MULTIGRID AND IMPROVED ACCURACY FOR THE EULER EQUATIONS

P.W. Hemker

CWI, Centre for Mathematics and Computer Science
P.O.Box 4079, 1009 AB Amsterdam, The Netherlands

ABSTRACT

In this paper we describe high resolution discretisations for the 2-dimensional steady Euler equations. Two new schemes (superbox and starbox) are presented. The second order discrete equations are solved with the help of a specially efficient first order multigrid solver. Moreover, a tau-extrapolation technique is applied to further improve the accuracy. In this way an iterative procedure is developed to find, during the solution process, an increasingly more accurate approximation to the solution (higher than 2nd order if an asymptotic expansion exists). A few basic multigrid cycles are usually sufficient to obtain the final approximation up to truncation error accuracy.

1. DISCRETISATION PROCEDURES

Neglecting heat conduction, the flow of an inviscid gas is described by the Euler equations. In two dimensions these equations are

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} f(q) + \frac{\partial}{\partial y} g(q) = 0 , \tag{1.1}$$

with

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix} , \quad f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u e + u p \end{bmatrix} , \quad g = \begin{bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v e + v p \end{bmatrix} , \tag{1.2}$$

where ρ , u , v , e and p represent density, velocity in x- and y- direction, specific energy and pressure respectively. The pressure is obtained from the equation of state, which - for a perfect gas - is given by

$$p=(\gamma-1)\rho(e-\tfrac{1}{2}(u^2+v^2)),$$

γ is the ratio of specific heats. $q(t,x,y)$ describes the state of the gas as a function of time and space and f and g are the fluxes in the x- and y- direction. We denote the open domain of definition of (1.1) by Ω^* .

To allow discontinuous solutions, (1.1) can be rewritten in its integral form

$$\frac{\partial}{\partial t}\iint_{\Omega} q\, dx\, dy \;+\; \int_{\partial\Omega}(f.n_x + g.n_y)\, ds \;=\; 0\,, \tag{1.3}$$

for all $\Omega\subset\Omega^*$;
$\partial\Omega$ is the boundary of Ω and (n_x,n_y) is the unit outward normal vector at the boundary $\partial\Omega$.

The form (1.3) of equation (1.1) shows the character of the system of conservation laws: the increase of q in Ω is caused only by the inflow of q over $\partial\Omega$. In symbolic form we write (1.3) as

$$q_t + N(q) = 0. \tag{1.4}$$

In this paper we are only interested in the solution of the steady state equation

$$N(q) = 0. \tag{1.5}$$

The solution of the weak form (1.3) of (1.1) is known to be non-unique and a physically realistic solution (which is the limit of a flow with vanishing viscosity) is known to satisfy the additional entropy condition (cf. [1,2]). Further, the equation (1.1) is hyperbolic, i.e. written in the form

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial y} = 0$$

the matrix

$$k_1\frac{\partial f}{\partial q} + k_2\frac{\partial g}{\partial q}$$

has real eigenvalues for all (k_1,k_2) and the 4 eigenvectors span the 4-D state space.

These eigenvalues are $(k_1u+k_2v)\pm c$ and (k_1u+k_2v) (a double eigenvalue), where $c = \sqrt{\gamma p/\rho}$ is the local speed of sound. The sign of these eigenvalues determines the direction in which the information about the solution is carried along the line (k_1,k_2) as time develops. It locates the domain of dependence. The entropy condition implies that characteristics do not emerge at a discontinuity in the flow.

In order to discretise (1.1) or (1.3) on a domain with an irregular grid, we use a finite volume technique. The domain of definition, Ω^*, is divided into a number of disjunct cells $\{\Omega_\alpha\}$ and equation (1.3) is required to hold on each $\bigcup\Omega_\alpha$. In this way the conservation character is preserved, provided that we take the same approximation for the flow quantities $\int_{\Gamma_{\alpha\beta}} f.n_x + g.n_y\, ds$, both for the computation of the outflow of Ω_α and for the inflow of Ω_β for any two

neighbouring cells Ω_α and Ω_β with $\Gamma_{\alpha\beta} = \partial\bar{\Omega}_\alpha \cap \partial\bar{\Omega}_\beta$. In that case (1.3) holds for any Ω which is the union of an arbitrary subset of $\{\Omega_\alpha\}$. In this method there is no need to transform the equations (1.1) or the boundary conditions with respect to the independent variables (x,y).

In order to generate a nested sequence of discretisations with decreasing meshwidth for our multigrid solution procedure, and to minimise the administrative overhead of the computer code, we found it most convenient to to divide the domain Ω^* in quadrilateral cells Ω_{ij}, in a way topologically equivalent with a regular division in squares. Then $\Omega_{i\pm1,j\pm1}$ are the only possible neighbours of Ω_{ij} . In this way a mapping is conceived from a rectangular and regular "computational domain" to the irregular physical domain. This mapping plays a role only for the generation of the sequence of refining irregular grids, and in the proofs of the accuracy of the resulting schemes. We take this mapping non-singular (i.e. with non-vanishing Jacobian J on $\bar{\Omega}^*$) and sufficiently smooth (with bounded partial derivatives of J). The mapping determines the vertices of the quadrilateral cells in the physical space, where all cell-edges are straight line segments. As a consequence any domain Ω^* with a curved boundary is approximated by a polygonal domain, but in subsequent refinements the curved lines are better approximated.

The discrete approximation q_h of $q(x,y)$ is represented by a (vector-) value q_{ij} for each Ω_{ij}. This q_{ij} is associated with the mean value of q over Ω_{ij}. Using (1.3), the space discretisation method is now completely determined by the method of approximation of

$$\int_{\Gamma_{ijk}} (f.n_x + g.n_y)\, ds\,, \qquad k = N,E,S,W, \tag{1.6}$$

at the four walls Γ_{ijk} of the cell Ω_{ij}. The wall Γ_{ijk} may be either a common boundary with another cell Ω_{ijk} or a part of the boundary $\partial\Omega^*$. In both cases the approximation of (1.6) is computed as

$$f^k(q^k_{ij}, q^k_{ijk}) \cdot \mathrm{meas}(\Gamma_{ijk}), \tag{1.7}$$

i.e. at each Γ_{ijk} we approximate $fn_x + gn_y$ by a constant value , which depends only on q^k_{ij}, an approximation to $q(x,y)$ in Ω_{ij} at the edge Γ_{ijk}, and also on q^k_{ijk}, an approximation to $q(x,y)$ in the neighbouring cell Ω_{ijk} at Γ_{ijk}. (Notice that we allow the approximation of q to be discontinuous over Γ_{ijk}.)

The semi-discretisation of the equation (1.5) is now the set of non-linear equations

$$N_h(q_h)\,|_{i,j} := \sum_{k=N,E,S,W} f^k(q^k_{ij}, q^k_{ijk})\ \mathrm{meas}(\Gamma_{ijk}) = 0\ , \tag{1.8}$$

for all (i,j) with $\Omega_{ij} \subset \Omega^*$.

The approximate flux $f^k(q^k_{ij}, q^k_{ijk})$ depends on q_h, the approximation of q, near Γ_{ijk} and also on the direction (n^k_x, n^k_y) of the edge Γ_{ijk}. However, by the rotation invariance of the Euler equations, we may relate $f^k(.,.)$ to a local coordinate system, rotated such that it is aligned with Γ_{ijk}. Then this computation comes down to

$$f^k(q_{ij}^k, q_{ijk}^k) = T_k^{-1} f(T_k q_{ij}^k, T_k q_{ijk}^k)$$

Here $f(.,.)$ is a *numerical flux function,* independent of the orientation of Γ_{ijk}. The operator T_k takes care of the local rotation of the coordinate system at Γ_{ijk}. If $(n_x^k, n_y^k) = (1,0)$, then T_k is the identity operator. In this way only a single function $f(.,.)$ is needed to approximate the flux between two cells.

The description of the discretisation is completed by the choice of the numerical flux function and by the determination of q_{ij}^k and q_{ijk}^k from $\{q_{ij} \mid \Omega_{ij} \subset \Omega^*\}$.

For consistency of the resulting scheme, $f(.,.)$ should satisfy $f(q,q) = f(q)$, cf. [3]. A usual representation of $f(.,.)$ is given by

$$f(q_0, q_1) = \tfrac{1}{2} f(q_0) + \tfrac{1}{2} f(q_1) - \tfrac{1}{2}\, d(q_0, q_1). \tag{1.9}$$

A central difference flux is defined by $d(q_0, q_1) = 0$. For reasons explained in [4], in our multigrid procedure we use a slight modification of an upwind numerical flux function that was proposed by Osher [5,6] ; viz. (1.9) with $d(.,.)$ defined by

$$d(q_0, q_1) = \int_{q_0}^{q_1} \left|\frac{\partial f}{\partial q}\right|(w)\, dw\,, \tag{1.10}$$

where the integration path in the state space follows three sub-paths along the eigenspaces of $\partial f / \partial q$. These sub-paths correspond to the eigenvalues $\lambda_1 = u - c$, $\lambda_2 = \lambda_3 = u$, and $\lambda_4 = u + c$ respectively. This implies that, except at a location of a shock, we can write $f(q_0, q_1) = f(q^*)$, where q^* is connected with q_0 by the Riemann invariants corresponding to $\lambda \geqslant 0$, and q^* is connected with q_1 by the Riemann invariants for $\lambda \leqslant 0$. In the case of a shock we find a numerical flux of the form $f(q_0, q_1) = f(q_1^*) - f(q_2^*) + f(q_3^*)$, for some proper q_k^*, $k = 1,2,3$.

At the boundary of the domain Ω^*, interpolation of q_h from the interior of Ω^* yields a value q_{ijk}^{IN} at that boundary, which corresponds to a mean value of q at Γ_{ijk} in Ω_{ij}. For well posed boundary conditions $B(q) = 0$ at Γ_{ijk}, a value q_{ijk}^{OUT} can be determined such that it satisfies the boundary conditions and is connected to q_{ijk}^{IN} by the Riemann invariants for $\lambda < 0$ at the left (or $\lambda < 0$ at the right) boundary. This implies that

$$f^k(q_{ijk}^{OUT}, q_{ijk}^{IN}) = T_k^{-1} f(T_k\, q_{ijk}^{OUT}), \tag{1.11a}$$

and

$$B(q_{ijk}^{OUT}) = 0\,, \tag{1.11b}$$

is satisfied at a point on Γ_{ijk}. For details see [4].

The basic first order scheme.

For our first order scheme we use a piecewise constant numerical approximation for q for each quadrilateral cell:

$$q_h(x,y) = q_{ij} \qquad \text{for} \quad (x,y) \in \Omega_{ij}. \tag{1.12a}$$

This uniform state in Ω_{ij} is assumed for all (i,j), and hence

$$f^k(q_{ij}^k, q_{ijk}^k) = f^k(q_{ij}, q_{ijk}) . \tag{1.12b}$$

The flux at Γ_{ijk} now corresponds with the flux at a discontinuity between two uniform states. Such a flux can be computed by solving the Riemann problem of gasdynamics (i.e. Godunov's method). However, this is a nontrivial nonlinear computation, and we approximate it by (1.9) - (1.10), which is essentially Osher's "approximate Riemann solver".

The order of accuracy of the resulting schemes on the nonuniform mesh is not immediate. It can be proved that, in this way, at most second order accuracy can be obtained for irregular grids, when q_{ij}^k, q_{ijk}^k are computed properly. With the approximation (1.12.b) the scheme is first order accurate.

For a function q, the truncation error is defined as

$$\tau_h(q) = N_h(R_h q) - \bar{R}_h N(q) , \tag{1.13}$$

where the restriction operators R_h and $\bar{R}_h$ take mean values over cells. For (1.12.b) and a smooth q it can be shown that $\tau_h(q) = \mathcal{O}(h)$ for $h \to 0$. We denote the first order discrete equations (1.8) - (1.12) in symbolic form by

$$N_h^1(q_h) = 0 . \tag{1.14}$$

This first order discretisation is conservative, satisfies an entropy condition, is monotonous and gives a sharp representation of discontinuities (shocks and contact discontinuities), as long as these are aligned with the mesh. Further it allows an efficient solution of the discrete equations by a multigrid method [7] . Disadvantages are the low order of accuracy and the fact that it is highly diffusive for oblique discontinuities. For a first order (upwind) scheme these are well known facts and it leads to the search for other grids and higher order methods.

The superbox scheme.

A property that we want to maintain in a second order scheme, is the conservation of q, because it allows discontinuities to be captured as weak solutions of (1.1). Therefore, we consider only schemes that are still based on (1.8), and we select $f^k(q_{ij}^k, q_{ijk}^k)$ to yield a better approximation to (1.6) than (1.12b).

A straightforward way to form a more accurate approximation is to replace the 1st order approximation (1.12) by a 2nd order one. Instead of the piecewise constant approximation for $q(x,y)$, we now use a piecewise bilinear function $q_h(x,y)$ on a set of 2×2 cells (a "superbox"). Such a superbox on the h-level corresponds with a single cell at the 2h-level. Over the boundaries of the superbox, the trial function $q_h(x,y)$ can be discontinuous again. Inside the superbox it is determined by q_{ij}, $q_{i+1,j}$, $q_{i,j+1}$ and $q_{i+1,j+1}$, the mean values over the sub-cells of the superbox (see figure 1). Using the bilinear function, we see that a central difference approximation is used for the four flux computations inside the superboxes; at superbox boundaries interpolation is made from the inside of the superbox by the bilinear interpolation. In this way the

values $\{q_{ij}^k\}$ are computed and the approximate Riemann solver is used to compute the numerical flux. We denote the corresponding discrete operator by N_h^{2S}. It is easily shown that this *superbox scheme* is 2nd order accurate in the sense that

$$\overline{R}_{2h,h}(N_h^{2S}(R_h q) - \overline{R}_h N(q)) = \mathcal{O}(h^2) .$$

We elucidate this superbox discretisation by its 1-dimensional analogue. Let $\{q_j\}_{j=1,\dots,2N}$ denote states in the cells $\{\Omega_j\}$ with barycentres at x_j and walls between cells at $x_{j+1/2}$. Superboxes are $\Omega_i^B = \Omega_{2i-1} \bigcup \Omega_{2i}$, $i = 1,2,\dots,N$. For the computed fluxes at $x_{j+1/2}$ we have to distinguish between odd and even j. At an internal wall $x_{2i-1/2}$ we find the interpolated value $q_{2i-1/2} = (q_{2i-1}+q_{21})/2$ and at a superbox wall $x_{2i+1/2}$ we have interpolated values from the left, $q_{2i+1/2}^L$, and from the right, $q_{2i+1/2}^R$. We define $\Delta q_{j+1/2} = q_{j+1} - q_j$ and find the 2nd order one-sided interpolated values

$$q_{2i+1/2}^L = q_{2i} + \frac{1}{2}\Delta q_{2i-1/2} , \tag{1.15}$$

$$q_{2i+1/2}^R = q_{2i+1} - \frac{1}{2}\Delta q_{2i+3/2} ,$$

The computed fluxes at $\{x_{j+1/2}\}$ now are $f_{2i-1/2} = f(q_{2i-1/2} + \Delta q_{2i-1/2})$ and $f_{2i+1/2} = f(q_{2i+1/2}^L , q_{2i+1/2}^R)$. We notice that the states $q_{2i+1/2}^L$ and $q_{2i+1/2}^R$ are well determined, even near the boundary $\partial\Omega\star$, so that the procedure to compute the boundary fluxes is also competely described by (1.11).

Star schemes

A disadvantage of the quadrilateral cells in a more complex flow configuration can be the existence of (only) two special directions in which possible discontinuities are allowed. Discontinuities aligned with the gridlines are easily represented by q_h, whereas discontinuities skew to the meshlines are hard to resolve. Although higher order approximations may relieve this situation [8] , they can lead to too large spurious extrema (a Gibbs-like effect). Some improvement of this situation may be expected if more mesh-line directions are introduced. For this purpose we constructed a mesh with triangular cells. In the "computational domain" these triangles are formed by refinement of the quadrilaterals as previously described, in such a way that each quadrangle is divided into 4 triangles by connecting its barycentre with the vertices (see figure 1). For the new first order scheme based on these triangles (the *1st-order star scheme*), uniform states are assumed in all triangular cells (i.e. $q_h(x,y)$ is a piecewise constant function over the triangles), and the discrete operator is defined by

$$N_h^{1*}(q_h)\,|_\alpha := \sum_{k=1,2,3} f^k(q_\alpha, q_{\alpha,k}) \operatorname{meas}(\Gamma_{\alpha,k}) = 0 , \tag{1.16}$$

where the summation is over the 3 sides of the triangular cell Ω_α; $q_{\alpha,k}$ is the state in cell $\Omega_{\alpha,k}$, the k-th neighbour triangle of Ω_α.

For the second order scheme based on these triangles (the *2nd order star-scheme*), we introduce a linear representation of $q_h(x,y)$ in each set of 4

triangles that were all formed in the same quadrangle. This interpolation is made such that

$$q_h(x_{2i},y_{2j}) = (q_{2i+2/3,2j} + q_{2i-2/3,2j} + q_{2i,2j+2/3} + q_{2i,2j-2/3})/4 ,$$

$$\frac{\partial q_h}{\partial x} = \frac{q_{2i+2/3,2j} - q_{2i-2/3,2j}}{x_{2i+2/3,2j} - x_{2i-2/3,2j}} ; \qquad \frac{\partial q_h}{\partial y} = \frac{q_{2i,2j+2/3} - q_{2i,2j-2/3}}{x_{2i,2j+2/3} - x_{2i,2j-2/3}} .$$

The indices correspond with the barycentres of the triangles in the computational domain. By this interpolation all the values at the cell walls, $\{ q_\alpha^k \mid \Omega_\alpha \subset \Omega ; k = 1,2,3\}$, are determined, and the discrete scheme reads

$$\begin{aligned} N_h^{2^*}(q_h) \mid_\alpha &:= \sum_{k=1,2,3} f^k(q_\alpha^k, q_{\alpha,k}^k) \operatorname{meas}(\Gamma_{\alpha,k}) \\ &= \sum_{k=1,2,3} T_k^{-1} f(T_k q_\alpha^k, T_k q_{\alpha,k}^k) \operatorname{meas}(\Gamma_{\alpha,k}) = 0 . \end{aligned} \tag{1.17}$$

2. MULTIGRID ALGORITHMS AND TAU-EXTRAPOLATION

The usual way to find the solution of the steady state equations

$$N_h(q_h) = 0 , \tag{2.1}$$

is to integrate for $t \to \infty$, the semi-discrete system of equations

$$(q_h)_t + N_h(q_h) = 0 , \tag{2.2}$$

i.e. to take an initial guess and to compute $q_h(t)$ until initial disturbances have sufficiently died out. The advantage is that, starting with a physically meaningful situation, we expect that a meaningful steady state will be reached, even when unicity of the steady equations is not guaranteed. The drawback is that many timesteps may be necessary before the solution has sufficiently converged. For the acceleration of the convergence, many devices have been developed such as local time stepping, residual smoothing, implicit residual averaging or enthalpy damping [9] .

Multigrid is also used as an acceleration device for (2.2) [10, 11, 9] . In that approach space-discretisations (2.1) are given on a sequence of grids. The coarse grids are used to move low frequency disturbances rapidly out of the domain Ω^* by large timesteps, whereas high frequency disturbances are locally damped on the fine grids by sufficiently dissipative timestepping. This accelerated procedure still may require a reasonably large number of steps (>100).

For the multigrid solution of (2.1), we take another approach [12, 13, 4] . We consider directly the steady state equations and we first construct a sequence of nested 1st order discretisations as described in the previous section. By the stability of the first order discretisation, a relatively simple relaxation method (Collective Symmetric Gauss Seidel (CSGS) iteration, i.e. a SGS relaxation where the 4 variables corresponding to a single cell Ω_{ij} are relaxed

collectively) is able to reduce the high frequency error components efficiently, and -therefore- a FAS multigrid algorithm with this relaxation is well suited to solve the discrete first order equations [7] .

Although no explicit artificial viscosity is added to the scheme, a suitable amount of "numerical diffusivity" is automatically introduced by the upwind discretisation. As $h \to 0$, this "artificial diffusion" vanishes and the sequence of discretisations converges to the Euler equations as the limit of an equation with vanishing viscosity. This also motivates us to apply a simple mesh-continuation procedure (FMG) for $h \to 0$, to find the initial estimates for the FAS multigrid iteration. It appears that, under quite general circumstances, the convergence factor of a FAS iteration cycle (with CSGS-relaxation and $p = q = s = 1$ as described in [7]) ranges between 0.4 and 0.85, so that from n=1 to n=8 of these FAS cycles suffice to compute the 1st order accurate solution up to truncation error accuracy (see e.g. Chapter 5 in [14]). The complete Full Multigrid (FMG) method to compute the 1st order approximation is given in the following ALGOL68 procedures, cf. [15] .

```
proc FMG = ( int n,L, ref [] field q, r) void:
begin
  for m to L
  do
    to n do FAS ( m,q,r) od;
    if m<L
    then q[m+1]:= interpolation q[m]
    fi
  od
end;

proc FAS = ( int m, ref [] field q,r) void:
begin
  to p do relax (q[m],r[m]) od;
  if m>1
  then
    field qcoarse = restriction q[m];
    q[m-1]:= qcoarse;
    r[m-1]:= euler q[m-1]+ restrict (r[m] - euler q[m]);
    to s do FAS ( m-1, q, r) od;
    q[m]:= q[m] + prolongation (q[m-1] - qcoarse)
  fi;
  to q do relax (q[m], r[m] ) od
end;
```

In these programs q[1],...,q[L], and r[1],...,r[L] are sequences of grid functions with increasingly finer meshes; q for the solution and r for the right-hand-side. A call to the procedure relax(q_h, r_h) has the effect of a relaxation sweep for the improvement of the approximate solution q_h of $N_h^1(q_h) = r_h$. The operators

euler , **restrict** , **restriction** and **prolongation** correspond with the operators N_h, $\overline{R}_{2h,h}$ or $R_{2h,h}$ and $P_{h,2h}$ in the mathematical notation.

The procedure FMG assumes an initial estimate q[1] of the solution on the coarsest grid. In r[1], ..., r[L] it takes as input the right-hand side of (2.1), i.e. it should be initialized at zero. After a call it delivers in q[L] an approximate solution on the finest grid. The procedure FAS updates the approximation at the level m in q[m] by a FAS-cycle. By a single call of FMG and -depending on the problem- a few subsequent calls of FAS at the level L we obtain the 1st order discrete solution up to truncation error accuracy.

If we try to solve the 2nd order discretisation (2.1) in the same manner as we do the first order equations, we may expect difficulties for two reasons. First, the construction of the set of 4 equations to be solved at each cell Ω_{ij} in the CSGS relaxation is much more complex . Secondly, the nonlinear equations (2.1) are less stable. The 2nd order discretisations are less diffusive, (in the case of central differences clearly "anti-diffusive" [16]). This may lead not only to non-monotonous solutions, but it can also cause a Gauss Seidel relaxation not to reduce sufficiently the rapidly varying error components.

To obtain 2nd order accurate solutions, we do not try to solve the system

$$N_h^2(q_h) = 0 \tag{2.3}$$

as such. We use the first order operator N_h^1 to find the higher order accurate approximation in a defect correction iteration:

$$N_h^1(q_h^{(1)}) = 0 \,, \tag{2.4.a}$$

$$N_h^1(q_h^{(i+1)}) = N_h^1(q_h^{(i)}) - N_h^2(q_h^{(i)}). \tag{2.4.b}$$

For an introduction to the defect correction principle see [17] . By a well-known technique [18] , it can be proved that -if the problem is smooth enough- the accuracy of $q_h^{(i)}$ is of order 2 for $i \geqslant 2$.

In fact we may use $q_h^{(i+1)} - q_h^{(i)}$ as an error indicator. In the smooth parts of the solution $q_h^{(1)} - q_h^{(1+i)} = \mathcal{O}(h)$, $q_h^{(2)} - q_h^{(2+i)} = \mathcal{O}(h^2)$; where these differences are larger, e.g. $\mathcal{O}(1)$, the solution is not smooth (relative to the the grid used). Then grid adaptation is to be considered rather than the choice of a higher order method, if a more accurate solution is wanted.

In a multigrid environment, where solutions on more grids are available, we can consider other approaches to compute more accurate solutions, such as (1) Richardson extrapolation or (2) τ-extrapolation. Both extrapolation methods can be well used to find a more accurate solution if the solution is smooth indeed. A drawback is that these methods rely on the existence of an asymptotic expansion of the (truncation) error for $h \to 0$, and -globally- no a-priori information about the validity of this assumption is available.

Since the evaluation of $N_h^2(q_h)$ is hardly more expensive than the evaluation of $N_h^1(q_h)$, the costs to compute the defect in (2.4.b) is of the same order as the evaluation of the relative truncation error $\tau_{2h,h}(q_h) = N_{2h}^1(R_{2h,h}q_h) - \overline{R}_{2h,h}N_h^1(q_h)$. This makes us to prefer (2.4.b) to τ-extrapolation for the computation of a 2nd order discrete system.

Another disadvantage of extrapolation is that the accurate solution (for Richardson) or the estimate for the truncation error (τ-extrapolation) is obtained at the one-but-finest level and no high resolution of local phenomena is obtained. Whereas we want not only a high order of accuracy, but also an accurate representation of possible discontinuities, in [8] we used Richardson extrapolation (only) as a possibility to find a higher order initial estimate for the iteration process (2.4.b). In the present paper we concentrate for a while on τ-extrapolation to improve the accuracy of the smooth components in the solution as they are obtained from the 2nd order scheme.

Tau extrapolation

Let the nonlinear equation

$$N_h(q_h) = r_h \,, \tag{2.5}$$

with $q_h \in X_h$, $r_h \in Y_h$, be a discretisation of

$$N(q) = r \,, \tag{2.6}$$

where $q \in X$, $r \in Y$, and let the discretisation be such that $r_h = \bar{R}_h r$ and let the operator $N_h : X_h \to Y_h$ satisfy

$$N_h(R_h q) = r_h + \tau_h(q) \,, \tag{2.7}$$

$$\tau_h(q) = h^p \bar{R}_h \tau(q) + \mathcal{O}(h^{\tilde{p}}), \quad \text{for } h \to 0 \,, \tag{2.8}$$

where $R_h : X \to X_h$ and $\bar{R}_h : Y \to Y_h$ are restrictions (linear surjections) and $\tau(q)$ is independent of h. The latter requirement with $\tilde{p} > p$ means that $\tau_h(q)$, the local truncation error for the solution q, satisfies an asymptotic expansion.

Further, assume that we have a sequence of nested discretisations for $h = 2^{-k} h_0$, $k = 1,2,\ldots$; this means that, for h and $2h$ from this sequence, restrictions $R_{2h,h} : X_h \to X_{2h}$ and $\bar{R}_{2h,h} : Y_h \to Y_{2h}$ exist such that $R_{2h,h} R_h = R_{2h}$ and $\bar{R}_{2h,h} \bar{R}_h = \bar{R}_{2h}$.

Then for the *relative truncation error*

$$\tau_{2h,h} := N_{2h} R_{2h,h} - \bar{R}_{2h,h} N_h \,, \tag{2.9}$$

we easily derive

$$\tau_{2h,h} R_h = \tau_{2h} - \bar{R}_{2h,h} \tau_h \,. \tag{2.10}$$

With the procedure as described in section 1 we construct a sequence of nested discretisations for the Euler equations. Under the assumption (2.8), with $\tilde{p} > p$, we can compute $\tau_{2h,h}(q_h)$ and apply τ-extrapolation [19, 14] to improve the accuracy of our solution.

Combining (2.8) and (2.10) we see

$$\begin{aligned} h^p \bar{R}_{2h} \tau &= \frac{1}{2^p - 1} \tau_{2h,h} R_h + \mathcal{O}(h^{\tilde{p}}) \\ &= \bar{R}_{2h,h} \tau_h + \mathcal{O}(h^{\tilde{p}}) \end{aligned} \tag{2.11}$$

$$= 2^{-p'} \tau_{2h} + \mathcal{O}(h^{\tilde{p}}) .$$

:e

$$N_h(R_h q) = r_h + h^p \overline{R}_h \tau(q) + \mathcal{O}(h^{\tilde{p}}) \tag{2.12}$$

$$= r_h + h^p \overline{P}_{h,2h} \overline{R}_{2h} \tau(q)$$

$$+ h^p (I_h - \overline{P}_{h,2h} \overline{R}_{2h,h}) \overline{R}_h \tau(q) + \mathcal{O}(h^{\tilde{p}})$$

$$= r_h + \frac{1}{2^p - 1} \overline{P}_{h,2h} \tau_{2h,h}(R_h q)$$

$$+ (I_h - \overline{P}_{h,2h} \overline{R}_{2h,h}) \tau_h(q) + \mathcal{O}(h^{\tilde{p}}).$$

:onclude that for $\overline{P}_{h,2h}$ and $\tau(q)$ such that
$_{,2h} \overline{R}_{2h,h}) \overline{R}_h \tau(q) = \mathcal{O}(h^{\tilde{p}-p})$ we obtain a discretisation scheme con-
$\mathcal{O}(h^{\tilde{p}})$ if we solve for q_h^* the equation

$$N_h(q_h^*) = r_h + \frac{1}{2^p - 1} \overline{P}_{h,2h} \tau_{2h,h}(q_h^*) . \tag{2.13}$$

hat, with the accuracy restriction, we still have some freedom in the
$\overline{P}_{2h,h}$.

'or a given $\tilde{q}_h \in X_h$ the $\tau_{2h,h}(\tilde{q}_h)$ is easily evaluated from (2.9), the solu-
2.13) is readily found by a defect correction iteration. For the accu-
·oximation of the Euler equations, we apply the τ-extrapolation to the
r discretisation (superbox or starbox)

$$N_h^2(q_h) = 0 . \tag{2.14}$$

, since the efficient solution procedure FMG is available only for the
r discrete system

$$N_h^1(q_h) = r_h ,$$

in an iterative procedurte the equations

$$N_h^1(q_h^{(n+1)}) = N_h^1(q_h^{(n)}) - N_h^2(q_h^{(n)}) + \frac{1}{3} \overline{P}_{h,2h} \tau_{2h,h}(q_h^{(n)}). \tag{2.15}$$

itial step the right-hand-side is taken equal to zero, so that approxi-
ıe first order accurate discrete system is solved. In the *next few* steps
erm in the right-hand-side of (2.15) is replaced by zero. At the end of
ion the formula (2.15) is applied in full.
15) converges, we find an approximate solution $q_h = R_h \tilde{q}$, that

$$N_h^2(R_h \tilde{q}) = \overline{P}_{h,2h} \overline{R}_{2h} \tau_h(\tilde{q}) + \mathcal{O}(h^{\tilde{p}}) . \tag{2.16}$$

thm with increasing accuracy

n at the approximate solution q_h of the discrete Euler equations (2.16)
:n fine mesh and we assume that also $L - 1$ coarser meshes exist. We

denote the level of refinement by m and the approximate solution at level m by q[m] = $q_{2^{L-m}h}$. The coarser grids, $m<L$, are not only used for the realisation of FAS-iteration steps, but also for the construction of the initial estimate for the iteration process. The algorithm used to obtain the initial estimate and further iterands in the defect correction process reads as follows:

```
proc IDEC TAU = ( int L,few,maxit, ref [] field q,r) void:
begin
  initialise at zero (r);
  initialise at zero (q);
  initialise (q[1]);
  FMG2 (2, L, q, r );
  r[L]:= euler1 q[L] - euler2 q[L];
  FMG2 (2, L, q, r );
  for f to maxit
  do
    r[L]:= euler1 q[L] - euler2 q[L];
    if f>few then
      field tau= euler2 restrict q[L] - restrict euler2 q[L];
      r[L]:= r[L] + prolon tau / 3
    fi;
    FAS (L, q, r)
  od
end
```

Here, q and r have the same meaning as in the procedure FMG. The operators **euler1** and **euler2** are respectively the 1st and the 2nd order accurate discrete operators N_h. The procedure FMG2 is a generalisation of FMG for the case that non-trivial initial estimates for q_h and r_h are known on all levels m=1,...,L.

```
proc FMG2 = ( int n,L, ref [] field q,r) void:
begin
  for m from L-1 by -1 to 1
  do r[m] := restrict r[m+1] od;
  for m to L
  do
    to n do FAS (m,q,r) od;
    if m<L
    then
      q[m+1]:= q[m+1] +
      + interpolation (q[m] - restriction q[m+1])
    fi
  od
end;
```

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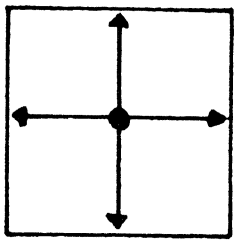


fig. 1a

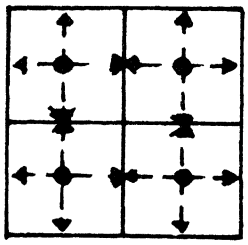
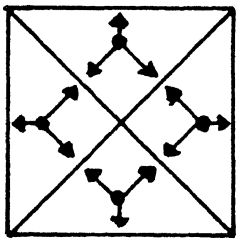


fig. 1b



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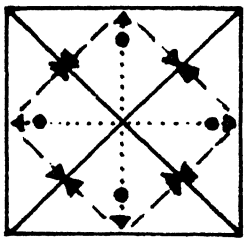

fig. 1d

Figure 1. The computation of q_{ij}^k from q_{ij} for the different schemes.

- ● : q_{ij} values,
- ◀— : q_{ij}^k values,
- — — — : lines of linear interpolation,
- ——— : lines of piecewise constant interpolation.

fig. 1a: the basic 1st order scheme,
fig. 1b: the superbox scheme,
fig. 1c: the 1st order starbox scheme,
fig. 1d: the 2st order starbox scheme.