)$ and such that A,B,C and D are tableaux on

$\{1,..,n\}$ with strictly increasing rows, is finite. Therefore it is sufficient to prove that each non-standard element in this set can be written as linear combination of elements in this set that are later in the order.

So let $[(A|B),(C|D)]$ be a non-standard pair of bitableaux of shape $[\sigma,\tau]$ in the set mentioned above. For $\sigma_1+\tau_1 > n$ it defines zero, thus we may assume that $\sigma_1+\tau_1 \leq n$. If $(A|B)$ or $(C|D)$ is not standard then we apply Proposition 1.1 and Lemma 1.3 and we are done. Now the only situation that remains is (in the notation of above)

$$\begin{pmatrix} \hat{c}_{11} & \cdots & \hat{c}_{1\mu_1} \\ a_{11} & \cdots & a_{1\sigma_1} \end{pmatrix} \text{ is not standard.}$$

Thus if $(a_1\ldots a_\ell|b_1\ldots b_\ell)$ and $(c_1\ldots c_k|d_1\ldots d_k)$ are the first rows of $(A|B)$ and $(C|D)$ respectively, $\ell = \sigma_1$ and $k = \tau_1$, and $\{\hat{c}_1<\ldots<\hat{c}_{n-k}\}$ is the complement of $\{c_1,\ldots,c_k\}$ in $\{1,\ldots,n\}$ then there exists a $1 \leq r \leq \ell$ such that $a_1 > \hat{c}_1,\ldots,a_{r-1} > \hat{c}_{r-1}$ and $a_r < \hat{c}_r$. Since $(Y_{ij})^t(X_{ij}) = 0$ over R , it follows from [3,prop.1.3.ii] that:

$$\sum_\pi \pm [(\pi(a_1)\ldots\pi(a_r)a_{r+1}\ldots a_\ell|b_1\ldots b_\ell),(\{\hat{c}_1\ldots\hat{c}_{r-1}\pi(\hat{c}_r)\ldots\pi(\hat{c}_k)\}^C|d_1\ldots d_k)] = 0 ,$$

where superscript C stands for taking the complement in $\{1,2,\ldots,n\}$ and the sum is taken over all cosets in

$$\mathrm{Symm}(a_1,\ldots,a_r,\hat{c}_r,\ldots,\hat{c}_{n-k})/\mathrm{Symm}(a_1,\ldots,a_r)\times\mathrm{Symm}(\hat{c}_1,\ldots,\hat{c}_{n-k}) .$$

Since $[(a_1\ldots a_\ell|b_1\ldots b_\ell),(c_1\ldots c_k|d_1\ldots d_k)]$ is smaller than each other summand we find an expression for it as linear combination of pairs of bitableaux that are strictly later in the order. After multiplying with the other minors associated to $[(A|B),(C|D)]$ we find the desired expression for this element. This finishes the proof of the second part of Proposition 1.2.

Before we proof the linear independence of the standard pairs of bitableaux, we recall some generalities on the representation theory of $GL(n,\mathbb{Q})$, which we will need below. Let T,U and $B \subseteq GL(n,\mathbb{Q})$ be the subgroups of diagonal matrices, the upper triangular unipotent matrices and the upper triangular matrices respectively. For each diagram σ with $\sigma_1 \leq n$ there is an unique irreducible and polynomial representation M_σ of $GL(n,\mathbb{Q})$ with highest weightvector $\check{\sigma}$ (with respect to B). It is well known that $\dim_{\mathbb{Q}} M_\sigma$ is equal to the number of standard tableau on $\{1,\ldots,n\}$ of shape σ, see [2] or [5]. Furthermore we denote by L^p the one dimensional representation of $GL(n,\mathbb{Q})$ with character $(\text{determinant})^p$, $p \in \mathbb{Z}$.

(Thus $L^1 \cong M_{(n)}$). In order to prove the first part of Proposition 1.2, it is enough to do it for $k = \mathbb{Q}$. We will show that $\dim_{\mathbb{Q}} R_{(p,q)} = d$, where $(p,q) \in \mathbb{N}^2$ and d is the number of standard pairs of bitableaux of shape $[\sigma,\tau]$ with degree $[\sigma,\tau] = (p,q)$. Note that d is the number of all standard tableaux of shape $\mu.\mu.\sigma.\sigma$, where $\mu = (n-\tau_q, n-\tau_q, \ldots, n-\tau_1, n-\tau_1)$ and σ,τ diagrams with $|\sigma| = p$, $|\tau| = q$ and $\sigma_1+\tau_1 \leq n$. Since in a standard tableaux on $\{1,\ldots,n\}$ a row of length n can only be $1\ 2 \ldots n$, we can replace μ by $(n-\tau_k, n-\tau_k, \ldots, n-\tau_1, n-\tau_1)$ with $k = \ell(\tau) = \check{\tau}_1$.

From the second part of the Proposition 1.2, which we already proved, follows $\dim_{\mathbb{Q}} R_{(p,q)} \leq d$. We will now prove the converse.

From each pair of diagrams $[\sigma,\tau]$ with $\sigma_1+\tau_1 \leq n$ we define the canonical element

$$k_{[\sigma,\tau]} = \left[\begin{pmatrix} 1 & 2 & \ldots & \sigma_1 & | & 1 & 2 & \sigma_1 \\ 1 & 2 & \ldots & \sigma_2 & | & 1 & 2 & \sigma_2 \\ \vdots & & & \vdots & | & \vdots & \vdots & \vdots \\ 1 & \vdots & \ldots & \sigma_\ell & | & 1 & 2 & \sigma_\ell \end{pmatrix}, \begin{pmatrix} n & n-1 & \ldots & n-\tau_1+1 & | & n & n-1 & \ldots & n-\tau_1+1 \\ n & n-1 & \ldots & n-\tau_2+1 & | & n & n-1 & \ldots & n-\tau_2+1 \\ \vdots & \vdots & & \vdots & | & \vdots & \vdots & & \vdots \\ n & n-1 & \ldots & n-\tau_k+1 & | & n & n-1 & \ldots & n-\tau_k+1 \end{pmatrix}\right]$$

$\in R_{(|\sigma|,|\tau|)} \subseteq R$, $\ell = \ell(\sigma)$ and $k = \ell(\tau)$.

$$k_{[\sigma,\tau]}\left(\left[\begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{\sigma_1} & \theta \\ \theta & \theta \end{pmatrix}, \begin{pmatrix} \theta & & \\ & \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{\tau_1} & \theta \\ \theta & & \end{pmatrix}\right]\right) = 1 \text{ implies } k_{[\sigma,\tau]} \neq 0 .$$

It is easy to see that $k_{[\sigma,\tau]}$ is an U-invariant vector. For $\operatorname{diag}(t_1,\ldots,t_n) \in T$ we have

$$\operatorname{diag}(t_1,\ldots,t_n)\cdot k_{[\sigma,\tau]} =$$

$$t_1^{2\cdot\check{\sigma}_1}\cdot\ldots\cdot t_n^{2\cdot\check{\sigma}_n}\cdot t_1^{-2\check{\tau}_n}\cdot\ldots\cdot t_n^{-2\check{\tau}_1}\cdot k_{[\sigma,\tau]} = t_1^{\check{\mu}_1}\cdot\ldots\cdot t_n^{\check{\mu}_n}\cdot(t_1\cdots t_n)^{-2\check{\tau}_1}\cdot k_{[\sigma,\tau]} ,$$

where

$$\mu = (n-\tau_k, n-\tau_k, n-\tau_{k-1}, \ldots, n-\tau_1, n-\tau_1, \sigma_1, \sigma_1, \ldots, \sigma_\ell, \sigma_\ell) ,$$

$$\ell = \ell(\sigma) \quad \text{and} \quad k = \ell(\tau) .$$

Thus in particular $k_{[\sigma,\tau]}$ is a weight vector for T.
These three consecutive facts together imply that the $\mathbb{Q}$-span of $GL(n,\mathbb{Q})\cdot k_{[\sigma,\tau]}$, denoted by $M_{[\sigma,\tau]}$, is an irreducible representation. In fact $M_{[\sigma,\tau]} \otimes L^{2\check{\tau}_1} \cong M_\mu$ and $2\check{\tau}_1$ is the smallest non-negative integer h such that $M_{[\sigma,\tau]} \otimes L^h$ is polynomial. ($\sigma_1+\tau_1 < n$ implies $\check{\sigma}_n = 0$ or $\check{\tau}_1 = 0$).
But from that it follows that for

$$[\sigma,\tau] \neq [\sigma',\tau'] \quad (\sigma_1+\tau_1<n \text{ and } \sigma_1'+\tau_1'<n) \quad M_{[\sigma,\tau]} \not\cong M_{[\sigma',\tau']} .$$

Since $GL(n,\mathbb{Q})$ is a linear reductive group, the sum $\Sigma\, M_{[\sigma,\tau]} \subseteq R_{(p,q)}$ taken over all $[\sigma,\tau]$ with $|\sigma| = p$, $|\tau| = q$ and $\sigma_1+\tau_1 < n$, is direct. Counting dimensions yields $\dim_{\mathbb{Q}} R_{(p,q)} \geq d$, hence the dimension equals d. □

In the proof of Proposition 1.2 we have obtained a description of R as G-module, which will be gathered in Proposition 1.4 below.

DEFINITION. $A_{[\sigma,\tau]}$ <u>is the k-span of all pairs of bitableaux with shape</u> $\geq [\sigma,\tau]$.
$A'_{[\sigma,\tau]}$ <u>is the k-span of all pairs of bitableaux with shape</u> $> [\sigma,\tau]$.

PROPOSITION 1.4. *Let* k *be a field of characteristic zero. Then* $R \cong \oplus M_{[\sigma,\tau]}$ *is a multiplicity free decomposition in irreducible components, the sum is taken over all pairs of diagrams* $[\sigma,\tau]$ *with* $\sigma_1+\tau_1 < n$. *Furthermore*
$M_{[\sigma,\tau]} = k\text{-span } \underline{\text{of}}\ GL(n)\cdot k_{[\sigma,\tau]} \cong A_{[\sigma,\tau]}/A'_{[\sigma,\tau]} \cong M_\mu \otimes L^{-2\check{\tau}_1}$,
where $\mu = (n-\tau_k, n-\tau_k, \ldots, n-\tau_1, \sigma_1, \sigma_1, \ldots, \sigma_\ell)$, $\ell = \ell(\sigma)$ *and* $k = \ell(\tau)$.

Proof. Since the representation theory for $k = \mathbb{Q}$ and k an arbitrary field of characteristic zero are actually the same [5], the proposition follows immediately from the proof of Proposition 1.2. □

We end this section with a lemma that will be useful in section 2.

LEMMA 1.5. $M_{[\sigma,\phi]}\cdot M_{[\phi,\tau]} = M_{[\sigma,\tau]}$ $(\sigma_1+\tau_1<n)$.

Proof. $k_{[\sigma,\phi]}\cdot k_{[\phi,\tau]} = k_{[\sigma,\tau]}$ implies $M_{[\sigma,\tau]} \subseteq M_{[\sigma,\phi]}\cdot M_{[\phi,\tau]}$.

Now assume

$$M_{[\sigma',\tau']} \subseteq M_{[\sigma,\phi]}\cdot M_{[\phi,\tau]} .$$

Then

$$(|\sigma'|,|\tau'|) = \text{degree}[\sigma',\tau'] = \text{degree}[\sigma,\phi] + \text{degree}[\phi,\tau] = (|\sigma|,|\tau|) ,$$

thus

$$|\sigma'| = |\sigma| = p \quad \text{and} \quad |\tau'| = |\tau| = q .$$

On the other hand $M_{[\sigma',\tau']}$ must be isomorphic to a direct summand of

$$M_{[\sigma,\phi]} \otimes M_{[\phi,\tau]} .$$

Then

$$M_{\mu'} = M_{[\sigma',\tau']} \otimes L^q , \ \mu' = (n-\tau'_q, n-\tau'_q, \dots, n-\tau'_1, \sigma'_1, \sigma'_1, \dots, \sigma'_\ell) ,$$

$$M_\mu = M_{[\phi,\tau]} \otimes L^q , \ \mu = (n-\tau_q, n-\tau_q, \dots, n-\tau_1)$$

and $M_{[\sigma,\phi]}$ are polynomial representations by Proposition 1.4 and $M_{\mu'}$ is isomorphic to a direct summand of $M_{[\sigma,\phi]} \otimes M_\mu$, so $\mu' \supseteq \mu$ (cf. the Littlewood-Richardson rule [6]). Combining this with $|\tau'| = |\tau|$ yields $\tau = \tau'$. By symmetry $\sigma = \sigma'$ thus $M_{[\sigma',\tau']} = M_{[\sigma,\tau]}$. □

2. The G-invariant ideals.

From now on we assume that k is a field of characteristic zero. Let $I_{[\sigma,\tau]}$, $\sigma_1+\tau_1 < n$, be the ideal generated by the submodule $M_{[\sigma,\tau]}$. By Proposition 1.4 $I_{[\sigma,\tau]}$ is the minimal G-invariant ideal containing $k_{[\sigma,\tau]}$. The next theorem tells us how this ideal decomposes as a direct sum of irreducible submodules.

<u>THEOREM 2.1.</u> $I_{[\sigma,\tau]} = \bigoplus_{[\sigma',\tau'] \supseteq [\sigma,\tau]} M_{[\sigma',\tau']}$.

<u>Proof.</u> By lemma 1.5. we have $M_{[\sigma,\tau]} = M_{[\sigma,\phi]} \cdot M_{[\phi,\tau]}$, thus

$$I_{[\sigma,\tau]} = R \cdot M_{[\sigma,\tau]} = R_{SX} \cdot R_{SY} \cdot M_{[\sigma,\tau]} \cdot M_{[\phi,\tau]} = R_{SX} \cdot M_{[\sigma,\phi]} \cdot R_{SY} \cdot M_{[\phi,\tau]} .$$

But in [1] it is proved that

$$R_{SX} \cdot M_{[\sigma,\tau]} = \bigoplus_{\sigma' \supseteq \sigma} M_{[\sigma',\phi]}$$

and equivalently

$$R_{SY} \cdot M_{[\phi,\tau]} = \bigoplus_{\tau' \supseteq \tau} M_{[\phi,\tau']} .$$

Hence

$$I_{[\sigma,\tau]} = \bigoplus_{\sigma' \supseteq \sigma} M_{[\sigma',\phi]} \cdot \bigoplus_{\tau' \supseteq \tau} M_{[\phi,\tau']} = \bigoplus_{[\sigma',\tau'] \supseteq [\sigma,\tau]} M_{[\sigma',\tau']}$$

by Lemma 1.5 again.

<u>Remark.</u> For $\sigma_1+\tau_1 > n$ we define $M_{[\sigma,\tau]} = 0$.

This theorem enables us to describe all G-invariant ideals in terms of diagrams.

DEFINITION. 1) $D_0 = \{[\sigma,\tau] \mid \sigma_1+\tau_1>n\}$.

2) <u>A set of pairs of diagrams</u> D <u>is called a D-ideal iff</u> $D_0 \subseteq D$ <u>and if</u> $[\sigma,\tau] \in D$ <u>and</u> $[\sigma',\tau'] \supseteq [\sigma,\tau]$ <u>then</u> $[\sigma',\tau'] \in D$.

For a finite set $\{[\sigma',\tau'],\dots[\sigma^m,\tau^m]\}$ we denote by $([\sigma^1,\tau^1],\dots,[\sigma^m,\tau^m])$ the D-ideal $D_0 \cup \{[\sigma,\tau] \mid [\sigma,\tau] \supseteq [\sigma^i,\tau^i]$ for some $1<i<m\}$.

By Proposition 1.4 R has multiplicity free decomposition as G-module, thus the same holds for each G-invariant ideal. Therefore an immediate consequence of Theorem 2.1 is.

PROPOSITION 2.2. <u>There is a</u> 1-1 <u>correspondence between G-invariant ideals and D-ideals given by:</u>

$$D \text{ a D-ideal} \quad \to I(D) = \bigoplus_{[\sigma,\tau]\in D} M_{[\sigma,\tau]}$$

$$I \text{ a G-invariant ideal} \quad \to D = D_0 \cup \{[\sigma,\tau] \mid k_{[\sigma,\tau]} \in I\} .$$

Furthermore this correspondence preserves containment and commutes with taking intersections. □

By this proposition and the fact that the ordering $<$ on pairs of diagrams extends the ordering $\subseteq$ follows that the sets

$$A_{[\sigma,\tau]} = \bigoplus_{[\sigma,\tau]\leq[\sigma',\tau']} M_{[\sigma',\tau']} \quad \text{and} \quad A'_{[\sigma,\tau]} = \bigoplus_{[\sigma,\tau]<[\sigma',\tau']} M_{[\sigma',\tau']}$$

are in fact ideals. The next proposition tells us that $A_{[\sigma,\tau]}$ is generated by certain product of minors of $X = (X_{ij})$ and $Y = (Y_{ij})$.

PROPOSITION.

$$A_{[\sigma,\tau]} = A_{[\sigma,\phi]} \cdot A_{[\phi,\tau]} = I_{[(\sigma_1),\phi]} \cdot \ldots \cdot I_{[(\sigma_\ell),\phi]} \cdot I_{[\phi,(\tau_1)]} \cdot \ldots \cdot I_{[\phi,(\tau_k)]} ,$$

<u>where</u> $\ell = \ell(\sigma)$, $k = \ell(\tau)$.

<u>Proof.</u> $A_{[\sigma,\tau]} = \bigoplus_{[\sigma',\tau']\geq[\sigma,\tau]} M_{[\sigma',\tau']}$, so the first identity follows by lemma 1.5. The identity

$$A_{[\sigma,\phi]} = I_{[(\sigma_1),\phi]} \cdot \ldots \cdot I_{[(\sigma_\ell),\phi]}$$

is one of the results on R_{SX} proved in [1]. □

Let D be a D-ideal, we will say that D is

prime if $[\sigma,\tau]\cdot[\sigma',\tau'] \in D$ implies $[\sigma,\tau] \in D$ or $[\sigma',\tau'] \in D$

primary if $[\sigma,\tau]\cdot[\sigma',\tau'] \in D$ implies $[\sigma,\tau] \in D$ or $[\sigma',\tau']^m \in D$ for some m

radical if $[\sigma,\tau]^m \in D$ for m implies $[\sigma,\tau] \in D$.

For any D-ideal D we write $\sqrt{D} = \{[\sigma,\tau] \mid [\sigma,\tau]^m \in D \text{ for some } m\}$, this is again a D-ideal.

Before we show that these notions correspond to the usual ones we describe these D-ideals in detail.

PROPOSITION 2.3.

1) The prime D-ideals are $([(p),\phi],[\phi,(q)])$ with $p+q < n+2$.

2) The radical D-ideals are $([(p_1),(q_1)],\dots,[(p_m),(q_m)])$, with $p_1 < p_2 < \dots < p_m$ and $q_1 > \dots > q_m$.

3) A D-ideal D is primary iff

   a) $p+q < n+2$, with $p = \min\{a\,|\,[(a),\phi]^m \varepsilon D$ for some $m\}$ and $q = \min\{b\,|\,[\phi,(b)]^m \varepsilon D$ for some $m\}$.

   b) $[(n-q+2),\phi]$, $[\phi,(n-p+2)] \;\varepsilon\; D$

   c) For each $[\sigma,\tau]$ in the (unique and finite) minimal set of generators for D holds $\sigma_\ell > p$ and $\tau_k > q$, $\ell = \ell(\sigma)$ and $k = \ell(\tau)$.

Proof. 1) A diagram $[\sigma,\tau]$ can be written as

$$[(\sigma_1),\phi]\cdot\ldots\cdot[(\sigma_\ell),\phi]\cdot[\phi,(\tau_1)]\cdot\ldots\cdot[\phi,(\tau_k)] ,$$

$$\ell = \ell(\sigma) \text{ and } k = \ell(\tau) .$$

Thus it is clear that a prime D-ideal D must be of the form $D = ([(p),\phi],[\phi,(q)])$. Then $[(p-1),(q-1)] = [(p-1),\phi]\cdot[\phi,(q-1)] \not\varepsilon D$ implies $(p-1)+(q-1) < n$, thus $p+q < n+2$. Since $\sigma_1+\tau_1 > n$ implies $\sigma_1 > p$ or $\tau_1 > q$ the converse is also clear.

2) Let $[\sigma,\tau] \;\varepsilon\; D$, D a radical D-ideal. $[(\sigma_1),(\tau_1)]^m \supseteq [\sigma,\tau]$ for m suffciently large, thus $[(\sigma_1),(\tau_1)] \;\varepsilon\; D$. Then it is clear that D must be of the stated form, whilst the converse is trivial.

3) Let D be a primary ideal. a) and b) follow by the same type of arguments used in 1). Now let $[\sigma,\tau] \;\varepsilon\; D$. We can write $[\sigma,\tau] = [\sigma^1,\tau^1][\sigma^2,\tau^2]$, with $[\sigma^1,\tau^1]$ of the form as indicated in c) and $\sigma_1^2 < p$ and $\tau_1^2 < q$. Then $[\sigma^2,\tau^2]^m \not\varepsilon D$ for all m , thus $[\sigma^1,\tau^1] \;\varepsilon\; D$.

The converse is a straightforward calculation. □

Remark. 1) For D a primary D-ideal we have in the notation of this proposition $\sqrt{D} = ([(p),\phi],[\phi(q)])$.

Remark. 2) Each radical D-ideal D can be written as intersection of prime D-ideals: For

$$D = ([(p_1),(q_1)],\dots,[(p_m),(q_m)])$$

as in the proposition we can write

$$D = ([(p_1),\phi],[\phi,(n+1)]) \cap ([(p_2),\phi],[\phi,(q_1)]) \cap \dots \cap ([(n+1),\phi],[\phi,(q_m)]) \ .$$

With the help of the identity $([(p),\phi],[\phi,(q)] = \cap([(p'),\phi],[\phi,(q')])$, where the intersection is taken over all $p' < p$, $q' < q$ and $p'+q' = n+2$ if $p+q > n+2$, we can refine this intersection to an irredundant intersection of prime D-ideals.

PROPOSITION 2.4. *The* 1-1 *correspondence of Proposition 2.2. preserves the notions prime, primary and radical.*

Proof. Since $k_{[\sigma,\tau]} \cdot k_{[\sigma',\tau']} = k_{[\sigma\sigma',\tau\tau']}$ it is clear that the transition of G-invariant ideals to D-ideals preserves these notions.

In order to prove the converse we need the following.

LEMMA 2.5. *Let* ϕ *be the projection* $M_{[\sigma,\tau]} \otimes M_{[\sigma',\tau']} \to M_{[\sigma\sigma',\tau\tau']}$, *with* $\max(\sigma_1,\sigma_1') + \max(\tau_1,\tau_1') < n$, *then*

$$\phi(f \otimes g) = 0 \ \text{implies} \ f = 0 \ \text{or} \ g = 0 \ .$$

Proof of the Lemma. By Proposition 1.4 we can write

$$M_{[\sigma,\tau]} \cong M_{\mu} \otimes L^{-2\check{\tau}_1} \ , \ M_{[\sigma',\tau']} \cong M_{\nu} \otimes L^{-2\check{\tau}_1'}$$

and

$$M_{\mu\nu} = M_{[\sigma\sigma',\tau\tau']} \otimes L^{-2(\check{\tau}_1+\check{\tau}_1')}$$

all three irreducible and polynomial representations of G , such that

$$\psi = \phi \otimes L^{+2(\check{\tau}_1+\check{\tau}_1')} : M_{\mu} \otimes M_{\nu} \to M_{\mu\nu}$$

is in fact the projection of $M_\mu \otimes M_\nu$ on its Cartan component (that is the irreducible component in $M_\mu \otimes M_\nu$ that contains the highest weightvector). It is thus the same to prove that $\phi(f \otimes g) = 0$ implies $f = 0$ or $g = 0$.

Let $v \in M_\mu$ and $w \in M_\nu$ be highest weightvectors, $f \in M_\mu$, $f \neq 0$ and $g \in M_\nu$, $g \neq 0$. There is an open subset $O_f \subseteq U$, where U is the subgroup of upper triangular unipotent matrices, such that for all $u \in O_f$ $u \cdot f = \alpha_u \cdot v +$ + (terms of lower weight) with $\alpha_u \neq 0$. Similarly there is an open subset O_g for g. Thus for $u \in O_f \cap O_g \neq \emptyset$ $uf \otimes ug = \alpha \cdot v \otimes w +$ + (terms of lower weight) and $\alpha \neq 0$. But $v \otimes w$ is the highest weightvector in $M_\mu \otimes M_\nu$, thus $\phi(v \otimes w) \neq 0$. Then $\phi(uf \otimes ug) \neq 0$ and also $\phi(f \otimes g) \neq 0$. □

Furthermore we define a total order $<_\ell$ on pairs of diagrams by $[\sigma,\tau] <_\ell [\sigma',\tau']$ iff σ is lexicografically smaller then σ' or if $\sigma = \sigma'$ then τ is lexicografically smaller then τ'. This order $<_\ell$ is an extension of the partial order $<$. Clearly $<_\ell$ satisfies the multiplication rule: if

$$[\sigma,\tau] <_\ell [\sigma',\tau'] \quad \text{then} \quad [\sigma,\tau]\cdot[\sigma'',\tau''] <_\ell [\sigma',\tau']\cdot[\sigma'',\tau'']$$

for any $[\sigma'',\tau'']$.

Now we go on with the proof of the propostion.

Let D be a D-ideal, $I = I(D)$ and write

$$I^C = \bigoplus_{[\sigma,\tau]\notin D} M_{[\sigma,\tau]}, \quad \text{so } R = I \oplus I^C .$$

First assume D is a prime D-ideal, say $D = ([(p),\phi],[\phi,(q)])$ $p+q < n+2$. In order to show that I is prime, it is sufficient to prove for $f,g \in I^C$ and $f \neq 0$, $g \neq 0$ that $f\cdot g \notin I$. We can write

$$f = \sum_{[\sigma,\tau]\notin D} f_{[\sigma,\tau]} \quad \text{and} \quad g = \sum_{[\sigma,\tau]\notin D} g_{[\sigma,\tau]}$$

with $f_{[\sigma,\tau]}, g_{[\sigma,\tau]} \in M_{[\sigma,\tau]}$. From Proposition 1.2 it follows that

$$f_{[\sigma,\tau]}\cdot g_{[\sigma',\tau']} \in A_{[\sigma\sigma',\tau\tau']} = \bigoplus_{[\sigma'',\tau'']\geq[\sigma\sigma',\tau\tau']} M_{[\sigma'',\tau'']} .$$

Now let $[\sigma,\tau]$ and $[\sigma',\tau']$ be minimal in the order $<_\ell$ such that $f_{[\sigma,\tau]} \neq 0$ and $g_{[\sigma',\tau']} \neq 0$ respectively. Using that $<_\ell$ extends $<$ and the multiplication rule holds for it yields that $f\cdot g \in A_{[\sigma\sigma',\tau\tau']}$ and in

$$f\cdot g = \sum_{[\sigma'',\tau'']\geq[\sigma\sigma',\tau\tau']} h_{[\sigma'',\tau'']} \quad , \quad h_{[\sigma'',\tau'']} \in M_{[\sigma'',\tau'']} \; ,$$

the only contribution to $h_{[\sigma\sigma',\tau\tau']}$ comes from $f_{[\sigma,\tau]}\cdot g_{[\sigma',\tau']}$. Since $f_{[\sigma,\tau]}$ and $g_{[\sigma',\tau']} \in I^C$ we have $\sigma_1,\sigma_1' < p$ and $\tau_1,\tau_1' < q$ thus $[\sigma\sigma',\tau\tau'] \notin D$ and $M_{[\sigma\sigma',\tau\tau']} \neq 0$. Then $h_{[\sigma\sigma',\tau\tau']} \neq 0$ by Lemma 2.5, so $fg \notin I$.

Now by Remark 2) to Proposition 2.3 and Proposition 2.2 it follows immediate that the radical D-ideals correspond to the radical G-invariant ideals. Finally let D be a primary D-ideal. Remark 1) to Proposition 2.3 says that $\sqrt{D} = \big([(p),\phi],[\phi,(q)]\big)$ for some p,q with $p+q \leq 2$. Because $\sqrt{I} \supseteq I(\sqrt{D}) \supseteq I$ and $I(\sqrt{D})$ is prime it follows that $\sqrt{I} = I(\sqrt{D})$. In order to show that I is primary it is sufficient to prove for f with $f \notin \sqrt{I}$ and $g \in I^C$, $g \neq 0$, that $f\cdot g \notin I$. We can write

$$f = \sum_{[\sigma,\tau]} f_{[\sigma,\tau]} \; , \; f_{[\sigma,\tau]} \in M_{[\sigma,\tau]} \; ,$$

and since $f \notin \sqrt{I}$ there is a minimal $[\sigma,\tau]$ with respect to the order $<_\ell$ with $\sigma_1 < p$, $\tau_1 < q$ and $f_{[\sigma,\tau]} \neq 0$. For g we choose $[\sigma',\tau']$ in the same way as is in the case where D was prime.

From Proposition 2.3 3) follows $\sigma_1' \leq n-q+1$, on the other hand we have $\tau_1 \leq q-1$, thus $\sigma_1'+\tau_1 \leq n$. Similarly we find $\tau_1'+\sigma_1 \leq n$. Then we may conclude that $[\sigma\sigma',\tau\tau'] \notin D_0$ and thus $M_{[\sigma\sigma',\tau\tau']} \neq 0$. As in the case D prime it follows that

$$h_{[\sigma\sigma',\tau\tau']} \neq 0 \quad \text{in} \quad f\cdot g = \sum_{[\sigma,\tau]} h_{[\sigma,\tau]} \; .$$

Because $[\sigma,\tau]^m \notin D$ for all m and $[\sigma',\tau'] \notin D$ we have $[\sigma\sigma',\tau\tau'] \notin D$ and thus $f\cdot g \notin I$. □

We now want to decompose each G-invariant ideal as finite intersection of primary ideals. By Proposition 2.2 and 2.4 it is equivalent to give an algorithm for D-ideals.
Let D be an arbitrary D-ideal. Since R is Noetherian, D is finitly generated say $D = ([\sigma^1,\tau^1],\dots[\sigma^m,\tau^m])$.

If for example $\sigma^1 = \sigma^{11} \cup \sigma^{12}$ and $\tau^1 = \tau^{11} \cup \tau^{12}$, then $[\sigma,\tau] \supseteq [\sigma^1,\tau^1]$ if and only if $[\sigma,\tau] \supseteq [\sigma^{11},\tau^{11}]$ and $\supseteq [\sigma^{12},\tau^{12}]$, so

$$D = \left([\sigma^{11},\tau^{11}],[\sigma^2,\tau^2],\dots[\sigma^m,\tau^m]\right) \cap \left([\sigma^{12},\tau^{12}],[\sigma^2,\tau^2],\dots[\sigma^m,\tau^m]\right) .$$

The first step of the algorithm is to write each σ^i and τ^i as union of diagrams of the form $(p)^a = (p,\dots p)$, a times p , and then, by repeating the argument, to decompose D as intersection of D-ideals of type

$$D' = \left([(p_1)^{a_1},\phi],\dots,[(p_s)^{a_s},\phi],[\phi,(q_1)^{b_1}],\dots,[\phi,(q_t)^{b_t}]\right)$$

with $p_1 < \dots < p_s$, $q_1 < \dots < q_t$, $a_1 > \dots > a_s$ and $b_1 > \dots > b_t$.
It is an easy calculation to see that

$$D' = [(p_1),\phi],[\phi,(q_1)]) \cap \left(D',[\phi,(n-p_1+1)]\right) \cap \left(D',[(n-q_1+1),\phi]\right) .$$

By Proposition 2.3 and Remark 3) we know for the first term of the right hand side a primary decomposition. The second step of the algorithm is to write the other two terms as intersection of primary ideals. We claim that

$$\left(D',[\phi,(n-p_1+1)]\right) = \bigcap_{n-p_1+1>\ell>1} [(p_1+\ell),\phi],D',[\phi,(n+2-p_1-\ell)])$$

and is a primary decomposition. Of course an analogous result holds for the third term.

The inclusion $\subseteq$ is trivial, on the other hand if $[\sigma,\tau] \notin \left(D',[\phi,(n-p_1+1)]\right)$, is in the intersection then either $\sigma_1 > p_1+\ell$ for all ℓ or $\sigma_1 = p_1+\ell$ and $\tau_1 > n+2-p_1-(\ell+1)$, so in both cases $\sigma_1+\tau_1 > n+1$ thus $[\sigma,\tau] \in D_0 \subseteq D$, contradiction. Now fix ℓ and define p and q as in Proposition 2.3 3), then

$$p+q < (p_1+\ell) + (n+2-p_1-\ell) < n+2 \ , \text{ and } \ n-p+2 > n+2-p_1+\ell \quad \text{and} \quad n-q+2 > p_1+\ell$$

so a) and b) of Proposition 2.3 3) are satisfied, while c) is trivial. Thus the decomposition is primary. To refine a primary decomposition into an irredundant decomposition the next result may be useful to intersect primary ideals that belong to the same prime ideal:

$$\big([\sigma^1,\tau^1],\dots,[\sigma^s,\tau^s]\big) \cap \big([\tilde\sigma^1,\tilde\tau^1],\dots,[\tilde\sigma^t,\tilde\tau^t]\big) = \big([\sigma^i \cup \tilde\sigma^j, \tau^i \cup \tilde\tau^j]\big)_{\substack{1\leq i\leq s\\ 1\leq j\leq t}}$$

Now we will describe the symbolic powers of G-invariant prime-ideals. Let

$$J_{p,q} = I\big([(p),\phi],[\phi,(q)]\big) \quad \text{with} \quad p+q < n+2 \ .$$

The m-th symbolic power of $J_{p,q}$ is defined as

$$J^{(m)}_{p,q} = \{f \in R \mid \exists\ s \notin J_{p,q} \text{ such that } s\cdot f \in J^m_{p,q}\} \ .$$

PROPOSITION 2.6. <u>The D-ideal</u> D <u>corresponding to</u> $J^{(m)}_{p,q}$ <u>is generated by</u>

$$[\phi,(n-p+2)] \ , \ [(n-q+2),\phi] \ \ \underline{\text{and all}} \ \ [\sigma,\tau] \ \ \underline{\text{with}} \ \ \gamma_p(\sigma) + \gamma_q(\tau) = m \ .$$

<u>Proof.</u> Let D' be the D-ideal corresponding to

$$J^m_{p,q} = \big(I_{[(p),\phi]}+I_{[\phi,(q)]}\big)^m = \sum_{i=0}^{m} I^i_{[(p),\phi]}\cdot I^{m-i}_{[\phi,(q)]} = \sum_{i=0}^{m} A_{[(p)^i,(q)^{m-i}]} \ .$$

Since $k_{[\sigma\sigma',\tau\tau']} = k_{[\sigma,\tau]}\cdot k_{[\sigma',\tau']} \in J^m_{p,q}$ and $k_{[\sigma,\tau]} \notin J_{p,q}$ implies $k_{[\sigma',\tau']} \in J^{(m)}_{p,q}$ it follows that $[(p-1),(q-1)]^d\cdot[\sigma,\tau] \in D'$ implies $[\sigma,\tau] \in D$. This gives immmediate $[(n-q+2),\phi]$ and $[\phi,(n-p+2)] \in D$, and taking d sufficiently large, that $[\sigma,\tau] \in D$ if $\gamma_p(\sigma)+\gamma_q(\tau) = m$. These elements span a D-ideal D" . It is easy to check that D" is primary. Since $D \supseteq D'' \supseteq D'$ and $J^{(m)}_{p,q}$ is the smallest $J_{p,q}$-primary ideal containing $J^m_{p,q}$ it follows that $D = D''$. □

We conclude with the description of integral closures of G-invariant ideals. First we state a special case:

PROPOSITION 2.7. The integral closure of $I_{[\sigma,\tau]}$ is $A_{[\sigma,\tau]}$.

Proof. In [1] it is proved for the ring R_{SX} that $A_\sigma = \bigoplus_{\sigma'>\sigma} M_{[\sigma',\phi]}$ is the integral closure of $I_\sigma = R_{SX}\cdot M_{[\sigma,\phi]}$ in R_{SX} . Because $I_\sigma \subseteq A_\sigma$ this implies $A_\sigma^m = I_\sigma\cdot A_\sigma^{m-1}$ for m large enough. After multiplying with R^m we get $A^m_{[\sigma,\phi]} = I_{[\sigma,\phi]}\cdot A^{m-1}_{[\sigma,\phi]}$. Similarly $A^m_{[\phi,\tau]} = I_{[\phi,\tau]}\cdot A^{m-1}_{[\phi,\tau]}$, for m large enough. Because $A_{[\sigma,\tau]} = A_{[\sigma,\phi]}\cdot A_{[\phi,\tau]}$ and $I_{[\sigma,\tau]} = I_{[\sigma,\phi]}\cdot I_{[\phi,\tau]}$ we get for m large enough $A^m_{[\sigma,\tau]} = I_{[\sigma,\tau]}\cdot A^{m-1}_{[\sigma,\tau]}$. But then, see [7], $A_{[\sigma,\tau]}$ is integral over $I_{[\sigma,\tau]}$.
In order to prove that $A_{[\sigma,\tau]}$ is the integral closure of $I_{[\sigma,\tau]}$ it is sufficient to prove that each

$$f = \sum_{[\sigma',\tau']\not\geq[\sigma,\tau]} f_{[\sigma'.\tau']} \neq 0 \ , \ f_{[\sigma',\tau']} \in M_{[\sigma',\tau']}$$

and almost all zero, cannot be integral over $I_{[\sigma,\tau]}$.
Suppose f is integral over $I_{[\sigma,\tau]}$, thus $f^m+a_1f^{m-1}+\ldots+a_{m-1}f+a_m=0$ for some $a_i \in I^i_{[\sigma,\tau]}$ $i=1,\ldots,m$. As in the prove of Proposition 2.4. there is a minimal $[\sigma',\tau']$ in the order $<_\ell$ with $f_{[\sigma',\tau']} \neq 0$, furthermore there exists a k such that $\gamma k(\sigma') < \gamma k(\sigma)$ or $\gamma k(\tau') < \gamma k(\tau)$. Then the only contribution in $M_{[\sigma',\tau']^m}$ of the sum $f^m+a_1f^{m-1}+\ldots+a_m$ comes from $f^m_{[\sigma',\tau']}$ and is non-zero by lemma 2.5, contradicting the assumption.

We will now define the integral closure of any D-ideal and prove that it agrees with the same notion on G-invariants ideals. As is [2] we order the $\mathbb{R}^n$ by the definition $(a_1,\ldots,a_n) < (b_1,\ldots,b_n)$ iff $a_i < b_i$ for all i. Define maps γ_1 and γ_2 on the set of pairs of diagrams $[\sigma,\tau]$ with $\sigma_1+\tau_1 < n$ into $\mathbb{R}^n$ by $\gamma_1[\sigma,\tau] = \big(\gamma_1(\sigma),\ldots,\gamma_n(\sigma)\big)$ and $\gamma_2[\sigma,\tau] = \big(\gamma_1(\tau),\ldots,\gamma_n(\tau)\big)$.

We define the integral closure $\overline{D}$ of a D-ideal by

$$\overline{D} = \{[\sigma,\tau] \mid \gamma_1[\sigma,\tau] > x \text{ for some } x \text{ in the convex hull of } \gamma_1(D\setminus D_0)\} \cap$$
$$\{[\sigma,\tau] \mid \gamma_2[\sigma,\tau] > y \text{ for some } y \text{ in convex hull of } \gamma_2(D\setminus D_0)\} .$$

PROPOSITION 2.8 *Let* D *be a D-ideal. The integral closure of* I(D) *is* $I(\overline{D})$.

Proof. Using $A_{[\sigma,\tau]} = A_{[\sigma,\phi]} \cdot A_{[\phi,\tau]}$, the proof runs along the same line as the proof in [2] of Theorem 8.2. □

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