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]iosyncratic Remarks by a Bibliomaniac: . A Random Sample of Structured Chaos

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his time my theme is fractals and chaos, universality phenomena, and, yes, regularity ı the midst of chaotic behaviour; moreover the chaos is deterministic, generated by the pithome of lawfully prescribed behaviour: a differential or a difference equation, the ort of thing which describes the laws of physics. The theme is also how the monsters f yesterday, those nowhere differentiable continuous curves and such, once thought to xist only as diseased imaginings in the minds of some pure fringe mathematicians, xcessively preoccupied with rigour and foundational matters, got tamed and put to ıse. Or, to put it another way, the tale is about the rescuing of beautiful monsters from lavering princesses - a phrase, which, I believe, comes from somewhere in one of)ouglas Adams' books; it sounds like him anyway.

As several times before, these lines are written as if I were sitting in a bookshop, humbing through a number of volumes and trying to decide which ones to buy, which ɔnes to recommend, and of which ones to make a note in a small book with a view :owards later purchase in the event of my salary or the library budget ever catching up with current book prices. This time also a sense of urgency is not entirely absent — other people might get in their recommendations first, or, God forbid, the next installment of worthwhile books might be even more expensive.

Let's start with a curious and outlandish problem. Is it possible to divide a plane into three regions in such a way that every boundary point between any two of the regions is also a boundary point of the third region? The first solution was given by L.E.J. Brouwer, and the positive answer is one of those weird infinite-repeat-limit constructions that make more applied (serious?) minded people (mathematicians and scientists?) throw up their hands in disgust at the silly games some people indulge in. Of course, they might already have done so at the question itself.

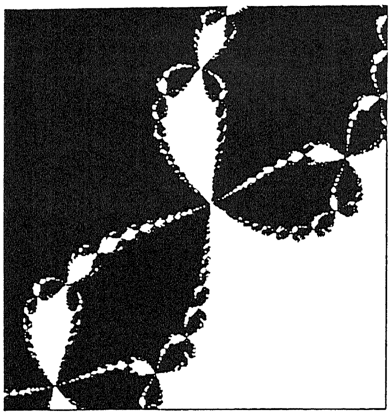
However, nature is extraordinarily kind — if one knows where to look, and what questions to ask —, and in this case provides us with an abundance of entirely natural processes which yield just such a division of the plane (for any n not just n=3). Indeed, let us have a look at Newton's well-known iteration procedure to find the roots of a polynomial:

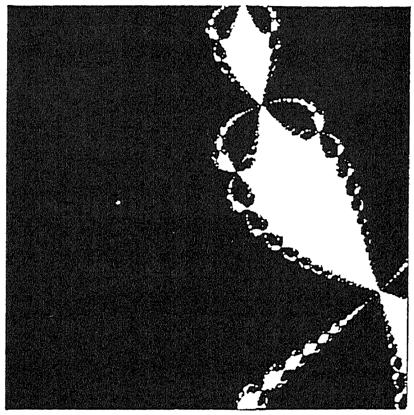
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{1}$$

And let's apply this not to the usual situation where one aims to find the real roots of a

[1] Previous instalments of this column can be found in the issues 1:3, 7:3, 10:3 (Ian Stewart), 11:3 (Bob Hermann).

real polynomial, but, instead, view equation (1) as an iteration procedure which can be applied to any initial complex number x_0 in the complex plane. Assuming that $f(x)$ has n different complex roots, there will result n basins of attraction of these roots and these form a division of the complex plane like the one we wanted to have above, for it is a general theorem of Julia that in this case the boundary of each of the n regions is also the boundary of each of the n-1 other regions. In particular, if the recipe is applied to a polynomial of degree three with three distinct complex roots, for instance the polynomial x^3-1, then there results such a division into three parts. The book **10** below contains a number of fascinating pictures of just this situation. The two pictures[2] below show (in white) two parts of the attractor of the point (1,0) for the Newton process (1) for the polynomial x^3-1. The right picture belongs on top of the left one.





The boundary of one of the basins of attraction — because of what has been said, it does not matter which one is taken — is called the *Julia set* of the corresponding iteration process given by a rational map. These Julia sets also tend to be fractals, which, whatever they are precisely — for that see several of the books listed or reviewed below; here I will content myself with a number of pictures — are in any case a magnificent describing tool, and provide the fundamental building blocks of a second 'dual' geometry to the more familiar geometry of circles, spheres, planes, and lines: while these last-named objects have high symmetry in space, 'ideal' fractals have scaling symmetry: small parts of them, when magnified, look exactly like the original. And, perhaps, just as the more general nice smooth surfaces of differential geometry are locally quite like planes, so are more general fractals locally (in some suitable sense) like the regular fractals constructed by the familiar generating processes which

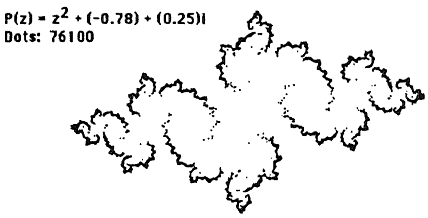
[2] The pictures in this review were produced with a Macintosh II + Laserwriter, using three public domain programs called Super Mandelzoom (by R.Munafo) and Julia and Fractal (by Richard Koch), the commercial paint program Graphics Works, and some ad hoc programming by my son Maarten and myself using Lightspeed Pascal; for the article itself I used the wordprocessor WriteNow and the formula processor Formulator.

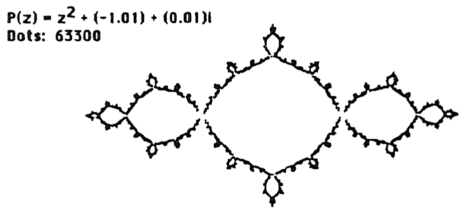
yield Koch islands and Sierpinsky cheeses.

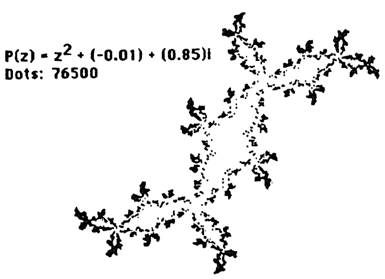
The four pictures below all come from the family of iterative procedures

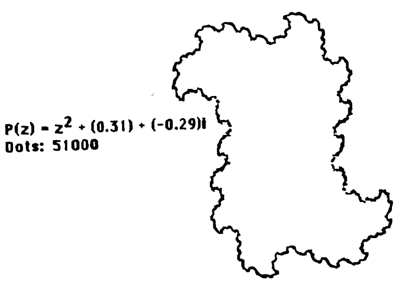
(2) $$x_{k+1} = x_k^2 + c$$

for various values of the parameter c.



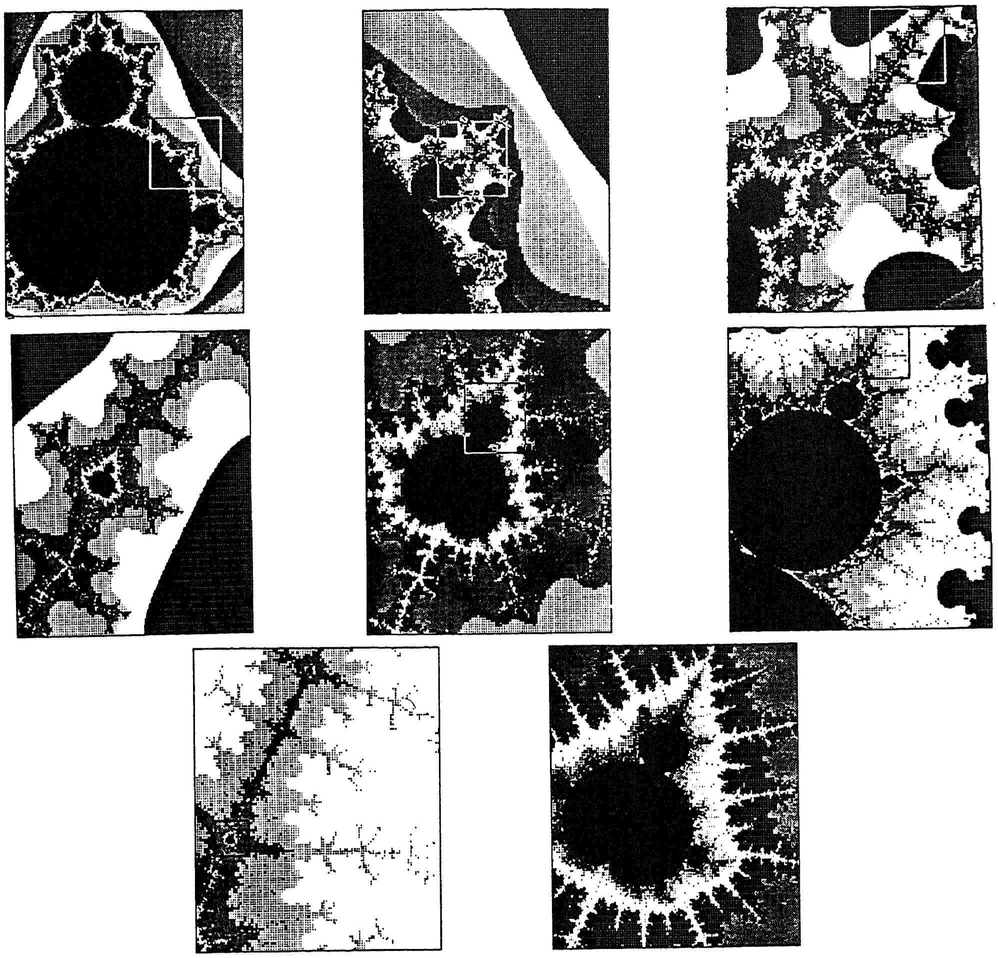






A great deal of information about the behaviour of the iterative process (2) is encoded in the famous Mandelbrot set M, occasionally referred to as the Mandelbrot brain. This set can be defined as the set of those values of the parameter c for which the corresponding Julia set is connected, or, equivalently, as the set of those c for which the orbit of the starting point 0 remains bounded. A few pictures of the Mandelbrot set can be found below. The first of these is a picture of the set M itself. Each of the other seven, reading from left to right — and using wrap around — is a magnification of the indicated area in the preceding picture. The Mandelbrot set is connected (Hubbard-Douady), though that is not very visible from the pictures. It is still unknown whether it is also locally connected. Each of the various baby Mandelbrot sets visible in the various magnification pictures below is connected to the main mass by very thin 'filaments'.

The name fractal, coined by Mandelbrot, derives from the property of these sets to have a (Hausdorff-Besicovitch) dimension which is not an integer. Trust the mathematicians to generalize just about everything, and to give meaning even to such a statement as the dimension of a set A is 1.61748... ; for that matter there are also negative dimensions (indexes) and ones that take their values, for instance, in a ring of characters of a suitable group. Several authors reserve the word fractal, however, exclusively for sets which also exhibit scaling self-similarity: under suitable magnification arbitrarily small parts of the set look just like the whole (or larger parts) of the set. The Mandelbrot set is certainly fractal in the original sense of something rough with rough scaling similarity and also in the sense that its boundary has a (fractal) dimension larger than its topological dimension; it is, however, not strictly self-similar: as can be sort of seen from the eight pictures below, the baby copies get hairier and hairier because of all the filaments in which they are embedded.



So far I have mainly talked about fractals and not about chaos. Roughly, chaos describes a situation where the typical solutions of a difference equation (such as the ones above), or a differential equation, or another deterministic process, do not converge to a stationary or periodic function of time, but continue to exhibit seemingly unpredictable behaviour such as in the case of the example given by

$$x_{k+1} = 2x_k \mod 1 \tag{3}$$

There is extreme sensitivity to initial conditions and there tend to be periodic orbits of all periods. Although there are many books which have both the words 'fractal' and 'chaos' in the title, this does not sound, a priori, as if that would be much of a relation.

There is, however. One is as follows. Let J be the Julia set of one of the iterative processes (1) and (2), then J is invariant under the iteration and on J the process exhibits chaotic behaviour. Also, of course, the family (2) includes the family

$$x_{k+1} = x_k^2 - c, \ c \text{ real} \tag{4}$$

which models the period doubling route to chaos discovered by Feigenbaum (and Coullet-Tresser), with its beautiful universality and scaling properties, by now far to often described in the literature, including many of the books below, to repeat here. In addition, the strange attractors, introduced by Ruelle and Takens to explain turbulent flow, and thought to be present in such equations as the Lorenz equations

$$\dot{x} = \sigma(y-x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz \tag{5}$$

seem to be fractal-like and to have non-integral dimensions. In the case of the Lorenz attractor the numerical evidence points to a dimension 2.06... .For more about attractors and dimension cf the article by Girault in **11** below.

Incidentally, it seems to have become clear that there may be more to symmetry than groups. Fractals, crystals with five-fold symmetry, quasi-crystals, incommensurable crystals, 'regular' finite geometries, Penrose universes, all seem to urge us to entertain the idea of 'symmetry without groups'. Possibly symmetry groups in higher dimensions of which we see only three-dimensional traces (projections) are part of the answer (as in gauge theories). And that brings me to a second, different duality, which does not yet come into the present setting (as far as I can see): a completely regular arrangement of points in the plane, say, is very symmetric, but so is a completely random distribution of points, in any case before a realization has taken place, but, I would argue, also after (most of the time).

After the fact it is remarkable that it was not seen earlier that there are many things in nature for which models with an approximate scaling symmetry are more suitable than models with an approximate translation or rotation symmetry.

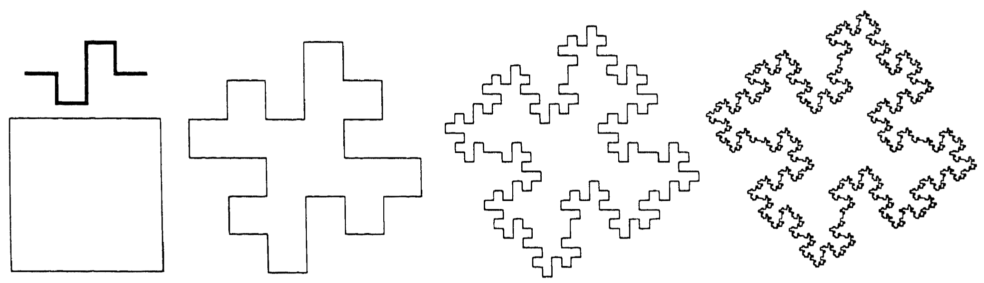
> "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

Thus writes Mandelbrot, to whom a great deal of credit is due in these matters, who, one gets the impression, would have liked to have discovered everything in the field himself, and who sometimes writes as if he did.

There is little doubt — at least in terms of the number of research papers published — of the importance of 'chaos' and 'fractals'; and, joking aside, definitely, this importance goes very far beyond providing a lot of scientists with a living. A quick search through the STN/FIZ database in Karlsruhe, invaluable for this sort of thing, found 1187 articles in the math database, 5992 in the phys database, and 6648 in the inspec database, with either the word 'fractal' or the word 'chaos' (or 'chaotic') in the title or abstract. These numbers are not to be taken too absolutely as all having to do with the topic of this review; in particular the word 'chaos' also has other meanings, for instance in the combination of the probabilistic 'Wiener chaos'. Still on the basis of a rather too small random sample I would say that about 2/3 or slightly more of all the articles thus found have in fact to do with our topic here. These are impressive numbers, and one wonders what we were doing about describing and analyzing the many natural phenomena dealt with in these papers in the years BC and BF.

Parenthetically, let me remark here that the field also benefits from good words and phrases like 'chaos', 'fractal'. Good notation is important (even crucial (Leibniz)), but so are good (sounding) words and phrases, which stick in one's mind and make thinking easier. And what about a lovely invention such as 'cantori', which refers to KAM tori breaking up into Cantor sets. (KAM refers to Kolmogorov-Arnold-Moser; cf also below for some more details.)

Thus, as has already been remarked above, there is no doubt at all that fractals — and also chaos, but that is not my topic in this paragraph — are a magnificent describing tool, and one wonders how we did without it not all that long ago. Still, that is not enough: one also needs to understand how nature can dynamically create fractals; for the world has dynamics and the usual way of looking at fractals seems to have little to relate to that aspect of our attempts to understand (and control) of what is going on around us. To appreciate what I am trying to get at consider one of the usual ways, indeed the most common way, of introducing fractals. Take, the recipe says, some sort of figure built out of segments, or triangles, or ... ; let us stick with the usual, and take segments; take some broken line, called a generator, like the one depicted at the top left of the figure below,and introduce the rule that each straight segment of a starting figure is to be replaced by the broken line appropriately (= linearly in this case) scaled; repeat the process with the figure thus obtained; ... and so on, ad infinitum. The first three stages of the process are illustrated below with as starting figure a square and with as generator the broken line shown at the top left. Another such process, with as starting figure a segment, and with as generator a segment with the middle third missing produces the famous Cantor set, that ubiquitous animal of certain parts of general topology.



Nice as the recipe is, and beautiful as the pictures are, it is hard to believe that nature, even approximately, will indulge herself in such procedures; the situation is slightly better with respect to the fractals which arise as the boundaries of such a thing as the Newton process described above: after all, every problem of finding a zero of a function can be cast as an optimization problem, and there seems to be a lot of that around us; but even so. Many see this as a very serious drawback to the whole fractal scenario. Fortunately, there are also much more believable activities which lead to (approximate) fractals. Two of my favourites are diffusion-limited aggregation, which may lead to fractal objects as first pointed out by Witten and Sander [38,39], cf especially the comprehensive and richly illustrated review by Meakin [40], and a process governed by (a boundary value problem for) the Laplace operator, as are so many processes in physics [42]. This last one requires to generalize the idea of a fractal but that has never stopped mathematicians and scientists before. Indeed, trying to stop a mathematician from trying to generalize something is about as profitable as listening for the echo of a rose petal dropped in the Grand Canyon (to paraphrase P.G.Wodehouse). And never mind if this process produces an occasional dehydrated elephant; those are probably just the thing to sell to an ETI (= Extra Terrestrial Intelligence).

It is probably about time to turn to the books themselves which caused this review to be written. Given all that has been mumbled above — and all that has been left out —, the specific remarks about each book can be short and, hopefully, to the point.

1. P.Cvitanovic (ed), Universality in chaos, Adam Hilger, 1984, £ 12.95 (paper), £30.00 (hardback), 511 pp.

Introductory articles: D.Ruelle, Strange attractors; M.J.Feigenbaum: Universal behaviour in nonlinear systems; R.M.May, Simple mathematical models with very complicated dynamics; J.-P.Eckmann, Roads to turbulence in dissipative dynamical systems; **Experiments:** A.Libchaber, J.Maurer, A Rayleigh Bénard experiment: helium in a small box; A.Libchaber, C.Laroche, S.Fauve, Period doubling cascade in mercury, a quantitative measurement; J.P.Gollub, H.L.Swinney, Onset of turbulence in a rotating fluid; M.Giglio, S.Musazzi, U.Perini, Transition to chaotic behaviour via a reproducible sequence of period doubling bifurcations; P.Bergé, M.Dubois, P.Manneville, Y.Pomeau, Intermittency in Rayleigh-Bénard convection; J.C.Roux, A.Rossi, S.Bachelart, C.Vidal, Representation of a strange attractor from an experimental study of chemical turbulence; J.L.Hudson, J.C.Mankin, Chaos in the Belousov-Zhabotinskii reaction; R.H.Simoyi, A.Wolf. H.L.Swinney, One dimensional dynamics in a multicomponent chemical reaction; Y.Pomeau, J.C.Roux, A.Rossi, S.Bachelart, C.Vidal, Intermittent behaviour in the Belousov-Zhabotinsky reaction; F.T.Arecchi, R.Meucci, G.Puccioni, J.Tredicce, Experimental evidence of subharmonic bifurcations, multistability, and turbulence in a Q-switched gas laser; J.Testa, J.Pérez, C.Jeffries, Evidence for universal chaotic behavior of a driven nonlinear oscillator; M.R.Guevara, L.Glass, A.Shrier, Phase locking, period doubling bifurcations, and irregular dynamics in periodically stimulated cardiac cells; **Theory:** M.Metropolis, M.L.Stein, P.R.Stein, On finite limit sets for transformations on the unit interval; M.J.Feigenbaum, The universal metric properties of nonlinear transformations; O.E.Lanford III, A computer assisted proof of the Feigenbaum conjectures; M.Nauenberg, J.Rudnick, Universality and the power spectrum at the onset of chaos; **Noise**: S.Grossmann, S.Thomae, Invariant distributions and stationary correlation functions of one dimensional discrete processes, E.N.Lorenz, Noisy periodicity and reverse bifurcation; B.A.Huberman, J.Rudnick, Scaling behaviour of chaotic flows; A.Wolf, J.Swift, Universal power spectra for the reverse bifurcation sequence; B.A.Huberman, A.B.Zisook, Power spectra of strange attractors; J.D.Farmer, Spectral broadening of period-doubling bifurcation sequences; J.P.Crutchfield, B.A.Huberman, Fluctuations and the onset of chaos; B.Shraiman, C.E.Wayne, P.C.Martin, Scaling theory for noisy period doubling transitions to chaos; J.Crutchfield, M.Nauenberg, J.Rudnick, Scaling for external noise at the onset of chaos; **Intermittency**: Y.Pommeau, P.Manneville, Intermittent transition to turbulence in dissipative dynamical systems; J.E.Hirsch, M.Nauenberg, D.J.Scalapino, Intermittency in the presence of noise: a renormalization group formulation; **Period doubling in higher**

dimensions: M.Hénon, A two dimensional mapping with a strange attractor; A.B.Zisook, Universal effects of dissipation in two dimensional mappings; P.Collet, J.-P.Eckmann, H.Koch, Period doubling bifurcations for families of maps on **R**(n); E.N.Lorenz, Deterministic nonperiodic flow; V.Francheschini, C.Tebaldi, Sequences of infinite bifurcations and turbulence in a five mode truncation of the Navier-Stokes equations; J.Crutchfield, D.Farmer, N.Packard, R.Shaw, G.Jones, Power spectral analysis of a dynamical system; **Beyond the one-dimensional theory:** S.J.Shenker, Scaling behaviour in a map of the circle onto itself: empirical results; R.S.MacKay, Period doubling as a universal route to stochasticity; R.H.G.Helleman, Self-generated chaotic behaviour in nonlinear mechanics; **References.**

This is simply an excellent collection of reprints of often seminal articles in the field up to about 1982. It is of necessity not representative of current knowledge, but that is simply because so much has happened since 1982. A most striking omission is the paper by Ruelle and Takens, 'On the nature of turbulence', Comm. Math. Phys. **20** (1971), 167–192; another one is the absence of anything on 1/f-noise, so important in electrical engineering, and it would have been nice and thoughtful to have included one of the papers of Coullet and Tresser, codiscoverers of some of the universality phenomena in the period doubling route to chaos. A very nice inclusion is the very useful survey paper by Helleman with its enormous list of references.

That same list of references, though, immediately loses much of its value because of the abominable habit of the physics community to: a) not give the titles of journal articles, b) not give the final page numbers of articles, and c) not to arrange them in alphabetical order. Surely, assuming that, on the average, the number of readers of a paper should exceed the number of authors, and that, at least occasionally, a paper will be looked at more than once by the same scientist, one should take a less 'once-I-have got-it-published-who-cares' attitude. It also seems a waste to spend a great amount of time to collect a very representative list of references and then to present them in such a way that they are perfectly useless to anybody who needs to look up some of them.

All in all, though, this is a very good book to have on your shelves if you are interested at all in the subject; and it is still a good buy.

2. H.G.Schuster, Deterministic chaos, VCH Verlag, 1988, 2-nd edition, DM 108.–, 273 pp.

Experiments and simple models: Experimental detection of deterministic chaos; The periodically kicked rotator; **Piecewise linear maps and deterministic chaos:** The Bernouilli shift; Characterization of chaotic motion; Deterministic diffusion; **Universal behavior of quadratic maps:** Parameter dependence of the iterates; Pitchfork bifurcations and the doubling transformation; Selfsimilarity, universal power spectrum, and the influence of external noise; Behavior of the logistic map; Parallels between period doubling and phase transitions; Experimental support for the bifurcation route; **The intermittency route to chaos:** Mechanisms for intermittency; Renormalization-group treatment of intermittancy; Intermittency and 1/f-noise; Experimental observation of the intermittency route; **Strange attractors in dissipative dynamical systems:** Introduction and definition of strange attractors; The Kolmogorov entropy; Characterization of the attractor by a measured signal; Pictures of strange attractors and fractal boundaries; **The transition from quasiperiodicity to chaos:** Strange attractors and the onset of turbulence; Universal properties of the transition from quasiperiodicity to chaos; Experiments and circle maps; Routes to chaos; **Regular and irregular motion in conservative systems:** Coexistence of regular and irregular motion; Strongly irregular motion and ergodicity; **Chaos in quantum systems?:** The quantum cat map; A quantum particle in a stadium; The kicked quantum rotator; **Outlook; Appendix;** Remarks and references.

It is no accident that this book is now in its second edition, a revised and expanded one. It is well-conceived, well-written, lucid, thorough, and quite complete. Though written by a physicist for physicists it is also excellently suited for mathematicians and other scientists. It contains moreover enough, and that of sufficient depth, for all but those who use chaos and fractals professionally so to speak. And to top it off, it also contains a number of beautiful colour pictures (of the by now familiar kinds).

3. W.Horsthemke, D.K.Kondepudi (eds), Fluctuations and sensitivity in nonequilibrium systems, Springer, 1984, DM 79.–, 273 pp.

Basic theory: I.Prigogine, Irreversibility and space-time structure; L.Arnold, Stochastic systems: qualitative theory and Lyapunov exponents, B.J.Matkovsky, Z.Schuss, C.Knessl, C.Tier, M.Mangel, First passage times for processes governed by master equations; **Pattern formation and selection**: H.S.Greenside, Three caveats for linear stability theory; Rayleigh-Bénard convection; D.Walgraef, P.Borckmans, G.Dewel, Pattern selection and phase fluctuations in chemical systems; J.P.Gollub, Experiments on patterns and noise in hydrodynamic systems; **Bistable systems**: A.T.Rosenberger, L.A.Orozco, H.J.Kimble, Optical bistability: steady-state and transient behavior; J.C.Roux, H.Saadaoui, P.de Kepper, J.Boissonade, Experimental studies of the transitions between stationary states in a bistable chemical system; E.Ben-Jacob, D.J.Bergman, B.J.Matkovsky, Z.Schuss, Noise-induced transitions in multi-stable systems; P.Hanggi, Bistable flows driven by colored noise; **Response to stochastic and periodic forcing**: W.Horsthemke, Noise-induced transitions; J.M.Sancho, M.San Miguel, Dynamical aspects of external nonwhite noise; S.M.Meerkov, Dynamic systems with fast parametric oscillations; F.Moss, P.V.E.McClintock, Experimental studies of noise-induced transitions; R.Lefever, J.W.Turner, Sensitivity of a Hopf bifurcation to external multiplicative noise; **Noise and deterministic systems**: N.B.Abraham, Noise and chaos in selected quantum optic systems; A.Brandstater, H.L.Swinney, Distinguishing low-dimensional chaos from random noise in a hydrodynamic experiment; J.D.Farmer, Sensitive dependence to parameters, fat fractals , and universal strange attractors; R.Kapral, E.Celarier, S.Fraser, Noise-induced transitions in discrete time systems; A.Arnéodo, Scaling for external excitations of a period-doubling system; **Sensitivity in nonequilibrium systems**: H.Rabitz, General sensitivity analysis of differential equation systems; D.K.Kondepudi, Nonequilibrium sensitivity; O.Decroly, A.Goldbeter, Patterns of nonequilibrium sensitivity in biological systems; R.Larter, B.L.Clarke, Chemical reaction network sensitivity analysis; **Contributed papers and posters.**

This is not really a book about chaos and/or fractals. Instead, as the editors of this proceedings of a March 1984 meeting in Austin, Texas, note, it deals with stochastic phenomena and sensitivity in nonequilibrium systems from the macroscopic point of view. Of course (part of) chaos is about extreme sensitivity to initial conditions and thus this workshop looked at the phenomenon within a larger context but in the framework of macroscopic systems often thought to be made up of many microscopic parts. Thus, also other aspects than the ones alluded to in the first part of the review occur, but the reader who takes the trouble to look at the table of contents listed above and compares it with the one of book 1 above will see many of the same words and phrases and will recognize several of the same (famous) authors. A useful volume for those who from the background of stochastic and/or nonequilibrium systems want to get a feel for chaos; and the chaotici among us can get from it some idea of other aspects and thoughts about (extreme) sensitivity.

4. P.Fisher, W.R.Smith (eds), Chaos, fractals, and dynamics, Marcel Dekker, 1985, 261 pp.

Part I: R.Abraham, Chaostrophes, intermittency, and noise; R.H.Abraham, Ch.C.Shaw, The outstructure of the Lorenz attractor; R.H.Abraham, H.Koçak, W.R.Smith, Chaos and intermittency in an indocrine system model; O.Gurel, An index for chaotic solutions in cooperative peeling; W.F.Langford, Unfoldings of degenerate bifurcations; O.E.Rössler, Example of an axiom A ODE; **Part II**: R.H.Abraham, Is there chaos without noise; R.H.Abraham, K.A.Scott, Chaostrophes of forced Van der Pol systems; M.E.Alexander, J.Brindley, I.M.Moroz, Numerical solution of the Lorenz equations with spatial inhomogeneity; S.-N. Chow, D.Green Jr, Some results on singular delay-differential equations; P.Fisher, Feigenbaum functional equations as dynamical systems; M.W.Hirsch, The chaos of dynamical systems; G.Ikegami, On network perturbations of electrical circuits and singular perturbation of dynamical systems; B.B.Mandelbrot, On the dynamics of iterated maps III–VII (The individual molecules of the M-set, selfsimilarity properties, the empirical n-squared rule and the n-squared conjecture; The notion of the 'normalized radical' R of the M-set, and the fractal dimension of the boundary of R; Conjecture that the boundary of the M-set has a fractal dimension equal to 2; Conjecture that certain Julia sets include smooth components; Domain-filling ('Peano') sequences of fractal Julia sets, and an intuitive rationale for the Siegel discs).

These are the papers delivered at two meetings in Guelph (Ontario, Canada) in 1981 and 1983. Judging from the number of different authors and, especially, the number of repeat authors, these could have been examples of those nice, cosy, small meetings, where one can get a good deal of work done in interaction with others. These papers are now somewhat older, but they are by people who have thought deeply, and, mostly, I find them still very valuable. The five papers by Mandelbrot are to be seen as sequels to his papers in [22] and [15] below and to his book [13]; the VIII-th paper in the series is the one in book **6** below.

5. Kunihiko Kaneko, Collapse of tori and genesis of chaos in dissipative systems, World Scientific, 1986, £ 23.85, 264 pp.

Introduction: Dawn of nonlinear nonequilibrium physics; Dawn of chaos physics; Onset of chaos; Transition from torus to chaos accompanied by lockings — outline of the book; **Instability of phase motion of tori**: Introduction; Structure of lockings; Similarity of the period-adding sequences of lockings (numerical results); Phenomenological theory of the similarity of the period-adding sequence; Classification of the period-adding sequences; Period-adding sequence as windows; Scaling properties at the collapse of tori — a brief review on a recent progress; Global properties of the Devil's staircase; Supercritical behavior of disordered orbits of the circle map; Discussion; **Transition from torus to chaos accompanied by frequency lockings with symmetry breaking**: Introduction; Phase diagram and general aspects of the coupled logistic map; Scaling of the period-adding sequence at the frequency locking; Frequency locking with symmetry breaking; Discussion; **Oscillation and fractalization of tori**: Introduction; Oscillation of torus in two dimensional mappings; Fractalization of torus; Summary and discussion; **Doubling of torus**: Discovery; Doubling stops by a finite number of times; Mechanism of the interruption of thedoubling cascade; Discussion; **Fates of the three torus**: Introduction; Double Devil's staircase in the modulated circle map; Chaos from the three torus in a coupled circle map; Summary and discussions; **Turbulence in coupled map lattices**: Introduction; Period-doublings of kink-antikink patterns; Zigzag instability and transition from torus to chaos; Spatiotemporal intermittency; Period-doubling in open flow; Cellular automata; Discussions; **Summary, future problems, and discussions**: Summary and future problems; What has chaos brought about and will bring about in science?; Towards a field theory of chaos.

Consider a completely integrable Hamiltonian system. Then there are action-angle coordinates which serve to partition phase space into tori; motion takes place on these tori. Now what happens if the system is perturbed. The famous KAM theorem (Kolmogorov-Arnol'd-Moser), very roughly, says that certain of these tori persist provided the frequencies involved are sufficiently irrational with respect to one another (are far enough from resonance) and some other conditions are met. As the perturbation is made larger these tori start to break up. This phenemenon, more precisely the collapse of tori with frequency lockings route to chaos, is the main topic of this book, which is largely based on the author's 1983 thesis. In addition there is another topic: that of studying coupled equations on a lattice; i.e. for each site of the lattice there is an evolution going on and there couplings between the equations describing what happens at the various sites. This subject has attracted further interest and was, for example, the subject of some reports at the Jan 87 Karpacz winter school (by Buminovich).

6. Y.Kuramoto (ed.), Chaos and statistical methods, Springer, 1984, DM 85.–, 272 pp.

General concepts: K.Tomita, Coarse graining revisited — the case of macroscopic chaos; Y.Takahashi, Gibbs variational principle and Fredholm theory for one-dimensional maps; T.Short, J.A.Yorke, Truncated development of chaotic attractors in a map when the Jacobian is not small; **Fractals in dynamical and stochastic systems**: B.B.Mandelbrot, On the dynamics of iterated maps VIII: The map $z \rightarrow \lambda(z + 1/z)$, from linear to planar chaos, and the measurement of chaos; H.Yoshida, Self-similar natural boundaries of nonintegrable dynamical systems in the complex t-plane; M.Widom, S.J.Shenker, Topological phase transitions; M.Kohmoto, Dynamical system related to an almost periodic Schrödinger equation; K.Kawasaki, M.Tokuyama, Mean field Hausdorf dimensions of diffusion-limited and related aggregates; **Onset of chaos**: P.Coullet, Stability of the scenarios towards chaos; B.Hu, Functional renormalization-group equations approach to the transition to chaos; K.Kaneko, Collapse of tori in dissipative mappings; H.Daido, Periodic forcing near intermittency threshold — resonance and collapse of tori; T.Shimizu, Perturbation theory analysis of bifurcations in a three-dimensional differential system; **One-dimensional mappings**: I.Tsuda, K.Matsumoto, Noise-induced disorder — complexity theoretic digression; Y.Aizawa, T.Kohyama, Symbolic dynamics approach to intermittent chaos — towards the comprehension of large scale self-similarity and asmptotic non-linearity; H.Fujisaka, Diffusion and generation of non-gaussianity in chaotic discrete dynamics; B.C.So, H.Okamoto, H.Mori, Analytic study of power spectra of intermittent chaos; **Bifurcations and normal forms**: S.Ushiki, Versal deformations of singularities and its applications to strange attractors; G.Iooss, Some codimension-two bifurcations for maps, leading to chaos; E.Knobloch, Bifurcations in doubly diffusive convection; Y.Ueda, H.Ohta, Strange attractors in a system described by nonlinear differential-difference equations; T.Yamada, H.Ohta, Coupled chaos; K.-C.Lee, S.Y.Kim, D.-I.Choi, Bifurcations in 2D area-preserving maps; **Soliton systems**: M.Imada, Chaotic behaviour induced by spatially inhomogeneous structures such as solitons; H.Nagashima, Chaotic behavior of quasi solitons in a nonlinear dispersive system; **Fluid dynamics**: H.Mori, K.Takayoshi, Inviscid singularity and relative diffusion in intermittent turbulence; N.J.Zabusky, Computational synergetics and innovation in wave and vortex dynamics; A.Pouquet, C.Gloaguen, J.Leorat, R.Grappin, A scalar model of MHD turbulence; U.Frisch, The analytic structure of turbulent flow; A.Libchaber, Low Prandtl number fluids, a paradigm for dynamical systems studies; M.Sano, Y.Sawada, Chaotic attractors in Rayleigh-Bénard systems; H.Yahata, Onset of chaos in some hydrodynamic model systems of equations; **Chemical and optical systems**: H.L.Swinney, R.H.Simoyi, J.C.Roux, Instabilities and chaos in a chemical reaction; K.Ikeda, O.Akimoto, Optical turbulence; **Anomalous fluctuations**: M.Suzuki, Scaling theory of relative diffusion in chaos and turbulence; M.Nelkin, 1/f resistance fluctuations.

The leaflet which came with this book states that chaos is a most remarkable as well as ubiquitous mode of motion in nature and that its study has developed rapidly in recent years. Certainly most true. It then goes on to state that this conference proceedings (of the 6-th Kyoto summer institute in 1983) contains 36 carefully selected papers which together give an authoritative review of the field at the time. That I find somewhat harder to believe. It has the looks of a straightforward conference proceedings consisting of short 6-7 page contributions, with its usual mix of potboilers and contributions which will appear in fuller form later elsewhere. That is not to say that there is not much here that is very worthwhile. For instance — if I may indulge in my personal taste for a moment — the paper by Takahashi (a sequel can be found in [41]), the papers by Mandelbrot and Yoshida, and the Ushiki paper. Certainly this volume goes a long way in making it clear just how universally occurring chaos is.

7. Dynamical chaos, The Royal Society, 1987, reprint of Proc. of the Royal Society London **A413**(1987), 1–199, £ 20.–, 199 pp.

E.C.Zeeman, Chairman's introduction; D.Ruelle, Diagnosis of dynamical systems with fluctuati parameters; L.Glass, A.L.Goldberger, A.Schreier, Nonlinear dynamics, chaos and complex cardi arrhytmias; R.M.May, Chaos and the dynamics of biological populations; D.A.Rand, Fractal bifurcati sets, renormalization strange sets and their universal invariants; A.Libchaber, From chaos to turbulence Bénard convection; N.O.Weiss, Dynamics of convection, E.A.Spiegel, Chaos: a mixed metaphor f turbulence; F.Vivaldi, Arithmetical theory of Anosov diffeomorphisms; J.Wisdom, Chaotic behavior the solar system; I.C.Percival, Chaos in Hamiltonian systems; B.V.Chirikov, Particle confinement and t adiabatic invariance; W.P.Reinhardt, I.Dana, Semi-classical quantization adiabatic invariants and classic chaos; C.Series, Some geometrical models of chaotic dynamics; M.V.Berry, Quantum chaology.

This definitely is an authoritative book on the current status of many aspects of th field of chaos. Anosov diffeomorphisms on the two torus are the most chaoti diffeomorphisms. Yet they can be well studied by arithmetic; cf. the paper by Vivaldi. there indeed a symmetry at some level between the (most) chaotic and the (mos regular. A related paper by Vivaldi appears in book **12** below. I personally especiall like the Rand paper in this collection.

8. P.Schuster (ed.), Stochastic phenomena and chaotic behavior in comple systems, Springer, 1984, Dm 85.–, 271 pp.

Gneral concepts: H.Haken, Some basic ideas on a dynamic information theory; Y.M.Ermoliev, Aspects optimization and adaptation; P.Whittle, Relaxed Markov processes, Jackson networks and polymerisatio **Chaotic dynamics theory:** O.E.Rössler, J.L.Hudson, J.D.Farmer, Noodle-map chaos — a simple exampl H.-O.Peitgen, A mechanism for spurious solutions of nonlinear boundary value problems; D.Maye Approach to equilibrium: Kuzmin's theorem for dissipative and expanding maps; P.Coullet, Comple behaviors in macrosystems near polycritical points; **Chaotic dynamics — real systems and experiment verification:** P.H.Richter, H.-J.Scholz, Chaos in classical mechanics: the double pendulum; J.L.Hudso J.C.Mankin, O.E.Rössler, Chaos in continuously stirred chemical reactors; R.M.Noyes, The interfac between mathematical chaos and experimental chemistry, L.F.Olsen, The enzyme and the strange attract — comparisions of experimental and numerical data for an enzyme reaction with chaotic motio J.S.Nicolis, G.Mayer-Kress, G.Haubs, Nonuniform information processing by strange attractors of chaoti maps; **Stability and instability in dynamical networks:** P.E.Phillipson, Generalized modes and nonline dynamical systems; J.Hofbauer, P.Schuster, Dynamics of linear and nonlinear autocatalysis an competition; K.Sigmund, P.Schuster, Permanence and uninvadability for deterministic population model **Stochasticity in complex systems:** P.Schuster , K.Sigmund, Random selection and the neutral theory – sources of stochasticity in replication; A.M.Rodriguez-Vargas, P.Schuster, The dynamics of catalyti hypercycles — a stochastic simulation; M.Rejmánek, Perturbation-dependent coexistence and specie diversity in ecosystems; P.M.Allen, M.Sanglier, G.Engelen, Chance and necessity in urban system K.F.Albrecht, V.Chetnikov, W.Ebeling, R.Funke, W.Mende, M.Peschel, Random phenomena in nonline systems in connection with the Volterra approach.

These are the full proceedings of the fourth meeting (in 1983) of the UNESCO Workin Group on System Analysis; i.e. all contributions are recorded here. A major topic of th meeting was the occurrence of chaos and its verification in chemistry, physics an information processing; other sections treat of stochastic phenomena in comple sytems in similar and other fields. It does not look that there has been much contac between the two groups, and, except for the very large vague theme of dynamics wit overtones of stochasticity, this proceedings lacks direction and a central theme o philosophy.

9. G.M.Zaslavsky, Chaos in dynamic systems, Harwood, 1985, $ 195.–, 370 pp.

Elements of dynamics and ergodic theory: Motion in phase space; Action-angle variables; Nonlinear resonance; Kolmogorov-Arnold-Moser theory, Ergodicity and mixing, Entropy, Historical background; **Stochasticity criterion**: Two models of mixing; Stochasticity criterion; Collisions of absolutely rigid spheres; Scattering billiards (Sinai's billiards); **Stochastic acceleration of particles (Fermi acceleration)**: Mechanism of stochastic acceleration; Gravitational engine; Mixing of skipping electrons; **Stochastic instability of oscillations**: Universal transformation (mapping) of nonlinear oscillations; Criterion of overlapping of resonances (Chirikov overlap criterion); Sine transform; **Stochastic layer: theory of formation**: Stochastic destruction of a separatrix; Stochastic layer: specifics of formation; General pattern of stochastic destruction in phase space; Homoclinic structure in the neighborhood of a separatrix; **Mixing and the kinetic equation**: Principles of kinetic description; Kinetics of the nonlinear oscillator; Diffusional model of a particle in a wave-packet field; **Nonlinear wave field**: The Fermi-Pasta-Ulam problem; Stochastization of a wave field; Kinetic description of a wave field; Kinetic equation for phonons; **Stochasticity of nonlinear wave**: Stationary dynamics of nonlinear waves; Perturbation of nonlinear waves; Nonlinear resonance; Stochastic instability of nonlinear waves; **Stochasticity of quantum systems. Nonstationary problems (part I)**: Quantum K systems; Quantum mappings; Projecting in the basis of coherent states; Spreading of the wave packets; 'T'-mapping and the stochasticity criterion; **Stochasticity of quantum systems. Nonstationary problems (part II)**: Quantum mapping of wave functions; Analysis of quantum mappings; Interaction of quantum resonances; **Kinetic description of quantum K-systems**: Equation for the density matrix; Derivation of the kinetic equation; **Destruction of integrals of motion in quantum systems**: Historical background; Formulation of the problem; Universality of K-systems and periodic orbits; Quantization rules; Distribution of spacings between neighboring levels; Some general remarks on quantum systems; Stochastic destruction of a bound state of atoms and radiation field; Intermolecular energy exchange; **Appendices**: Mixing billiards; Arnold diffusion; Stochasticity in dissipative dynamic systems; **References**.

This is a translation of the book that appeared in 1984 with Nauka under the (transliterated) title ' Stokhastichnost dinamicheskikh sistem', though, besides the fact that it is a translation, these data are not mentioned. The translation was therefore done quickly; this is definitely a lot faster than is usual. It was also done competently except for a few odd things such as the consistent misspelling of Hausdorff as Housdorff. The book is devoted to the systematic exposition of the phenomenon of randomness or chaos which arises under certain conditions in nonlinear systems. Besides the basic theory it contains many applications from physics (mechanics, optics, plasma theory, hydrodynamics). The topic of quantum chaos, which is rather different from that of the deterministic (classical) chaos discussed so far in this review is given a lot of space (about two fifths of the book). Compared to book 2 there is much more stress on the physics side of things, and, in fact, I do not think that without a good physics background the book will be digestible. This one also goes quite a bit deeper in certain directions — it is after all more a research monograph than a textbook —, for instance in the direction of mixing and ergodicity (for instance it discusses Sinai's stochasticity criterion); and of course it pays special attention to the many results of the various Soviet schools, such as obtained by Sinai, Chirikov, the author himself, Buminovich, Izrailev, Berman, ... There is a great deal in this book, which, given the price, is only right; but it is not easy to dig it out of there. As is so often the case, the index is laughable, and the 209 item bibliography suffers from all the defects I complained about a few pages back; in fact even more so beause of the Soviet penchant for acronyms, making it for not absolute insiders often quite difficult to guess what institute, or journal, or ... is meant. A very valuable but also a very difficult book.

10. H.-O.Peitgen, P.H.Richter, The beauty of fractals, Springer, 1986, DM 78.-, 199 pp.

Frontiers of chaos; *Special sections*: Verhulst dynamics; Julia sets and their computergraphical generation; Sullivan's classification of critical points; The Mandelbrot set; External angles and Hubbard trees; Newton's method for complex polynomials: Cayley's problem; Newton's method for real equations; A discrete Lotka-Volterra system; **Magnetism and complex boundaries;** *Special sections*: Yang-Lee zeroes; Renormalization; **References;** *Invited contributions*: B.B.Mandelbrot, Fractals and the rebirth of iteration theory; A.Douady, Julia sets and the Mandelbrot set; G.Eilenberger, Freedom, science and aesthetics, H.W.Franke, Refractions on science and art; **Do it yourself; Documentation.**

With its 184 pictures, over half of them in colour, its large size, and its glossy heavy quality paper, this book looks like a coffee-table book. And certainly it can be used for that purpose (and not only by scientists). The book is based, as far as its pictures go, on three exhibitions, and having been present at the first one in Bremen in Jan. 1984, I can testify that many of these pictures of Julia sets and such are extraordinarily beautiful, especially when seen in large, high quality formats. Moreover, the underlying (simple) mathematical ideas/processes give the pictures a coherence and inner strength (a hidden internal symmetry if one wishes), which I find very appealing. One notices a similar harmony in many of the drawings of Anatoli Fomenko.

Yet this is not a coffee-table book; for that it contains far too much mathematics. I have heard complaints from the experts that the book is sort of halfway in that respect: too little mathematics — no proofs — for the mathematician, too much for others. I disagree; I think the mathematics has been done just right. A main purpose of a text is to explain, to bring understanding, and to develop intuition and feeling for the phenomena. Proofs can explain — though often they fail to do just that — but surely there are other cognitive tools in our arsenal. All the same there is a lot of explanatory (this is how things are) mathematics in the text, and there is also a do-it-yourself section, just a few pages, for those who would like to generate such pictures themselves on their Macintoshes or PC's. (You need reasonable processing power, but nothing excessive as such things go nowadays, and good graphics capabilities.)

The book has become a bestseller: over 50 000 copies sold. This, perhaps naturally, made me immediately sceptical; as it turned out, that was totally unjustified; I cannot do otherwise than recommend this book very strongly.

11. G.Cherbit (ed.), Dimensions non entières et applications, Masson, 1987, 362 pp.

G.Cherbit, Introduction; B.Mandelbrot, Propos à bâtons rompus; S.Dubuc, Modèles de courbes irrégulières; G.Deslauriers, S.Dubuc, Interpolation dyadique; M.Weber, Pocessus stochastiques et procédés de recouvrement; P.Girault, Attracteurs et dimensions, F.M.Dekking, Constructions de fractals et problèmes de dimension; J.Peyriere, Introduction aux mésures et dimensions de packing, J.-L.Jonot, Remarques sur la dimension de Hausdorff; A.Le Mehauté, Fractals, matériaux et énergie; M.Keddam, Problématique autour du concept de fractal en électrochimie; P.Mills, Quelques remarques concernant la structure des amas galactiques et la constante de Hubble; G. Cherbit, Désordre, hasard et fractals en biologie; J.-P.Rigault, Fractals, sémi-fractals et biométrie, N.de Beaucoudray, L.Garnero, J.-P.Hugonin, Reconstruction d'images à partir de projections; M.Rosso, B.Sapoval, J.-F.Gouyet, J.-F.Colonna, Création d'objets fractals par diffusion; J.Chanu, Irréversibilité et flèche du temps; M.Courbage, Entropie thermodynamique et information; M.Mendès-France, Dimensions et entropie des courbes irrégulières; G.Cherbit, Dimension locale, quantité de mouvement et trajectoires; G.Cherbit, Dimensionalité spatio-temporelle.

This book is based on a series of seminars organized by the 'Groupe de recherche biophysique' of the Univ. of Paris VII. It is a very stimulating book in my opinion, and, possibly true to French tradition, is much more solid mathematically than one would perhaps expect from its biophysical origin.

12. E.R.Pike, L.A.Lugiato (eds), Chaos, noise and fractals, Adam Hilger, 1987, £ 19.50, 249 pp.

F.T.Arecchi, Hyperchaos and 1/f spectra in nonlinear dynamics; D.S.Broomhead, R.Jones, G.P.King, E.R.Pike, Singular system analysis with application to dynamical systems; G.Casati, A review of progress in the kicked rotator problem; B.Eckhardt, Fractals in quantum mechanics?; M.Feingold, Ergodic semiclassical quantum dynamics; T.Geisel, G.Radons, J.Rubner, Cantori and quantum mechanics; L.A.Lugiato, M.Brambilla, G.Strini, L.M.Narduci, Influence of chaos noise and driven optical systems; P.Meystre, E.M.Wright, Chaos in the micromaser; H.J.Mikesda, H.Frahm, Chaos in a driven quantum spin system; J.V.Moloney, H.Adachihara, D.W.McLaughlin, A.C.Newell, Fixed points and chaotic dynamics of an infinite dimensional map; F.Vivaldi, The arithmetic of chaos; P.L.Knight, S.J.D.Phoenix, Limitations of the Rabi model for Rydberg transitions; J.S.Satchell, S.Sarkar, H.J.Carmichael, Quasi-probability distributions in astable dissipative quantum systems.

This volume constitutes the proceedings of a pre-meeting held in 1986 in Como, just before a NATO ARW on Quantum Chaos. That explains possibly the fact that 10 of the 13 contributions in fact have to do with quantum chaos, a fact that is (regrettably) not reflected in the title of this book. The central question concerns the the stability (robustness) of fractal and chaotic behaviour in the presence of classical and quantum noise. A useful collection by mostly well known authors.

For those who are (relatively) new to the field, and those who would like to have some of the beautiful pictures on tap which have become traditional if not obligatory, books **2** and **10** are the clear winners; book **1**, but not only book **1**, is for those who would do as Abel recommended; for breadth of mathematics and freshness of ideas I like **4**, **11**; **5** seems to me to be important but not very accessible and **9**, though comprehensive and authoritative and a mine of information not so easily found elsewhere, will be very hard going, especially for those who have not had a comprehensive training as a physicist; the other books above will appeal to various more specialized scientists and collections. All this, however, is subjective; your correspondent has much to learn and ponder; and is now better equipped to try to do so.

To help complete the picture let me list here the other books on chaos and fractals which happen to be present in my personal collection.

13. B.M.Mandelbrot, The fractal geometry of nature, Freeman, 1982
14. B.M.Mandelbrot, Fractals: form, chance, and dimension, Freeman, 1977
15. D.Campbell, H.Rose (eds), Order in chaos, North-Holland, 1983
16. A.J.Lichtenberg, M.A.Liebermann, Regular and stochastic motion, Springer, 1983

17. P.Collet, J.-P.Eckmann, Iterated maps on the interval as dynamical systems, Birkhäuser, 1980.
18. R.L.Devaney, Chaotic dynamical systems, Benjamin/Cummings, 1986. (Reviewed in this issue by F.Takens)
19. L.Gumovski, C.Mira, Dynamique chaotique: transformations ponctuelles, transition ordre-désordre, Cépadues, 1980.
20. C.Sparrow, The Lorenz equations: bifurcations, chaos, and strange attractors, Springer, 1982.
21. Hao Bai-lin (ed.), Chaos, World Scientific, 1984.
22. R.H.G.Helleman (ed.), Nonlinear dynamics, Ann. N.Y. Acad. Sci. **357**, 1980.
23. J.Gleick, Chaos: making a new science, Viking, 1987.
24. K.J.Falconer, The geometry of fractal sets, Cambridge Univ. Press, 1985.
25. R.Thibault (ed.), Théorie de l' itération et ses applications, CNRS, 1982.

And, finally, here are some more books on chaos and/or fractals which I happen to know about.

26. H.E.Stanley, N.Ostrowsky, On growth and form. Fractal and nonfractal patterns in physics, KAP, 1985.
27. M.F.Shlesinger, B.B.Mandelbrot, R.J.Rubin (eds), Fractals in the physical sciences, J. Stat. Phys. **36**:5/6, Plenum, 1984.
28. U.Beck, Computer-Graphik. Bilder and Programme zu Fraktalen, Chaos und Selbstähnkichkeit, Birkhäuser, 1988.
29. F.V.Atkinson, W.F.Langford, A.B.Mangarelli (eds), Oscillation, bifurcation, and chaos, AMS, 1987.
30. J.Chandra (ed.), Chaos in nonlinear dynamical systems, SIAM, 1984.
31. A.Lasota, M.C.Mackey, Probailistic properties of deterministic systems, Cambridge Univ. Press, 1985. (Reviewed earlier in this journal by J.de Vries, **10**, 312-314.)
32. J.M.Thompson, H.B.Stewart, Nonlinear dynamics and chaos. Geometrical methods for scientists and engineers, Wiley, 1986.
33. A.Kunick, W.-H.Steeb, Chaos in dynamischen Systemen, Bibl. Inst., 1986.
34. M.F.Barsley, S.G.Demko(eds), Chaotic dynamics and fractals, Acad. Pr., 1986.
35. G.Mayer-Kress (ed.), Dimensions and entropies in chaotic systems, Springer, 1986.
36. H.Lauwerier, Fractals. Meetkundige figuren in oneindige herhaling, Aramith, 1987.

and to this list I should probably add the book

37. S.P.Novikov (ed.), Mathematical physics reviews Vol. 2, Harwood, 1982

which contains four long survey articles on deterministic and quantum chaos: Ya.B.Pesin, Ya.G.Sinai, Hyperbolicity and stochasticity of dynamical systems; O.I.Bogoyavlenskii, Geometrical methods of the qualitative theory of dynamical systems in problems of theoretical physics; A.S.Pikovskii, M.I.Rabinovich, Stochastic behaviour of dissipative systems; B.V.Chirikov, F.M.Izrailev, D.L.Shepelyansky, Dynamical stochasticity in classical and quantum mechanics.

References

38. T.A.Witten, L.M.Sander, Diffusion limited aggregation, Phys. Rev. **B27** (1983), 5686–5697.
39. T.A.Witten, L.M.Sander, Diffusion limited aggregation, a kinetic critical phenomenon,Phys. Rev.Lett. **47** (1981), 1400–1403.
40. P.Meakin, Fractal aggregates and their fractal measures, preprint, E.I.du Pont de Nemours and Cy, Wilmington, DE 19898, 1987.
41. S.Albeverio, Ph.Blanchard, M.Hazewinkel, L.Streit (eds), Stochastic processes in physics and engineering, Reidel, 1988.
42. L.Pietronero, C.Evertsz, A.P.Siebesma, Fractal and multifractal structures in kinetic critical phenomena, In [41], 253–278.