# Wavelets: First Steps

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**Abstract.** This chapter gives a short elementary introduction to wavelets. We give a few properties of continuous wavelets, a few remarks on multiresolution analysis, and the construction for the first few spline wavelets as solutions of dilation equations. We also describe an example of the compactly supported orthonormal Daubechies wavelets. All this will be discussed in more depth in the following chapters.

## §1 About the history of wavelets

Wavelets were introduced at the beginning of the 'eighties by J. Morlet, a French geophysicist at Elf-Aquitane, as a tool for signal analysis in view of applications for the analysis of seismic data. The numerical success of Morlet prompted A. Grossmann to make a more detailed study of the wavelet transform, which resulted in a paper giving the mathematical foundations (see Grossmann & Morlet [7]), where the title of the paper still shows the name *wavelets of constant shape*. In 1985, the harmonic analyst Y. Meyer became aware of this theory and he recognized many classical results inside it. Meyer pointed out to Grossmann and Morlet that there was a connection between their signal analysis methods and existing, powerful techniques in the mathematical study of singular integral operators. Then Ingrid Daubechies became involved, and all this resulted in the first construction of a special type of frames (see Daubechies, Grossmann & Meyer [3]), (the concept *frame* generalizes the concept *basis* in a Hilbert space). It also was the start of a cross-fertilization between the signal analysis applications and the purely mathematical aspects of techniques based on dilations and translations.

In 1988 Daubechies provided a major breakthrough by constructing families of orthonormal wavelets with compact support (see Daubechies [4]). In this she was inspired by work of Mallat and Meyer in the field of *multiresolution analysis*,



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and by Mallat's algorithms in which he used this analysis for decomposition and reconstruction of images (see Mallat [9, 10]).

All these activities have created quite a stir among mathematicians. Apart from applications to signal analysis, orthonormal wavelets should be useful in physics also. A first application, to quantum field theory, can be found in Battle & Federbush [1]. From the point of view of numerical analysis, interest arose in *fast* techniques (by analogy with fast Fourier transformation), with which certain integral operators can be transformed into other operators with dominant diagonals; (see Beylkin, Coifman & Rokhlin [2]).

An excellent recent book on wavelets is Daubechies [5]. In this chapter we have used the introductory paper of Strang [11].

## §2 The continuous wavelet transform

Wavelets constitute a family of functions derived from one single function, and indexed by two labels, one for position and one for frequency. More explicitly, when we start with the function $g$, we define

$$g_{a,b}(x) = \frac{1}{\sqrt{|a|}} g\left(\frac{x-b}{a}\right), \quad a \neq 0, \quad b \in \mathbb{R}.$$

In the theory some conditions on $g$ are needed. We request $g \in L^2(\mathbb{R})$ such that

$$C_g = \int_{-\infty}^{\infty} |\xi|^{-1} |\widehat{g}(\xi)|^2 \, d\xi < \infty;$$

$\widehat{g}$ is the Fourier transform defined by

$$\widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-ix\xi} \, dx.$$

Since we require that $C_g$ is finite, the integrand defining $C_g$ should be integrable at $\xi = 0$. This implies that $\widehat{g}(0) = 0$, which says that the mean value of the wavelet $g$ should be zero: $\int_{-\infty}^{\infty} g(x) \, dx = 0$. So $g$ must change its signs on $\mathbb{R}$; $g(x)$ will also decay to 0 as $x$ tends to $\pm\infty$.

In Figure 1, we take the wavelet

$$g(x) = (1 - x^2) e^{-\frac{1}{2}x^2},$$

the *Mexican hat*. The graph of $g$ looks a bit like a transverse section of a Mexican hat, whence the name. Up to a constant, $g$ is the second derivative of the Gaussian $\exp(-\frac{1}{2}x^2)$. We know that

$$\widehat{g}(\xi) = \sqrt{2\pi} \, \xi^2 \, e^{-\frac{1}{2}\xi^2}$$

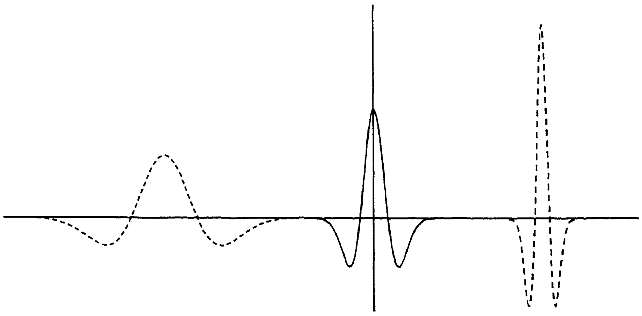

**Figure 1.** The Mexican hat and dilated shifts.

and that $C_g = 2\pi$.

The parameter $b$ in $g_{a,b}$ gives the position of the wavelet, while the dilation parameter $a$ governs its frequency. For $|a| \ll 1$, the wavelet $g_{a,b}$ is a very highly concentrated "shrunken" version of $g$, with frequency content mostly in the high frequency range. Conversely, for $|a| \gg 1$, the wavelet $g_{a,b}$ is very much spread out and has mostly low frequencies.

Now one can define the *continuous wavelet transform* of a function $f$ by writing

$$F(a,b) = \int_{-\infty}^{\infty} f(x)\overline{g_{a,b}(x)}\,dx.$$

For all $f \in L^2(\mathbb{R})$ we have an *inversion formula* for the continuous wavelet transform, that is we can recover $f$ when $F(a,b)$ is given. The inversion formula is a double integral over the parameters $a$ and $b$. It reads:

$$f(x) = C_g^{-1} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} F(a,b) g_{a,b}(x)\,db \right] \frac{da}{a^2}$$

where $C_g$ is introduced earlier.

In Grossmann, Kronland-Martinet & Morlet [8] one can find how to interpret the role of the parameters $a$ and $b$, by depicting horizontal and vertical lines in the $(a,b)$−plane, and by identifying these with quantities from Fourier analysis.

An important topic in wavelet theory is the discretization of $F(a,b)$. We would like to have the wavelet $g$ such that $f$ already be can recovered from $F$−values on a certain grid in the $(a,b)$−plane, that is from the values

$$F(2^{-j}, 2^{-j}k), \quad j,k \in \mathbb{Z}.$$

In particular it is very satisfactory if $g$ equals a function $\psi$ which has the property that the wavelets

$$2^{j/2}\psi\left(2^j x - k\right), \quad j,k \in \mathbb{Z}$$

constitute an orthonormal basis of $L^2(\mathbb{R})$. The Mexican hat does not have this property.

A wavelet $\psi$ that does have this property is called the *mother of the wavelets*. Often, prior to the construction of $\psi$, one constructs a function $\phi$ such that, among others, the functions $\{\phi(x-k)\}$, $k \in \mathbb{Z}$ constitute an orthonormal system. One calls $\phi$ the *father of the wavelets*, when this orthonormal system can be supplemented to a full orthonormal basis of $L^2(\mathbb{R})$ with the functions

$$2^{j/2}\psi\left(2^j x - k\right), \quad j \in \mathbb{Z}_+, \quad k \in \mathbb{Z}$$

(for some mother wavelet $\psi$).

## §3 A few remarks on multiresolution analysis

This theory can be nicely described in the framework of the so-called *multiresolution analysis of* $L^2(\mathbb{R})$. A multiresolution analysis consists in breaking up $L^2(\mathbb{R})$ into a ladder of closed subspaces $V_j$:

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots,$$

with $V_m \to L^2(\mathbb{R})$ as $m \to -\infty$. The subspaces $V_m$ are all obtained from $V_0$ by a *dilation rule*.

To explain this, we assume that we have chosen the function space $V_0$ with an appropriate basis. We require next that the larger space $V_{-1}$ contains all functions of $V_0$, according to the following rule:

$$f \in V_0 \Longleftrightarrow f(2\cdot) \in V_{-1}.$$

So when $f$ is an oscillating function in $V_0$, the function that oscillates 'twice as fast' is an element of $V_{-1}$. Intuitively, $V_{-1}$ is 'twice as large' as $V_0$. On the other hand, we prescribe that $V_1$ contains all functions $f$ of $V_0$ that oscillate 'twice as slow':

$$f \in V_0 \Longleftrightarrow f(2^{-1}\cdot) \in V_1.$$

All other spaces are constructed in the same way. We prescribe

$$f \in V_m \Longleftrightarrow f(2\cdot) \in V_{m-1}, \quad m \in \mathbb{Z}.$$

In wavelet theory it will be assumed that $V_0$ is generated by the integer translates $\phi_{0n}(x) = \phi(x-n)$ of one single function $\phi$, the father. When indeed the set of functions $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ constitute an orthonormal basis of $V_0$ (this can be made more general into the direction of a Riesz basis), then each $f \in V_0$ can be written as

$$f = \sum_{n=-\infty}^{\infty} a_n\,\phi_{0n}, \quad \phi_{0n}(x) = \phi(x-n).$$

Now, since $\phi \in V_0$, and $V_0 \subset V_{-1}$, we have $\phi \in V_{-1}$. But the above dilation property then implies that $\phi(2^{-1}\cdot) \in V_0$. It follows that we can expand

$$\phi(\tfrac{1}{2}x) = \sum_{n=-\infty}^{\infty} c_n \, \phi(x-n), \quad x \in \mathbb{R},$$

for some coefficients $\{c_n\}$. We can rewrite this in the form

$$\boxed{\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \, \phi(2x-n), \quad x \in \mathbb{R},}$$

where the square root in front of the sum is for normalization. The numbers $h_n$ are called the *filter coefficients* of the function $\phi$.

We infer that, after a few assumptions on the function $\phi$ in order to generate a multiresolution analysis, we arrive at an exciting property of this function: it satisfies the above two-scale difference equation. This name reflects the fact that the equation relates translates of scaled versions of the same function, involving two different scales.

When we require a normalization, say $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$, then we obtain

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} \phi(x)\,dx &= \sqrt{2} \sum h_n \int_{-\infty}^{\infty} \phi(2x-n)\,dx \\ &= \frac{1}{\sqrt{2}} \sum h_n \int_{-\infty}^{\infty} \phi(2x-n)\,d(2x-n) = \frac{1}{\sqrt{2}} \sum h_n, \end{aligned}$$

yielding

$$\sum_{n=-\infty}^{\infty} h_n = \sqrt{2}.$$

The associated wavelet $\psi$ (the mother of the wavelets) is then generated through $\phi$ by the definition

$$\boxed{\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_n \, \phi(2x-n), \quad g_n = (-1)^n \overline{h_{1-n}}}$$

Other related functions are

$$\phi_{m,n}(x) = 2^{-\frac{1}{2}m} \phi(2^{-m}x - n), \quad m, n \in \mathbb{Z},$$

and the associated wavelets

$$\psi_{m,n}(x) = 2^{-\frac{1}{2}m} \psi(2^{-m}x - n), \quad m, n \in \mathbb{Z}.$$

In the previous section, we assumed that the wavelet $g$ has zero mean value; requiring this for the present wavelet $\psi$, we observe that the coefficients $h_n$ should satisfy

$$\sum_{n=-\infty}^{\infty} (-1)^n h_n = 0.$$

Under certain conditions on $\phi$, the wavelet $\psi$ inherits the mean zero value from its generator $\phi$.

## §4 Simple solutions of dilation equations

The two-scale difference equation ( *i.e.*, the *dilation equation*) for $\phi$ of the previous section becomes especially interesting when only a finite number of the coefficients $h_n$ are non-zero. This has important consequences for the construction of *compactly supported orthogonal wavelets.* We have obtained the dilation equation through the multiresolution approach. We can start differently, and take the equation

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n \,\phi(2x-n), \quad x \in \mathbb{R},$$

as the first relation available. Note that we use $c_n = \sqrt{2}h_n$, where the numbers $h_n$ are introduced in the previous section; in later chapters it is more convenient to work with $h_n$.

We can look for solutions of the dilation equation for a given set $\{c_n\}$. We assume that $\sum c_n = 2$, as a normalization. It appears that, when a solution of the dilation equation exists, it is unique. The function $\phi$ solving the dilation equation is also called the *scaling function*, and, as we learned in the previous section, the father of the wavelets.

In Daubechies [4] the first constructions of orthonormal bases of wavelets with compact support are given. The theory for finite equations is given in a more general setting in Daubechies & Lagarias [6]. That is, for equations of the form

$$f(x) = \sum_{n=0}^{N} c_n \, f(\alpha x - \beta_n),$$

where $\alpha > 1$ and $\beta_0 < \beta_1 < \cdots < \beta_n$ are real constants, and $x$ takes real values, while the $c_n$ are complex constants.

We mention a few solutions for the dilation equation

$$\phi(x) = \sum_n c_n \,\phi(2x-n), \quad \sum_n c_n = 2,$$

and we give the corresponding wavelet

$$\psi(x) = \sum_n (-1)^n c_{1-n}\, \phi(2x-n),$$

(for real $c_n$), where only a finite number of $h_n$ is assumed to be different from zero. The following examples and the pictures are taken from Strang [11].

**Example 1.** Take $c_0 = 2$ and all remaining $c_n$ equal to zero. The Dirac delta function satifies $\delta(x) = 2\delta(2x)$, and hence is a solution. The Dirac function is not a regular function; the idea that we have is a 'function' with compact support (of length zero): a needle at the origin.

**Example 2.** Take $c_0 = c_1 = 1$ and the remaining $c_n$ equal to zero. A solution of the dilation equation is the box function

$$\phi(x) = \begin{cases} 1, & \text{if } 0 \le x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

with a support of unit length. The corresponding wavelet is $\psi(x) = \phi(2x) - \phi(2x-1)$, the so-called *Haar wavelet*, explicitly given by

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The box function and the Haar wavelet are orthogonal with respect to their own translates:

$$\int_{-\infty}^{\infty} \phi(x)\, \phi(x-n)\, dx = 0, \quad \int_{-\infty}^{\infty} \psi(x)\, \psi(x-n)\, dx = 0, \quad n \in \mathbb{Z} \setminus \{0\}.$$

The resulting $\psi_{mn}$,

$$\psi_{m,n}(x) = 2^{-\frac{1}{2}m} \psi\left(2^{-m}x - n\right), \quad m, n \in \mathbb{Z},$$

have the desired property: they constitute an orthonormal basis for $L^2(\mathbb{R})$. Historically the Haar function was the original wavelet (but with poor approximation).

**Example 3.** Take $c_0 = \frac{1}{2}, c_1 = 1, c_2 = \frac{1}{2}$ and the remaining $c_n$ equal to zero. A solution of the dilation equation is the hat function

$$\phi(x) = \begin{cases} x, & \text{if } 0 \le x \le 1, \\ 2-x, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise,} \end{cases}$$

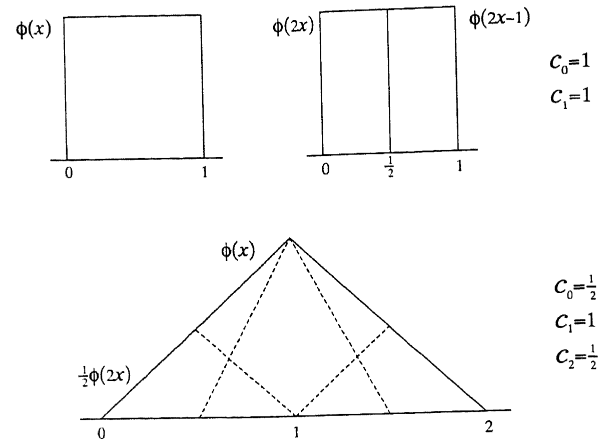


**Figure 2.** The box function and the hat function.

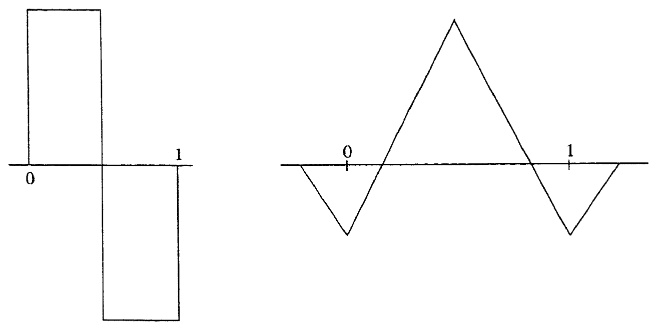


**Figure 3.** The wavelets for the box function and the hat function.

which has a support of two unit lengths. It is the linear spline function. The corresponding wavelet is continuous, and given by

$$\psi(x) = \phi(2x) - \tfrac{1}{2}\phi(2x-1) - \tfrac{1}{2}\phi(2x+1).$$

The support of this function is $[-1, 2]$. From a picture of the hat function it is easily seen that $\phi(x)$ and $\phi(x \pm 1)$ are not orthogonal on $\mathbb{R}$. Hence the translates of $\phi$ cannot constitute an orthogonal set. We remark that the box function and the hat function are related by convolution. That is, let us denote the *convolution* of two functions $f, g$ by $(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$, provided that the integral exists. Let us denote the box function and the hat function of the previous examples by

$\phi_B, \phi_H$, respectively. Then we have

$$\int_{-\infty}^{\infty} \phi_B(x-y)\phi_B(y)\,dy = \int_0^1 \phi_B(x-y)\,dy = \int_{x-1}^{x} \phi_B(u)\,du = \phi_H(x).$$

In other words: $\phi_H = \phi_B * \phi_B$.

**Example 4.** Take $c_0 = c_3 = \frac{1}{4}, c_1 = c_2 = \frac{3}{4}$ and the remaining $c_n$ equal to zero. A solution of the dilation equation is the function

$$\phi(x) = \begin{cases} x^2, & \text{if } 0 \le x \le 1, \\ 2 - 2x^2 + 6x - 3, & \text{if } 1 \le x \le 2, \\ (3-x)^2, & \text{if } 2 \le x \le 3, \\ 0, & \text{otherwise,} \end{cases}$$

which has a support of three unit lengths. It is the quadratic spline function, a $C^1$–function. The corresponding wavelet is also a $C^1$–function, and given by

$$\psi(x) = \tfrac{3}{4}\phi(2x) - \tfrac{1}{4}\phi(2x-1) - \tfrac{3}{4}\phi(2x+1) + \tfrac{1}{4}\phi(2x+2).$$

The support of this function is $[-1, 2]$.

Again, this spline follows from convolution. Now we have $\phi = \phi_B * \phi_B * \phi_B = \phi_B * \phi_H$, a three-fold convolution of the box function, or a simple convolution of box and hat function. In a similar way higher order splines can be constructed by taking $n$–fold convolutions of the box function. All these splines are solutions of a corresponding dilation equation, of which the coefficients easily follow from the above pattern. The next one, the cubic spline, has coefficients $1/8, 4/8, 6/8, 4/8, 1/8$; note the binomial structure.

Observe from the examples that, when $n$ successive coefficients are given, with the remaining equal to zero, the solution $\phi$ is compactly supported on an interval $[0, n-1]$ of length $n-1$. Starting from spline functions one can, in fact, construct wavelets with an arbitrarily high number of continuous derivatives. In these constructions the initial function $\phi$ is compactly supported, but the $\phi(\cdot - n)$ are not orthogonal, as illustrated by Example 3 and Example 4. It is possible (see Daubechies [4]) to construct an associated function $\tilde{\phi}$, of which the shifts $\tilde{\phi}(\cdot - n)$ constitute an orthonormal set, but the function $\tilde{\phi}$ is not compactly supported, resulting in a non-compactly supported wavelet $\psi$.

### §5 How to construct solutions of dilation equations?

In the above examples we assigned the $c_n = \sqrt{2}\, h_n$ and we proposed the solution, of the dilation equation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n\, \phi(2x-n), \quad x \in \mathbb{R}$$

without construction. The question may arise, how to construct the solution $\phi$ from a given set of $h_n$. The answer is obtained by Fourier analysis. Define

$$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) e^{-ix\xi}\, dx,$$

then from the dilation equation we obtain

$$\begin{aligned}\widehat{\phi}(\xi) &= \sqrt{2}\sum_n h_n \int_{-\infty}^{\infty} \phi(2x-n) e^{-ix\xi}\, dx \\ &= H\left(\tfrac{1}{2}\xi\right) \int_{-\infty}^{\infty} \phi(y) e^{-iy\xi/2}\, dy \\ &= H\left(\tfrac{1}{2}\xi\right) \widehat{\phi}\left(\tfrac{1}{2}\xi\right).\end{aligned}$$

The symbol $H(\xi) = \frac{1}{\sqrt{2}} \sum h_n e^{-in\xi}$ is the crucial function in this theory. Note that $H(0) = 1$. The above result may be iterated, and we find

$$\widehat{\phi}(\xi) = \left[\prod_{j=1}^{N} H\left(2^{-j}\xi\right)\right] \widehat{\phi}\left(2^{-N}\xi\right), \quad N = 1, 2, \ldots .$$

Taking $N \to \infty$ and observing that $\widehat{\phi}(0) = \int \phi(x)\, dx = 1$, we find

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} H\left(2^{-j}\xi\right).$$

Take, as in Example 1, $c_0 = 2, h_0 = \sqrt{2}$, then we find $H(\xi) = 1$ and $\widehat{\phi}(\xi) = 1$, indeed, the transform of the Dirac delta function. For $c_0 = c_1 = 1, h_0 = h_1 = 1/\sqrt{2}$ (Example 2) the products of the $H$−function are geometric series:

$$H(\tfrac{1}{2}\xi)H(\tfrac{1}{4}\xi) = \tfrac{1}{4}(1+e^{-i\xi/2})(1+e^{-i\xi/4}) = \frac{1-e^{-i\xi}}{4(1-e^{-i\xi/4})}.$$

Taking a product with $N$ such $H$−functions, and letting $N \to \infty$, this approaches $[1-\exp(-i\xi)]/(i\xi)$. So we obtain

$$\widehat{\phi}(\xi) = \int_0^1 e^{-ix\xi}\, dx,$$

the transform of the box function. Observe that the above methods lead to the curious formula

$$\prod_{j=1}^{\infty} \cos(2^{-j}x) = \frac{\sin x}{x}.$$

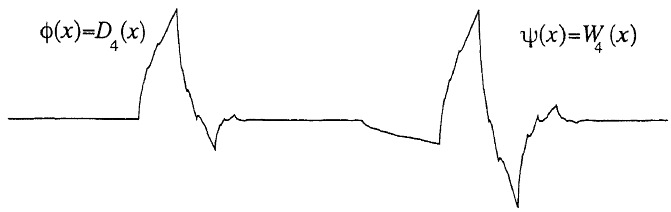


**Figure 4.** The Daubechies $D_4$ and corresponding wavelet $W_4$.

## §6 Daubechies' wavelets

The breakthrough in the construction of compactly supported wavelet, providing at the same time an orthonormal set of wavelets for $L^2(\mathbb{R})$, came in Daubechies [4]; (see also [5]). We give her solution $D_4$, which reads

$$\phi(x) = \sqrt{2}\,[h_0\phi(2x) + h_2\phi(2x-1) + h_2\phi(2x-2) + h_3\phi(2x-3)]$$

with

$$h_0 = \frac{\sqrt{2}}{8}\left(1+\sqrt{3}\right),\; h_1 = \frac{\sqrt{2}}{8}\left(3+\sqrt{3}\right),$$
$$h_2 = \frac{\sqrt{2}}{8}\left(3-\sqrt{3}\right),\; h_3 = \frac{\sqrt{2}}{8}\left(1-\sqrt{3}\right).$$

To draw a picture of this, and of the other scaling functions that solve a dilation equation of the form $\phi(x) = \sqrt{2}\sum h_n\phi(2x-n)$, one may iterate

$$\phi_j(x) = \sqrt{2}\sum h_n\phi_{j-1}(2x-n), \quad j = 1, 2, \ldots,$$

with the box function as starting function $\phi_0(x)$. Then $\phi_j(x) \to \phi(x)$ as $j \to \infty$. In Figure 4, we give a picture of $D_4$ and of the corresponding wavelet $W_4$, defined by

$$\psi(x) = \sqrt{2}\,[h_1\phi(2x) - h_0\phi(2x-1) - h_2\phi(2x+1) + h_3\phi(2x+2)]\,.$$

The coefficients $h_0, h_1, h_2$, and $h_3$ satisfy the following set of equations

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1,$$
$$h_0h_2 + h_1h_3 = 0,$$
$$h_3 - h_2 + h_1 - h_0 = 0,$$
$$0h_3 - 1h_2 + 2h_1 - 3h_0 = 0.$$

In later chapters the use and construction of $D_4$, and of other Daubechies wavelets, will receive detailed attention.

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