From Number Guessing Games

## to
Security Protocol Analysis

and
Back

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Abstract

The talk will explain what goes on in number guessing games, and present a new formal representation (in terms of Kripke semantics) of information update in such games. This can be used in real life settings: one application is the analysis of cryptographic security protocols. I will look at a protocol for secret key distribution over an insecure network, and show how this is analysed with Kripke semantics. Finally, I address the question whether there are lessons to be learned for natural language analysis.
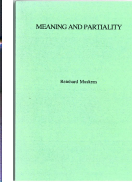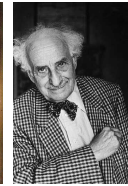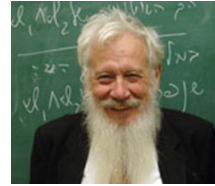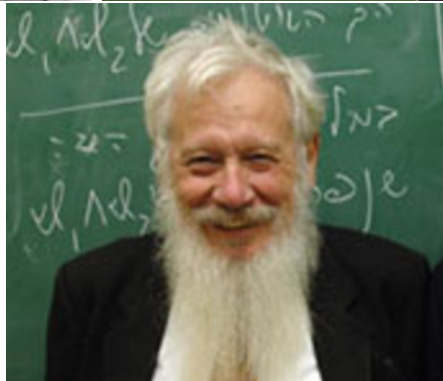
Language and Communication

No essential difference between formal and natural languages for communication.

What does it mean to know a proposition?

How to Model Ignorance?

Probability, Betting, Ignorance

Probability, Betting, Ignorance

Christiaan Huygens, 1629–1695

Van
Rekeningh
in
Spelen van Geluck

1660

beroemden Heer Christianvs Hvgenivs
aengaende 't reeckenen in Spelen van Geluck uyt-
gevonden ende my in schrift van hem mede gedeelt
is, alhier met deffelfs brief, in plaets van 'tgeene my
overig was, by-voegde. Welck sijn Tractaet ick dan
UE. des te aengenamer acht te sullen wesen, als
'tgeen daer in verhandelt wordt te subtijlder en on-
gemeender sal bevonden worden; insonderheyt
door dien hy tot deffelfs vinding deselve *Analysis*,
welckers fondamenten hy eertijts van my geleert
heeft, als ick, gebruyckt; en alsoo de bevlytigers
van die de weg baent om diergelijcke Voorstellen
te ontbinden. Waer in, soo ick nevens mijnen an-
dren arbeyt aen U, Beminde Leser, in dese soort van
Studie genoegsame stof van oeffening gegeven heb-
be; soo sult ghy daer uyt (gelijck ick hoop) mij-
ne bereytwilligheyt t'uwaerts konnen afnemen, en
dienvolgende oock mijnen arbeyt, t'uwen en
der Studien besten aengenomen, ten
goeden duyden. Vaert wel.

Aen

FRANCISCUS van SCHOOTEN.

Mijn Heer,

Naer dien ick weet dat V E., de loffelijcke vruchten van
sijn vernuft ende arbeyt in 't licht gevende, onder anderen
dit oogmerck heeft: namentlijck, om door de verscheyden-
heyt der verhandelde stoffen te betoonen hoe wijt ende uytge-
nuende Konst van Algebra sich uytstreckt; soo en twijffele ick oock
niet, of het geene ick van de Rekeningh in Spelen van geluck beschre-
ven heb, sal tot V E. opset niet ondienstig zijn. Want soo veel te
swaerder als het scheen, door reden te konnen bepalen het geene onseker
is ende het geval onderworpen, soo veel te meer verwonderinghs
waerdigh sal die weetenschap schijnen, waer door sulcx kan werden
te weege gebracht. Dewijl ick den oop V E. versoeck ende aen-maeninghe,
dese Rekening eerst heb beginnen by geschrift te stellen, ende V E.
deselve waerdigh acht om te gelijck met sijne diepsinnige vonden in
't licht te komen; soo en sal ick niet alleen het selve gaerne toestaen,
maer oock tot mijn voordeel duyden, dat die op dese maniere te voor-
schijn werdt gebracht. Want of sommige mochten denken dat ick
ontrent geringhe dingen en van weynigh gewichte mijn moeyte be-
steedt hadde, soo en sullen sij nochtans niet 't eenemael voor onnut en-
de onprijselijck houden, het geene V E. in dier voegen als voor het
sijne is aen-nemende, en niet sonder arbeyt uyt onse spraeck in de La-
tijnsche heeft overgeset. Alhoewel ick wil gelooven, soo iemandt dese
dinghen wat naerder begint in te sien, dat hy haest sal bevinden, geen
enckel spel te zijn het geene hier wert verhandelt, maer datter de be-
ginselen en gronden geleyt werden van een seer aerdige en diepe spe-
culatie. Soo sullen oock, meyn ick, de Voorstellen die in dese Ma-
terie voorvallen, geensins lichter als die van Diophantus geacht
werden, doch wel vermaeckelijcker misschien, door dienst iets meer
inhouden als bloote eygenschappen der getallen. Voorts is te weeten,

dat

Huygens on the Foundations of Probability

"Ick neeme tot beyder fondament, dat in het speelen de kansse, die yemant ergens toe heeft, even soo veel weerdt is als het geen, het welck hebbende hy weder tot deselfde kansse kan geraecken met rechtmatigh spel, dat is, daer in niemandt verlies geboden werdt. By exempel. So yemandt sonder mijn weeten in d'eene handt 3 schellingen verbergt, en in d'ander 7 schellingen, ende my te kiesen geeft welck van beyde ick begeere te hebben, ick segge dit my even soo veel weerdt te zijn, als of ick 5 schellingen seecker hadde. Om dat, als ik 5 schellingen hebbe, ick wederom daer toe kan geraecken, dat ick gelijcke kans sal hebben, om 3 of 7 schellingen te krijgen, en dat met rechtmatigh spel: gelijck hier naer sal betoont werden."

Translation

"I take as the foundation of both [calculating what non-finished hazard games are worth, and calculating winning chances in such games] that in playing the chance that someone has in some matter, is worth just as much as the amount that, if he possesses it, will give him the same chances in a fair game, that is a game where no loss is offered to anyone. For instance. Suppose someone without my knowing hides in one hand 3 shillings, and in the other 7 shillings, and he offers me the choice between the two hands. Then I would say that this offer is worth the same as having 5 shillings for sure. Because, if I have 5 shillings, I can wager them in such manner that I have equal chances of getting 3 or 7 shillings, and that in a fair game, as will be explained hereafter."

Huygens' First Proposition

"Als ick gelijcke kans hebbe om a of b te hebben, dit is my so veel weerdt als $(a + b)/2$"

"If I have equal chances of getting a and b, then to me this is worth $(a + b)/2$."

Next, Huygens gives an explanation in terms of a game, and uses this explanation to prove the proposition.

Explanation and Proof in Terms of a Game

"Want (a+b)/2 hebbende, soo kan ick dat tegen een ander waegen die mede (a+b)/2 sal insetten, ende bedingen dat die het spel wint, den anderen sal a geven. Waer door ick gelijcke kans sal bekomen om a te hebben, te weeten, als ick verlies, of b als ick win; want alsdan soo treck ick a+b dat in-geset is, ende geef hem daer van a."

Tr:

"For having $(a+b)/2$, I can wager that against someone else who also stakes $(a + b)/2$, with the stipulation that the one who wins the game will give the other a. This will give me equal chance of getting a, namely if I lose, or b if I win; because in that case I get the stake of $a + b$, and from this I give him a."

Example in Numbers

"[…] In getaelen. Indien ick gelijcke kans heb om 3 te hebben of 7, soo is door dit Voorstel mijn kansse 5 weerdt; ende het is seecker dat ick 5 hebbende weder tot de selfde kansse kan geraecken. Want speelende om de selve tegen een ander die daer 5 tegen set, met beding dat de geene die wint den anderen 3 sal geven; soo is dit rechtmaetig spel, ende het blijckt dat ick gelijcke kans hebbe om 3 te hebben, te weeten, als ick verlies, of 7 indien ick win; want alsdan treck ick 10, daer van ick hem 3 geef."

Example in Numbers: Translation

"[...] In numbers. If I have equal chances to have 3 or 7, then by my Proposal this chance is worth 5; and it is sure that if I have 5, I will get to the same chance. Because putting 5 at stake against someone who stakes 5 against it, with condition that the one who wins will give the other 3, one has a fair game, and it becomes clear that I have equal chance of getting 3, namely, if I lose, or 7 if I win; because if I win I draw 10, of which I give 3 to him."
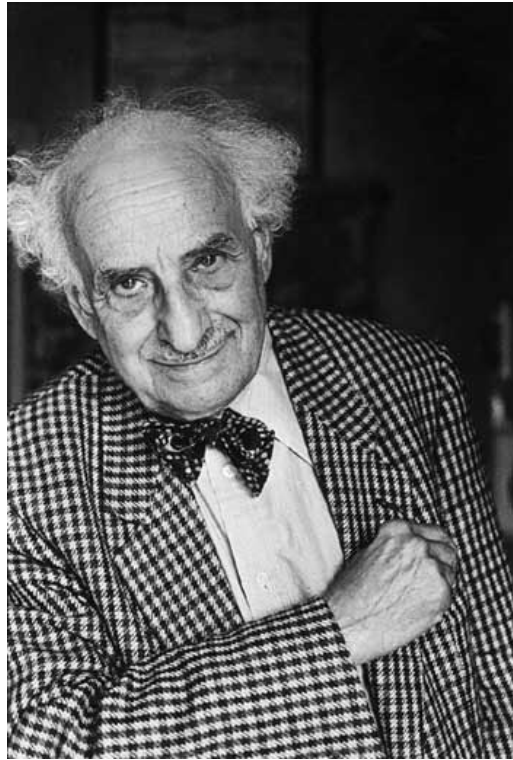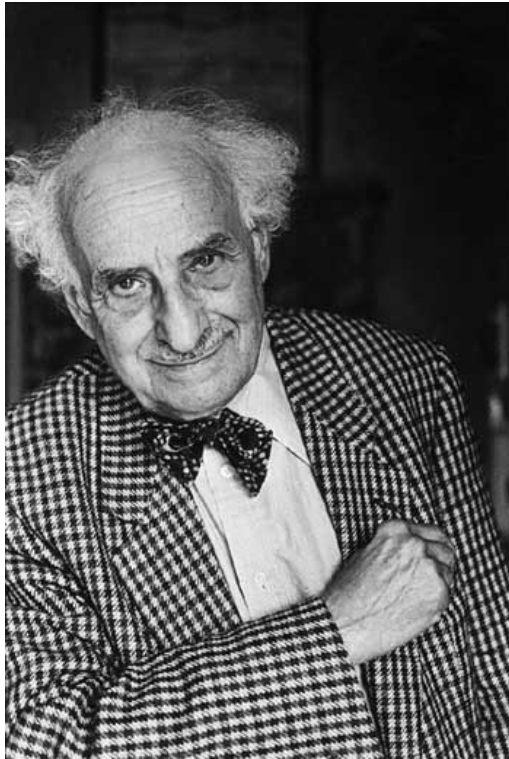
Hans Freudenthal, 1905–1990

Comment of Hans Freudenthal

"This is not a trivial transformation. The original situation for which the expectation has to be calculated is that of an individual in a game-like situation, yet with no adversary identified (say a lottery). It is reconstructed as an n persons' game (if n is the number of chances proposed) with equal stakes and the appropriate stipulations, which by virtue of their mutuality are fair. The value of the stakes under consideration is just the expectation.

"Equal chance," as used by Huygens, does not beg the question either. In Huygens' model, where the player is given the free choice to take one hand or the other, or, more generally, to decide himself which one of a number of possibilities he chooses, equal chance is validly defined by the complete ignorance and the free will of the player."
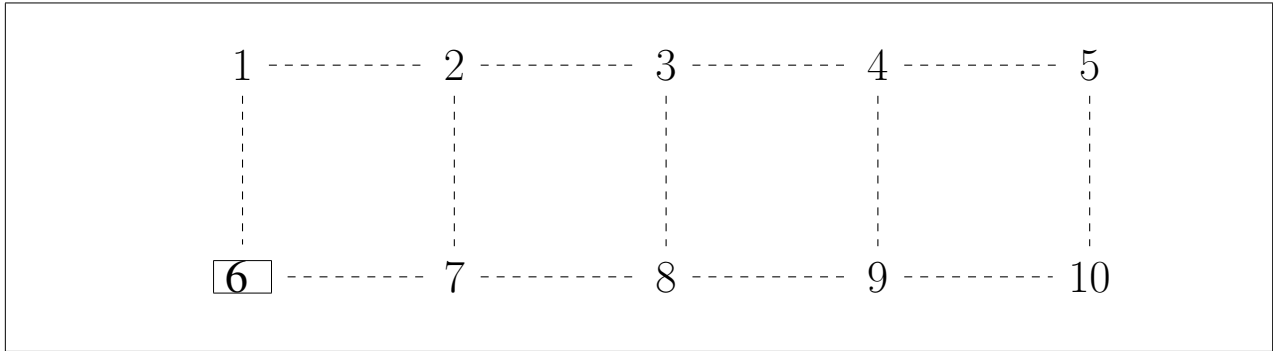
[Fre80]

What does it mean to know a number?

Jan, Gaia and Rosa Play the Number Guessing Game
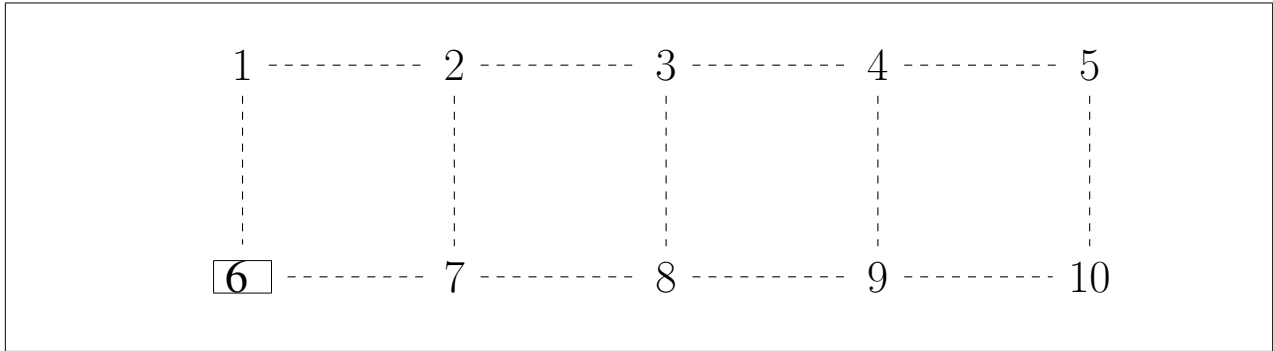
Number Guessing Game, Naive Version

- Jan says: "I have a number in mind, in the range from one to ten. You may take turns guessing. Whoever guesses the number first gets the toy. It is Gaia's turn to start, for last time we played this game Rosa started the guessing."

- Gaia and Rosa agree . . .

- After a number of rounds, Jan announces: "Rosa, you have won."
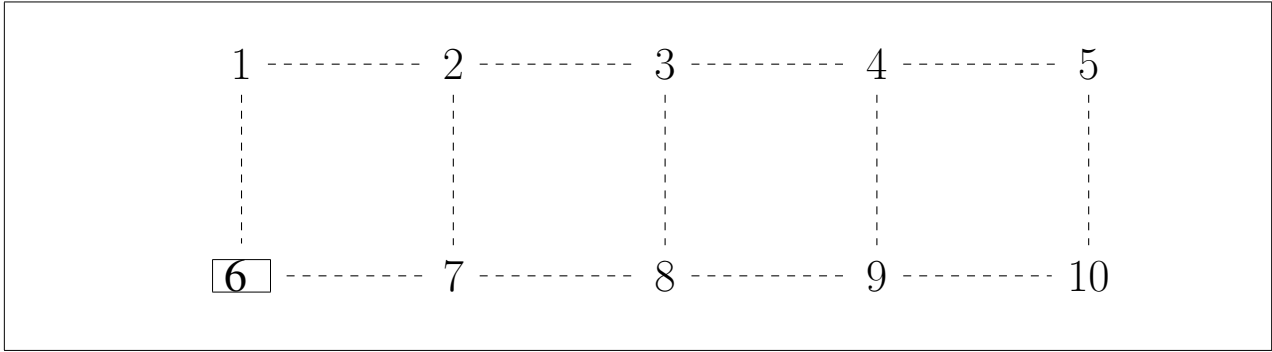
Number Guessing: Naive Representation
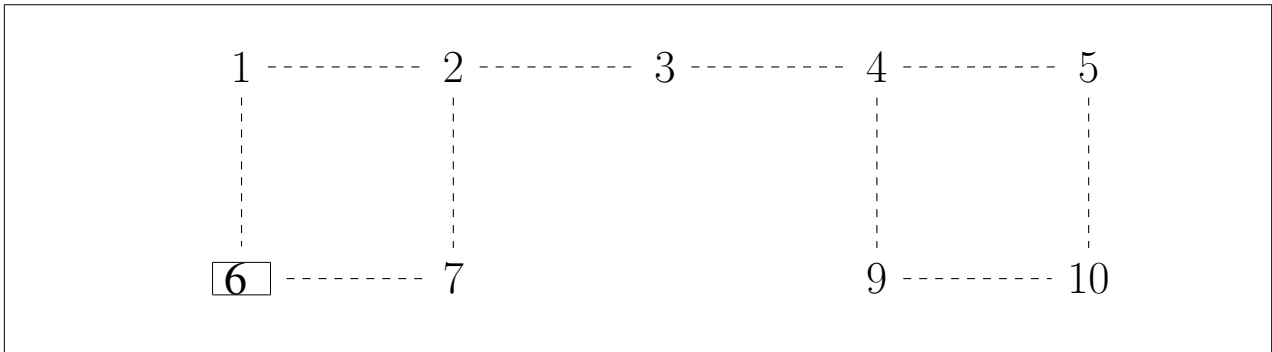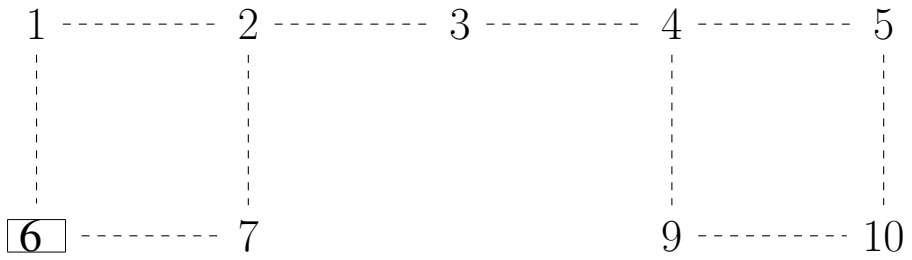
Number Guessing: Naive Representation

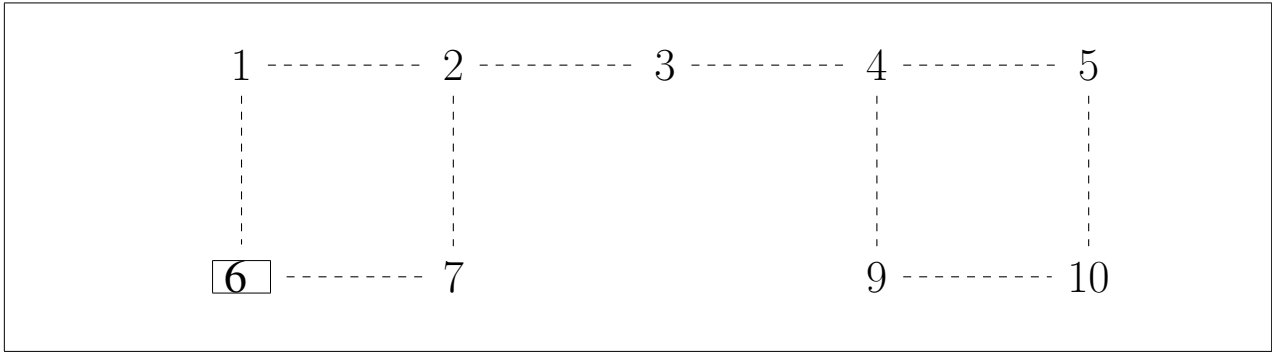1 ----- 2 ----- 3 ----- 4 ----- 5

6 ----- 7 ----- 8 ----- 9 ----- 10

"eight" ... "no"

Number Guessing: Naive Representation


```
1 -------- 2 -------- 3 -------- 4 -------- 5
|          |          |          |          |
[6] ------ 7 -------- 8 -------- 9 -------- 10
```

"eight" ... "no"


```
1 -------- 2 -------- 3 -------- 4 -------- 5
|          |                     |          |
[6] ------ 7                     9 -------- 10
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1 ------ 2 ------ 3 ------ 4 ------ 5

6 ------ 7   9 ------ 10

1 --------- 2 --------- 3 --------- 4 --------- 5

6 --------- 7   9 --------- 10

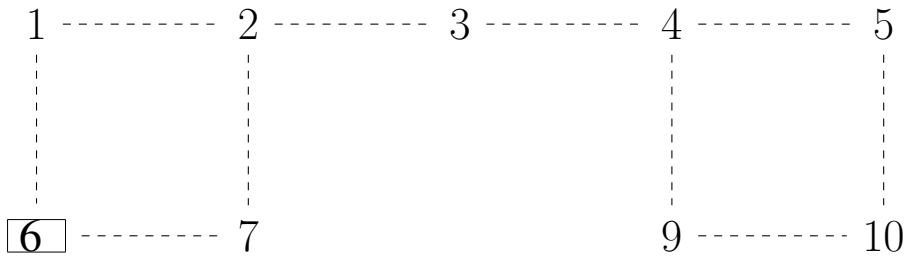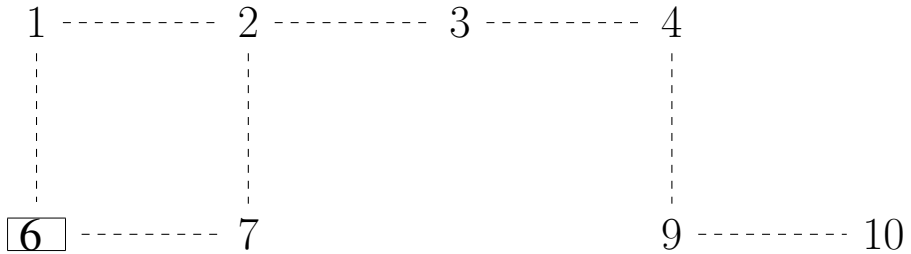"five" ... "no"

"five" ... "no"

Number Guessing Game, More Sophisticated Version

- Jan says: "I have a number in mind, in the range from one to ten. You may take turns guessing. Whoever guesses the number first gets the toy. It is Gaia's turn to start, for last time we played this game Rosa started the guessing."

- Gaia does not agree. "How can we know you are not cheating on us? Please write down the number, so you can show it to us afterwards as a proof."

- Jan writes a number on a piece of paper, hidden from Gaia and Rosa.

- After a number of rounds, Jan announces: "Rosa, you have won."

- Jan shows the piece of paper as a proof.

Jan, Gaia and Rosa Play the Number Guessing Game (again)

Jan, Gaia and Rosa Play the Number Guessing Game (again)

register

Registers, Fixing a Number

At the point where either Gaia or Rosa demands that the secret number gets written down, and that Jan shows it as a proof that the procedure was honest, the important notion of a register arises.

The register allows Jan to prove that he knew (had fixed) the number beforehand.

What Does it Mean to Know a Number?

We are working in the context of epistemic logic.

> To know an integer number n is to be able distinguish a world where n is written in some register from all possible worlds where an integer number n' different from n is written in that register.

This can be made more precise, by clearly distinguishing between a register and its contents.

And the representation can be made more concise, by lumping together all situations where the register does not contain the number.

> To know an integer number n is to know the contents $\dot{n}$ of register n, not to know a number is not to have access to register n.

Jan knows a number

Jan knows n, the two children do not know n. We use n for the register, $\dot{n}$ for the value of that register (its contents).

This leads to the following register model.

$$0 : n \qquad 1 : \overline{n} \qquad \begin{array}{l} n \mapsto 6 \\ \overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\} \end{array}$$
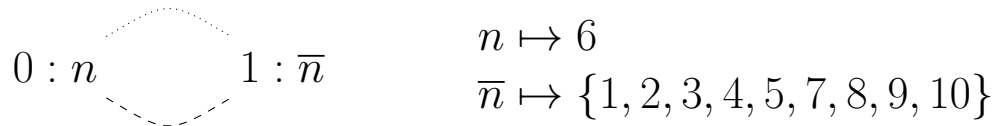
0 is a world that has register n, with number $\dot{n} = 6$ stored in it, while 1 differs from 0 in that it also has this register, but with all the possible values in it that differ from $\dot{n}$.
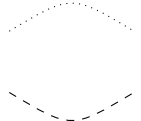
Gaia (dots) and Rosa (dashes) do not have access to the register.
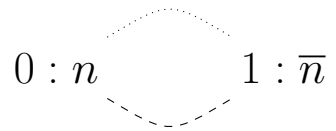
$$0 : n \qquad 1 : \overline{n}$$

$$n \mapsto 6$$
$$\overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$$

$$0 : n \qquad 1 : \overline{n}$$

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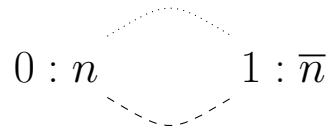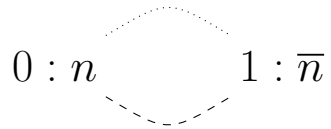
"ten" ... "no"

$$0 : n \qquad 1 : \overline{n}$$

$$n \mapsto 6$$
$$\overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$$

"ten" ... "no"

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$$0 : n \qquad 1 : \overline{n}$$

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"ten" ..."no"

$$0 : n \qquad 1 : \overline{n}$$

$$n \mapsto 6$$
$$\overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9\}$$

"six" ..."yes"

$$0 : n \qquad 1 : \overline{n}$$

$$n \mapsto 6$$
$$\overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$$

"ten" ... "no"

$$0 : n \qquad 1 : \overline{n}$$
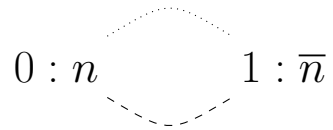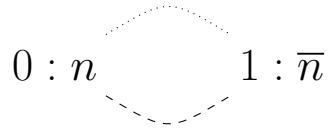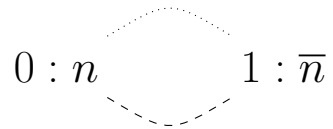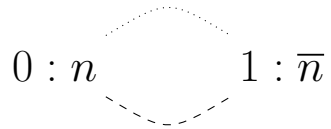
$$n \mapsto 6$$
$$\overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9\}$$

"six" ... "yes"

$$0 : n \qquad\qquad n \mapsto 6$$

Gaia Makes a Guess

Use g for the guess of Gaia. Gaia knows her guess, but has not yet revealed it to the others: solid lines for Jan, dotted lines for Gaia, dashed lines for Rosa.

$$0 : ng \qquad 1 : \overline{n}g$$

$$2 : n\overline{g} \qquad 3 : \overline{ng}$$

$$n \mapsto 6, \overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$$
$$g \mapsto 5, \overline{g} \mapsto \{1, 2, 3, 4, 6, 7, 8, 9, 10\}$$
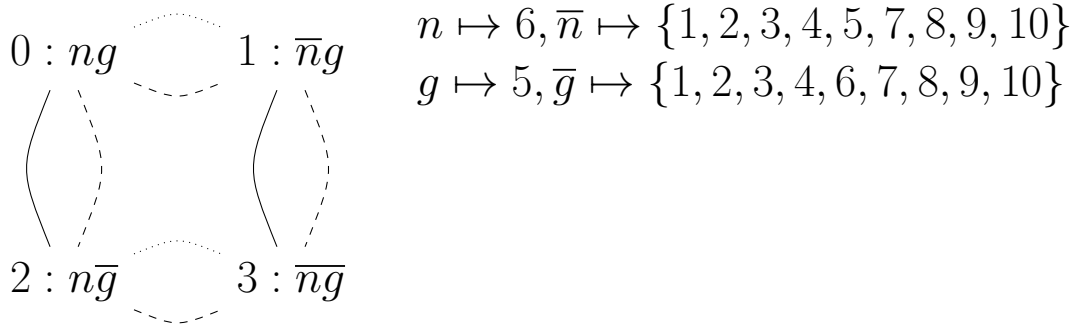
Without registers, this blows up to a model with 100 worlds.

k registers, each of m bits, blow up to $(2^m)^k$ possibilities.

Gaia Reveals Her Guess

Gaia reveals the contents of her register: $g = 5$.

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$$0 : ng \qquad 1 : \overline{n}g \qquad n \mapsto 6, \overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$$
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Gaia Reveals Her Guess

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$$0 : ng \qquad 1 : \overline{n}g \qquad \begin{aligned} &n \mapsto 6, \overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\} \\ &g \mapsto 5 \end{aligned}$$

Jan states that the guess is wrong: $n \neq g$.

Gaia Reveals Her Guess

Gaia reveals the contents of her register: $g = 5$.

$$0 : ng \qquad 1 : \overline{n}g \qquad \begin{aligned} &n \mapsto 6, \overline{n} \mapsto \{1, 2, 3, 4, 5, 7, 8, 9, 10\} \\ &g \mapsto 5 \end{aligned}$$

Jan states that the guess is wrong: $n \neq g$.

$$0 : ng \qquad 1 : \overline{n}g \qquad \begin{aligned} &n \mapsto 6, \overline{n} \mapsto \{1, 2, 3, 4, 7, 8, 9, 10\} \\ &g \mapsto 5 \end{aligned}$$

Names and the Meaning of Equality Statements [Mus89]

Names and the Meaning of Equality Statements [Mus89]

MEANING AND PARTIALITY

Reinhard Muskens

Names and the Meaning of Equality Statements [Mus89]

MEANING AND PARTIALITY

Reinhard Muskens
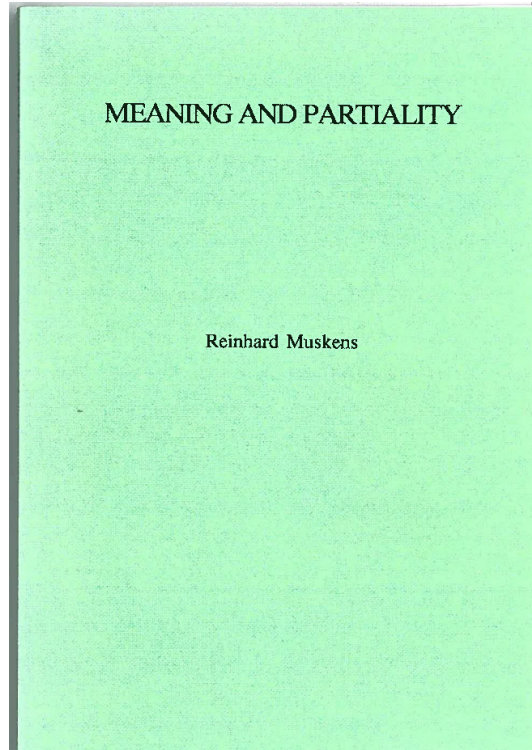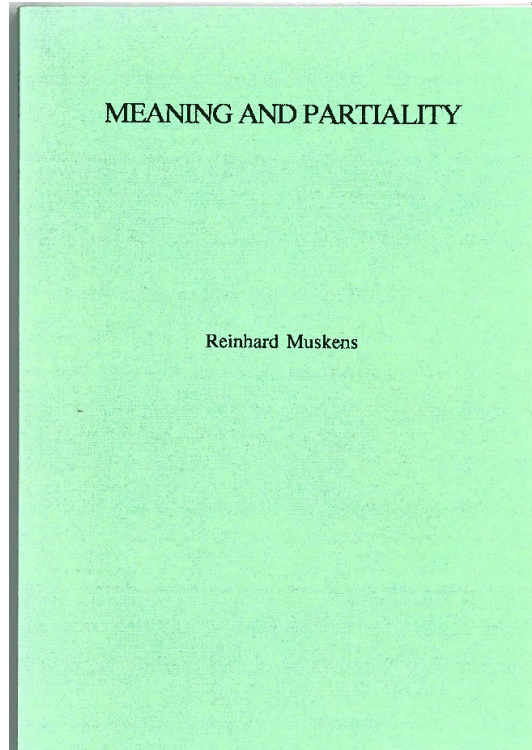
Chapter 8: Names.

Registers are like Names for Numbers

- Creating a register and writing a number in it can be viewed as a baptism.

- Register equality statements are like name equality statements.

- Compare $n = g$ with "Hesperus is Phosphorus".

- Reread Chapter 8 of Reinhard's PhD Thesis.

Truth of Equality Statements in Register Models

- It seems reasonable to represent the announcement of Gaia's guess as the announcement of the equality statement $g = \dot{g}$.

- Since $\overline{g}$ means that the register does contain a different value from $\dot{g}$, the statement $g = \dot{g}$ is false in both $\overline{g}$ worlds, so these drop out of the picture.

- Jan can respond to Gaia's guess with $n = g$ (this is what "you guessed it" means: your guess equals my number).

- $n = g$ is different from $n = \dot{n}$.

Truth of (In)equality Statements in Register Models

- The difference between $n = g$ and $n = \dot{n}$ can perhaps be seen more clearly in the negative case.

- "You have it wrong" is paraprased as $n \neq g$, not as $n \neq \dot{n}$.

- In a world where g and n are both true, the equality $n = g$ can be checked by checking whether $\dot{g} = \dot{n}$.

- In a world where g is true and n false, the equality $n = g$ means that every value of n except for $\dot{n}$ equals g, which is certainly false (except in a domain with just two numbers, in case $\dot{n}$ and $\dot{g}$ are different).

- In a world where g and n are both false the equality $n = g$ means that every value for g except $\dot{g}$ equals every value of n except $\dot{n}$, which is certainly false.

Register Language for Guessing Games

Let p range over $\mathbf{P}$ and let N range over $\mathbb{Z}$. Let i range over the set of agents I.

$$\varphi ::= \top \mid p \mid p = E \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid [!p = E]\varphi \mid [!p \neq E]\varphi$$

$$E ::= p \mid N$$

Update Operations

Revealing positive or negative information about p:

$! \, p = E$

"You guessed it"

$! \, p \neq E$

"Your guess was wrong"

Interpretations of $! \, p = E$ and $! \, p \neq E$ are action models in the sense of [BMS98].

Interpretation in Register Models

- A register assignment function is a function from registers to integers.

- A register assignment function h agrees with a possible w (notation $w \multimap h$) if $q \in V_w$ iff $h(q) = \dot{q}$.

- Examples of (Dis-)Agreement:

$$0 : ng \qquad 1 : \overline{n}g \qquad\qquad n \mapsto 6, \overline{n} \mapsto \{1..4, 7..10\}$$
$$g \mapsto 5$$
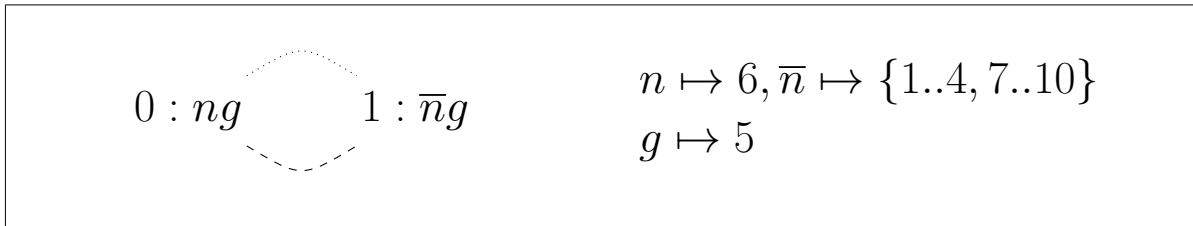
The function $h = \{n \mapsto 6, g \mapsto 5\}$ agrees with 0, but not with 1.

The function $h' = \{n \mapsto 3, g \mapsto 5\}$ agrees with 1 but not with 0.

Interpretation in Register Models: $\mathcal{M}, w, h \models \varphi$

Let h be a register assignment that agrees with w.

$$\mathcal{M}, w, h \models \top \quad \text{always}$$
$$\mathcal{M}, w, h \models p \quad \text{iff} \quad p \in V(w)$$
$$\mathcal{M}, w, h \models p_1 = p_2 \quad \text{iff} \quad h(p_1) = h(p_2)$$
$$\mathcal{M}, w, h \models p = N \quad \text{iff} \quad h(p) = N$$
$$\mathcal{M}, w, h \models \neg\varphi \quad \text{iff} \quad \text{not } \mathcal{M}, w, h \models \varphi$$
$$\mathcal{M}, w, h \models \varphi_1 \wedge \varphi_2 \quad \text{iff} \quad \mathcal{M}, w, h \models \varphi_1 \text{ and } \mathcal{M}, w, h \models \varphi_2$$
$$\mathcal{M}, w, h \models K_i\varphi \quad \text{iff} \quad (w, w') \in R_i \text{ and } w' \multimap h' \text{ imply } \mathcal{M}, w', h' \models \varphi$$
$$\mathcal{M}, w, h \models [!\, p = E]\varphi \quad \text{iff} \quad \mathcal{M}, w, h \models p = E \text{ implies } \mathcal{M}^{p=E}, w, h \models \varphi$$
$$\mathcal{M}, w, h \models [!\, p \neq E]\varphi \quad \text{iff} \quad \mathcal{M}, w, h \models p \neq E \text{ implies } \mathcal{M}^{p\neq E}, w, h \models \varphi$$
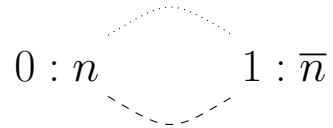
Truth and Falsity at a World

From truth at a register assignment to truth and falsity at a world:

$$\mathcal{M}, w \models \varphi \text{ iff } \forall h \text{ with } w \multimap h : \mathcal{M}, w, h \models \varphi.$$

$$\mathcal{M}, w =\!\!\mid \varphi \text{ iff } \forall h \text{ with } w \multimap h : \mathcal{M}, w, h \models \neg\varphi.$$

Truth Value Gaps

$$0 : n \qquad 1 : \overline{n}$$

$$n \mapsto 6$$
$$\overline{n} \mapsto \{1..5, 7..10\}$$

$n = 7$ is false in 0.

$n = 7$ is neither true nor false in 1.

$n \neq 7$ is true in 0.

$n \neq 7$ is neither true nor false in 1.

Now Let's Look at Probabilities

$$1 \qquad 1 \qquad\qquad 9 \qquad 4294967295$$

$$0 : n \quad\text{⋯}\quad 1 : \overline{n}$$

Numbers of agreeing assignments for register size 10.

Numbers of agreeing assignments for register size of 32 bits.

Now Let's Look at Probabilities

$$1 \qquad 1 \qquad\qquad\qquad 9 \qquad 4294967295$$

$$0 : n \qquad\qquad 1 : \overline{n}$$

Numbers of agreeing assignments for register size 10.

Numbers of agreeing assignments for register size of 32 bits.

With 64 bit size: $2^{64} - 1 = 18446744073709551615$ agreeing assignments for world 1.

Back to Christiaan Huygens

- Suppose possible guesses run from 1 to 10.

- A fair guessing game would have 10 participants. Each guess corresponds with a ticket. Winning ticket gets the stake.

- Then a ticket is worth $\frac{1}{10}$th of the stake.

- World 0 corresponds to a single ticket, world 1 corresponds to a set of 9 tickets.

- With a register size of 64 bits, a ticket is worth $1/18446744073709551616$ of the stake, and world 0 corresponds to a single ticket, world 1 corresponds to 18446744073709551615 tickets.

- This can be worked out in a logic that combines probability theory with epistemic operators [Hal03, ES].

Monte Carlo Checking of (In)Equality Statements

Suppose the register size is large enough, say 64 bits.

Then in a world where n is false, there are

$$2^{64} - 1 = 18446744073709551615$$

possibilities for the value of n.

Thus, in a world where n is false, equality statements $n = X$ will be false for almost all values X.

Equality and inequality statements can be checked by means an approximate (Monte Carlo) model checking algorithm.

Monte Carlo Style Model Checking

$$\mathcal{M}, w \overset{\cdot}{\models} \varphi \text{ iff for "enough" } h \text{ with } w \multimap h : \mathcal{M}, w, h \models \varphi.$$

Method:

- Randomly generate a list of n agreeing assignments $h_1, \ldots, h_n$ for w.

- Check $\mathcal{M}, w, h_i \models \varphi$ for each h_i in the list.

- The probability that there is disagreement among the outcomes can be made arbitrarily small.

Truth Value Gaps Almost Closed

$$1$$
$$0 : n \mapsto 6 \qquad 1 : \overline{n} \mapsto \{M = -2^{63}..2^{63} - 1\} - \{6\}$$
$$18446744073709551615$$

$n = 7$ is false in 0.

$n = 7$ is almost false in 1.

$n \neq 7$ is true in 0.

$n \neq 7$ is almost true in 1.

Article of faith of cryptanalysis: "Brute force attacks on number secrets are impossible."

Register Creation

Fix a large register size. Let $M..K$ be the bounds of the register (say $M = -2^{63}, K = 2^{63} - 1$).

Add to the language:

$$\varphi ::= [p \xleftarrow{i} N]\varphi$$

$p \xleftarrow{i} N$ is the command to link p to the integer number N with the link known only to agent i. It is assumed that N fits the register size.

Interpretation: $[\![p \xleftarrow{i} N]\!]$ as the action model (in the sense of [BMS98]):

$$\boxed{e} : p := \top, \dot{p} := N \overset{\displaystyle I - \{i\}}{\rule{6cm}{0.5pt}} e : p := \bot$$

Update Example

Start with the blissful ignorance unit model:

$$0$$

Update Example

Start with the blissful ignorance unit model:

$$0$$

$$n \xleftarrow{j} 6$$

Update Example

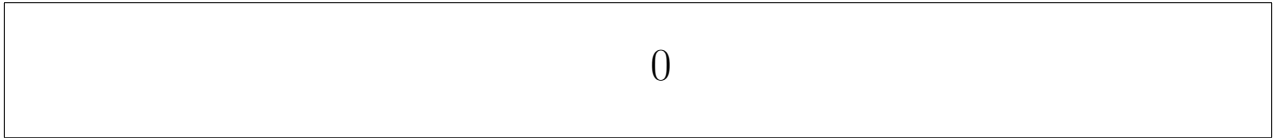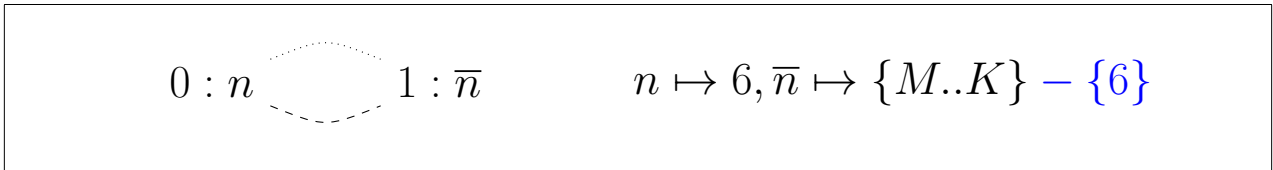Start with the blissful ignorance unit model:

$$0$$

$$n \xleftarrow{j} 6$$

$$0 : n \qquad 1 : \overline{n} \qquad\qquad n \mapsto 6, \overline{n} \mapsto \{M..K\} - \{6\}$$

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$$g \xleftarrow{g} 5$$

Update Example

Start with the blissful ignorance unit model:

$$0$$

$n \xleftarrow{j} 6$

$$0 : n \qquad 1 : \overline{n} \qquad\qquad n \mapsto 6, \overline{n} \mapsto \{M..K\} - \{6\}$$

$g \xleftarrow{g} 5$

$$0 : ng \qquad 1 : \overline{n}g$$

$$2 : n\overline{g} \qquad 3 : \overline{ng}$$

$$n \mapsto 6, \overline{n} \mapsto \{M..K\} - \{6\}$$
$$g \mapsto 5, \overline{g} \mapsto \{M..K\} - \{5\}$$

$$0 : ng \qquad 1 : \overline{n}g$$

$$2 : n\overline{g} \qquad 3 : \overline{ng}$$
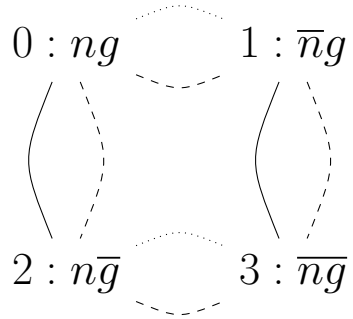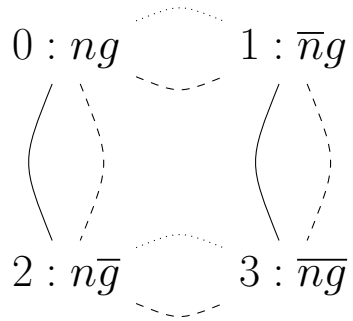
$$n \mapsto 6, \overline{n} \mapsto \{M..K\} - \{6\}$$
$$g \mapsto 5, \overline{g} \mapsto \{M..K\} - \{5\}$$

$$n \neq g$$
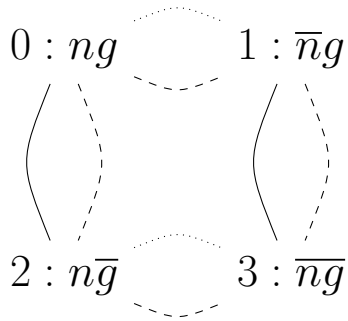
$$n \mapsto 6, \overline{n} \mapsto \{M..K\} - \{6\}$$
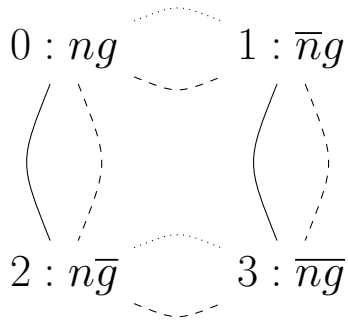$$g \mapsto 5, \overline{g} \mapsto \{M..K\} - \{5\}$$

$$n \neq g$$

$$n \mapsto 6, \overline{n} \mapsto \{M..K\} - \{6\}$$
$$g \mapsto 5, \overline{g} \mapsto \{M..K\} - \{5\}$$

Communication: "a sends φ to b"

Sequence of commands:

$?K_a\varphi$; **Open** b; $!\varphi$; **Close** b

- $?K_a\varphi$ tests whether $K_a\varphi$ is true in the actual world.

- **Open** b adds b to the set of agents that are listening.

- $!\varphi$ is the announcement of φ.

- **Close** b removes b from the set of agents that are listening.

- As far as b is concerned, the information φ could come from anywhere: no authentication.

- Anyone who is listening receives the info: the channel is insecure.

Fast Modular Arithmetic
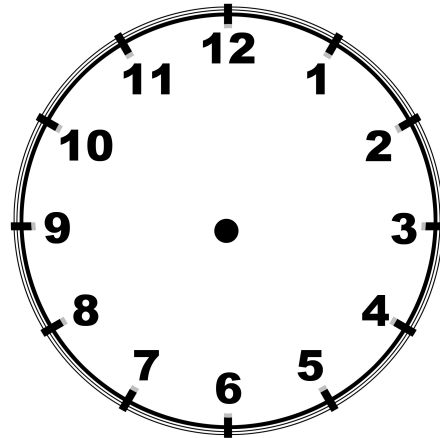
$$11 + 4 = 15 \equiv 3 \pmod{12}.$$

Modular division, modular inverse

Modular division is the same as multiplication by modular inverse. Modular inverses only exist for numbers that are co-prime with their modulus.

```
invM :: Integer -> Integer -> Integer
invM x n = let
    (u,v) = fct_gcd x n
    copr  = x*u + v*n == 1
    i     = if signum u == 1 then u else u+n
  in
    if copr then i else error "no inverse"
```

This uses the extended Euclidean algorithm for GCD.

The Extended Euclidean Algorithm

```
fct_gcd :: Integer -> Integer
         -> (Integer,Integer)
fct_gcd a b =
  if b == 0
  then (1,0)
  else
    let
      (q,r) = quotRem a b
      (s,t) = fct_gcd b r
    in (t, s - q*t)
```

Fast Modular Exponentiation

Modular exponentiation of x^y by repeatedly squaring modulo N.

E.g., $x^{33} \bmod 5$ can be computed by means of

$$x^{33} \bmod 5 = x^{32} \bmod 5 \times x \bmod 5.$$

$x^{32} \pmod{N}$ is computed in five steps by means of repeatedly squaring modulo N:

$$x \bmod N \to x^2 \bmod N \to x^4 \bmod N \to \ldots \to x^{32} \bmod N.$$

Fast Modular Exponentiation

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$$x \mod N \to x^2 \mod N \to x^4 \mod N \to \ldots \to x^{32} \mod N.$$

```
exM _ 0 _ = 1
exM x y n = let
    z = exM x (y `div` 2) n
    w = rem (z*z) n
  in
    if even y then w else rem (x*w) n
```

Feasible Computation

- Fast algorithms for primality testing (e.g., the probabilistic Miller-Rabin test [Mil76, Rab80]).

- Fast algorithms for co-primality testing (Euclid's GCD algorithm) and for finding modular inverses (Euclid's extended GCD algorithm, see above).

- Fast algorithms for addition and multiplication modulo, and for exponentiation modulo (see above, and compare [DK02, PP09]).

- + Articles of faith: factorisation, discrete logarithm, ..., are not feasible.

- Now refine the meaning of "knowing a number": I know (i) the numbers that I can look up in an accessible register, and (ii) the numbers that I can feasibly compute from numbers I know.

Application: Diffie-Hellman Key Exchange [DH76]
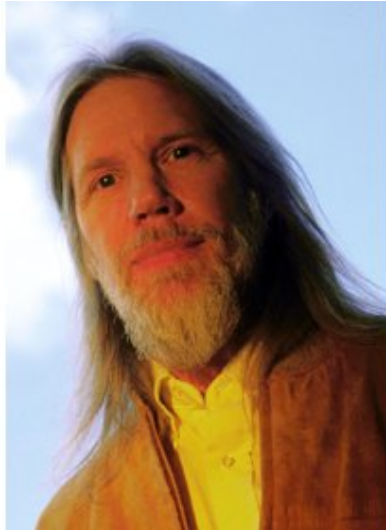
Whitfield Diffie – Martin Hellman

Diffie-Hellman Key Exchange Protocol

Key Exchange Over Insecure Channel

1. Alice and Bob agree on a large prime p and a base $g < p$ such that g and $p - 1$ are co-prime.

2. Alice picks a secret a and sends $g^a \bmod p = A$ to Bob.

3. Bob picks a secret b and sends $g^b \bmod p = B$ to Alice.

4. Alice calculates $k = B^a \bmod p$.

5. Bob calculates $k = A^b \bmod p$.

6. They now have a shared key k. This is because $k = (g^a)^b = (g^b)^a \bmod p$.

Use of a Shared Secret Key for Secure Communication

Let p be the prime that Alice and Bob have agreed on, and let k be their shared key. Then message m is encoded as

$$m \times k \bmod p.$$

Such messages can be decoded by both Alice and Bob.

Alice knows p, k, g^b and a. She decodes cipher c with

$$c \times (g^b)^{(p-1)-a} \bmod p.$$

Bob knows p, k, g^a and b. He decodes cipher c with

$$c \times (g^a)^{(p-1)-b} \bmod p.$$

Behind this: Fermat's Little Theorem

Pierre de Fermat (1601–1665)

If p is prime, then for every $1 \leq a < p$: $a^{p-1} \equiv 1 \pmod{p}$.

Example

```
Prelude> [ a^28 'mod' 29 | a <- [1..28] ]
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]
```

Explanation

We have:

$$(g^a)^{(p-1)-b} = g^{a((p-1)-b)} = g^{a(p-1)} \times g^{-ab} = (g^{p-1})^a \times g^{-ab}$$
$$\stackrel{Fermat}{=} 1^a \times g^{-ab} = g^{-ab} \bmod p,$$

and therefore:

$$c \times (g^a)^{(p-1)-b} = (m \times g^{ab}) \times g^{-ab} = m \times (g^{ab} \times g^{-ab}) = m \bmod p.$$

Diffie-Hellman Key Exchange as a Register Language Protocol

$$a \xleftarrow{i} N \; ; \; A \xleftarrow{i} g^a \bmod p \; ;$$

Open j ; $!x = A$; **Close** j ;

$$b \xleftarrow{j} M \; ; \; B \xleftarrow{j} g^b \bmod p \; ;$$

Open i ; $!y = B$; **Close** i ;

$$k \xleftarrow{i} y^a \bmod p \; ;$$

$$k' \xleftarrow{j} x^b \bmod p \; ;$$

$$?k = k'$$

Program for Epistemic Crypto Logic

- Formulate complete logics. A complete calculus for register logic has axioms for propositional logic, S5 axioms for the K_i operators, K axioms for the A operators, Modus Ponens, necessitation for K_i and A, and reduction axioms for the command operators, in the style of [BvEK06].

- Build an efficient model checker for this language, using register models as defined in this talk. See [Gat13] for a first prototype.

- Use the model checker for modelling cryptographic protocols, plus attacks on them.

Lesson For Formal Communication Studies

- Semantics of Natural Language $\Rightarrow$ Formal Communication Studies

- Focus on Representation of Update

- Parallels Between Natural and Programming Languages

- Haskell as Computational PTQ

```
Prelude> :t all
all :: (a -> Bool) -> [a] -> Bool
Prelude> :t any
any :: (a -> Bool) -> [a] -> Bool
```

References

[BMS98] A. Baltag, L.S. Moss, and S. Solecki. The logic of public announcements, common knowledge, and private suspicions. In I. Bilboa, editor, Proceedings of TARK'98, pages 43–56, 1998.

[BvEK06] J. van Benthem, J. van Eijck, and B. Kooi. Logics of communication and change. Information and Computation, 204(11):1620–1662, 2006.

[DH76] W. Diffie and M. Hellman. New directions in cryptography. IEEE Transactions on Information Theory, 22(6):644–654, 1976.

[DK02] Hans Delfs and Helmut Knebl. Introduction To Cryptography — Principles and Applications. Springer, 2002.

[ES] Jan van Eijck and François Schwarzentruber. Epistemic probability logic simplified. Under Submission.

[Fre80] Hans Freudenthal. Huygens' foundations of probability. Historia Mathematica, 7:113–117, 1980.

[Gat13] Malvin Gattinger. Epistemic crypto logic — functional programming and model checking of cryptographic protocols. Technical report, ILLC, Amsterdam, 2013. Exam paper for the course 'Functional Specification of Algorithms'.

[Hal03] J. Halpern. Reasoning About Uncertainty. MIT Press, 2003.

[Huy60] Christiaan Huygens. Van Rekeningh in Spelen van Geluck. 1660. About Calculation in Hazard Games; Latin: De ratociniis in Ludo aleae.

[Mil76] Gary L. Miller. Riemann's hypothesis and tests for primality. Journal of Computer and System Sciences, 13(3):300–317, 1976.

[Mus89] R. Muskens. Meaning and Partiality. PhD thesis, University of Amsterdam, 1989.

[PP09] Christof Paar and Jan Pelzl. Understanding Cryptography — A Textbook for Students and Practitioners. Springer, 2009.

[Rab80] Michael O. Rabin. Probabilistic algorithm for testing primality. Journal of Number Theory, 12(1):128–138, 1980.