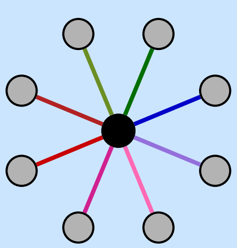
# **Graph Parameters** and **Invariants** of the **Orthogonal Group**

$$g^T g = I$$

$$h \otimes h$$

**Guus Regts**

Graph Parameters and Invariants of the Orthogonal Group
Guus Regts



# Graph Parameters and Invariants of the Orthogonal Group–Errata

Guus Regts

## Errata published on 10 December 2013

All references in this text are to the document 'Graph Parameters and Invariants of the Orthogonal Group'.

**p.17, l.4:** $\sigma(u)$ should be $\sigma(v)$.
**p.17, l.7:** 'term' should be 'factor'.
**p.19, above and below (3.7):** $\mathcal{G}$ should be $\mathcal{G}'$.
**p.21, (3.13):** $\phi(i)$ should be $\phi(e)$.
**p.22, l.6:** 'maps' should be 'functionals'.
**p.26, l.5:** No comma before 'denote'.
**p.27, l.1 from Section 4.2:** Add 'group' after 'orthogonal'.
**p.27, line above (4.4):** $2m$ should be $m$.
**p.28 l.7 from below:** Add 'that' after 'fact'.
**p.29, l.10:** There is a superfluous '(' after ':='.
**p.31, l.4:** $S_n$ should be $S_m$.
**p.31, l.6 below Theorem 4.3:** End(V) should be End(W).
**p.32, l.1:** Replace 'is' by 'induces'.
**p.37, first line below (5.5):** Add 'distinct' before $u_1, u_2$.
**p.37, third line below (5.5):** $\phi \circ \rho$ should be $\rho \circ \pi$.
**p.38, l.-8:** Remove 'it' after $f_{-2}$.
**p.40,l.5:** Add 'not' before crossing.
**p.41,l.-4:** Schur's Lemma actually only implies that $S^\lambda \subseteq \text{Im } A_n$.
**p.42, l.6:** $[2l]$ should be $[l]$.
**p.48, in line 3 of (5.42):** $y_{\phi(\delta(u)\cup\delta(s(\pi(v))))}$ should be $y_{\phi(\delta(u)\cup\delta(s((v)))}$.
**p.61, l.3,5,6:** $F$ should be $H_1$ and $H$ should be $H_1$.
**p.62 in (6.38):** Replace $F$ by $H$ (two times).
**p.63, l.6,-4:** Replace $A^{-1}$ by $A^{-2}$ (also on p.64, l.2,3).
**p.64, l.1:** $K_1^\bullet \cdot K_1^\bullet$ should be $K_1^\bullet \otimes K_1^\bullet$.
**p.65, 5th line in the proof of Theorem 6.15:** Replace $\subseteq$ by $\supseteq$.
**p.70, second and third line below the proof of Lemma 7.1:** $\mathbb{C}^k$ should be $\mathbb{C}$ and $\mathbb{C}$ should be $\mathbb{C}^k$.
**p.70, l.-10:** $p_{a,B}$ should be $p_{\mathbb{1},B}$.
**p.75:** add $\dim(\text{span}(\{u_1, \ldots, u_n\})) = \dim(\text{span}(\{w_1, \ldots, w_n\}))$ in the statement of Proposition 7.6.
**p.75:** In the proof of Theorem 7.7 we assume that $u_1$ is orthogonal to all $u_i$, but this not completely correct. Here is fix: In case none of the $u_i$ is orthogonal to all of the $u_i$, we can find, by degeneracy, a nonzero linear combination of the $u_i$, which is orthogonal to all of the $u_i$, and call this $u_{n+1}$. Let $U = \text{span}\{u_1, \ldots, u_n\}$ and write $U = \mathbb{F}u_{n+1} \oplus U'$ (for some algebraic complement $U'$ of $u_{n+1}$). Next we find for each $\varepsilon > 0$, $g(\varepsilon) \in O_k$ such that $gu_{n+1} = \varepsilon u_{n+1}$ by letting $g(\varepsilon)$ map $U'$ identically onto $U'$. Then $\lim_{\varepsilon \to 0} g(\varepsilon)(u_1, \ldots, u_n) = (u_1', \ldots, u_n')$ for certain $u_i' \in U$. Let $h' = \sum_{i=1}^n a_i \text{ev}_{u_i'}$.

Then $\lim_{\varepsilon \to 0} g(\varepsilon) h_{\leq e} = h'_{\leq e}$. Hence by (7.6) $h'_{\leq e}$ is not contained in the orbit of $h_{\leq e}$ (as $\dim(U') < \dim(U)$). This implies that the orbit of $h_{\leq e}$ is not closed.
**p.84, l.7:** The term 'graphon' is first used in [7].
**p.84, l.8:** In fact an equivalence class of almost everywhere equal functions $W$.
**p.84, (8.3):** $W_H$ should be $W_G$.
**p.88/p.95:** In Examples 8.2, 8.3 and 8.4 we implicitly use $C = \mathbb{N}$.
**p.90, l.4:** There is a superfluous 'a' before 'any'.
**p.94:** in (8.27) $\pi_F$ should be $\pi_H$ and in line 2 of (8.29) the sum is over $\phi : E(H') \to C$.

## Acknoledgements

I thank Tom Koornwinder for pointing out some of the errata.

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and
# Invariants of the Orthogonal Group

**Guus Regts**

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ACADEMISCH PROEFSCHRIFT

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ingestelde commissie,
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*voor Roos*

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# Preface

After finishing my master's thesis at the University of Amsterdam in 2009, I started a PhD-project at the Centrum Wiskunde & Informatica (CWI) under supervision of Lex Schrijver, who was supported by a Spinoza grant. The aim of this project was to apply and develop algebraic techniques to and for combinatorial optimization. Despite the aim of the project, Lex gave me a lot of freedom to work on almost anything in the field of discrete mathematics. So in the beginning I tried to work on various topics. In particular, together with Dion Gijswijt, we showed that matroid base polytopes have integer Carathéodory-like properties. However, about nine months after I started at the CWI, Lex gave a talk about a question of Balazs Szegedy, which got me interested in so-called partition functions of edge-coloring models. This eventually resulted in this thesis entitled 'Graph Parameters and Invariants of the Orthogonal Group', in which these partition functions are the main characters. Incidentally, this thesis even fits within the original aim of the project, as it contains a significant interaction between combinatorics (in the form of graph theory and graph parameters) and algebra (in the form of invariant theory and some basic algebraic geometry).

Needless to say that without the help, support and advice of Lex, this thesis would not exist. It has been a great honor and pleasure to have been supervised by Lex. The way Lex approaches mathematics has been really inspiring. Not only is he open to learn new things in mathematics (and in life probably), he also always seeks ways to simplify proofs, always wondering whether things can be made more insightful. Thanks for the help support, advice and inspiration Lex!

Besides Lex there are a few other people I want to thank, starting with Monique. Both Lex and she were alway very helpful and supportive with both work related issues and other issues that I encountered in the past four years. Thanks to my office mate Antonis for pleasant conversations and for introducing me to some great music. The pictures in this thesis would not have

looked so good If it was not for Sunil; thanks for introducing me to TikZ. It was always nice to be able to talk to Dion about any kind of math topic; thanks for keeping an open door. Thanks to my coauthors, Dion, Jan, Laci, Lex, Ross and Viresh for sharing their knowledge with me; it has been a great pleasure and honor to have been working with them. Moreover, thanks to Aida, Anargyros, Bart, Evan, Fred, Jop, Tobias and Xavier for interesting discussions, helpful comments, doing homework together and many other things.

I would like to thank all members of the PNA1 and C&O group over the past four years for the friendly atmosphere they provided and the pleasant conversations over lunch and coffee we had. Organizing the 'Barvinok reading group' and later the 'Graph Limits and Flag Algebras reading group' was a great pleasure. Thanks to all participants for their contributions. In particular, my thanks goes out to Evan, who shared my interest for graph limits and gave a lot of talks on that topic.

I have met a number of great, friendly, interesting and inspiring people the past four years. It was a privilege to have had this opportunity.

Finally, much thanks to my family and friends for their support. In particular, for their support on issues that were not of any mathematical nature.

Guus Regts
Zwolle, October 2013

# Chapter 1

# Introduction

This thesis deals with connections between graph parameters and invariants of the orthogonal group and some of its subgroups. We consider basically two types of graph parameters: partition functions of vertex-coloring models and partition functions of edge-coloring models.

An edge-coloring model over a field $\mathbb{F}$ is a map $h$ from the set of all multisets on $k$ elements (for some $k \in \mathbb{N}$) to $\mathbb{F}$. Given a coloring of the edges of a graph $H$ with $k$ colors, at every vertex $v$ of $H$ we obtain a multiset of colors by looking at the colors of the edges incident with $v$; applying $h$ to it we get a number in $\mathbb{F}$; taking the product of these numbers over all vertices of $H$ one obtains the *weight* associated to the coloring of the edges. The *partition function* of $h$, $p_h$, is the map from the set of all graphs to $\mathbb{F}$ given by the sum over all $k$-colorings of the edges of a graph $H$ of the weights associated to the coloring:

$$p_h(H) = \sum_{\phi:E(H)\to\{1,\ldots,k\}} \prod_{v\in V(H)} h(\phi(\delta(v))), \tag{1.1}$$

where $\phi(\delta(v))$ denotes the multiset of colors of the edges incident with $v$. The partition function of an edge-coloring model can be seen as a generalization of the number of homomorphism of the linegraph of $H$ into the linegraph of a graph $G$. See section 3.3 for more details.

A vertex-coloring model over $\mathbb{F}$ is a symmetric $n \times n$ matrix. Given a coloring of the vertices of a graph $H$ with $n$ colors, it associates to the colors of the endpoints of each edge a number (the entry of the matrix corresponding to the pair of colors at the endpoints of the edge). The partition functions of a vertex-coloring model is a similar expression as (1.1), except that the role of vertices and edges is interchanged. For a graph $H$, the partition function of a vertex-coloring model can be seen as a generalization of the number of homomorphisms of $H$ into a graph $G$. See section 3.2 for more details.

Edge- and vertex-coloring models can be viewed as generalizations of statistical models. Their partition functions were introduced as graph parameters by de la Harpe and Jones [28], where they are called vertex models and spin models respectively. We choose to call them edge- and vertex coloring models to emphasize their connection with graph and linegraph coloring. We refer to Section 3.1 for more about the statistical mechanics origin.

One of the things we are concerned with in this thesis is the question: which graph parameters are partition functions of edge-coloring models? It turns out that there is an action of the orthogonal group on the set of edge-coloring models which leaves the partition function invariant. To give an answer to this question and other related questions, we need to use tools from (classical) invariant theory (the First and Second Fundamental Theorem for the orthogonal group), algebraic geometry (Hilbert's Nullstellensatz) and the theory of (affine) algebraic groups. This gives an interesting connection between partition functions of edge-coloring models and invariants of the orthogonal group. Before we say more about this we will first sketch some background and motivation for our work.

## 1.1 Background and motivation

The study of partition functions of edge-coloring models is part of the recently emerged field of graph limits and graph partition functions. This is an exciting new area of research which has connections to extremal graph theory, probability theory, topology, basic commutative algebra and invariant theory. This, and many more things, have been described by Lovász in his beautiful recent book entitled: 'Large Networks and Graph Limits' [40].

**Characterization of partition functions of vertex-coloring models**

As Lovász writes in the introduction of his book [40], about ten years ago Freedman, who was interested in applications to quantum computing, asked the question: which graph parameters are partition functions of (real) vertex-coloring models? This question was solved a few months later by Freedman, Lovász and Schrijver [24]. To prove their characterization, Freedman, Lovász and Schrijver equipped the set of labeled graphs with the structure of a semigroup and introduced the concept of graph algebras and vertex-connection matrices (infinite matrices indexed by the set of all labeled graphs). The results were later generalized by Lovász and Schrijver to semidefinite functions on semigroups [42] and to semidefinite functions on certain categories [43]. The tools they developed turned out to be very useful to tackle other related questions. In particular, they were used by Lovász and Sós to characterize so-called generalized quasi random graphs [44]. Moreover, the structure of these graph algebras has connections to algebras of tensors invariant under certain subgroups of the symmetric group [41].

At around the same time Razborov [51] introduced the concept of a flag algebra, which he used to solve a longstanding problem in extremal graph theory [52]. These flag algebras turned out to be closely related to the graph algebras introduced by Freedman, Lovász and Schrijver (cf. [40] and [43]).

**Characterization of partition functions of edge-coloring models**
Motivated by a question of Freedman, Lovász and Schrijver, Szegedy [66] proved a characterization of partition functions of real edge-coloring models quite similar to the characterization of partition functions of real vertex-coloring models. Szegedy's characterization is based on a different type of infinite matrices; edge-connection matrices. However, whereas the proof of Freedman, Lovász and Schrijver is based on basic properties of finite dimensional commutative algebras, Szegedy used the First Fundamental Theorem for the orthogonal group and the Positivstellensatz. This proof method inspired Schrijver [58] to give a characterization of algebras of tensors that are invariant under a subgroup of the orthogonal group. Szegedy [66, 67] also studied connections between partition functions of vertex-coloring models and complex edge-coloring models. In particular, he showed that the first are a special case of the latter, which led him to ask for which vertex-coloring models the edge-coloring model can be taken to be real-valued.

**Large networks and graph limits**
Parallel to the characterization of partition functions, in [45], which as awarded the Fulkerson prize in 2012, Lovász and Szegedy initiated a theory of limits of dense graphs. In particular, they defined a notion of convergence for a sequence of simple graphs based on homomorphism densities and exhibited a natural limit object for a convergent sequence of graphs. This was further developed by Borgs, Chayes, Lovász, Sós and Vesztergombi in [7], where they study various variants of the cut-metric related to the topology on the space of graphs as defined in [45].

The theory of graph limits has deep connections to Szemerédi's regularity Lemma [46], to graph property testing [5] and to exchangeable random graphs [18]. We refer to the book by Lovász [40] for many more details.

## 1.2 Contributions

Motivated by questions that arose in the field of graph limits and graph partition functions, we present in this thesis various results about partition functions of edge-coloring models. In particular, we characterize which graph parameters are such partition functions. Furthermore, we determine the rank of the edge-connection matrices of partition functions of edge-coloring models (this is done by giving a combinatorial parametrization of certain algebras of tensors), and we characterize which partition functions of vertex-coloring models are partition functions of real edge-coloring models. Furthermore, we develop, analogues to the theory of graph limits, the first step for limits of edge-coloring models. For a more detailed overview of this thesis see below.

One can view our contributions as deepening the connection between the field of graph limits and partition functions and that of the invariant theory of the orthogonal group. Szegedy [66] was the first to observe that partition functions of (real) edge-coloring models and the invariant theory of the (real) orthogonal are intimately connected. In this thesis we will make use of this important observation and use various techniques from classical and geometric invariant theory. One could say that some of the results in this thesis are merely an application of these invariant-theoretical techniques. For example, from the point of view of invariant theory, one could consider the characterization of partition functions of edge-coloring models as a purely invariant-theoretical statement about the action of the orthogonal group acting on some polynomial ring with infinitely many variables. However, we rather speak of an interesting interaction between combinatorics and invariant theory. For example, the result by Schrijver [58] about characterizing algebras of tensors invariant under subgroups of the orthogonal group was motivated and inspired by its connection to combinatorics. Moreover, using combinatorial objects such as graphs or fragments to parametrize certain polynomial or tensor invariants might be the most natural way of looking at them.

## 1.3 Outline of this thesis

This thesis is roughly organized as follows: in Chapters 2-4 we state definitions, preliminaries and some more background; Chapters 5-8 contain the heart of this thesis. We will now give a more detailed outline.

**Chapter 2. Preliminaries**
Here we introduce important concepts such as labeled graphs, fragments and

connection matrices. These concepts are probably not so well known and will be used in several parts of this thesis. We will also state some basic definitions and set up some notation about graphs.

**Chapter 3. Partition functions of edge- and vertex-coloring models**
In this chapter we define partition functions of edge- and vertex-coloring models and we say something about their statistical mechanics background. Furthermore, we show how the orthogonal group acts on edge-coloring models and we explain why this action leaves the partition function invariant. We also give some further background on partition functions of edge- and vertex-coloring models.

**Chapter 4. Invariant theory**
In this chapter we give a very brief introduction to invariant theory. In particular, we state the First and Second Fundamental Theorem for the orthogonal group, and we state a theorem about the uniqueness and existence of so-called closed orbits.

**Chapter 5. Characterizing partition functions of edge-coloring models**
In this chapter we give a characterization of partition functions of edge-coloring models with values in an algebraically closed field of characteristic zero. Furthermore, we characterize when the edge-coloring model can be taken to be of finite rank. To do so we use Hilbert's Nullstellensatz and the First and Second Fundamental Theorem for the orthogonal group. Our proof is much inspired by Szegedy's proof [66] for real edge-coloring models. This is based on joint work with Jan Draisma, Dion Gijswijt, Laci Lovász and Lex Schrijver, which appeared in the Journal of Algebra [19].

**Chapter 6. Connection matrices and algebras of invariant tensors**
Here we consider the rank of edge-connection matrices and relate this to the dimension of algebras of tensors that are invariant under certain subgroups of the orthogonal group. In particular, we give a combinatorial parametrization of such algebras. For real edge-coloring models this is based on [53], which appeared in the European Journal of Combinatorics. For edge-coloring model with values in an algebraically closed field of characteristic zero this is based on joint work with Jan Draisma, which appeared in the Journal of Algebraic Combinatorics [20]. The proofs are based on, and inspired by, the aforementioned result of Schrijver, characterizing algebras of invariant tensors [58].

**Chapter 7. Edge-reflection positive partition functions of vertex-coloring models**
In this chapter, we give an answer to the question by Szegedy asking for which vertex-coloring model its partition function is the partition function of a real edge-coloring model. We use ideas from Kempf and Ness [33] and a generalization of the Hilbert-Mumford theorem, a deep result from geometric invariant theory. Except for Section 7.2, this is based on [54] of which an extend abstract appeared in The Seventh European Conference on Combinatorics, Graph Theory and Applications, Eurocomb 2013. Section 7.2 is based on joint work with Jan Draisma [20, Section 6].

**Chapter 8. Compact orbit spaces in Hilbert spaces and limits of edge-coloring models**
Motivated by the theory of graph limits, we construct limits of edge-coloring models in this chapter. To do so we prove a general result about compact orbit spaces in Hilbert spaces. This result can be applied to construct limit objects for certain sequences of edge-coloring models as well as limit objects for convergent sequences of graphs. This is based on joint work with Lex Schrijver [55].

# Chapter 2

# Preliminaries

In this chapter we introduce some important and probably not so well-known concepts such as labeled graphs, fragments and connection matrices. Moreover, we set up some basic notation.

## 2.1 Some notation and conventions

We set up some basic notation and conventions used throughout the thesis. We moreover give a few basic definitions.

**Fields and sets**
By $\mathbb{R}, \mathbb{C}$ we denote the set of real numbers and complex numbers respectively. Throughout this thesis, $\mathbb{F}$ denotes any field of characteristic zero, unless indicated otherwise. (Many definitions in this thesis make sense also in characteristic $p$, but for simplicity we just stick to characteristic zero throughout this thesis.) By $\overline{\mathbb{F}}$ we denote the algebraic closure of $\mathbb{F}$.

By $\mathbb{N}$ we denote the set of natural numbers including zero; $\mathbb{N} := \{0, 1, 2 \ldots\}$ and for $n \in \mathbb{N}$ we set

$$[n] := \{1, \ldots, n\}. \tag{2.1}$$

Note that $[0]$ denotes the empty set. For $\alpha \in \mathbb{N}^k$, we denote by $x^\alpha \in \mathbb{F}[x_1, \ldots, x_k]$ the monomial $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$. Furthermore, for $\alpha \in \mathbb{N}^k$ we set $|\alpha| := \sum_{i=1}^k \alpha_i$.

We will not only use $\delta$ to denote the set of edges incident with a vertex in a graph but also to define a certain set function: for a set $S$ and $s_1, s_2 \in S$, $\delta_{s_1,s_2} = 1$ if $s_1 = s_2$ and 0 otherwise; it is also known as the *Kronecker delta* function.

**Linear algebra**
For a vectorspace $V$ over $\mathbb{F}$ we denote by $V^*$ its dual space, the space of $\mathbb{F}$-linear functions $f : V \to \mathbb{F}$. However, by $\mathbb{F}^*$ we denote the nonzero entries of $\mathbb{F}$. The set $\text{End}(V)$ denotes the set of linear maps from $V$ to itself. By $I_V \in \text{End}(V)$ we denote the identity map; sometimes we just write $I$. For a (finite or infinite) matrix $M$ with values in $\mathbb{F}$ we denote by $M^T$ its transpose and by $M^*$ its conjugate transpose (if $\mathbb{F} = \mathbb{C}$). Moreover, we denote by $\text{rk}\,(M)$ the rank of the matrix $M$. For a subset $S$ of $V$, we denote by $\text{span}(S)$ the subspace of $V$ spanned by $S$.

**Graphs**
A *graph* $H$ is a pair $(V, E)$, with $V$ a finite set and $E$ a finite multiset of unordered pairs of elements of $V$. Elements of $V$ are called *vertices* and elements of $E$ are called *edges*. A *loop* of $H$ is an edge of the form $\{v, v\}$ for $v \in V$. A *simple graph* is a graph without loops and where each edge has multiplicity one. For a graph $H$ we denote by $V(H)$ its vertices and by $E(H)$ its edges. For $u, v \in V(H)$ we usually denote by $uv$ the set $\{u, v\}$. We say that $u, v$ are *adjacent* in $H$ if $uv \in E(H)$. For $v \in V(H)$, $\delta(v) \subset E(H)$ denotes the set of edges incident with $v$ (loops are counted twice); $d(v) := |\delta(v)|$ is the *degree* of $v$.

Let $\mathcal{G}$ denote the set of all graphs. By $\bigcirc$ we denote the *circle* (or vertex less loop). More precisely, $\bigcirc = (\emptyset, \{1\})$. According to the definition it is not a graph, but it will be convenient to think of it as a graph. We will write $\mathcal{G}'$ for the set consisting of elements that are the disjoint union of a graph and finitely many circles. A map $f : \mathcal{G}' \to \mathbb{F}$ is called a *graph parameter or graph invariant* if $f$ assigns the same values to isomorphic graphs[1]. Sometimes $f$ will only be defined on a subset of $\mathcal{G}'$, but we will then nevertheless call $f$ a graph parameter.

## 2.2 Labeled graphs and fragments

In this section we introduce the concept of labeled graphs and fragments.

### 2.2.1 Labeled graphs

For $l \in \mathbb{N}$, an *$l$-labeled graph* is a graph $H = (V, E)$ with an injective map $\lambda : [l] \to V$. For an $l$-labeled graph $H = (V, E)$ we think of $[l]$ as a subset of $V$, identifying $1, \ldots, l$ with the labeled vertices of $H$. See Figure 2.1 for some

[1]Two graphs $H_1, H_2$ are *isomorphic* if there exists a bijection $\tau : V(H_1) \to V(H_2)$ such that $\tau(u)\tau(v) \in E(H_2)$ if and only if $uv \in E(H_2)$.

examples. We denote the set of $l$-labeled graphs by $\mathcal{G}_l$ and we identify $\mathcal{G}$ with $\mathcal{G}_0$.

Schrijver [60] introduced a different kind of labeled graphs, where the map $\lambda : [l] \to V$ is not required to be injective. See Lovász [40] for other examples.

When $f : \mathcal{G} \to \mathbb{F}$ is a graph parameter, we can simply extend $f$ to $\mathcal{G}_l$ for any $l$ by letting $f(H) := f([\![H]\!])$ for $H \in \mathcal{G}_l$, where $[\![H]\!]$ is the graph obtained from $H$ by deleting its labels.

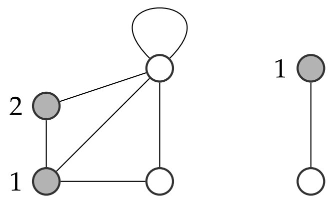



Figure 2.1: Some examples of labeled graphs.

The labeled vertex will be denoted by $K_1^\bullet$, the labeled loop will be denoted by $C_1^\bullet$ and the 2-labeled edge will be denoted by $K_2^{\bullet\bullet}$. See Figure 2.2.

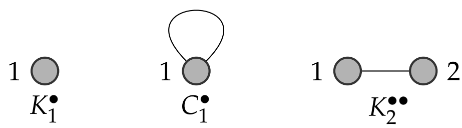



Figure 2.2: The labeled graphs $K_1^\bullet, C_1^\bullet$ and $K_2^{\bullet\bullet}$.

Let $H_1$ and $H_2$ be two $l$-labeled graphs. We define their *gluing product* $H_1 \cdot H_2$ by taking their disjoint union, and then identifying nodes with equal labels. See Figure 2.3 for an example. We sometimes just write $H_1H_2$ instead of $H_1 \cdot H_2$. In particular, for two ordinary (unlabeled) graphs $H_1, H_2$, $H_1H_2$ denotes their disjoint union. Note that with this gluing product, $\mathcal{G}_l$ becomes a semigroup for any $l$.

### 2.2.2 Fragments

For $l \in \mathbb{N}$, an *$l$-fragment* is an $l$-labeled graph such that all the labeled vertices have degree one. (Lovász [40] calls them $l$-broken graphs.) These labeled vertices are called *open ends* and the edge connected to an open end is called a *half edge*. So in Figure 2.1, the first labeled graph is not a fragment whereas the

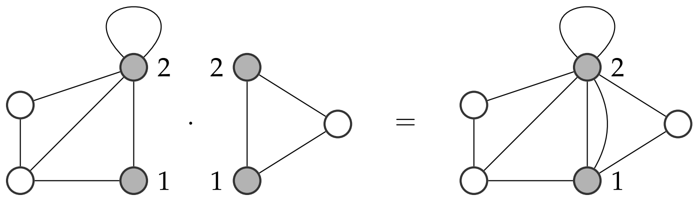



Figure 2.3: Gluing two 2-labeled graphs.

second one is. We sometimes refer to $K_2^{\bullet\bullet}$ as the *open edge*. By $\mathcal{F}_l$ we denote the set of all $l$-fragments. We will identify $\mathcal{G}'$ with $\mathcal{F}_0$. Define a *gluing operation* $* : \mathcal{F}_l \times \mathcal{F}_l \to \mathcal{G}'$ as follows: for $F_1$ and $F_2 \in \mathcal{F}_l$, take their disjoint union and connect the half edges incident with open ends with equal labels to form single edges (with the labeled vertices erased); the resulting graph is denoted by $F_1 * F_2 \in \mathcal{G}'$. See Figure 2.4 for an example. Note that $K_2^{\bullet\bullet} * K_2^{\bullet\bullet} = \bigcirc$. This explains why it is useful to consider $\bigcirc$ as a graph.

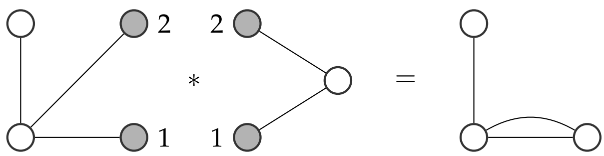



Figure 2.4: Gluing two 2-fragments into a graph.

Note that the gluing operation does not make $\mathcal{F}_l$ into a semigroup for $l \geq 1$. We can however make $\mathcal{F}_{2l}$ into a (noncommutative) semigroup as follows. Consider $F_1, F_2 \in \mathcal{F}_{2l}$. Think of the labels $1, \ldots, l$ as the *left* labels and $l + 1, \ldots, 2l$ as the *right* labels. Define $F_1 \cdot F_2$ to be the $2l$-fragment obtained from the disjoint union of $F_1$ and $F_2$ by gluing the right open end of $F_1$ labeled $l + i$ to the left open end of $F_2$ labeled $i$, for $i = 1, \ldots, l$. This operation should not be confused with the gluing product for labeled graphs. Note that the identity element in $\mathcal{F}_{2l}$ is the matching connecting $i$ to $l + i$ for $i \in [l]$. See Figure 2.5 for an example of this gluing product.

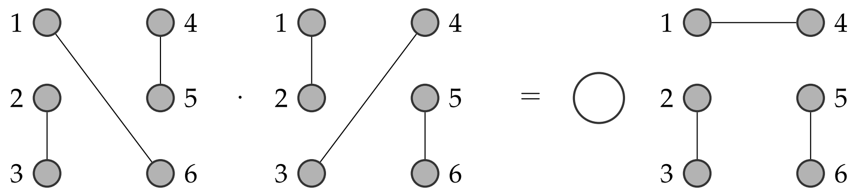


Figure 2.5: Gluing two 6-fragments into a 6-fragment.

## 2.3 Connection matrices

Let $f : \mathcal{G}' \to \mathbb{F}$ be a graph parameter. The *l-th vertex-connection matrix of f* is the $\mathcal{G}_l \times \mathcal{G}_l$ matrix defined by[2]

$$N_{f,l}(H_1, H_2) = f(H_1 \cdot H_2), \tag{2.2}$$

for $H_1, H_2 \in \mathcal{G}_l$. Vertex-connection matrices were introduced by Freedman Lovász and Schrijver [24], to characterize partition functions of real vertex-coloring models (cf. Theorem 5.1).

The *l-th edge-connection matrix of f* is the $\mathcal{F}_l \times \mathcal{F}_l$ matrix defined by[3]

$$M_{f,l}(F_1, F_2) = f(F_1 * F_2), \tag{2.3}$$

for $F_1, F_2 \in \mathcal{F}_l$. The edge-connection matrices were used by Szegedy [66] to characterize partition functions of edge-coloring models over $\mathbb{R}$ (cf. Theorem 5.2).

Clearly, these connection matrices contain a lot of information about the graph parameter $f$. There are various other ways to define connection matrices based on different kinds of labeling and gluing. Makowski [47] introduced several variants of gluing operations which he used to study questions about definability of graph parameters in monadic second order logic.

For $H_1, H_2 \in \mathcal{G}'$, we will refer to the disjoint union of $H_1$ and $H_2$, $H_1H_2$ as the *product* of $H_1$ and $H_2$. Let $f : \mathcal{G}' \to \mathbb{F}$ be a graph parameter. Let $\mathcal{S} \subset \mathcal{G}'$ be subset that is closed under multiplication and that contains $\emptyset$. We call $f$ *multiplicative on* $\mathcal{S}$ if $f(H_1H_2) = f(H_1)f(H_2)$ for all $H_1, H_2 \in \mathcal{S}$ and $f(\emptyset) = 1$. Equivalently, $f$ is multiplicative on $\mathcal{S}$ if the submatrix of $M_{f,0}$ indexed by $\mathcal{S}$

[2]Of course we could also index this matrix by isomorphism classes of $l$-labeled graphs, but for our purposes it does not make a difference.

[3]Of course we could also index this matrix by isomorphism classes of $l$-fragments, but for our purposes it does not make a difference.

has rank 1 and $f(\emptyset) = 1$. We will often omit the reference to $\mathcal{S}$ and just call $f$ multiplicative.

When $\mathbb{F} = \mathbb{R}$, we call $f$ *reflection positive* if $N_{f,l}$ is positive semidefinite for all $l$; we call $f$ *edge-reflection positive* if $M_{f,l}$ is positive semidefinite for all $l$. An infinite matrix is positive semidefinite if all its finite principal submatrices are positive semidefinite. So $N_{f,l}$ is positive semidefinite if and only if $\sum_{i=1,j=1}^{n} \lambda_i \lambda_j f(H_i H_j) \geq 0$ for all $H_1, \ldots, H_n \in \mathcal{G}_l$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Similarly, $M_{f,l}$ is positive semidefinite if and only if $\sum_{i=1,j=1}^{n} \lambda_i \lambda_j f(F_i * F_j) \geq 0$ for all $F_1, \ldots, F_n \in \mathcal{F}_l$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Let us end this section with an example to illustrate these definitions.

**Example 2.1.** For $x \in \mathbb{R}$, define $f_x : \mathcal{G} \to \mathbb{R}$ by

$$f_x(H) = \begin{cases} x^{c(H)} \text{ if } H \text{ is 2-regular} \\ 0 \text{ otherwise,} \end{cases} \tag{2.4}$$

where $c(H)$ denotes the number of connected components of $H$.

Note that $f_x$ is clearly multiplicative for any $x$. However, for any nonzero $x$, $f_x$ is not reflection positive. As $f_x$ is only nonzero on 2-regular graphs, $f_x((K_1^\bullet \pm C_1^\bullet)^2) = \pm 2 f_x(K_1^\bullet C_1^\bullet) = \pm 2x$. So $N_{f_x,1}$ is not positive semidefinite and hence $f_x$ is not reflection positive. For $x \in \mathbb{N}$, $f_x$ is edge-reflection positive however. This follows from the fact that for $x \in \mathbb{N}$, $f_x$ is the partition function of an $x$-color edge-coloring model over $\mathbb{R}$, as we will see in Section 5.2. Combined with Szegedy's characterization (cf. Theorem 5.2) it follows that $f_x$ is edge-reflection positive (this is actually the easy part of Szegedy's theorem and we recover this in Section 6.2).

Clearly, $f_x$ is not edge-reflection positive for $x < 0$. Indeed, consider the the path on three vertices with both its endpoint labeled and denote it by $K_{1,2}^{\bullet\bullet}$. Then $f_x(K_{1,2}^{\bullet\bullet} * K_{1,2}^{\bullet\bullet}) = f(C_2) = x < 0$. In fact, for any $x \in \mathbb{R} \setminus \mathbb{N}$, $f_x$ is not edge-reflection positive. As, by Proposition 5.6, $f_x$ is not the partition function of any complex-valued edge-coloring model. Hence by Szegedy's theorem, $f_x$ is not edge-reflection positive.

## 2.4 Graph algebras

With the gluing product, the set of all $l$-labeled graphs $\mathcal{G}_l$ becomes a semigroup with unit element the disjoint union of $l$ copies of $K_1^\bullet$. Let $\mathbb{F}\mathcal{G}_l$ be the semigroup algebra of $(\mathcal{G}_l, \cdot)$, i.e., elements of $\mathbb{F}\mathcal{G}_l$ are finite formal $\mathbb{F}$-linear combinations of $l$-labeled graphs; they are called *$l$-labeled quantum graphs* (if $l = 0$ they are just called *quantum graphs*.[4])

Let $f : \mathcal{G} \to \mathbb{F}$ be a graph parameter. Extend $f$ linearly to $\mathbb{F}\mathcal{G}$. Note that $f : \mathbb{F}\mathcal{G} \to \mathbb{F}$ is multiplicative if and only if $f$ is a homomorphism of algebras. (In this thesis a homomorphism of algebras always maps the unit to the unit).

Let $\mathcal{I}_l(f)$ be the ideal in $\mathbb{F}\mathcal{G}_l$ generated by the kernel of $f$, i.e.,

$$\mathcal{I}_l(f) := \{x \in \mathbb{F}\mathcal{G}_l \mid f(x \cdot y) = 0 \text{ for all } y \in \mathbb{F}\mathcal{G}_l\}. \tag{2.5}$$

Equivalently, $\mathcal{I}_l(f)$ is the kernel of $N_{f,l}$. Then define the quotient algebra by

$$\mathcal{Q}_l(f) := \mathbb{F}\mathcal{G}_l / \mathcal{I}_l(f). \tag{2.6}$$

We will indicate elements of $\mathcal{Q}_l(f)$ by representatives in $\mathbb{F}\mathcal{G}_l$. We say that $x, y \in \mathcal{Q}_l(f)$ are equivalent modulo $f$ if $x - y \in \mathcal{I}_l(f)$. These algebras were introduced by Freedman, Lovász and Schrijver [24] and they are called *graph algebras*.

These graph algebras carry the same information about the parameter $f$ as the vertex connection matrices, but they provide more structure and are somehow more convenient to work with. In particular, we have:

**Proposition 2.1.** *The dimension of $\mathcal{Q}_l(f)$ is equal to the rank of $N_{f,l}$.*

We can of course define similar objects for fragments. In particular, $\mathbb{F}\mathcal{F}_{2l}$ denotes the semigroup algebra of $(\mathcal{F}_{2l}, \cdot)$. In Section 6.2 we will show that we can equip the space of linear combinations of all fragments with the structure of an associative algebra.

[4]In the terminology we follow Freedman, Lovász and Schrijver. It should be noted that the term quantum graph has been used elsewhere in mathematics with a different meaning.

# Chapter 3

# Partition functions of edge- and vertex-coloring models

In this chapter we give formal definitions of edge- and vertex-coloring models and their partition functions. We will also discuss the action of the orthogonal group on edge-coloring models and explain why this action leaves the partition function invariant.

Whereas our intention in this thesis is for example to characterize partition functions of edge-coloring models and study connections between invariant theory of the orthogonal group, partition functions of edge-coloring models have also been studied in different contexts. We will briefly say a few things about that in this chapter.

## 3.1 Graph parameters from statistical models

The graph parameters that we study in this thesis were introduced by de la Harpe and Jones [28] in 1993 and are motivated by statistical models. Although we will not discus any relation of our work to statistical mechanics, it makes sense to say a few words about the origin of these graph parameters. Statistical physics is a huge area of research and we will not make any attempt to say much about it. We refer to [2, 29] for an introduction to statistical physics and its connections to graph theory.

We will now introduce the Ising model and show how it can be generalized to obtain interesting graph parameters. This introduction is based on [40].

**The Ising model**
Let $H$ be a finite lattice; for example the $n \times n$ grid. We can think of the vertices of $H$ as the atoms of some crystal, where each of the atoms can have an up or down spin. This will be modeled by a 1 and a $-1$ respectively. An assignment of spins to the vertices of $H$ is called a *state*. This is described by a map $\sigma : V(H) \to \{-1, 1\}$. Two vertices that are adjacent in $H$ have an *interaction energy*, which, in the Ising model, is equal to some real number $-J$, if the atoms have the same spin and which is equal to $J$, if the atoms have opposite spin. For a state $\sigma \in \{-1, 1\}^{V(H)}$, the total energy of this state is given by

$$H(\sigma) = -\sum_{uv \in E(H)} J\sigma(u)\sigma(v). \tag{3.1}$$

The probability of a system to be in state $\sigma$ is proportional to $e^{-H(\sigma)/kT}$, where $T$ denotes the temperature, and $k$ the Boltzmann constant. As probabilities add up to one, these values need to be normalized; the normalizing factor $Z$ is the *partition function* of the system:

$$\sum_{\sigma:V(H)\to\{-1,1\}} e^{-H(\sigma)/kT} = \sum_{\sigma:V(H)\to\{-1,1\}} \exp\big(\frac{1}{kT} \sum_{uv\in E(H)} J\sigma(u)\sigma(v)\big). \tag{3.2}$$

The physical behavior of the system depends very much on the signature of $J$, but we will however not discuss this. Instead we will show how we can obtain interesting graph parameters from partition functions of generalizations of the Ising model.

**The spin model**
Let

$$S := \begin{pmatrix} \exp(J/kT) & \exp(-J/kT) \\ \exp(-J/kT) & \exp(J/kT) \end{pmatrix}. \tag{3.3}$$

This allows to rewrite (3.2) as follows:

$$Z = \sum_{\sigma:V(H)\to\{1,2\}} \prod_{uv\in E(G)} S_{\sigma(u),\sigma(v)}. \tag{3.4}$$

From a mathematical point of view, the parameters $k$, $T$ and $J$ are just constants; so we might as well replace $S$ by an arbitrary symmetric $2 \times 2$ matrix. Then (3.4) might not have any physical interpretation, but it still assigns a number to the graph $H$. Of course we can calculate (3.4) for any graph $H$, it need not represent any crystal structure. In other words (3.4) describes a *graph parameter*.

The next step is to generalize it to symmetric $n \times n$ matrices for arbitrary $n$. We then end up with what de la Harpe and Jones [28] call a *spin model*. The partition function of a spin model is a similar expression as (3.4), except that the sum is now taken over all maps from $V(H)$ to $[n]$.

From a physical point of view it is natural to equip the Ising model with an external magnetic field. In the simplest case we just need to add $-\sum_{v\in V(H)} h\sigma(u)$ to (3.1), for some number $h$, to obtain the total energy of the state $\sigma$. For (3.4) this implies we would, for each $\sigma$, have to multiply $\prod_{uv\in E(G)} S_{\sigma(u),\sigma(v)}$ by the term $\prod_{v\in V(H)} \exp(-h\sigma(v)/kT)$ to obtain the partition function of the model. Again, this can be generalized by replacing the vector $(\exp(-h/kT), \exp(h/kT)$ by any nonnegative vector (or even any complex one). See Section 3.2 for the formal definition.

**The vertex model**

By viewing the edges of the graph $H$ as particles or atoms and thinking of the vertices of $H$ as the interaction between them, we get a different physical model. Since a vertex can be incident with an arbitrary large number of edges, a symmetric matrix does not suffice to describe the energy of a system. If we allow each particle to be in $k$ possible states, we need for each multiset of colors $\{c_1, \ldots, c_d\}$, with $c_1, \ldots, c_d \in \{1, \ldots, k\}$, a real number. That is we have a map $h : \mathbb{N}^k \to \mathbb{R}$. De la Harpe and Jones call $h$ a *vertex model*. The partition function of $h$ is a similar expression as (3.4), but the role of edges and vertices is interchanged (cf. (1.1)). See section 3.3 for the formal definition.

**Edge-and vertex-coloring models**

As the partition functions of the respective generalizations of the Ising model generalize graph and linegraph coloring (cf. Sections 3.2 and 3.3 below), we choose to call them *vertex-coloring models* instead of spin models and *edge-coloring models* instead of vertex models to emphasize their combinatorial interpretation. This is consistent, at least for edge-coloring models, with the book by Lovász [40], but in the literature (including the work of the author) both terminologies have been used.

## 3.2 Partition functions of vertex-coloring models

Let $a \in (\mathbb{F}^*)^n$ and let $B \in \mathbb{F}^{n\times n}$ be a symmetric matrix. We call the pair $(a, B)$ an *n-color vertex-coloring model over* $\mathbb{F}$. We think of $n$ as the number of *colors* of the model (or *states* from the physical point of view). When talking about a vertex-coloring model, we will sometimes omit the number of colors or the field of definition. In case $(a, B)$ is defined over $\mathbb{F} = \mathbb{R}$ and $a_i > 0$ for each $i = 1, \ldots, n$, we will call $(a, B)$ a *real vertex-coloring model*. The *partition function* of an $n$-color vertex coloring model $(a,B)$ is the graph invariant $p_{a,B} : \mathcal{G} \to \mathbb{F}$ defined by

$$p_{a,B}(H) := \sum_{\phi:V(H)\to[n]} \prod_{v\in V(H)} a_{\phi(v)} \cdot \prod_{uv\in E(H)} B_{\phi(u),\phi(v)} \tag{3.5}$$

for $H \in \mathcal{G}$. Clearly, $p_{a,B}$ is multiplicative.

If one takes $a = \mathbb{1}$, the all ones vector, and $B$ the adjacency matrix of a graph $G$, then $p_{\mathbb{1},B}(H) = \hom(H,G)$, the number of homomorphisms from $H$ to $G$ (adjacency preserving maps from $V(H)$ to $V(G)$). In particular, for $G = K_n$, the complete graph on $n$ vertices, $\hom(H,G)$ counts the number of proper vertex-colorings of $H$ with $n$ colors.

For general $(a,B)$, we can view $p_{a,B}$ in terms of weighted homomorphisms. Let $G(a,B)$ be the complete graph on $n$ vertices (including loops) with vertex weights given by $a$ and edge weights given by $B$. Then $p_{a,B}(H)$ can be viewed as counting the number of weighted homomorphisms of $H$ into $G(a,B)$. In this context $p_{a,B}$ is denoted by $\hom(\cdot, G(a,B))$. In this thesis we will use both $p_{a,B}$ and $\hom(\cdot, G(a,B))$ to denote the same graph parameter.

**Twins**

Let $(a,B)$ be an $n$-color vertex-coloring model. We say that $i,j \in [n]$ are *twins of* $(a,B)$ if $i \neq j$ and the $i$th row of $B$ is equal to the $j$th row of $B$. If $(a,B)$ has no twins we call the model *twin free*. Suppose now $i,j \in [n]$ are twins of $(a,B)$. If $a_i + a_j \neq 0$, let $B'$ be the matrix obtained from $B$ by removing row and column $i$ and let $a'$ be the vector obtained from $a$ by setting $a'_j := a_i + a_j$ and then removing the $i$th entry from it. In case $a_i + a_j = 0$, we remove the $i$th and the $j$th row and column from $B$ to obtain $B'$ and we remove the $i$th and the $j$th entry from $a$ to obtain $a'$. Then $p_{a',B'} = p_{a,B}$. So for every vertex-coloring model with twins, we can construct a vertex-coloring model with fewer colors which is twin free and which has the same partition function.

At several points in this thesis we will assume that an $n$-color vertex-coloring model is twin free. By the above we do not lose any graph parameters in this way.

## 3.3 Partition functions of edge-coloring models

Let

$$R(\mathbb{F}) := \mathbb{F}[x_1, \ldots, x_k] \tag{3.6}$$

denote the polynomial ring in $k$ variables. We will usually just write $R$ instead of $R(\mathbb{F})$. Note that there is a one-to-one correspondence between elements of $R^*$

and maps $h : \mathbb{N}^k \to \mathbb{F}$; $\alpha \in \mathbb{N}^k$ corresponds to the monomial $x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k} \in R$ and the monomials form a basis for $R$. Moreover, $\alpha \in \mathbb{N}^k$ corresponds to a multisubset of $[k]$; $\alpha_i$ is the multiplicity of $i$.

We call any $h \in R^*$ a *$k$-color edge-coloring model over* $\mathbb{F}$. We think of $k$ as the number of *colors* of the model (or *states* from the physical point of view). When talking about an edge-coloring model, we will sometimes omit the number of colors or the field of definition. In case $h$ is defined over $\mathbb{F} = \mathbb{R}$, we will sometimes call $h$ a *real edge-coloring model*. The *partition function* of a $k$-color edge-coloring model $h$ is the graph parameter $p_h : \mathcal{G} \to \mathbb{F}$ defined by

$$p_h(H) := \sum_{\phi: E(H) \to [k]} \prod_{v \in V(H)} h\Big( \prod_{e \in \delta(v)} x_{\phi(e)} \Big) \tag{3.7}$$

for $H \in \mathcal{G}$. Here $\delta(v)$ is the multiset of edges incident with $v$. Note that, by convention, a loop is counted twice. Moreover, observe that $p_h(\bigcirc) = k$, as the empty product is equal to 1 by definition. Clearly, $p_h$ is multiplicative.

Many interesting graph parameters are partition functions of edge-coloring models. Let us give a few examples.

**Example 3.1** (Counting perfect matchings)**.** Let $k = 2$. Define the the edge-coloring model $h : \mathbb{F}[x_1, x_2] \to \mathbb{F}$ by $h(x_1^{\alpha_1} x_2^{\alpha_2}) = \delta_{\alpha_1,1}$. Then $p_h(H)$ is equal to the number of perfect matchings of $H$. (A *perfect matching* in a graph $H$ is a set of edges that covers each vertex exactly once.) To see this, note that for an assignment of the colors to the edges of $H$ there is a contribution in the sum (3.7) if and only if at each vertex there is a unique edge which is colored with 1, that is, if and only if the edges colored with 1 form a perfect matching.

**Example 3.2** (Counting proper $k$-edge-colorings)**.** Let $k \in \mathbb{N}$. Define the $k$-color edge-coloring model $h$ by

$$h(x_1^{\alpha_1} \cdots x_k^{\alpha_k}) = \begin{cases} 1 \text{ if } \alpha_i \leq 1 \text{ for all } i \in [k], \\ 0 \text{ otherwise.} \end{cases} \tag{3.8}$$

Then $p_h(H)$ is equal to the number of proper $k$-edge-colorings of $H$.

**Example 3.3** (Counting linegraph homomorphisms)**.** Let $G = (V, E)$ be a simple graph with $k$ edges. Identify $E$ with $[k]$ and define the $k$-color edge-coloring model $h$ by

$$h(x_1^{\alpha_1} \cdots x_k^{\alpha_k}) = \begin{cases} 1 & \text{if all } \alpha_i \leq 1 \text{ and if the edges } i \in [k] \text{ such that} \\ & \alpha_i = 1 \text{ meet in a unique vertex in } G, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Then $p_h(H)$ is equal to the number of homomorphisms from the linegraph[1] $L(H)$ of $H$ to the linegraph $L(G)$ of $G$.

We will see later that partition functions of vertex-coloring models over $\mathbb{C}$ are partition functions of edge-coloring models over $\mathbb{C}$ (cf. Lemma 7.1). So also the number of (ordinary) homomorphisms is the partition function of an edge-coloring model.

## 3.4 Tensor networks

We will introduce tensor networks in this section and show that they allow to give a more conceptual interpretation of the partition function of an edge-coloring model $h$.

Let $V$ be $k$-dimensional vectorspace over $\mathbb{F}$ equipped with a symmetric non-degenerate bilinear form $(\cdot,\cdot)$, i.e., $(\cdot,\cdot) : V \times V \to \mathbb{F}$ is a symmetric bilinear map such that for each nonzero $v \in V$ there exists $v' \in V$ such that $(v, v') \neq 0$. The bilinear form induces a nondegenerate symmetric bilinear form on the $l$-th tensor power, $V^{\otimes l}$, of $V$, for any $l$, via

$$(v_1 \otimes \cdots \otimes v_l, u_1 \otimes \cdots \otimes u_l) := \prod_{i=1}^{l} (v_i, u_i) \tag{3.10}$$

for $u_1, v_1, \ldots, u_l, v_l \in V$.

Let $H = ([n], E)$ be a graph. Let for $i \in [n]$, $h_i \in V^{\otimes d(i)}$. (Recall that $d(i)$ denotes the degree of vertex $i$.) Assume that we have some specific ordering of $\delta(i)$ for each $i \in [n]$. Then $(H, h_1, \ldots, h_n)$ is called a *tensor network*. To a tensor network we can associate an element of $\mathbb{F}$ by contracting the network, as we will now describe.

For $1 \leq i < j \leq l \in \mathbb{N}$ the *contraction* $C^l_{i,j}$ is the unique linear map

$$C^l_{i,j} : V^{\otimes l} \to V^{\otimes l-2} \text{ satisfying} \tag{3.11}$$
$$v_1 \otimes \ldots \otimes v_l \mapsto (v_i, v_j) v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \ldots \otimes v_{j-1} \otimes v_{j+1} \otimes \ldots \otimes v_l.$$

Let $h = h_1 \otimes \cdots \otimes h_n \in V^{\otimes 2m}$, where $m$ denotes the cardinality of $E$. An edge $e \in E$ gives rise to a unique contraction $C^{2m}_{i,j}$ for some $i, j \in [2m]$. Then the *contraction of* $(H, h)$ *along* $e$ is the pair $(H', h')$ where $H'$ is the graph obtained from $H$ by removing $e$ and identifying the endpoints of $e$ (an edge parallel to

[1] The linegraph $L(H)$ of a graph $H = (V, E)$ is the graph with vertex set $E$; $e_1, e_2 \in E$ are adjacent in $L(H)$ if and only if $e_1$ and $e_2$ share a vertex.

$e$ becomes a loop in this way) and $h' := C^{2m}_{i,j}(h)$. Note that $h'$ is again of the form $h' = h'_1 \otimes \cdots \otimes h'_{n'}$, with $n' = n$ if $e$ is a loop and $n' = n-1$ otherwise. If $m = 1$, this contraction just gives an element of $\mathbb{F}$. The *contraction* of the tensor network $(H, h_1, \ldots, h_n)$ is the element of $\mathbb{F}$ obtained by recursively contracting $(H, h)$ along any sequence of edges $e_1, \ldots, e_m$. (This does not depend on the chosen sequence as the form is symmetric and bilinear.)

Suppose that $V$ has an orthonormal basis $e_1, \ldots, e_k$ with respect to $(\cdot,\cdot)$. (If $\mathbb{F}$ is algebraically closed, an orthonormal basis always exists, but for example for $\mathbb{F} = \mathbb{R}$ such a basis need not exist.) Define for $\phi : [d] \to [k]$, $e_\phi := e_{\phi(1)} \otimes \cdots \otimes e_{\phi(d)}$ and note that the $e_\phi$ form an orthonormal basis for $V^{\otimes d}$. Let $x_1, \ldots, x_k$ be the associated dual basis for $V^*$ and let $h \in \mathbb{F}[x_1, \ldots, x_k]^*$. Let $H([n], E)$ be a graph and let for $v \in [n]$, $h_v$ be the restriction of $h$ to the space of homogeneous polynomials of degree $d(v)$. We can view $h_v$ as a symmetric tensor in $V^{\otimes d(v)}$. That is, for each $\phi : [d(v)] \to [k]$,

$$(h_v, e_\phi) = h(x_{\phi(1)} \cdots x_{\phi(d(v))}). \tag{3.12}$$

Then it is easy to see that $p_h(H)$ is equal to the contraction of the tensor network $(H, h_1, \ldots, h_n)$ (taking any ordering of $\delta(i)$ for $i \in [n]$, as the $h_i$ are symmetric). This will be spelled out in Section 6.2. For completeness, we will now sketch a proof of this fact here. Using (3.12) we find that

$$\begin{aligned} p_h(H) &= \sum_{\phi:E\to[k]} \prod_{i=1}^{n} h\Big(\prod_{e\in\delta(i)} x_{\phi(e)}\Big) = \sum_{\phi:E\to[k]} \prod_{i=1}^{n} \Big(h_i, \bigotimes_{e\in\delta(i)} e_{\phi(i)}\Big) \\ &= \sum_{\phi:E\to[k]} \Big(h_1 \otimes \cdots \otimes h_n, \bigotimes_{e\in E} e_{\phi(e)} \otimes e_{\phi(e)}\Big). \end{aligned} \tag{3.13}$$

The last line of (3.13) is equal to the contraction of $(H, h_1, \ldots, h_n)$, as for any $v \in V^{\otimes 2}$, $C^2_{1,2}(v) = \sum_{i=1}^{k} (v, e_i \otimes e_i)$.

*Remark.* Using tensor networks, we obtain a coordinate-free definition of partition function of edge-coloring models. Many results in this thesis have analogues that are coordinate-free, but for the sake of concreteness, we will mostly work with a fixed orthonormal basis and formulas such as (3.7).

## 3.5 The orthogonal group

As in the previous section, let $V$ be a $k$-dimensional vector space over $\mathbb{F}$ that is equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$. The *orthogonal group* $\mathrm{O}_k(\mathbb{F})$ is the group of invertible linear maps $g : V \to V$ such that

$(gu, gv) = (u, v)$ for all $u, v \in V$. We do not indicate the dependency on the bilinear form. Moreover, we will usually write $\mathrm{O}_k$ instead of $\mathrm{O}_k(\mathbb{F})$.

If $V$ admits an orthonormal basis $e_1, \ldots, e_k$ with respect to $(\cdot, \cdot)$, then, with respect to this basis, $\mathrm{O}_k$ is the set of $\mathbb{F}$-valued $k \times k$ matrices $g$ such that $g^T g = I$, where $I$ denotes the $k \times k$ identity matrix.

By $\mathcal{O}(V)$ we denote the algebra generated by the linear maps on $V$. If we choose a basis $x_1, \ldots, x_k$ for $V^*$, this is just the polynomial ring $R$. The group $\mathrm{O}_k$ acts on $\mathcal{O}(V)$ as follows:

$$\text{for } p \in \mathcal{O}(V),\ g \in \mathrm{O}_k \text{ and } v \in V,\ (gp)(v) := p(g^{-1}v). \tag{3.14}$$

This action induces an action on $\mathcal{O}(V)^*$ (and hence on edge-coloring models) as follows:

$$\text{for } h \in \mathcal{O}(V)^*,\ p \in \mathcal{O}(V) \text{ and } g \in \mathrm{O}_k,\ (gh)(p) = h(g^{-1}p). \tag{3.15}$$

The orthogonal group acts linearly on $V^{\otimes l}$:

$$\text{for } v = v_1 \otimes \cdots \otimes v_l \in V^{\otimes l} \text{ and } g \in \mathrm{O}_k,\ gv := gv_1 \otimes \cdots \otimes gv_l. \tag{3.16}$$

It is a well-known fact that $V$ and $V^*$ are isomorphic as $\mathrm{O}_k$-modules. Indeed, define the map $\tau : V \to V^*$ by $\tau(v)(u) = (v, u)$ for $v, u \in V$. Then for $g \in \mathrm{O}_k$ and $u, v \in V$,

$$\tau(gv)(u) = (gv, u) = (v, g^{-1}u) = (g\tau(v))(u). \tag{3.17}$$

So for the $\mathrm{O}_k$-action it does not matter whether we think of a $k$-color edge-coloring model $h$ as a linear function on the polynomial ring or as a collection of symmetric tensors. Sometimes it will be more convenient to think of $h$ as an element of $\mathcal{O}(V)^*$ (or $R^*$) and sometimes it is more convenient to think of $h$ as a collection of symmetric tensors.

As contractions are by definition $\mathrm{O}_k$-invariant, the tensor network interpretation of the partition function function of $h$ immediately implies that it is invariant under the action of $\mathrm{O}_k$:

$$\text{for each } g \in \mathrm{O}_k \text{ and any graph } H,\ p_{gh}(H) = p_h(H). \tag{3.18}$$

## 3.6 Computational complexity

It is by all means not a surprise that computing the partition functions of edge- or vertex-coloring models is generally hard, as already deciding whether a graph can be properly colored with $k$ colors is NP-complete for $k \geq 3$. In fact, computing the number of $k$-colorings of a graph is known to be #P-complete for $k \geq 3$.

There are so-called dichotomy results about the the complexity of evaluating $p_{a,B}$ for certain classes of vertex-coloring models $(a, B)$. This roughly means that $p_{a,B}$ can be computed in polynomial time if $(a, B)$ has a special structure and that it is #P-complete otherwise. We refer to [21, 10, 11] for more details.

Lovász [38] (see also [40]) showed that if for some $l \in \mathbb{N}$, the vertex-connection matrix $N_{f,l}$ has finite rank, then there exists a polynomial time algorithm that computes $f$ on graphs of treewidth bounded by $l$. This implies that for graphs of bounded treewidth, $p_{a,B}(H)$ can be computed in polynomial time.

Partition functions of edge-coloring models can be seen as a special case of a so-called Holant problems (cf. [12, 13]). In [12, 13] Cai, Lu and Xia prove a dichotomy result for a particular class of Holant problems. As far as we know no complete classification for the complexity of computing partition functions of edge-coloring models has been obtained. But the main message is that these are generally hard problems, unless the edge-coloring model has some special structure.

We should remark however, that using the interpretation of partition functions of edge-coloring models as contractions of tensor networks, we can conclude by a result of Markov and Shi [48], who used tensor networks in the field of quantum computing to simulate quantum computation, that for graphs of bounded degree and bounded treewidth the partition function of a $k$-color edge-coloring model can be computed in polynomial time. Using the fact that for a $k$-color edge-coloring model $h$ such that $h(x^\alpha) = 0$ if $|\alpha| > d$, the rank of its $l$-th vertex-connection matrix is bounded by $(k^d)^l$ (as follows from Proposition 5.5), this also follows from the result of Lovász.

# Chapter 4

# Invariant theory

Throughout this thesis we will use the language of, and some important results from, invariant theory. We will give a short introduction to invariant theory in this chapter. It is by all means not a complete introduction. We refer to [35, 25] for more details; see [50] for some advanced topics and we refer to [36] for background on algebra. This introduction is partly based on the manuscript by Kraft and Procesi [35].

## 4.1 Representations and invariants

**Basic definitions**

Let $G$ be a group and let $W$ be a vector space. We say that $G$ *acts linearly* (sometimes we omit linearly) on $W$ if there exists a homomorphism of groups $\rho : G \to \mathrm{GL}(W)$, where $\mathrm{GL}(W)$ denotes the group of invertible linear maps from $W$ to $W$. The pair $(W, \rho)$ is called a *representation* of $G$; $W$ is sometimes called a *G-module* [1]. We will usually just write $gw$ instead of $\rho(g)w$ for $w \in W$ and $g \in G$. Let $V, W$ be two $G$-modules. A linear map $\phi : V \to W$ is called *G-equivariant* if $\phi(gv) = g\phi(v)$ for all $g \in G$ and $v \in V$.

A subspace $W'$ of a $G$-module $W$ is called *G-stable* if $gw' \in W'$ for all $w' \in W'$. A $G$-module $W$ is called *completely reducible* if for each $G$-stable subspace $W' \subset W$ there exists a $G$-stable subspace $U$ such that $W' \oplus U = W$. By Maschke's Theorem (cf. [36, XVIII, §1] or [57, Section 1.5]), if $G$ is a finite group

[1] More specifically, $W$ is a left module of the group algebra of $G$. A *left module* $W$ of a ring $A$ is an abelian group with an action of $A$ on $W$ satisfying $(a+b)w = aw + bw$, $a(v+w) = av + aw$ and $(ab)w = a(bw)$ for all $v, w \in W$ and $a, b \in A$.

and if $W$ is a finite-dimensional $G$-module, then $W$ is completely reducible. By $W^G$ we denote the subspace of *$G$-invariants*, i.e.,

$$W^G := \{w \in W \mid gw = w \text{ for all } g \in G\}. \tag{4.1}$$

**Hilbert's theorem**

Suppose that $G$ acts on a $n$-dimensional vectorspace $W$. Let $W^*$, denote the space of linear functions $f : W \to \mathbb{F}$ and let $\mathcal{O}(W)$ denote the space of regular functions on $W$ (the algebra generated by $W^*$). The action of $G$ on $W$ induces an action on $\mathcal{O}(W)$ via $(gf)(w) := f(g^{-1}w)$ for $g \in G$, $f \in \mathcal{O}(W)$ and $w \in W$. Note that $\mathcal{O}(W)$ has natural grading coming from the homogenous functions. This grading is respected by the group action. So $\mathcal{O}(W)$ splits into an infinite sum of finite dimensional $G$-modules.

The next theorem is due to Hilbert.

**Theorem 4.1.** *Let $W$ be a $G$-module and assume that the representation of $G$ on $\mathcal{O}(W)$ is completely reducible. Then the invariant ring $\mathcal{O}(W)^G$ is finitely generated.*

We will not prove this result here; see [35] or [9] for a proof. We want however to highlight an important idea from the proof.

**The Reynolds operator**

Let $W$ be a $G$-module and suppose that $\mathcal{O}(W)$ is completely reducible. Let $\rho_G : \mathcal{O}(W) \to \mathcal{O}(W)^G$ denote the $G$-equivariant linear projection onto $\mathcal{O}(W)^G$. This map is usually called the *Reynolds operator of $G$*. (More generally, if $V$ is any completely reducible $G$-module, then the projection onto $V^G$ is called the Reynolds operator.) Then $\rho_G$ satisfies

$$\rho_G(pq) = p\rho_G(q) \quad \text{for } p \in \mathcal{O}(W)^G \text{ and } q \in \mathcal{O}(W). \tag{4.2}$$

To see this, let $Q \subset \mathcal{O}(W)$ denote a $G$-stable complement to $\mathcal{O}(W)^G$. It is convenient to first prove the following:

$$\text{for } p \in \mathcal{O}(W)^G \text{ and } q \in Q,\ pq \in Q. \tag{4.3}$$

To see this, define $\phi : Q \to \mathcal{O}(W)$ by $q \mapsto pq$. Then $\phi$ is $G$-equivariant. Indeed, since $p$ is $G$-invariant, $g(pq) = gp \cdot gq = p \cdot gq$. Suppose now that $pq \notin Q$ for some $q \in Q$. Since the Reynolds operator is also $G$-equivariant we may assume that $\phi(q) = p'$ for some nonzero $p' \in \mathcal{O}(W)^G$. Moreover, by restricting $\phi$, we may assume that $\phi(Q) = \mathbb{F}p'$. Now note that $\operatorname{Ker}\phi$ is $G$-stable and moreover that $G$ acts on $Q/\operatorname{Ker}\phi$. Then $\phi$ induces an $G$-equivariant isomorphism $\phi : Q/\operatorname{Ker}\phi \to \mathbb{F}p'$. But this implies that $G$ acts trivially on $Q/\operatorname{Ker}\phi$. Hence $q \in \mathcal{O}(W)^G + \operatorname{Ker}\phi$. A contradiction. This proves (4.3).

To prove (4.2), write $q = q_1 + q_2$ with $q_1 \in \mathcal{O}(W)^G$ and $q_2 \in Q$. Note that $q_1 = \rho_G(q)$. Then $\rho_G(pq) = \rho_G(pq_1) + \rho_G(pq_2) = pq_1$ by (4.3).

The proof of (4.2) revealed a special case of Schur's lemma which will be convenient to record.

**Lemma 4.2.** *Suppose $G$ acts on a space $W$ and suppose that $W$ admits a direct sum decomposition $W = W^G \oplus W'$, with $W'$ stable under $G$. Let $\phi : W \to \mathbb{F}$ be a linear map such that $\phi(gw) = \phi(w)$ for all $g \in G$ and $w \in W$. Then $\phi(W') = 0$.*

**Classical invariant theory**

By Theorem 4.1 we know that there exists finitely many $f_1, \ldots, f_m \in \mathcal{O}(W)$ that generate $\mathcal{O}(W)^G$. In classical invariant theory one is interested in finding an explicit set of generators for $\mathcal{O}(W)^G$ and determining relations between them. In the next section we will state some results about this for the orthogonal group acting on $W = \mathbb{F}^{k\times n}$.

The results in Chapter 5 can be viewed from the perspective of classical invariant theory: describing generators for a certain algebra of (polynomial) functions invariant under the action of the orthogonal group and describing relations between them.

## 4.2 FFT and SFT for the orthogonal group

In this section we consider the natural action of the orthogonal $\mathrm{O}_k$ on $\mathbb{F}^{k\times n}$. The theorem describing generators of $\mathcal{O}(\mathbb{F}^{k\times n})$ is called the First Fundamental Theorem (FFT) for the orthogonal group and the theorem describing the relations between these generators is called the Second Fundamental Theorem (SFT) for the orthogonal group. In this section we will state these theorems. We will however start with the natural action of $\mathrm{O}_k$ on $V^{\otimes n}$, where $V := \mathbb{F}^k$, and describe a generating set for the $\mathrm{O}_k$-invariants. This is usually referred to as the Tensor FFT for $\mathrm{O}_k$.

Let $\mathcal{M}_m$ be the set of perfect matchings on $[2m]$, i.e., $M \in \mathcal{M}_m$ is the disjoint union of $2m$ edges. Define for $M \in \mathcal{M}_m$ the tensor $t_M \in V^{\otimes 2m}$ by

$$t_M := \sum_{\substack{\phi:[2m]\to[k],\ \phi(u)=\phi(v) \\ \text{for each } uv\in E(M)}} e_{\phi(1)} \otimes \cdots \otimes e_{\phi(2m)}. \tag{4.4}$$

**Theorem 4.3** (Tensor FFT for $\mathrm{O}_k$)**.** *If $n$ is odd, then $(V^{\otimes})^{\mathrm{O}_k} = 0$ and if $n = 2m$ for some $m$, then*

$$(V^{\otimes 2m})^{\mathrm{O}_k} = span\{t_M \mid M \in \mathcal{M}_m\}. \tag{4.5}$$

For a proof of Theorem 4.3 see [25, Section 5.2], (the proof there is given for $\mathbb{F} = \mathbb{C}$, but it is valid for arbitrary algebraically closed fields of characteristic zero and hence it is also valid for $\mathbb{F}$ as $\mathrm{O}_k(\mathbb{F})$ is Zariski dense in the orthogonal group over the algebraic closure of $\mathbb{F}$ (cf. [35, §10 Exercise 5])). We will prove it in Section 4.4 using a different approach.

**Theorem 4.4** (FFT for $\mathrm{O}_k$). *The $\mathrm{O}_k$-invariants in $\mathcal{O}(\mathbb{F}^{k\times n})$ are generated by the polynomials $\sum_{l=1}^{k} x_{l,i}x_{l,j}$ for $i, j = 1, \ldots, n$.*

Theorem 4.4 can easily be derived from the Tensor FFT (cf. [25, Section 5.4]). For a direct proof see [35, Section 10.3].

Now for the relations between the generators of $\mathcal{O}(\mathbb{F}^{k\times n})^{\mathrm{O}_k}$. Let $S\mathbb{F}^{n\times n}$ denote the space of symmetric $n \times n$ matrices in $\mathbb{F}^{n\times n}$. Define

$$\tau : \mathcal{O}(S\mathbb{F}^{n\times n}) \to \mathcal{O}(\mathbb{F}^{k\times n}) \quad \text{by} \quad z \mapsto (M \mapsto z(M^T M)). \tag{4.6}$$

Then Theorem 4.4 says that $\tau(\mathcal{O}(S\mathbb{F}^{n\times n})) = \mathcal{O}(\mathbb{F}^{k\times n})^{\mathrm{O}_k}$.

**Theorem 4.5** (SFT for $\mathrm{O}_k$). *The kernel of $\tau$ is the ideal generated by the $(k+1) \times (k+1)$ minors of $S\mathbb{F}^{n\times n}$.*

For a proof of Theorem 4.5 see [25, Section 11.2]. The proof there is for $\mathbb{F} = \mathbb{C}$, but it is valid for any algebraically closed field of characteristic zero; it is quite technical. We now sketch an outline for a different proof. Assume first that $\mathbb{F}$ is algebraically closed. Define $t : \mathbb{F}^{k\times n} \to S\mathbb{F}^{n\times n}$ by $M \mapsto M^T M$, for $M \in \mathbb{F}^{k\times n}$. Then $\operatorname{Ker} \tau \subseteq \mathcal{O}(S\mathbb{F}^{n\times n})$ is the ideal defined by those polynomials that vanish on the image of $t$. The image of the map $t$ is equal to the space of all symmetric matrices of rank at most $k$ (cf. [25, Lemma 5.2.4]). As the image of $t$ is determined by the vanishing of the $(k+1) \times (k+1)$ minors, it follows by the Nullstellensatz (see below) that if these minors generate a radical ideal, then this ideal equals the kernel of $\tau$. Unfortunately, it is not easy to prove that the minors generate a radical ideal. It can be proved using Gröbner bases; Conca [14] proved that the minors form a Gröbner basis. Combined with the fact each monomial in a minor is square free, this implies that they generate a radical ideal. (See [16] for an introduction to Gröbner bases.)

To see that the SFT is also valid for non-algebraically closed fields $\mathbb{F}$, note that $\mathbb{F}^{k\times n}$ is Zariski dense in $\overline{\mathbb{F}}^{k\times n}$, implying that the same holds for the image of $t$. So the vanishing ideals of $t(\mathbb{F}^{k\times n})$ and $t(\overline{\mathbb{F}}^{k\times n})$ are the same (when viewed as ideals of $\mathcal{O}(S\overline{\mathbb{F}}^{n\times n})$). As the minors are defined over $\mathbb{F}$, it follows that the SFT also holds over $\mathbb{F}$.

## 4.3 Existence and uniqueness of closed orbits

Finite groups and also the orthogonal group are examples of linear algebraic groups. Using the algebraic structure, one can obtain some useful results such as the existence and uniqueness of closed orbits. We will state this result here; we will however only do this for affine algebraic groups. For more details on linear algebraic groups we refer to [4, 30]. Throughout this section we will work with algebraically closed fields. So $\mathbb{F} = \overline{\mathbb{F}}$ throughout this section.

**Zariski topology and the Nullstellensatz**
Let $V := \mathbb{F}^n$. A set $A \subseteq V$ is called *Zariski closed* if is is the zero set of finitely many polynomials, i.e., if there exists polynomials $f_1, \ldots, f_m$ such that $A = \mathcal{V}(\{f_1, \ldots, f_m\}) := (\{v \in V \mid f_i(v) = 0 \mid i = 1, \ldots, m\}$. Clearly, we can replace the set $\{f_1, \ldots, f_m\}$ by the ideal they generate. With this definition, $V$ becomes a topological space. The *Zariski closure* of a set $A \subset V$ is the set $\overline{A}$ defined by all the zeros of all the polynomials vanishing on $A$. A Zariski closed set is sometimes called an *affine variety*. Define for an ideal $I \subset R = \mathbb{F}[x_1, \ldots, x_n]$ its *radical* by $\sqrt{I} := \{f \mid f^k \in I \text{ for some } k \in \mathbb{N}\}$. We will now state a fundamental result in algebraic geometry.

**Theorem 4.6** (Hilbert's Nulstellensatz)**.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $I$ be an ideal in $R$. Then $\{f \mid f(v) = 0 \text{ for all } v \in \mathcal{V}(I)\} = \sqrt{I}$. In particular, if $I \neq R$, then there exists $v \in \mathbb{F}^n$ such that $f(v) = 0$ for all $f \in I$.*

See [36, IX, §1] for a proof of the Nullstellensatz.

**Orbits of affine algebraic groups**
An *affine algebraic group* is an affine variety $G \subset \mathbb{F}^n$ with a group structure such that the group operations are given by polynomial maps in the coordinates of $\mathbb{F}^n$. The orthogonal group $\mathrm{O}_k$ is an example of an affine variety; $\mathrm{O}_k$ is determined by $g^t g = I$ for $g \in \mathbb{F}^{k \times k}$. Clearly, the group operation and taking the inverse are polynomial maps in the coordinates of $\mathbb{F}^{k \times k}$.

A representation $(W, \rho)$ of an affine algebraic group $G \subset \mathbb{F}^n$ is called *polynomial* if the map $\rho : G \to \mathrm{GL}(W)$ is given by polynomial maps in the coordinates of $\mathbb{F}^n$. All representations we will encounter in this thesis are polynomial. An affine algebraic group is called *reductive* if each finite dimensional polynomial representation is completely reducible. It is a well-known fact that the orthogonal group is reductive (cf. [25, Theorem 3.3.12]). We will see a proof of this fact in the next section.

Suppose $(W, \rho)$ is a finite dimensional polynomial representation of a reductive affine algebraic group $G$. Recall from Theorem 4.1 that $\mathbb{F}[W]^G$ is finitely generated. Let $f_1, \dots, f_m$ be generators of $\mathbb{F}[W]^G$. Define $\pi : W \to \mathbb{F}^m$ by

$$\pi(w)_j = f_j(w) \text{ for } j = 1, \dots, m. \tag{4.7}$$

The map $\pi$ is called the *quotient map*. (The quotient map of course depends on the choice of generators, but it can be shown that $\pi(W)$ is an affine variety and that for different choices of generators these varieties are isomorphic; $\pi(W)$ is usually denoted by $W//G$.) Note that for each $v \in \pi(W)$, $\pi^{-1}(v)$ is Zariski closed. Furthermore, it is $G$-stable; so it is union of $G$-orbits. (A $G$-orbit is a set $Gw := \{gw \mid g \in G\}$ for some $w \in W$.) Then there is a unique Zariski-closed $G$-orbit (which is the orbit of minimal Krull-dimension) contained in $\pi^{-1}(v)$ which is contained in the Zariski closure of each orbit in $\pi^{-1}(v)$. (We will often just say closed orbit instead of Zariski-closed orbit.) We will record it as a theorem.

**Theorem 4.7.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $\pi : W \to \mathbb{F}^m$ be the quotient map. Then for each $v \in \pi(W)$, the fiber $\pi^{-1}(v)$ contains a unique Zariski-closed $G$-orbit $C$. Moreover, if $C'$ is another $G$-orbit contained in $\pi^{-1}(v)$, then $C \subseteq \overline{C'}$.*

To prove existence in Theorem 4.7, one can proceed as follows: let $w \in \pi^{-1}(v)$. If $Gw$ is not closed, then $Gw$ is open in its closure $\overline{Gw}$ and so $\overline{Gw} \setminus Gw$ is the union of $G$-orbits of strictly lower Krull-dimension. Hence an orbit of minimal Krull-dimension must be closed. See [30, Section 8.3] for details. See [9] or [34, II.3.2-3] for a proof of both existence and uniqueness.

## 4.4 Proof of the Tensor FFT

The proof of Theorem 4.3 in [25] is quite technical. Here we will prove it using a different approach, but we will not include all details. We consider the case $\mathbb{F} = \mathbb{C}$. (The proof is valid for arbitrary algebraically closed fields of characteristic zero as we will point out later.) First we need some preparations.

Write $W := V^{\otimes m}$. Then we have a representation $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$ defined by

$$v_1 \otimes \cdots \otimes v_m \mapsto gv_1 \otimes \cdots \otimes gv_m \tag{4.8}$$

for $g \in \mathrm{GL}(V)$ and $v_1, \dots, v_m \in V$. We moreover have a representation $\tau : S_m \to \mathrm{GL}(W)$ defined by

$$v_1 \otimes \cdots \otimes v_m \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)} \tag{4.9}$$

for $\sigma \in S_n$ and $v_1, \dots, v_m \in V$. (The group $S_m$ is the *symmetric group*; it consists of all permutations of the set $[m]$.)

For a subset $S \subseteq \mathrm{End}(W)$ we define its *commutant* by

$$\mathrm{Comm}(S) := \{x \in \mathrm{End}(W) \mid xs = sx \text{ for all } s \in S\}. \tag{4.10}$$

Let $\mathcal{A}$ be the span of the $\rho(g) \in \mathrm{End}(W)$ for $g \in \mathrm{GL}(V)$ and let $\mathcal{S}$ be the span of the $\tau(\sigma)$ for $\sigma \in S_n$. Schur (cf. [25, Section 4.2.4] ) proved that these algebras are each others commutant.

**Theorem 4.8** (Schur)**.** *$Comm(\mathcal{S}) = \mathcal{A}$ and $Comm(\mathcal{A}) = \mathcal{S}$.*

See [25, Section 4.2.4] or [35, Section 3.1] for a proof. The proofs are based on the so-called Double Commutant Theorem:

**Theorem 4.9** (Double Commutant Theorem)**.** *Let $W$ be a finite dimensional vector space and let $A$ be a subalgebra of $End(W)$ containing $I_W$. Set $B := Comm(A)$. If $W$ is a completely reducible $A$-module, then $Comm(B) = A$. Moreover, $W$ is a completely reducible $B$-module.*

See [35, Section 3.2] for a proof of the Double Commutant Theorem; see [25, Section 4.1.5] for a proof of the first statement only. We will use Theorem 4.8 combined with the Double Commutant Theorem to prove Theorem 4.3.

**Theorem 4.3** (Tensor FFT for $\mathrm{O}_k$)**.** *If $n$ is odd, then $(V^{\otimes n})^{\mathrm{O}_k} = 0$ and if $n = 2m$ for some $m$, then*

$$(V^{\otimes n})^{\mathrm{O}_k} = span\{t_M \mid M \in \mathcal{M}_m\}. \tag{4.11}$$

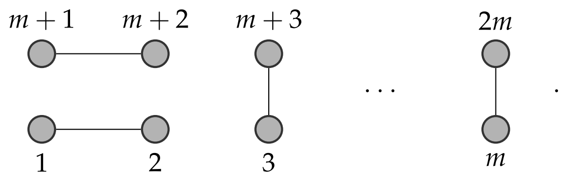
*Proof.* Since $-I \in \mathrm{O}_k$ if follows that if $n$ is odd, the only invariant is 0. Now suppose $n = 2m$ for some $m$. We may assume that $m \geq 2$, as the case $m = 1$ directly follows from the general case.

First identify $V$ with $V^*$ as $\mathrm{O}_k$-modules through the bilinear form. Let $W := V^{\otimes m}$ and make $\mathrm{End}(W)$ a $\mathrm{GL}(V)$-module by setting $gx := \rho(g)x\rho(g^{-1})$ for $g \in \mathrm{GL}(V)$ and $x \in \mathrm{End}(V)$. We then have a canonical isomorphism $\mathrm{End}(W) \cong V^{\otimes m} \otimes (V^*)^{\otimes m}$ as $\mathrm{GL}(V)$-modules. So $\mathrm{GL}(V)$-invariant tensors in $V^{\otimes m} \otimes (V^*)^{\otimes m}$ correspond uniquely to elements of $\mathrm{Comm}(\mathcal{A})$. Similarly, $\mathrm{O}_k$-invariant tensors in $V^{\otimes 2m}$ correspond uniquely to elements of the commutant of the space spanned by the $\rho(g)$ for $g \in \mathrm{O}_k$.

For $\phi : [2m] \to [k]$ we think of $e_{\phi(1)}, \dots e_{\phi(m)}$ as living in $V^{\otimes m}$ and $e_{\phi(m+1)}, \dots,$ $e_{\phi(2m)}$ as living in $(V^*)^{\otimes m}$. For a perfect matching $M$ on $2m$ points we can therefore naturally view $t_M$ as an element of $\mathrm{End}(W)$. Thinking of $M \in \mathcal{M}_m$ as a $2m$-fragment, $\mathbb{C}\mathcal{M}_m \subset \mathbb{C}\mathcal{F}_{2m}$ becomes an algebra if we replace each $\bigcirc$ in $M_1 \cdot M_2$ (the gluing product of the $2m$-fragments) by $k$. Recall that the identity element is the matching connecting $i$ to $m + i$ for $i \in [m]$. This algebra was introduced by Brauer [8] and is called the *Brauer algebra*. Let $\mathcal{B}_m \subset \mathrm{End}(W)$ be the span of the $t_M$ for $M \in \mathcal{M}_m$. Note that the linear map sending $M$ to $t_M$ is a surjective homomorphism of algebras from $\mathbb{C}\mathcal{M}_m$ to $\mathcal{B}_m$.

**Proposition 4.10.** *For $m > 1$, Comm$(\mathcal{B}_m)$, is equal to the span of the $\rho(g)$ for $g \in O_k$.*

*Proof.* Consider the matchings whose edges run between $\{1, \ldots, m\}$ and $\{m+1, \ldots, 2m\}$. Note that each such a matching uniquely defines an element $\sigma \in S_m$. It follows that $\mathcal{S} \subseteq \mathcal{B}_m$. This implies by Theorem 4.8 that Comm$(\mathcal{B}_m)$ is contained in $\mathcal{A}$. Now let $g \in \mathrm{GL}(V)$ such that $\rho(g)b\rho(g^{-1}) = b$ for all $b \in \mathcal{B}_m$. In other words, $gt_M = t_M$ for all $M \in \mathcal{M}_m$. Consider the matching $M$ defined as

$M :=$ 


Write $f_1, \ldots, f_k$ for the basis of $V^*$ (dual to $e_1, \ldots, e_k$). Then we obtain that $g$ should satisfy:

$$\Big(\sum_{i=1}^{k} ge_i \otimes ge_i\Big) \otimes \Big(\sum_{j=1}^{k} gf_j \otimes gf_j\Big) = \Big(\sum_{i=1}^{k} e_i \otimes e_i\Big) \otimes \Big(\sum_{j=1}^{k} f_j \otimes f_j\Big). \tag{4.12}$$

One directly obtains from (4.12) that

$$\sum_{i=1}^{k}\sum_{j=1}^{k} g_{l,i}g_{h,i}(g^{-1})^T_{l',j}(g^{-1})^T_{h',j} = \delta_{h,l}\delta_{h',l'} \text{ for all } l, h, l', h' = 1, \ldots, k, \tag{4.13}$$

where we write $g = (g_{i,j})$ and $g^{-1} = (g^{-1}_{i,j})$ for $g$ and $g^{-1}$ relative to the basis $e_1, \ldots, e_k$. This implies that $gg^T = aI$ for some nonzero $a$. Hence $\rho(g)$ is contained in the span of the $\rho(g')$ for $g' \in O_k$. This finishes the proof. □

The next thing we need is that $W$ is a completely reducible $\mathcal{B}_m$-module. This follows from the following observation. Define an inner product on End$(W)$ by $\langle x, y \rangle := \mathrm{tr}(xy^*)$ for $x, y \in \mathrm{End}(W)$. (Here tr$(x)$ denotes the trace of $x \in \mathrm{End}(W)$ and by $x^*$ we denote the conjugate transpose of $x$.) Now note that for each $x \in \mathcal{B}$, $x^* \in \mathcal{B}$, since $t_M^T = t_{M'}$, where $M'$ is the matching obtained from $M$ by interchanging vertex $i$ with $m+i$ for $i = 1, \ldots m$. This implies that $\mathcal{B}$ is a semisimple algebra[2]. (See for example [36, XVII, §7 Exercise 1-7, ].) From this we conclude that $W$ is completely reducible. So by the Double Commutant

[2]That is, $\mathcal{B}$ is completely reducible as a $\mathcal{B}$-module.

Theorem, we conclude that $\mathcal{B}_m$ is equal to $\mathrm{Comm}\{\rho(g) \mid g \in \mathrm{O}_k\}$. This finishes the proof. □

*Remark.* The proof remains valid for arbitrary algebraically closed fields $\mathbb{F}$ of characteristic zero. A field is called *real* if $-1$ is not a sum of squares. We just need to find a real subfield of index 2 in $\mathbb{F}$ (whose existence is granted by Zorn's Lemma cf. [36, XI, §2]) and define a 'complex conjugation' to be able to define the inner product.

Note that the Double Commutant Theorem also implies that $W = V^{\otimes m}$ is completely reducible as an $\mathrm{O}_k$-module. (In case $m = 1$, $W$ is irreducible.) Together with the observation in [35, Section 5.3] that any polynomial representation of $\mathrm{O}_k$ occurs in a sum $\oplus_{i=1}^{t} V^{\otimes n_i}$, this implies that $\mathrm{O}_k$ is reductive.

**Corollary 4.11.** *The orthogonal group $\mathrm{O}_k$ is reductive.*

# Chapter 5

# Characterizing partition functions of edge-coloring models

In this chapter we characterize which graph invariants are partition functions of edge-coloring models over an algebraically closed field of characteristic zero.

This chapter is based on joint work with Jan Draisma, Dion Gijswijt, Laci Lovász and Lex Schrijver [19] except for Section 5.2, which is based on unpublished joint work with Lex Schrijver and Dion Gijswijt.

## 5.1 Introduction

Motivated by a question of Freedman (see the preface of the book by Lovász [40]), Freedman, Lovász and Schrijver characterized partitions functions of real vertex-coloring models in terms of rank and positive semidefiniteness conditions for the vertex-connection matrices.

**Theorem 5.1** (Freedman, Lovász and Schrijver [24]). *Let $f : \mathcal{G} \to \mathbb{R}$ be a graph invariant. Then there exists $a \in \mathbb{R}^n_{>0}$ and a symmetric matrix $B \in \mathbb{R}^{n\times n}$ such that $f(H) = p_{a,B}(H)$ for all $H \in \mathcal{G}$ if and only if $f$ is multiplicative, reflection positive and* $\text{rk}\,(N_{f,l}) \leq n^l$ *for all $l \in \mathbb{N}$.*

In an earlier version of their paper Freedman, Lovász and Schrijver conjectured that a similar characterization holds for partition functions of real edge-coloring models. This was proved by Szegedy [66]. The characterization is as follows.

**Theorem 5.2** (Szegedy [66]). *Let $f : \mathcal{G}' \to \mathbb{R}$ be a graph invariant. Then there exists a real edge-coloring model $h$ such that $f = p_h$ if and only if $f$ is multiplicative and edge-reflection positive.*

Whereas the proof of Theorem 5.1 in [24] makes use of some basic properties of finite dimensional commutative algebras, Szegedy [66] proved Theorem 5.2 using the First Fundamental Theorem of invariant theory for the orthogonal group and the Positivestellensatz (real Nulstellensatz). This connection with invariant theory and algebra has been further developed by Schrijver [59], giving an alternative (and shorter) proof of Theorem 5.2. He also used this idea to characterize partition functions of vertex-coloring models with $a = \mathbb{1}$, the all-ones vector, [60, 61].

In this chapter we give a characterization of partition functions of edge-coloring models with values in an algebraically closed field of characteristic zero. So throughout this chapter $\mathbb{F} = \overline{\mathbb{F}}$. Moreover, we characterize when the edge-coloring model can be taken to be of finite rank (see definition below). To state our results we need to introduce some definitions.

For a graph $H = (V, E)$, $U \subseteq V$ and any $s : U \to V$, define

$$E_s := \{us(u) \mid u \in U\} \quad \text{and} \quad H_s := (V, E \cup E_s) \tag{5.1}$$

(adding multiple edges if $E$ intersects $E_s$). Let $S_U$ denote the group of permutations of $U$.

**Theorem 5.3.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $f : \mathcal{G} \to \mathbb{F}$ be a graph invariant. Then $f = p_h$ for some $k$-color edge-coloring model over $\mathbb{F}$ if and only if $f$ is multiplicative and for each graph $H = (V, E)$ and each $U \subseteq V$ of size $k + 1$ and each $s : U \to V$,*

$$\sum_{\pi \in S_U} sgn(\pi) f(H_{s\circ\pi}) = 0. \tag{5.2}$$

We will prove Theorem 5.3 in Section 5.4. Recently, based on Theorem 5.3, Schrijver [62] found a characterization of partition functions of complex edge-coloring models in terms of rank growth of the edge-connection matrices.

For a $k$-color edge-coloring model $h$, its *moment matrix* $M_h$ is defined by

$$M_h(\alpha, \beta) = h(x^{\alpha+\beta}), \text{ for } \alpha, \beta \in \mathbb{N}^k. \tag{5.3}$$

Abusing language we say that $h$ has *rank* $r$ if $M_h$ has rank $r$. For any graph $H = (V, E)$, $U \subseteq V$ and $s : U \to V$, let $H/s$ be the graph obtained by contracting all edges in $E_s$.

**Theorem 5.4.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $f : \mathcal{G} \to \mathbb{F}$ be the partition function of a k-color edge-coloring model over $\mathbb{F}$. Then $f = p_h$ for some k-color edge-coloring model over $\mathbb{F}$ of rank at most r if and only if for each graph $H = (V, E)$ and each $U \subseteq V$ of size $r + 1$ and each $s : U \to V \setminus U$,*

$$\sum_{\pi \in S_U} sgn(\pi) f(H/s \circ \pi) = 0. \tag{5.4}$$

We will prove Theorem 5.4 in Section 5.5. The conditions in Theorem 5.4 imply those in Theorem 5.3 for $k := r$. Indeed for each $u \in U$ we can add a new vertex $u'$ and a new edge $uu'$ to $H$, thus obtaining a graph $H'$. Then (5.4) for $H', U'$ and $s'(u') = s(u)$ gives (5.2) for $H, U, s$. This implies that if a graph parameter $f : \mathcal{G} \to \mathbb{F}$ is multiplicative and satisfies (5.4), for all $H, U$ and $s$, then $f$ is the partition function of an $r$-color edge-coloring model over $\mathbb{F}$.

Let us illustrate Theorem 5.4 by showing that it implies that the partition function of a vertex-coloring model is also the partition of an edge-coloring model. This was already shown by Szegedy in [66], where he even constructs the edge-coloring model from the vertex-coloring model (cf. Lemma 7.1). Let $(a, B)$ be an $n$-color vertex-coloring model over $\mathbb{F}$. Let $H = (V, E)$ be a graph, take $U \subset V$ of size $n + 1$ and let $s : U \to V \setminus U$. Then

$$\begin{aligned} &\sum_{\pi \in S_U} \mathrm{sgn}(\pi) p_{a,B}(H/s \circ \pi) = \\ &\sum_{\phi : V\setminus U \to [n]} \sum_{\pi \in S_U} \mathrm{sgn}(\pi) \prod_{v \in V \setminus U} a_{\phi(v)} \prod_{uv \in E(H/s\circ\pi)} B_{\phi(u),\phi(v)}. \end{aligned} \tag{5.5}$$

For fixed $\phi : V \setminus U \to [n]$ there exists $u_1, u_2 \in U$ such that $\phi(s(u_1)) = \phi(s(u_2))$. Let $\rho \in S_U$ be the transposition interchanging $u_1$ and $u_2$. Then the contribution of $\pi$ and $\pi \circ \rho$ will cancel each other. Hence (5.5) is zero.

Our proofs of both Theorem 5.3 and 5.4 are based on the First and Second Fundamental Theorem of invariant theory for the orthogonal group and Hilbert's Nulstellensatz. They are much inspired by Szegedy's proof of Theorem 5.2.

The rest of this chapter is organized as follows. In Section 5.2 we discuss a question of Szegedy concerning finite rank edge-coloring models, which has motivated the results in this chapter. In Section 5.3 we develop the invariant-theoretical framework necessary to prove both Theorems 5.3 and 5.4. The proofs of these theorems are given in the subsequent sections. Finally, in Section 5.6, we state analogues results for directed graphs.

## 5.2 Finite rank edge-coloring models

Using an explicit description of finite rank edge-coloring models, Szegedy [67] showed that partition functions of finite rank edge-coloring models over $\mathbb{C}$ can be seen as limits of partition functions of vertex-coloring models over $\mathbb{C}$. In particular, the vertex-connection matrices of these partition functions have exponentially bounded rank growth.

**Proposition 5.5** (Szegedy [67])**.** *Let $h$ be a $k$-color edge-coloring model over $\mathbb{C}$ of rank $r$. Then $\text{rk}\,(N_{p_h,l}) \leq r^l$ for all $l$.*

Let us give a short proof.

*Proof.* Define the $(\mathbb{N}^k)^l \times \mathcal{G}_l$ matrix $A$ by

$$A(\alpha_1, \dots, \alpha_l, H) := \sum_{\substack{\psi: E \to [k] \\ \psi(\delta(i)) = \alpha_i \text{ for all } i \in [l]}} \prod_{v \in V \setminus [l]} h(\psi(\delta(v))) \tag{5.6}$$

for $H = (V, E) \in \mathcal{G}_l$ and $(\alpha_1, \dots, \alpha_l) \in (\mathbb{N}^k)^l$. Then $N_{p_h,l} = A^T M_h^{\otimes l} A$. Hence $\text{rk}\,(N_{p_h,l}) \leq \text{rk}\,(M_h^{\otimes l}) = r^l$. □

This result made Szegedy ask the question whether there exists a graph parameter $f : \mathcal{G} \to \mathbb{C}$ whose vertex-connection matrices have exponentially bounded rank growth and which is not the partition function of an edge-coloring model. The answer to this question turns out to be positive as we will describe below.

Recall the graph parameter $f_x : \mathcal{G} \to \mathbb{C}$ for $x \in \mathbb{C}$ from Example 2.1;

$$f_x(H) = \begin{cases} x^{c(H)} \text{ if } H \text{ is 2-regular,} \\ 0 \text{ otherwise,} \end{cases} \tag{5.7}$$

where $c(H)$ denotes the number of connected components of $H$. We will show below that $\text{rk}\,(N_{f_{-2},l}) \leq 4^l$, but first we will show that $f_{-2}$ it is not the partition function of an edge-coloring model.

**Proposition 5.6.** *The graph parameter $f_x$ is the partition function of an edge-coloring model over $\mathbb{C}$ if and only if $x \in \mathbb{N}$.*

*Proof.* Suppose first that $x = k \in \mathbb{N}$. Define $h : \mathbb{C}[x_1, \dots, x_k] \to \mathbb{C}$ by

$$h(x^\alpha) = \begin{cases} 1 \text{ if } x^\alpha = x_i^2 \text{ for some } i \in [k], \\ 0 \text{ otherwise.} \end{cases} \tag{5.8}$$

Then it is easy to see that $f_k = p_h$.

We will now show that $f_x$ is not the partition function of a $k$-color edge-coloring model for any $k \in \mathbb{N}$, if $x \in \mathbb{C} \setminus \mathbb{N}$. Fix any $k \in \mathbb{N}$. Consider the graph $H = ([k+1], \emptyset)$ and define $s : [k+1] \to [k+1]$ by $s(i) = i$ for all $i$. Then for $\pi \in S_{k+1}$, $H_{s\circ\pi}$ consists of exactly $\mathrm{o}(\pi)$ cycles, where $\mathrm{o}(\pi)$ denotes the number of orbits of the permutation $\pi$. So

$$\sum_{\pi \in S_{k+1}} \mathrm{sgn}(\pi) f_x(H_{s\circ\pi}) = \sum_{\pi \in S_{k+1}} \mathrm{sgn}(\pi) x^{\mathrm{o}(\pi)}, \tag{5.9}$$

which is a polynomial $p$ in $x$ of degree $k+1$ with leading coefficient 1. As by the above and by Theorem 5.3, $p(x) = 0$ for $x = 0, \ldots, k$, it follows that $p(x) = x(x-1)\ldots(x-k)$. Hence $p(x) \neq 0$ for $x \notin \mathbb{N}$ and so Theorem 5.3 implies that $f_x$ is not the partition function of a $k$-color edge-coloring model over $\mathbb{C}$. □

Note that the proof of Proposition 5.6 actually shows that if $x \in \mathbb{F} \setminus \mathbb{N}$, then $f_x$ is not the partition function of any edge-coloring model over $\mathbb{F}$ for any algebraically closed field $\mathbb{F}$ of characteristic zero.

**Proposition 5.7.** *The rank of $N_{f_{-2},l}$ is bounded by $4^l$ for all $l$.*

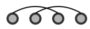
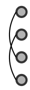
*Proof.* Write $\mathcal{Q}_l := \mathcal{Q}_l(f_{-2})$. The first thing to note is that $\mathcal{Q}_l$ is spanned by labeled graphs that are disjoint unions of $K_1^{\bullet}$'s, $C_1^{\bullet}$'s and $K_2^{\bullet\bullet}$'s. Indeed, since $f_{-2}$ is only nonzero on 2-regular graphs, this already implies that we can restrict ourselves to disjoint unions of $K_1^{\bullet}$'s, $C_1^{\bullet}$'s and paths with both endpoints labeled. Since any path with two endpoints labeled is equivalent modulo $f_{-2}$ to a multiple of $K_2^{\bullet\bullet}$, the claim follows.

For $i \in \mathbb{N}$, let $A_i$ be the submatrix of $N_{f_{-2},2i}$ indexed by $2i$-labeled graphs on $2i$ vertices that are disjoint unions of labeled edges (these are exactly the fully labeled perfect matchings on $2i$ vertices). Using that the submatrix of $N_{f_{-2},l}$ indexed by disjoint unions of $K_1^{\bullet}$'s, $C_1^{\bullet}$'s and $K_2^{\bullet\bullet}$'s, has a special block structure, it follows that

$$\mathrm{rk}\,(N_{f_{-2},l}) = \sum_{i=0}^{\lfloor l/2 \rfloor} \binom{l}{2i} 2^{l-2i} \mathrm{rk}\,(A_i). \tag{5.10}$$

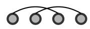
Next, note that $A_2$ is given by

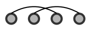
$A_2 =$

| |  |  |  |
|---|---|---|---|
|  | 4 | −2 | −2 |
|  | −2 | 4 | −2 |
|  | −2 | −2 | 4 |

. (5.11)

So $\mathrm{rk}\,(A_2) = 2$ and we we see that

 $+$  $+$  $= 0 \quad \text{in } \mathcal{Q}_l.$ (5.12)

We will refer to  as a *crossing pair*. By (5.12) we can replace a crossing pair by a linear combination of pairs of edges that are crossing. We will refer to this as *uncrossing*. Note that after uncrossing a crossing pair in a perfect matching, the two new matchings obtained both contain fewer crossing pairs than the original one. This implies that the row space of $A_i$ is spanned by the perfect matchings that do not contain crossing pairs. We will call these matchings *noncrossing*. The number of such matchings is bounded by $\binom{2i}{i} \leq 4^i$, as each noncrossing perfect matching uniquely determines a subset of $[2i]$ of size $i$ by looking at the left points of each edge. So, as $\mathrm{rk}\,(A_i) \leq 4^i$, (5.10) implies that $\mathrm{rk}\,(N_{f_{-2},l}) \leq 4^l$. □

### 5.2.1 Catalan numbers and the rank of $N_{f_{-2},l}$

Using representation theory of the symmetric group, we determine the rank of $N_{f_{-2},l}$ exactly. We will see that it is exactly the Catalan number $C_l$. In fact, an explicit computation of the rank of the vertex-connection matrices of $f_x$ can be determined in this way for any $x \in \mathbb{Z}$. It can be derived from [26, Theorem 3.1]. We refer to [57] for an introduction to the representation theory of the symmetric group.

The Catalan numbers form a sequence of natural numbers that occur in various counting problems. In his book, Stanley [64] gave a list of exercises with 66 possible interpretations of the Catalan numbers. The list of interpretations keeps on growing. Currently, there are 207; see [65]. For $n \geq 0$, the $n$-th *Catalan* *number* is defined as

$$C_n := \frac{1}{n+1}\binom{2n}{n}. \tag{5.13}$$

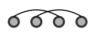
Let us give an interpretation.

**Lemma 5.8** (Exercise 19(o) in [64]). *$C_n$ is equal to the number of noncrossing perfect matchings on $[2n]$.*

To compute $C_{n+1}$ from $C_0, \ldots, C_n$, we can by Lemma 5.8 do the following. We start by putting an edge from 1 to any $i \in [2n+2]$. Then, since edges are not allowed to cross, we are left with finding the number of noncrossing perfect matchings under the first edge times the number of noncrossing perfect matchings right from the endpoint of the first edge. This in particular implies that $i$ should be even. Hence

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}. \tag{5.14}$$

The symmetric group $S_{2n}$ acts on the set of perfect matchings on $[2n]$, $\mathcal{M}_n$, by permuting the endpoints of the edges. For example for $n = 2$ and $\tau = (23) \in S_4$,

$\tau($  $) =$  (5.15)

Now note that the matrix $A_n$ is $S_{2n}$-equivariant, i.e., for each $N, M \in \mathcal{M}_n$ and $\tau \in S_{2n}$, we have $A_n(\tau N, \tau M) = A_n(N, M)$. Let $M_0$ be the matching on $[2n]$ with edges $12, 34, \ldots, (2n-1)n$, let $S_n \subset S_{2n}$ be the subgroup permuting the odd positions and let $v \in \mathbb{F}\mathcal{M}_n$ be defined by

$$v := \sum_{\tau \in S_n} \operatorname{sgn}(\tau)\tau M_0. \tag{5.16}$$

We claim that $A_n v \neq 0$. Indeed,

$$(A_n v)(M_0) = \sum_{\tau \in S_n} \operatorname{sgn}(\tau)(-2)^{c(M_0 \cup \tau M_0)} = \sum_{\tau \in S_n} \operatorname{sgn}(\tau)(-2)^{o(\tau)}. \tag{5.17}$$

So by the proof of Proposition 5.6 $A_n v \neq 0$.

As $v$ is a generator of the Specht module $S^\lambda$, where $\lambda$ is the partition of $2n$ given by $(2, 2, \ldots, 2)$, and as $A_n$ is $S_{2n}$-equivariant, Schur's lemma implies that $v \in \operatorname{Im} A_n$. Hence the rank of $A_n$ is at least the dimension of $S^\lambda$.

The dimension of $S^\lambda$ is equal to the number of ways to place the numbers $1, 2, \ldots, 2n$ in a $n \times 2$ array such that both the columns and the rows are increasing. (The number of standard Young tableaux of shape $\lambda$). This number is known to be equal to $C_n$ (cf. [64, Exercise 19(ww)]). It follows that $\mathrm{rk}\,(A_n) \geq \dim(S^\lambda) = C_n$. As by the proof of Proposition 5.7, $\mathrm{rk}\,(A_n)$ is bounded by the number of noncrossing perfect matchings, Lemma 5.8 implies that $\mathrm{rk}\,(A_n)$ is equal to $C_n$.

Viewing $C_1^\bullet$ as a matching of a vertex to itself we may say that the dimension of $\mathcal{Q}_l$ is equal to the number of noncrossing matchings on $[2l]$. So to compute the dimension of $\mathcal{Q}_l$ for $l \geq 1$, we can choose to put on the first position an isolated vertex, a loop or the left vertex of an edge and then continue recursively. Setting $\dim(\mathcal{Q}_{-1}) = \dim(\mathcal{Q}_0) = 1$, this gives rise to the following recurrence relation for $\dim(\mathcal{Q}_l)$:

$$\begin{aligned}\dim(\mathcal{Q}_l) &= 2\dim(\mathcal{Q}_{l-1}) + \sum_{i=0}^{l-2} \dim(\mathcal{Q}_i)\dim(\mathcal{Q}_{l-2-i}) \\ &= \sum_{i=0}^{l} \dim(\mathcal{Q}_{i-1})\dim(\mathcal{Q}_{l-1-i}). \end{aligned} \tag{5.18}$$

Now note that $\dim(\mathcal{Q}_l)$ satisfies the same recurrence relation as $C_{l+1}$ in (5.14). As $\dim(\mathcal{Q}_{-1}) = \dim(\mathcal{Q}_0) = C_0 = C_1$, it follows that $\dim(\mathcal{Q}_l) = C_{l+1}$ for all $l$. We will summarize it as a theorem.

**Theorem 5.9.** *The rank of $N_{f_{-2},l}$ is equal to $C_{l+1}$.*

As a corollary to the proof of Theorem 5.9 and (5.10), we obtain the following recurrence relation for the Catalan numbers, previously obtained by Xin and Xu [68].

**Corollary 5.10.** *The Catalan numbers satisfy the following recurrence equation:*

$$C_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 2^{n-2i} C_i. \tag{5.19}$$

## 5.3 Framework

Here we develop the framework used for the proof of both Theorem 5.3 and 5.4.

Let $k \in \mathbb{N}$. Introduce a variable $y_\alpha$ for each $\alpha \in \mathbb{N}^k$ and define the ring $T$ of polynomials in these (infinitely) many variables:

$$T := \mathbb{F}[y_\alpha \mid \alpha \in \mathbb{N}^k]. \tag{5.20}$$

Note that there is a bijection between the variables $y_\alpha$ and the monomials $x^\alpha = \prod_{i=1}^k x_i^{\alpha_i}$ in $R = \mathbb{F}[x_1,\dots,x_k]$. In this way functions $h : \mathbb{N}^k \to \mathbb{F}$ correspond to elements of $R^*$. The action of $\mathrm{O}_k$ on $R$ induces an action of $\mathrm{O}_k$ on $T$ via the bijection between the variables of $T$ and the monomials of $R$. Equivalently, using the action of $\mathrm{O}_k$ on $R^*$, define $gq(h) = q(g^{-1}h)$ for $g \in \mathrm{O}_k, q \in T$ and $h : \mathbb{N}^k \to \mathbb{F}$.

Define $p : \mathcal{G} \to T$ by

$$p(H) := \sum_{\phi: E(H)\to[k]} \prod_{v\in V(H)} y_{\phi(\delta(v))}, \tag{5.21}$$

where we view $\phi(\delta(v))$ as a multisubset of $[k]$, which we identify with its characteristic vector in $\mathbb{N}^k$. Note that $p(H) = p(H')$ for isomorphic graphs $H$ and $H'$. Now extend $p$ linearly to $\mathbb{F}\mathcal{G}$ to obtain an algebra homomorphism $p : \mathbb{F}\mathcal{G} \to T$. (Recall that $\mathbb{F}\mathcal{G}$ is the semigroup algebra of $(\mathcal{G},\cdot)$, where the product of two graphs is just their disjoint union.) Using the First and Second Fundamental Theorem for the orthogonal group we characterize the kernel $\operatorname{Ker} p$ and the image $\operatorname{Im} p$ of $p$. The characterization of $\operatorname{Im} p$ is similar to the one give by Szegedy [66].

To characterize $\operatorname{Ker} p$, let $\mathcal{I}$ be the subspace of $\mathbb{F}\mathcal{G}$ spanned by the quantum graphs

$$\sum_{\pi\in S_U} \operatorname{sgn}(\pi) H_{s\circ\pi}, \tag{5.22}$$

where $H = (V,E)$ is a graph, $U \subseteq V$ with $|U| = k+1$, and $s : U \to V$.

**Proposition 5.11.** *We have* $\operatorname{Im} p = T^{\mathrm{O}_k}$ *and* $\operatorname{Ker} p = \mathcal{I}$.

*Proof.* For $n \in \mathbb{N}$, let $\mathcal{G}_n$ be the collection of graphs with vertex set $[n]$. Let $S\mathbb{F}^{n\times n}$ be the set of symmetric matrices in $\mathbb{F}^{n\times n}$. For any linear space $X$ let $\mathcal{O}(X)$ denote the space of regular functions on $X$ (the algebra generated by the linear functions on $X$). Then $\mathcal{O}(S\mathbb{F}^{n\times n})$ is spanned by the monomials $\prod_{ij\in E} x_{i,j}$ in the variables $x_{i,j}$, where $([n],E)$ is a graph. Here $x_{i,j} = x_{j,i}$ are the standard coordinate functions on $S\mathbb{F}^{n\times n}$, while taking $ij$ as unordered pair.

Let $\mathbb{F}\mathcal{G}_n$ be the space of formal $\mathbb{F}$-linear combinations of elements of $\mathcal{G}_n$. Let $T_n$ be the set of homogenous polynomials in $T$ of degree $n$. We set $p_n := p|_{\mathbb{F}\mathcal{G}_n}$. So $p_n : \mathbb{F}\mathcal{G}_n \to T_n$. Hence it suffices to prove, for each $n$,

$$\operatorname{Im} p = T_n^{\mathrm{O}_k} \text{ and } \operatorname{Ker} p_n = \mathcal{I} \cap \mathbb{F}\mathcal{G}_n. \tag{5.23}$$

To show (5.23), we define linear functions $\mu, \sigma$ and $\tau$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{F}\mathcal{G}_n & \xrightarrow{p_n} & T_n \\ \uparrow \mu & & \uparrow \sigma \\ \mathcal{O}(S\mathbb{F}^{n\times n}) & \xrightarrow{\tau} & \mathcal{O}(\mathbb{F}^{k\times n}) \end{array} . \tag{5.24}$$

Define $\mu$ by

$$\mu(\prod_{ij\in E} x_{i,j}) := H \tag{5.25}$$

for any graph $H := ([n], E)$. Define $\sigma$ by

$$\sigma(\prod_{j=1}^{n}\prod_{i=1}^{k} z_{i,j}^{\alpha_{i,j}}) := \prod_{j=1}^{n} y_{\alpha_j} \tag{5.26}$$

for $\alpha \in \mathbb{N}^{k\times n}$, where $z_{i,j}$ are the standard coordinate functions on $\mathbb{F}^{k\times n}$ and where $\alpha_j := (\alpha_{1,j}, \ldots, \alpha_{k,j}) \in \mathbb{N}^k$. Then $\sigma$ is $\mathrm{O}_k$-equivariant for the natural action of $\mathrm{O}_k$ on $\mathcal{O}(\mathbb{F}^{k\times n})$. Finally, define $\tau$ by

$$\tau(q)(z) := q(z^T z) \tag{5.27}$$

for $q \in \mathcal{O}(S\mathbb{F}^{n\times n})$ and $z \in \mathbb{F}^{k\times n}$. Now (5.24) commutes; in other words,

$$p_n \circ \mu = \sigma \circ \tau. \tag{5.28}$$

To prove it, consider any monomial $q := \prod_{ij\in E} x_{i,j}$ in $\mathcal{O}(S\mathbb{F}^{n\times n})$, where $H = ([n], E)$ is a graph. Then for any $z \in \mathbb{F}^{k\times n}$,

$$\tau(q(z)) = q(z^T z) = \prod_{ij\in E}\sum_{h=1}^{k} z_{h,i} z_{h,j} = \sum_{\phi: E\to[k]} \prod_{i\in[n]} \prod_{e\in\delta(i)} z_{\phi(e),i}. \tag{5.29}$$

So, by definition (5.26) of $\sigma$ and (5.25) of $\mu$,

$$\sigma(\tau(q)) = \sum_{\phi: E\to[k]} \prod_{i\in[n]} y_{\phi(\delta(i))} = p_n(H) = p_n(\mu(q)). \tag{5.30}$$

This proves (5.28).

Note that $\tau$ is an algebra homomorphism, but $\sigma$ and $\mu$ generally are not. ($\mathbb{F}\mathcal{G}_n$ and $T_n$ are not algebras.) The latter two functions are surjective. Moreover, $\mu$ is bijective and restricted to the $S_n$-invariant part of its domain, $\sigma$ is bijective.

By the FFT for $\mathrm{O}_k$ (cf. Theorem 4.4), $\operatorname{Im}\tau = (\mathcal{O}(\mathbb{F}^{k\times n}))^{\mathrm{O}_k}$. Hence, as $\mu$ and $\sigma$ are surjective, and as $\sigma$ is $\mathrm{O}_k$-equivariant,

$$\begin{aligned} \operatorname{Im} p_n &= p_n(\mathbb{F}\mathcal{G}_n) = p_n(\mu(\mathcal{O}(S\mathbb{F}^{n\times n}))) = \sigma(\tau(\mathcal{O}(S\mathbb{F}^{n\times n}))) \\ &= \sigma((\mathcal{O}(\mathbb{F}^{k\times n})^{\mathrm{O}_k}) = T_n^{\mathrm{O}_k}. \end{aligned} \tag{5.31}$$

The last equality follows from the fact that $\sigma$ is $\mathrm{O}_k$-equivariant, so that we have $\subseteq$. To see $\supseteq$, take any $q \in T_n^{\mathrm{O}_k}$, as $\mu$ is surjective, $q = \sigma(r)$ for some $r \in \mathcal{O}(\mathbb{F}^{k\times n})$. Then, by Lemma 4.2, $q = \sigma(\rho_{\mathrm{O}_k}(r))$, where $\rho_{\mathrm{O}_k}$ is the Reynolds operator of $\mathrm{O}_k$. This proves the first statement in (5.23).

To see that $\mathcal{I} \cap \mathbb{F}\mathcal{G}_n \subseteq \operatorname{Ker} p_n$, let $H = ([n], E)$ be a graph, $U \subset [n]$ with $|U| = k+1$, and $s : U \to [n]$. Then $\sum_{\pi\in S_U} \operatorname{sgn}(\pi) H_{s\circ\pi}$ belongs to $\operatorname{Ker} p_n$, as

$$p(\sum_{\pi\in S_U} \operatorname{sgn}(\pi) H_{s\circ\pi}) = \sum_{\phi: E\cup E_s\to[k]} \sum_{\pi\in S_U} \operatorname{sgn}(\pi) \prod_{i\in[n]} y_{\phi(\delta_{H_{s\circ\pi}}(i))}. \tag{5.32}$$

For fixed $\phi$, there exist distinct $u_1, u_2 \in U$ such that $\phi(u_1, s(u_1)) = \phi(u_2, s(u_2))$. So if $\rho$ is the permutation of $U$ interchanging $u_1$ and $u_2$, we have that the terms in (5.32) corresponding to $\pi$ and $\pi\circ\rho$ cancel. Hence (5.32) is zero.

We finally show $\operatorname{Ker} p_n \subseteq \mathcal{I}$. By the SFT for $\mathrm{O}_k$ (cf. Theorem 4.5), (as $\mathbb{F}$ is algebraically closed) $\operatorname{Ker}\tau$ is the ideal in $\mathcal{O}(S\mathbb{F}^{n\times n})$ generated by the $(k+1)\times(k+1)$ minors of $S\mathbb{F}^{n\times n}$. Then

$$\mu(\operatorname{Ker}\tau) \subseteq \mathcal{I}. \tag{5.33}$$

To prove (5.33), it suffices to show that for any $(k+1)\times(k+1)$ submatrix $N$ of $\mathbb{F}^{n\times n}$ and any graph $H = ([n], E)$ one has

$$\mu(\det(N) \prod_{ij\in E} x_{i,j}) \in \mathcal{I}. \tag{5.34}$$

There is a subset $U$ of $[n]$ with $|U| = k+1$ and an injective function $s : U \to [n]$ such that $\{(u, s(u)) \mid u \in U\}$ forms the diagonal of $N$. So

$$\det(N) = \sum_{\pi\in S_U} \operatorname{sgn}(\pi) \prod_{u\in U} x_{u,s\circ\pi(u)}. \tag{5.35}$$

Then

$$\begin{aligned} \mu(\det(N) \prod_{ij\in E} x_{i,j}) &= \sum_{\pi\in S_U} \operatorname{sgn}(\pi) \mu(\prod_{u\in U} x_{u,s\circ\pi(u)} \cdot \prod_{ij\in E} x_{i,j}) \\ &= \sum_{\pi\in S_U} \operatorname{sgn}(\pi) H_{s\circ\pi} \in \mathcal{I}, \end{aligned} \tag{5.36}$$

by definition of $\mathcal{I}$. This proves (5.33).

To prove $\operatorname{Ker} p_n \subseteq \mathcal{I}$, let $\gamma \in \operatorname{Ker} p_n$. Then $\gamma = \mu(q)$ for some $q \in \mathcal{O}(S\mathbb{F}^{n\times n})$. Hence $\sigma(\tau(q)) = p(\mu(q)) = p(\gamma) = 0$. We may assume that $q$ is $S_n$-invariant since $p$ is isomorphism-invariant (cf. Lemma 4.2). As $\sigma$ is bijective on $(\mathcal{O}(\mathbb{F}^{k\times n}))^{S_n}$, this implies that $\tau(q) = 0$. Hence $\gamma = \mu(q) \in \mu(\operatorname{Ker}\tau) \subseteq \mathcal{I}$. This finishes the proof of the second statement in (5.23). □

## 5.4 Proof of Theorem 5.3

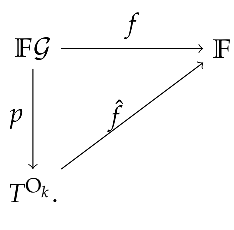
**Theorem 5.3.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $f : \mathcal{G} \to \mathbb{F}$ be a graph invariant. Then $f = p_h$ for some $k$-color edge-coloring model over $\mathbb{F}$ if and only if $f$ is multiplicative and for each graph $H = (V, E)$ and each $U \subseteq V$ of size $k + 1$ and each $s : U \to V$,*

$$\sum_{\pi\in S_U} sgn(\pi) f(H_{s\circ\pi}) = 0. \tag{5.2}$$

*Proof.* We fix $k$. Necessity of the conditions (5.2) follows from the fact $\operatorname{Ker} p = \mathcal{I}$ by Proposition 5.11.

To prove sufficiency, we must show that the polynomials $p(H) - f(H)$ have a common zero. Here $f(H)$ denotes the constant polynomial with value $f(H)$. A common zero means an element $y : \mathbb{N}^k \to \mathbb{F}$, with for all $H \in \mathcal{G}$ $(p(H) - f(H))(y) = 0$, equivalently, $p_y(H) = f(H)$, as required.

As $f$ is multiplicative, $f$ extends linearly to an algebra homomorphism $f : \mathbb{F}\mathcal{G} \to \mathbb{F}$. By the condition in Theorem 5.3, $f(\mathcal{I}) = 0$. So by Proposition 5.11, $\operatorname{Ker} p \subseteq \operatorname{Ker} f$. Hence there exists an algebra homomorphism $\hat{f} : p(\mathbb{F}\mathcal{G}) \to \mathbb{F}$ such that $\hat{f} \circ p = f$; that is such that the following diagram commutes:




(5.37)

Let $I$ be the ideal in $T$ generated by the polynomials $p(H) - f(H)$ for $H \in \mathcal{G}$. Let $\rho_{\mathrm{O}_k}$ denote the Reynolds operator of $\mathrm{O}_k$ on $T$. (This exists by reductiveness of $\mathrm{O}_k$ and the fact that $T$ has a canonical direct sum decomposition into finite dimensional $\mathrm{O}_k$-modules.) By Proposition 5.11, and the fact that $\rho_{\mathrm{O}_k}(qr) = \rho_{\mathrm{O}_k}(q)r$ for $q \in T$ and $r \in T^{\mathrm{O}_k}$ (cf. (4.2)), $\rho_{\mathrm{O}_k}(I)$ is the ideal in $p(\mathbb{F}\mathcal{G}) = T^{\mathrm{O}_k}$

generated by the polynomials $p(H) - f(H)$. This implies, as $\hat{f}(p(H)) - f(H) = 0$, that

$$\hat{f}(\rho_{O_k}(I)) = 0, \tag{5.38}$$

hence $1 \notin I$.

If $|\mathbb{F}|$ is uncountable (e.g. if $\mathbb{F} = \mathbb{C}$), the Nullstellensatz for countably many variables (Lang [37]) yields the existence of a common zero $y$.

To prove the existence of a common zero $y$ for general algebraically closed fields $\mathbb{F}$ of characteristic 0 let, for any $d \in \mathbb{N}$, $\mathbb{N}^k_{\leq d} := \{\alpha \in \mathbb{N}^k \mid |\alpha| \leq d\}$, where $|\alpha| := \sum_{i=1}^k \alpha_i$ and let

$$Y_d := \{z : \mathbb{N}^k_{\leq d} \to \mathbb{F} \mid q(z) = \hat{f}(q) \text{ for each } q \in \mathbb{F}[y_\alpha \mid \alpha \in \mathbb{N}^k_{\leq d}]^{O_k}\}. \tag{5.39}$$

So $Y_d$ consists of the common zeros of the polynomials $p(H) - f(H)$, where $H$ ranges over the graphs of maximum degree $d$.

By the Nullstellensatz, as $\mathbb{N}^k_{\leq d}$ is finite, $Y_d \neq \emptyset$. Note that $Y_d$ is a fiber of the quotient map

$$\pi : \mathbb{F}^{\mathbb{N}^k_{\leq d}} \to \mathbb{F}^{\mathbb{N}^k_{\leq d}}//O_k. \tag{5.40}$$

So by Theorem 4.7, $Y_d$ contains a unique Zariski-closed $O_k$-orbit $C_d$.

Let $\text{pr}_d$ be the projection $z \mapsto z_{\leq d} := z|_{\mathbb{N}^k_{\leq d}}$ for $z : \mathbb{N}^k_{\leq d'} \to \mathbb{F}$ with $d' \geq d$. (It is convenient to allow $d' = \infty$ here.) Note that if $\infty > d' \geq d$, then $\text{pr}_d(C_{d'})$ is an $O_k$-orbit contained in $Y_d$. Hence

$$\dim(C_d) \leq \dim(\text{pr}_d(C_{d'})) \leq \dim(C_{d'}), \tag{5.41}$$

where dim denotes the Krull-dimension. As $\dim(C_d) \leq \dim(O_k)$ for all $d \in \mathbb{N}$, there is $d_0$ such that for each $d \geq d_0$, $\dim(C_d) = \dim(C_{d_0})$. Hence we have equality throughout in (5.41).

By uniqueness of the orbit of minimal Krull-dimension, this implies that for each $d' \geq d \geq d_0$, $C_d = \text{pr}_d(C_{d'})$. Hence there exists $y : \mathbb{N}^k \to \mathbb{F}$ such that $y_{\leq d} \in C_d$ for each $d \geq d_0$. This $y$ is as required. □

## 5.5 Proof of Theorem 5.4

**Theorem 5.4.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $f : \mathcal{G} \to \mathbb{F}$ be the partition function of a k-color edge-coloring model over $\mathbb{F}$. Then $f = p_h$ for some k-color edge-coloring model over $\mathbb{F}$ of rank at most r if and only if for each graph $H = (V, E)$ and each $U \subseteq V$ of size $r + 1$ and each $s : U \to V \setminus U$,*

$$\sum_{\pi \in S_U} sgn(\pi) f(H/s \circ \pi) = 0. \tag{5.4}$$

*Proof.* Necessity can be seen as follows. Choose $y : \mathbb{N}^k \to \mathbb{F}$ with $\text{rk}\,(M_y) \leq r$ and let $H = (V, E)$ be a graph. Choose $U \subseteq V$ with $|U| = r+1$ and $s : U \to V \setminus U$. Then

$$\begin{aligned} & \sum_{\pi \in S_U} \text{sgn}(\pi) p_y(H/s \circ \pi) \qquad (5.42) \\ = \quad & \sum_{\phi : E \to [k]} \sum_{\pi \in S_U} \text{sgn}(\pi) \prod_{u \in U} y_{\phi(\delta(u) \cup \delta(s(\pi(u))))} \cdot \prod_{v \in V \setminus (U \cup s(U))} y_{\phi(\delta(v))} \\ = \quad & \sum_{\phi : E \to [k]} \det\big((y_{\phi(\delta(u) \cup \delta(s(\pi(v))))})_{u,v \in U}\big) \prod_{v \in V \setminus (U \cup s(U))} y_{\phi(\delta(v))} = 0. \end{aligned}$$

To see sufficiency, let $\mathcal{J}$ be the ideal in $\mathbb{F}\mathcal{G}$ be the ideal spanned by the quantum graphs

$$\sum_{\pi \in S_U} \text{sgn}(\pi) H/s \circ \pi, \qquad (5.43)$$

where $H = (V, E)$ is a graph, $U \subseteq V$ with $|U| = r+1$ and $s : U \to V \setminus U$. Let $J$ be the ideal in $R$ generated by the polynomials $\det(N)$ where $N$ is an $(r+1) \times (r+1)$ submatrix of $M_y$.

**Proposition 5.12.** $\rho_{O_k}(J) \subseteq p(\mathcal{J})$.

*Proof.* It suffices to prove that for any $(r+1) \times (r+1)$ submatrix $N$ of $M_y$ and any monomial $a \in T$, $\rho_{O_k}(a \det(N)) \in p(\mathcal{J})$. Let $a$ have degree $d$, and let $n := 2(r+1) + d$. Let $U := [r+1]$ and let $s : U \to [n] \setminus U$ be defined by $s(i) = r + 1 + i$ for $i \in U$.

We use the framework of Proposition 5.11, with $\tau$ as in (5.27). For each $\pi \in S_{r+1}$ we define linear functions $\mu_\pi$ and $\sigma_\pi$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{F}\mathcal{G}_m & \xrightarrow{p_m} & T_m \\ \uparrow \mu_\pi & & \uparrow \sigma_\pi \\ \mathcal{O}(S\mathbb{F}^{n \times n}) & \xrightarrow{\tau} & \mathcal{O}(\mathbb{F}^{k \times n}) \end{array}, \qquad (5.44)$$

where $m := r + 1 + d = n - (r+1)$.

The function $\mu_\pi$ is defined by

$$\mu_\pi\Big(\prod_{ij \in E} x_{i,j}\Big) := H/s \circ \pi \qquad (5.45)$$

for any graph $H = ([n], E)$. It implies that for each $q \in \mathcal{O}(S\mathbb{F}^{n\times n})$,

$$\sum_{\pi \in S_{r+1}} \mathrm{sgn}(\pi)\mu_\pi(q) \in \mathcal{J}, \tag{5.46}$$

by definition of $\mathcal{J}$.

Next, $\sigma_\pi$ is defined by

$$\sigma_\pi(\prod_{j=1}^{n}\prod_{i=1}^{k} z_{i,j}^{\alpha_{i,j}}) := \prod_{j=1}^{r+1} y_{\alpha_j+\alpha_{r+1+\pi(j)}} \cdot \prod_{j=2r+3}^{n} y_{\alpha_j} \tag{5.47}$$

for any $\alpha \in \mathbb{N}^{k\times n}$. So

$$a\det(N) = \sum_{\pi \in S_{r+1}} \mathrm{sgn}(\pi)\sigma_\pi(u) \tag{5.48}$$

for some monomial $u \in \mathcal{O}(\mathbb{F}^{k\times n})$. Note that $\sigma_\pi$ is $\mathrm{O}_k$-equivariant.

Now one directly checks that the diagram (5.44) commutes, that is,

$$p \circ \mu_\pi = \sigma_\pi \circ \tau. \tag{5.49}$$

By the FFT, $\rho_{\mathrm{O}_k}(u) = \tau(q)$ for some $q \in \mathcal{O}(S\mathbb{F}^{n\times n})$. Hence $\sigma_\pi(\rho_{\mathrm{O}_k}(u)) = \sigma_\pi(\tau(q)) = p(\mu_\pi(q))$. Therefore, using (5.48) and (5.46),

$$\rho_{\mathrm{O}_k}(a\det(N)) \in p(\mathcal{J}), \tag{5.50}$$

as required. □

Since $f$ is the partition function of a $k$-color edge-coloring model, there exists an algebra homomorphism $\hat{f} : T \to \mathbb{F}$, such that $\hat{f} \circ p = f$ (cf. (5.37)). If the conditions in Theorem 5.4 are satisfied, then $f(\mathcal{J}) = 0$, and hence with Proposition 5.12

$$\hat{f}(\rho_{\mathrm{O}_k}(J)) \subseteq \hat{f}(p(\mathcal{J})) = f(\mathcal{J}) = 0. \tag{5.51}$$

With (5.38) this implies that $1 \notin I + J$, where $I$ is again the ideal generated by the polynomials $p(H) - f(H)$ for graphs $H$. The proof of Theorem 5.3 now shows that $I + J$ has a common zero, as required. Indeed, we just have to replace $Y_d$ by

$$Y_d' := \{z \in Y_d \mid \mathrm{rk}\,(M_z) \leq r\}, \tag{5.52}$$

where for $z : \mathbb{N}^k_{\leq d} \to \mathbb{F}$, we set $M_z(\alpha, \beta) = 0$ if $|\alpha + \beta| > d$. Then $Y_d' \neq \emptyset$, by the Nullstellensatz, since $1 \notin I + J$. As $\mathrm{rk}\,(M_{gz}) = \mathrm{rk}\,(M_z)$ for all $g \in \mathrm{O}_k$, it follows that $Y_d'$ is closed and $\mathrm{O}_k$-stable. So the unique Zariski-closed orbit $C_d \subseteq Y_d$ is by Theorem 4.7 contained in $Y_d'$. The rest of the proof can now be copied from the proof of Theorem 5.3. □

## 5.6 Analogues for directed graphs

Similar results hold for directed graphs, with similar proofs, now by applying the FFT and SFT for $\mathrm{GL}(\mathbb{F}^k)$ (cf. [25, Section 5.2] and [25, Section 11.2] respectively). The corresponding models were also considered by de la Harpe and Jones [28]. We only state the results.

Let $\mathcal{D}$ denote the collection of all directed graphs. Directed graphs are finite and may have loops and multiple edges. A map $f : \mathcal{D} \to \mathbb{F}$ is called a *directed graph parameter* if it assigns the same value to isomorphic directed graph. The *directed partition function* of a $2k$-color edge-coloring model $y$ is the directed graph parameter $p_y : \mathcal{D} \to \mathbb{F}$ defined for any directed graph $D = (V, A)$ by

$$p_y(D) := \sum_{\kappa:A\to[k]} \prod_{v\in V} y_{(\kappa(\delta^-(v)),\kappa(\delta^+(v)))}. \tag{5.53}$$

Here $\delta^-(v)$ and $\delta^+(v)$ denote the sets of arcs entering $v$ and leaving $v$, respectively. Moreover, $(\kappa(\delta^-(v)), \kappa(\delta^+(v)))$ stands for the concatenation of the vectors $\kappa(\delta^-(v))$ and $\kappa(\delta^+(v)) \in \mathbb{N}^k$, so as to obtain a vector in $\mathbb{N}^{2k}$.

Call a function $f : \mathcal{D} \to \mathbb{F}$ *multiplicative* if $f(\emptyset) = 1$ and $f(D_1D_2) = f(D_1)f(D_2)$ for all $D_1, D_2 \in \mathcal{D}$. Again, $D_1D_2$ denotes the disjoint union of $D_1$ and $D_2$. Moreover, for any directed graph $D = (V, A)$, any $U \subseteq V$, and any $s : U \to V$, define

$$A_s := \{(u, s(u)) \mid u \in U\} \quad \text{and} \quad D_s := (V, A \cup A_s). \tag{5.54}$$

**Theorem 5.13.** *Let $\mathbb{F} = \overline{\mathbb{F}}$. A directed graph parameter $f : \mathcal{D} \to \mathbb{F}$ is the directed partition function of some $2k$-color edge-coloring model over $\mathbb{F}$ if and only if $f$ is multiplicative and for each directed graph $D = (V, A)$, each $U \subseteq V$ with $|U| = k + 1$, and each $s : U \to V$:*

$$\sum_{\pi\in S_U} sgn(\pi) f(D_{s\circ\pi}) = 0. \tag{5.55}$$

For any directed graph $D = (V, A)$, $U \subseteq V$, and $s : U \to V$, let $D/s$ be the directed graph obtained from $D_s$ by contracting all arcs in $A_s$.

**Theorem 5.14.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $f$ be the directed partition function of a $2k$-color edge-coloring model over $\mathbb{F}$. Then $f$ is the directed partition function of a $2k$-color edge-coloring model over $\mathbb{F}$ of rank at most $r$ if and only if for each directed graph $D = (V, A)$, each $U \subseteq V$ with $|U| = r + 1$, and each $s : U \to V \setminus U$:*

$$\sum_{\pi\in S_U} sgn(\pi) f(D/s \circ \pi) = 0. \tag{5.56}$$

# Chapter 6

# Connection matrices and algebras of invariant tensors

This chapter deals with a connection between connection matrices of partition functions of edge- and vertex-coloring models and algebras of tensors that are invariant under certain subgroups of the orthogonal group. Based on characterizations of these invariant algebras we characterize the rank of edge-connection matrices of partition functions of edge-coloring models as the dimension of the algebras of tensors invariant under the subgroup of the orthogonal group stabilizing the edge-coloring model. The corresponding result for the rank of vertex-connection matrices of partition functions of vertex-coloring models was proved by Lovász [41] using different ideas.

This chapter is based on joint work with Jan Draisma [20] and on [53].

## 6.1 Introduction

Let $(a, B)$ be a real twin-free $n$-color vertex-coloring model (i.e. $B$ has no two equal rows and $a_i > 0$ for all $i \in [n]$). In [41] Lovász determined the rank of the vertex-connection matrices of $p_{a,B}$. To describe his result we need some definitions.

Let $\text{Aut}(a, B) \subseteq S_n$ be the *automorphism group* of the weighted graph $G(a, B)$, i.e., the subgroup of the group of all permutations of $[n]$ preserving both the vertex- and edge weights of $G(a, B)$. The group $S_n$ has a natural action on $[n]^l := \{\phi : [l] \to [n]\}$, for any $l$, via $(\pi \cdot \phi)(i) = \pi(\phi(i))$, for $\pi \in S_n$ and $\phi \in [n]^l$.

**Theorem 6.1** (Lovász [41]). *Let $(a, B)$ be a real twin-free n-color vertex-coloring model. Then*

$$\text{rk}\,(N_{p_{a,B},l}) = \textit{ the number of orbits of the action of } \text{Aut}(a,B) \textit{ on } [n]^l. \tag{6.1}$$

Theorem 6.1 has applications in the study of generalized quasi-random graphs (see [44, 40]). It is natural to ask whether a similar result holds for the rank of edge-connection matrices of partition functions of (real) edge-coloring models. This question was posed by Szegedy [66] and by Borgs, Chayes, Lovász, Sós and Vesztergombi [6].

In this chapter we will show that a similar result indeed holds for the rank of edge-connection matrices of partition functions of both real and complex edge-coloring models. To state our results we need to introduce some definitions.

Let $V := \mathbb{F}^k$. (Recall that $\mathbb{F}$ denotes any field of characteristic zero.) Let $e_1, \ldots, e_k$ denote the standard basis for $V$ and let $(\cdot, \cdot)$ denote the standard symmetric bilinear form on $V$; i.e., $(e_i, e_j) = \delta_{i,j}$. The orthogonal group $\text{O}_k = \text{O}_k(\mathbb{F})$ is the group of $k \times k$ matrices over $\mathbb{F}$ that leave this bilinear form invariant, i.e., $g \in \text{O}_k$ if and only if $g^T g = I$. For an edge-coloring model $h \in R^*$ (recall that $R = \mathbb{F}[x_1, \ldots, x_k]$), define

$$\text{Stab}(h) := \{g \in \text{O}_k(\mathbb{F}) \mid gh = h\}. \tag{6.2}$$

The action of $\text{O}_k$ on $V$ extends naturally to $V^{\otimes l}$ for any $l \in \mathbb{N}$. Let $G \subseteq \text{O}_k$ be a subgroup. Recall that

$$(V^{\otimes l})^G = \{v \in V^{\otimes l} \mid gv = v \text{ for all } g \in G\}. \tag{6.3}$$

Now we can state our characterization. For real valued edge-coloring models the following result holds.

**Theorem 6.2.** *Let $h$ be a $k$-color edge-coloring model over $\mathbb{R}$. Then, for any $t \in \mathbb{N}$,*

$$\text{rk}\,(M_{p_h,t}) = \dim\left((V^{\otimes t})^{\text{Stab}(h)}\right). \tag{6.4}$$

Theorem 6.2 will be proved in Section 6.2.

To see the similarity between Theorem 6.2 and Theorem 6.1, let $e_1, \ldots, e_n$ be the standard basis of $W := \mathbb{R}^n$. Then the set $[n]^l$ corresponds to the standard basis of $W^{\otimes l}$ via $[n]^l \ni \phi \leftrightarrow e_\phi := e_{\phi(1)} \otimes \cdots \otimes e_{\phi(l)}$ and the action of $S_n$ on $[n]^l$ induces an action on $W^{\otimes l}$. With these definitions, (6.1) now reduces to

$$\text{rk}\,(N_{p_{a,B},t}) = \dim(W^{\otimes t})^{\text{Aut}(a,B)}, \tag{6.5}$$

showing the similarity between Theorem 6.2 and Theorem 6.1.

For edge-coloring models with values in an algebraically closed field $\mathbb{F}$ of characteristic zero a similar result as Theorem 6.2 holds.

**Theorem 6.3.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $h$ be a $k$-color edge-coloring model over $\mathbb{F}$. Then there exists a $k$-color edge-coloring model $h'$ over $\mathbb{F}$ such that $p_h = p_{h'}$, and such that for any $t \in \mathbb{N}$,*

$$\text{rk}\left(M_{p_h,t}\right) = \dim\left((V^{\otimes t})^{\text{Stab}(h')}\right). \tag{6.6}$$

We will prove Theorem 6.3 in Section 6.2.

We cannot simply take $h' = h$ in Theorem 6.3, as the following example shows.

**Example 6.1.** Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $\mathrm{i} \in \mathbb{F}$ be a square root of $-1$ and set $k := 2$. Consider the edge-coloring model $h : \mathbb{F}[x_1, x_2] \to \mathbb{F}$ given by

$$h(x_1^a x_2^b) = \begin{cases} 1 \text{ if } a = 1 \text{ and } b = 0, \\ \mathrm{i} \text{ if } a = 0 \text{ and } b = 1, \\ 0 \text{ otherwise.} \end{cases} \tag{6.7}$$

Note that for any graph $G$ with at least one vertex we have $p_h(G) = 0$. Indeed, if $G$ contains an isolated vertex or a vertex of degree at least 2, then $p_h(G) = 0$. Otherwise, $G$ is a perfect matching. Since $p_h(K_2) = h(x_1)^2 + h(x_2)^2 = 0$, the claim follows. So the rank of $M_{p_h,1}$ is equal to zero. It is not difficult to see that that $\text{Stab}(h) = \{I\}$. Hence $\text{rk}\left(M_{p_h,1}\right) \neq \dim\left(V^{\text{Stab}(h)}\right) = 2$. More generally, the following holds: $\text{rk}\left(M_{p_h,t}\right) = \dim\left((V^{\otimes t})^{\text{O}_2}\right)$. The edge-coloring model $h' \equiv 0 \in \mathbb{F}[x_1, x_2]^*$ does the job.

There is however a class of edge-coloring models for which we can take $h = h'$.

**Theorem 6.4.** *Let $\mathbb{F} = \overline{\mathbb{F}}$, let $u_1, \ldots, u_n \in V$ be distinct vectors that span a non-degenerate subspace of $V$ and let $a_1, \ldots, a_n \in \mathbb{F}^*$. Let $h$ be the edge-coloring model defined by $h(p) := \sum_{i=1}^n a_i p(u_i)$, for $p \in R$. Then, for any $t \in \mathbb{N}$,*

$$\text{rk}\left(M_{p_h,t}\right) = \dim\left((V^{\otimes t})^{\text{Stab}(h)}\right). \tag{6.8}$$

The proof of Theorem 6.4 depends on a result from Section 7.2, but we will prove it in Section 6.2.

The outline for the rest of this chapter is as follows. In the next section we develop the necessary framework to prove Theorem 6.2 and Theorem 6.3. Based on this framework, we use a theorem of Schrijver [58], characterizing algebras of the form $T(V)^G$ for subgroups of the (real) orthogonal group, to prove Theorem 6.2. In the algebraically closed case we cannot use Schrijver's result as it uses the compactness of the real orthogonal group. Instead, we prove an algebraic version of this result (cf. Theorem 6.11) and use the framework of Section

5.3 and the existence and uniqueness of closed orbits to prove Theorem 6.3. In Section 6.3 we will use this approach, based on a characterization of tensors invariant under subgroups of $S_n$ (cf. Theorem 6.16), to give different (but not necessarily simpler) proof of Theorem 6.1. Finally, in Section 6.4 we provide proofs of Theorem 6.11 and Theorem 6.16.

## 6.2 The rank of edge-connection matrices

As follows from Example 6.1, the real and the algebraically closed case are different. However, the proofs of Theorem 6.2 and Theorem 6.3 have the same structure. We first develop the common framework for both cases and then we will specialize to $\mathbb{F} = \mathbb{R}$ and algebraically closed fields separately. Throughout this section we let $V := \mathbb{F}^k$ and we let $h$ denote any $k$-color edge-coloring model over $\mathbb{F}$ unless indicated otherwise.

### 6.2.1 Algebra of fragments

Recall from Section 2.2 that $\mathcal{F}_l$ is the set of all $l$-fragments. Let $\mathbb{F}\mathcal{F}_l$ denote the linear space consisting of (finite) formal $\mathbb{F}$-linear combinations of $l$-fragments; they are called *quantum fragments*. Extend the gluing operation, $*$, bilinearly to $\mathbb{F}\mathcal{F}_l \times \mathbb{F}\mathcal{F}_l$. Let

$$\mathcal{A} := \bigoplus_{l=0}^{\infty} \mathbb{F}\mathcal{F}_l. \tag{6.9}$$

Make $\mathcal{A}$ into a graded associative algebra by defining, for $F \in \mathcal{F}_l$ and $H \in \mathcal{F}_t$, the *tensor product* $F_1 \otimes F_2$ to be the disjoint union of $F_1$ and $F_2$, where the open end of $F_2$ labeled $i$ is relabeled to $l + i$.

Set

$$\mathcal{I}_l(h) := \{x \in \mathbb{F}\mathcal{F}_l \mid p_h(x * F) = 0 \text{ for all } l\text{-fragments } F\} \tag{6.10}$$

and let $\mathcal{I}(h) := \bigoplus_{k=0}^{\infty} \mathcal{I}_l(h)$. Note that $\mathcal{I}_l(h)$ is the kernel of the $l$-th edge-connection matrix of $p_h$. Observe that

$$\operatorname{rk}(M_{p_h,l}) = \dim(\mathbb{F}\mathcal{F}_l / \mathcal{I}_l(h)). \tag{6.11}$$

Let $T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the tensor algebra of $V$ (with product the tensor product). For $\phi : [n] \to [k]$ define $e_\phi := e_{\phi(1)} \otimes \cdots \otimes e_{\phi(n)}$. The $e_\phi$ form a basis for $V^{\otimes n}$. We will write $(\cdot, \cdot)$ to denote the nondegenerate symmetric bilinear form on $V^{\otimes n}$ induced by $(\cdot, \cdot)$ for any $n$. We will now exhibit a natural homomorphism from $\mathcal{A}$ to $T(V)$.

For an $l$-fragment $F$ we denote its edges (including half edges) by $E(F)$ and its vertices (not including open ends) by $V(F)$. Moreover, we will identify the half edges of $F$ with the set $[l]$. Let $F \in \mathcal{F}_l$ and let $\phi : [l] \to [k]$. Define

$$h_\phi(F) := \sum_{\substack{\psi: EF \to [k] \\ \psi(i)=\phi(i)\ i=1,\ldots,l}} \prod_{v \in VF} h\Big( \prod_{e \in \delta(v)} x_{\psi(e)} \Big). \tag{6.12}$$

We can now extend the map $p_h : \mathcal{G} \to \mathbb{F}$ to a linear map $p_h : \mathcal{A} \to T(V)$ by defining

$$p_h(F) = \sum_{\phi:[l]\to[k]} h_\phi(F) e_\phi, \tag{6.13}$$

for $F \in \mathcal{F}_l$, for $l \geq 0$.

Note that for $F_1, F_2 \in \mathcal{F}_l$,

$$p_h(F_1 * F_2) = \sum_{\phi:[l]\to[k]} h_\phi(F_1) h_\phi(F_2) = \big(p_h(F_1), p_h(F_2)\big). \tag{6.14}$$

For $\mathbb{F} = \mathbb{R}$, (6.14) implies that for $\gamma = \sum_{i=1}^n \lambda_i F_i \in \mathbb{R}\mathcal{F}_l$,

$$p_h(\gamma * \gamma) = \sum_{\phi:[l]\to[k]} \sum_{i,j=1}^{n} \lambda_i \lambda_j h_\phi(F_i) h_\phi(F_j) \geq 0, \tag{6.15}$$

showing the easy part of Theorem 5.2.

It is not difficult to see that $p_h$ is a homomorphism of algebras. By (6.14) it follows that $\operatorname{Ker} p_h \subseteq \mathcal{I}(h)$. This gives rise to the following definition: we call an edge-coloring model $h$ *nondegenerate* if $\operatorname{Ker} p_h = \mathcal{I}(h)$. Equivalently, $h$ is nondegenerate if the algebra $p_h(\mathcal{A})$ is nondegenerate with respect to the bilinear form on $T(V)$ (induced by that on $V$). So for nondegenerate $h$ we have $\mathcal{A}/\mathcal{I}(h) \cong p_h(\mathcal{A})$. In particular, by (6.11), we have the following lemma.

**Lemma 6.5.** *Let $h$ be a nondegenerate $k$-color edge-coloring model. Then, for any $t \in \mathbb{N}$,*

$$\operatorname{rk}(M_{p_h,t}) = \dim(p_h(\mathcal{A}) \cap V^{\otimes t}). \tag{6.16}$$

### 6.2.2 Contractions

In this subsection we introduce contractions for tensors and fragments, and we show that $p_h$ preserves these.

For $1 \leq i < j \leq l \in \mathbb{N}$ the *contraction* $C^l_{i,j}$ is the unique linear map

$$C^l_{i,j} : V^{\otimes l} \to V^{\otimes l-2} \text{ satisfying} \tag{6.17}$$

$$v_1 \otimes \ldots \otimes v_l \mapsto (v_i, v_j) v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \ldots \otimes v_{j-1} \otimes v_{j+1} \otimes \ldots \otimes v_l.$$

A subspace $A$ of $T(V)$ is called *graded* if $A = \bigoplus_{l=0}^{\infty}(V^{\otimes l} \cap A)$. A graded subspace $A$ of $T(V)$ is called *contraction closed* if $C^l_{i,j}(a) \in A$ for all $a \in A \cap V^{\otimes l}$ and $i < j \leq l \in \mathbb{N}$. Note that for any subgroup $G \subseteq \mathrm{O}_k$, $T(V)^G = \bigoplus_{l=0}^{\infty}(V^{\otimes l})^G$ is a graded and contraction closed subalgebra of $T(V)$ as, by definition, contractions are $\mathrm{O}_k$-invariant.

We now define a contraction operation for fragments. For $1 \leq i < j \leq l \in \mathbb{N}$, the *contraction* $\mathcal{C}^l_{i,j} : \mathcal{F}_l \to \mathcal{F}_{l-2}$ is defined as follows: for $F \in \mathcal{F}_l$, $\mathcal{C}^l_{i,j}(F)$ is the $(l-2)$-fragment obtained from $F$ by connecting the half edges incident with the open ends labeled $i$ and $j$ into one single edge (deleting these open ends), and then relabeling the remaining open ends $1, \ldots, l-2$ such that the order is preserved. See Figure 6.1 for an example.

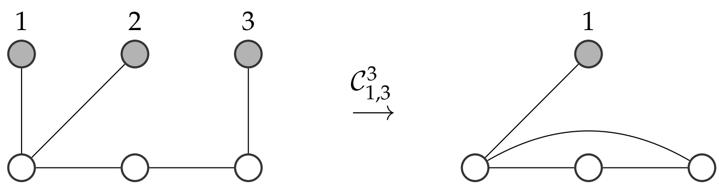



Figure 6.1: Contraction of a 3-fragment.

Besides being a homomorphism of algebras, $p_h$ also preserves contractions. Indeed, let $1 \leq i < j \leq l$ and let $F \in \mathcal{F}_l$. Note that for $\phi : [l] \to [k]$, the contraction of $e_\phi$ is contained in $\{e_\psi \mid \psi : [l-2] \to [n]\}$ if $\phi(i) = \phi(j)$ and is zero otherwise. Then

$$C^l_{i,j}(p_h(F)) = \sum_{\phi:[l]\to[n]} h_\phi(F) C^l_{i,j}(e_\phi) = \sum_{\substack{\phi:[l]\to[n] \\ \phi(i)=\phi(j)}} h_\phi(F) C^l_{i,j}(e_\phi)$$

$$= \sum_{\psi:[l-2]\to[n]} h_\psi(\mathcal{C}^l_{i,j}(F)) e_\psi = p_h(\mathcal{C}^l_{i,j}(F)). \tag{6.18}$$

The *basic l-fragment* $F_l$ is the $l$-fragment that contains one vertex and $l$ open ends connected to this vertex, labeled 1 up to $l$. Recall that $K_2^{\bullet\bullet}$ denotes the edge which has exactly two open ends and note that $p_h(K_2^{\bullet\bullet}) = \sum_{i=1}^k e_i \otimes e_i$. By relabeling $(K_2^{\bullet\bullet})^{\otimes m}$ for $m \in \mathbb{N}$, we see that by the Tensor FFT for $\mathrm{O}_k$ (cf. Theorem 4.3), the image of $p_h$ contains all $\mathrm{O}_k$-invariant tensors.

Let $F$ be an $l$-fragment without circles with $V(F) = [n]$ and $|E(F)| = m$, such that its underlying graph is connected. Then either $F = K_2^{\bullet\bullet}$ or $F$ can be obtained from the fragment $\bigotimes_{i=1}^n F_{d(i)}$ by applying $m - l$ contractions to it; see Figure 6.2.

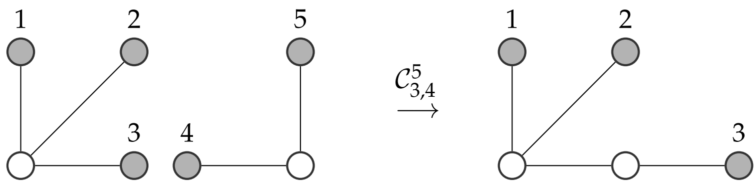



Figure 6.2: Obtaining a 3-fragment by contracting the product of two basic fragments.

Let us summarize the properties of the map $p_h$.

**Proposition 6.6.** *The image of $p_h$ is a graded contraction-closed algebra that contains $T(V)^{O_k}$. Moreover, $p_h(\mathcal{A})$ is generated by the images of the basic fragments and $K_2^{\bullet\bullet}$ as a contraction-closed algebra.*

### 6.2.3 Stabilizer subgroups of the orthogonal group

For $l \in \mathbb{N}$, we write $h_l$ for the restriction of $h$ to the space of homogenous polynomials of degree $l$. We think of $h_l$ as a symmetric tensor as follows:

$$(h_l, e_\phi) = h_l(x^\phi), \tag{6.19}$$

where for a map $\phi : [l] \to [k]$, we define the monomial $x^\phi \in \mathbb{F}[x_1, \ldots, x_k]$ by $x^\phi := \prod_{i=1}^k x_{\phi(i)}$. This gives a natural $\mathrm{O}_k$-equivariant embedding of $\mathbb{F}^{\mathbb{N}_l^k}$ into $V^{\otimes l}$. Indeed, as $V$ and $V^*$ are isomorphic $\mathrm{O}_k$-modules (cf. (3.17)), we have for any $\phi : [l] \to [k]$:

$$(gh_l)(x^\phi) = h_l(g^{-1}x^\phi) = (h_l, g^{-1}e_\phi) = (gh_l, e_\phi). \tag{6.20}$$

For a subset $A \subseteq T(V)$, define the *pointwise stabilizer* of $A$ by

$$\mathrm{Stab}(A) := \{g \in \mathrm{O}_k \mid ga = a \text{ for all } a \in A\}. \tag{6.21}$$

The next proposition shows that $\mathrm{Stab}(h)$ is equal to $\mathrm{Stab}(p_h(\mathcal{A}))$.

**Proposition 6.7.** *Let $h$ be an edge-coloring model. Then* $\mathrm{Stab}(h) = \mathrm{Stab}(p_h(\mathcal{A}))$.

*Proof.* Let $l \in \mathbb{N}$. Then $p_h(F_l) = h_l$ (viewing $h_l$ as a symmetric tensor). So in particular, $gp_h(F_l) = p_{gh}(F_l)$ for each $g \in \mathrm{O}_k$. Since $p_h(\mathcal{A})$ is generated, as a contraction-closed algebra, by $K_2^{\bullet\bullet}$ and the basic fragments and since contractions are by definition $\mathrm{O}_k$-invariant, it follows that for any $l$-fragment $F$, $p_{gh}(F) = gp_h(F)$ for each $g \in \mathrm{O}_k$. This implies that $g \in \mathrm{Stab}(h)$ if and only if $g \in \mathrm{Stab}(p_h(\mathcal{A}))$. □

### 6.2.4 The real case

Here we will give a proof of Theorem 6.2. So $\mathbb{F} = \mathbb{R}$ (and $h$ denotes a $k$-color edge-coloring model over $\mathbb{R}$).

First note that, by (6.15), $h$ is clearly nondegenerate. So by Lemma 6.5, it suffices to prove the following combinatorial parametrization of the tensors invariant under $\text{Stab}(h)$.

**Theorem 6.8.** *Let $h$ be a $k$-color edge-coloring model over $\mathbb{R}$. Then*

$$p_h(\mathcal{A}) = T(V)^{\text{Stab}(h)}. \tag{6.22}$$

A crucial ingredient in the proof of Theorem 6.8 is the characterization by Schrijver [58] of subalgebras of the tensor algebra that are of the form $T(V)^G$ for subgroups $G$ of the real orthogonal group.

**Theorem 6.9** (Schrijver [58])**.** *Let $A \subseteq T(V)$. Then $A = T(V)^G$ for some subgroup $G \subseteq O_k$ if and only if $A$ is a graded contraction-closed subalgebra of $T(V)$ that contains $T(V)^{O_k}$.*

We can now give a proof of Theorem 6.8

*Proof of Theorem 6.8.* By Proposition 6.6, $p_h(\mathcal{A})$ is a graded contraction-closed subalgebra of $T(V)$ that contains $T(V)^{O_k}$. So we can apply Theorem 6.9, to see that $p_h(\mathcal{A}) = T(V)^G$, for some subgroup $G$ of $O_k$. Now note that $G \subseteq \text{Stab}(p_h(\mathcal{A}))$, implying that $T(V)^{\text{Stab}(p_h(\mathcal{A}))} \subseteq T(V)^G$. Moreover, $T(V)^G = p_h(\mathcal{A}) \subseteq T(V)^{\text{Stab}(p_h(\mathcal{A}))}$. Hence $T(V)^{\text{Stab}(p_h(\mathcal{A}))} = T(V)^G$. As $\text{Stab}(h) = \text{Stab}(p_h(\mathcal{A})$ by Proposition 6.7 , this proves the theorem. ☐

### 6.2.5 The algebraically closed case

Here we will give a proof of Theorem 6.3. So $\mathbb{F}$ denotes an algebraically closed field from now on.

Just as in the real case, we will state a combinatorial parametrization of the tensors invariant under $\text{Stab}(h')$, for certain nondegenerate edge-coloring models $h'$ over $\mathbb{F}$, which implies Theorem 6.3 by Lemma 6.5.

**Theorem 6.10.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $h$ be a $k$-color edge-coloring model over $\mathbb{F}$. Then there exists a nondegenerate $k$-color edge-coloring model $h'$ over $\mathbb{F}$ such that $p_h(H) = p_{h'}(H)$ for all $H \in \mathcal{G}$ and such that*

$$p_{h'}(\mathcal{A}) = T(V)^{\text{Stab}(h')}. \tag{6.23}$$

We cannot proceed in the same way as in Section 6.2.4 for two reasons. The first reason being that any edge-coloring model over $\mathbb{F}$ is not automatically nondegenerate (cf. Example 6.1). To circumvent this issue, we will find and edge-coloring model $h'$ such that $h'_{\leq d}$ is contained in the unique closed orbit in $\overline{\mathrm{O}_k h_{\leq d}}$ for $d$ large enough and show that $h'$ is nondegenerate. The second reason is that the proof of Theorem 6.9 in [58] uses the compactness of the real orthogonal group and hence it does not apply to $\mathrm{O}_k(\mathbb{F})$, as it is not compact. Derksen (private communication, 2006) completely characterized which subalgebras of $T(V)$ are the algebras of $G$-invariant tensors for some reductive group $G \subseteq \mathrm{O}_k$, but we do not need the full strength of his result to prove Theorem 6.10. Instead, we state a sufficient condition for a subalgebra of $T(V)$ to be the algebra of $G$-invariants for some reductive group $G \subseteq \mathrm{O}_k$.

**Theorem 6.11.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $A \subseteq T(V)$ be a graded contraction closed subalgebra containing $T(V)^{O_k}$. If $\mathrm{Stab}(A) = \mathrm{Stab}(w)$ for some $w \in A$ whose $O_k$-orbit is closed in the Zariski topology, then $A = T(V)^{\mathrm{Stab}(A)}$ and moreover $\mathrm{Stab}(A)$ is a reductive group.*

We will prove this theorem in Section 6.4. Now we will use it to prove Theorem 6.10.

*Proof of Theorem 6.10.* The proof consists basically of checking the conditions in Theorem 6.11. It is based upon the framework developed in Section 5.3 and the proof of Theorem 5.3.

Let

$$Y_d := \{z : \mathbb{N}^k_{\leq d} \to \mathbb{F} \mid p_z(H) = p_h(H) \text{ for each graph } H \text{ of maximum degree at most } d\}. \quad (6.24)$$

Then $Y_d$ is the fiber of $h_{\leq d}$ under the quotient map $\pi : \mathbb{F}^{N^k_{\leq d}} \to \mathbb{F}^{N^k_{\leq d}}//O_k$ (cf. (5.40)). In the same way as in the proof of Theorem 5.3 we choose $h'$ such that $h'_{\leq d}$ is in the unique closed $\mathrm{O}_k$-orbit $C_d$ in $Y_d$ for each $d \geq d_0$ for $d_0$ large enough.

We will now show that this $h'$ is as required. First note that $\mathrm{Stab}(h') = \cap_{e\geq 0}\mathrm{Stab}(h'_{\leq e})$. Since the ring of regular functions of $\mathrm{O}_k$ is Noetherian it follows that there exists $e$ such that $\mathrm{Stab}(h') = \mathrm{Stab}(h'_{\leq e})$. We may assume that $e \geq d_0$. Let $F = \sum_{0\leq k\leq e} F_k$, the sum in $\mathcal{A}$ of the first $e+1$ basic fragments. Write $w := p_{h'}(F)$ and note that $w$ is the image of $h'_{\leq e}$ under the natural $\mathrm{O}_k$-equivariant embedding of $\mathbb{F}^{\mathbb{N}^k_{\leq e}}$ into $\bigoplus_{k=0}^{e} V^{\otimes k}$. Then

$$\mathrm{Stab}(w) = \mathrm{Stab}(h'). \quad (6.25)$$

Moreover, as we can view $C_e$ and $Y_e$ as subvarieties of $\bigoplus_{k=0}^{e} V^{\otimes k}$, it follows that the $O_k$-orbit of $w$ is Zariski closed. By Proposition 6.7, $\mathrm{Stab}(p_{h'}(\mathcal{A})) = \mathrm{Stab}(w)$. By Proposition 6.6, $p_h(\mathcal{A})$ is a graded contraction-closed subalgebra that contains $T(V)^{O_k}$. So we can apply Theorem 6.11 to find that $p_{h'}(\mathcal{A}) = T(V)^{\mathrm{Stab}(h')}$. Moreover, we find that $\mathrm{Stab}(h')$ is reductive. From this we conclude that $h'$ is nondegenerate.

Indeed, suppose that $p_{h'}(x) \neq 0$ for some $x \in \mathcal{A}$. Then there exists $y \in T(V)$ such that $(p_{h'}(x), y) \neq 0$. Since $\mathrm{Stab}(h')$ is reductive we can write $T(V) = T(V)^{\mathrm{Stab}(h')} \oplus W$ with $W$ stable under $\mathrm{Stab}(h')$. Write $y = v + w$ with $v \in T(V)^{\mathrm{Stab}(h')}$ and $w \in W$. As $p_{h'}(x) \in T(V)^{\mathrm{Stab}(h')}$, we have for each $g \in \mathrm{Stab}(h')$ and $u \in T(V)$,

$$(p_{h'}(x), gu) = (g^{-1} p_{h'}(x), u) = (p_{h'}(x), u). \tag{6.26}$$

So Lemma 4.2 implies that $(p_{h'}(x), w) = 0$. It follows that $h'$ is nondegenerate. □

Using a result from Section 7.2, the proof of Theorem 6.4 is now basically done.

**Theorem 6.4.** *Let $\mathbb{F} = \overline{\mathbb{F}}$, let $u_1, \ldots, u_n \in V$ be distinct vectors that span a nondegenerate subspace of $V$ and let $a_1, \ldots, a_n \in \mathbb{F}^*$. Let $h$ be the edge-coloring model defined by $h(p) := \sum_{i=1}^{n} a_i p(u_i)$, for $p \in R$. Then, for any $t \in \mathbb{N}$,*

$$\mathrm{rk}\,(M_{p_h,t}) = \dim\left((V^{\otimes t})^{\mathrm{Stab}(h)}\right). \tag{6.27}$$

*Proof.* By Theorem 7.7, for $d \geq 3n$, the orbit of $h_{\leq d}$ is closed. It follows by the proof of Theorem 6.10 and by Lemma 6.5 that $\mathrm{rk}\,(M_{p_h,t}) = \dim(V^{\otimes d})^{\mathrm{Stab}(h)}$. □

## 6.3 The rank of vertex-connection matrices

In this section we will give a proof of Theorem 6.1, using the ideas from the previous section. Since the groups we are dealing with are finite, we do not have to differentiate between fields that are algebraically closed or not. Throughout this section, $(a, B)$ will denote any $n$-color vertex-coloring model over $\mathbb{F}$ unless indicated otherwise. Moreover, we set $W := \mathbb{F}^n$.

### 6.3.1 Another algebra of labeled graphs

Recall from Section 2.2 that $\mathcal{G}_l$ denotes the set of $l$-labeled graphs. Let $\mathbb{F}\mathcal{G}_l$ be the vectorspace consisting of finite formal linear combinations of $l$-labeled graphs. Let

$$\mathcal{Q} := \bigoplus_{l=0}^{\infty} \mathbb{F}\mathcal{G}_l, \tag{6.28}$$

and make it into an associative algebra by defining for $H \in \mathcal{G}_l$ and $F \in \mathcal{G}_k$, $H_1 \otimes H_2$ to be the disjoint union of $H_1$ and $H_2$ where we add $l$ to all the labels of $H_2$ so that $F \otimes H \in \mathcal{G}_{l+k}$ and extend this bilinearly to $\mathcal{Q} \times \mathcal{Q}$. Note that $FH$ and $F \otimes H$ are different if the number of labels is positive.

Let $e_1, \ldots, e_n$ be the standard basis for $W = \mathbb{F}^n$. Let for any $w \in W$, $(\cdot, \cdot)_w$ be the symmetric bilinear form on $W \times W$ defined by

$$(e_i, e_j)_w := w_i \delta_{i,j}. \tag{6.29}$$

Note that taking $w$ the all ones vector, we obtain the standard bilinear form.

Write $G := G(a, B)$ and extend $p_{a,B}$ to a linear map $p_{a,B} : \mathcal{Q} \to T(V)$ by defining, for $H \in \mathcal{G}_l$,

$$p_{a,B}(H) = \sum_{\phi:[l]\to[n]} \hom_\phi(H, G) e_\phi, \tag{6.30}$$

where for $\phi : [l] \to [n]$ and $H \in \mathcal{G}_l$ we define

$$\hom_\phi(H, G) := \sum_{\substack{\psi:V(H)\to[n] \\ \psi(i)=\phi(i)\ \forall i\in[l]}} \prod_{v\in V(H)\setminus[l]} a_{\phi(v)} \cdot \prod_{uv\in E(H)} B_{\phi(u),\phi(v)}. \tag{6.31}$$

Recall from Section 2.2.1 that we extended graph parameters to labeled graphs by setting $f(H) := f([\![H]\!])$ for $H \in \mathcal{G}_l$ and a graph parameter $f$. So to avoid confusion, we will write $\hom(H, G)$ if we mean $p_{a,B}([\![H]\!])$; by $p_{a,B}(H)$ we mean an $l$-tensor as defined by (6.30). Now note that for any $H_1, H_2 \in \mathcal{G}_l$,

$$\hom(H_1 \cdot H_2, G) = \sum_{\phi:[l]\to[n]} \prod_{i\in[l]} a_{\phi(i)} \hom_\phi(H_1, G) \hom_\phi(H_2, G). \tag{6.32}$$

Note that when $\mathbb{F} = \mathbb{R}$ and $a_i > 0$ for each $i \in [n]$, (6.32) implies, similarly to (6.15), that $\hom(\cdot, G)$ is reflection positive.

Clearly, $p_{a,B}$ is a homomorphism of algebras. We call the pair $(a, B)$ *nondegenerate* if the image of $p_{a,B}$ is nondegenerate with respect to $(\cdot, \cdot)_a$. As in the edge-coloring model case we have the following result.

**Lemma 6.12.** *Let $(a, B)$ be a nondegenerate twin-free $n$-color vertex-coloring model. Then, for any $l \in \mathbb{N}$,*

$$\mathrm{rk}\,(N_{p_{a,B},l}) = \dim(p_{a,B}(\mathbb{F}\mathcal{G}_l)). \tag{6.33}$$

### 6.3.2 Some operations on labeled graphs and tensors

We define some operations on labeled graphs and tensors and show how they are related via the map $p_{a,B}$.

Let $\circ : W^{\otimes 2} \times W^{\otimes 2} \to W^{\otimes 2}$ be the linear map defined by $(C \circ D)_{i,j} = C_{i,j} D_{i,j}$, for $C, D \in W^{\otimes 2}$. This operation is called the *Schur product*. Note that for two 2-labeled graphs $H_1$ and $H_2$ we have

$$p_{a,B}(H_1 \cdot H_2) = p_{a,B}(H_1) \circ p_{a,B}(H_2). \tag{6.34}$$

We next define contraction-like operations for labeled graphs and tensors. For $i < j \leq l \in \mathbb{N}$ define the *labeled contraction* $\mathcal{K}^l_{i,j} : \mathcal{G}_l \to \mathcal{G}_{l-1}$ by identifying for $H \in \mathcal{G}_l$, the labeled vertices $i$ and $j$ as one vertex, giving the vertex label $i$ and relabeling the remaining labeled vertices $1, \ldots, i-1, i+1, \ldots, l-1$ in the same order. Note that if $i$ and $j$ are connected by an edge one creates a loop at vertex $i$. We now define the corresponding operation for tensors. For $l \in \mathbb{N}$ and $i < j \leq l$,

$$\begin{aligned} &K^l_{i,j} : W^{\otimes l} \to W^{\otimes l-1} \text{ is the unique linear map defined by} \\ &\quad e_{t_1} \otimes \cdots \otimes e_{t_l} \mapsto \delta_{t_i,t_j} e_{t_1} \otimes \cdots \otimes e_{t_{j-1}} \otimes e_{t_{j+1}} \otimes \cdots \otimes e_{t_l}. \end{aligned} \tag{6.35}$$

Then it is easy to see that

$$K^l_{i,j}(p_{a,B}(H)) = p_{a,B}(\mathcal{K}^l_{i,j}(H)) \tag{6.36}$$

for each $l \in \mathbb{N}$, $i < j \leq l$ and $H \in \mathcal{G}_l$.

We now define an unlabeling operation for labeled graphs and for tensors. For any $l$ and $i \in [l]$ define $\mathcal{U}^l_i : \mathcal{G}_l \to \mathcal{G}_{l-1}$ by unlabeling the $i$-th vertex and then relabeling the remaining vertices in the same order. Moreover, define

$$\begin{aligned} &U^l_i : W^{\otimes l} \to W^{\otimes l-1} \text{ to be the unique linear map satisfying} \\ &\quad v_1 \otimes \cdots \otimes v_l \mapsto (v_i, \mathbb{1})_a v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes v_l. \end{aligned} \tag{6.37}$$

Then it is easy to see that $p_{a,B}$ preserves unlabeling, that is for all $H \in \mathcal{G}_l$ and any $i \in [l]$ we have

$$U^l_i(p_{a,B}(F)) = \mathcal{U}^l_i(p_{a,B}(F)). \tag{6.38}$$

We define one more operation on two-tensors (i.e. matrices). Let $A$ be the diagonal matrix defined by $A_{i,i} = a_i$ for $i \in [n]$. For $C, D \in W^{\otimes 2}$ we define

$$C * D := CAD, \tag{6.39}$$

the (ordinary matrix) product of the matrices $C, A$ and $D$. Note that $C * D$ is equal to $C^4_{2,3}(C, D)$, the contraction of $C \otimes D$ with respect to $(\cdot,\cdot)_a$. Let $J \in W^{\otimes 2}$ denote the all-ones matrix and let $I$ denote the identity matrix.

**Lemma 6.13.** *Let $\mathcal{B} \subset W^{\otimes 2}$ be an algebra with $*$-product, generated by $B$ and $J$ and which is closed under taking the Schur product. If $(a, B)$ is twin free and if $\sum_{i\in S} a_i \neq 0$ for all $S \subseteq [n]$, then $I$ and $A^{-1}$ are contained in $\mathcal{B}$.*

*Proof.* Put an equivalence relation on $[n] \times [n]$ by $(i, j) \sim (i', j')$ if and only if $C_{i,j} = C_{i',j'}$ for all $C \in \mathcal{B}$. Let $M_1, \ldots, M_t$ be the incidence matrices of the equivalence classes of $\sim$. Then

$$M_i \in \mathcal{B} \text{ for } i = 1, \ldots, t. \tag{6.40}$$

To see (6.40), let $C = \sum_{i=1}^t c_i M_i \in \mathcal{B}$ be a matrix for which the number of distinct coefficients is maximal. Then all $c_i$ are distinct. For suppose this is not true. We may assume that $c_1 = c_2$. By definition of the equivalence relation, there exists $D = \sum_{i=1}^t d_i M_i \in \mathcal{B}$ such that $d_1 \neq d_2$. Pick a nonzero number $x$ such that if $c_i \neq c_j$, then $xc_i + d_i \neq xc_j + d_j$. Then $xC + D \in \mathcal{B}$ contains more distinct coefficients than $C$. A contradiction.

Now pick interpolating polynomials $p_1, \ldots, p_t$ such that $p_i(c_j) = \delta_{i,j}$ (cf. [17, Lemma 2.9]). Then, since $\mathcal{B}$ is closed under the Schur product, $p_i(C) = M_i \in \mathcal{B}$. This proves (6.40).

Observe that for each $i$, $M_i = M_j^T$ for some $j$, since $B$ and $J$ are symmetric. Moreover, as $J \in \mathcal{B}$ we have $\sum_{i=1}^t M_i = J$. Now suppose that $I \notin \mathcal{B}$. Then there exists $i \neq j$ and $k$ such that $C_{i,j} = C_{k,k}$ for all $C \in \mathcal{B}$. As no two rows of $B$ are equal, there exist $s, t$ such that $(M_s)_{i,t} = 0$ and $(M_s)_{j,t} = 1$. Since the $M_i$ sum up to $J$, there exists $l \neq s$ such that $(M_l)_{i,t} = 1$. So $(M_l * M_s^T)_{i,j} \neq 0$. (Here we use that $\sum_{i\in S} a_i \neq 0$ for all $S \subseteq [n]$.) But since $C_{i,j} = C_{k,k}$ for all $C \in \mathcal{B}$, we have that

$$(M_l * M_s^T)_{i,j} = (M_l * M_s^T)_{k,k} = 0, \tag{6.41}$$

since the rows of $M_s$ and $M_l$ have disjoint support. A contradiction. So we conclude that $I \in \mathcal{B}$.

Now observe that $A = I * I \in \mathcal{B}$. Hence, as $\mathcal{B}$ contains the $M_i$, we find that $A^{-1} \in \mathcal{B}$. □

We now summarize the properties of the image of $p_{a,B}$.

**Proposition 6.14.** *If $(a, B)$ is twin free and if $\sum_{i\in S} a_i \neq 0$ for each $S \subseteq [n]$, then the image of $p_{a,B}$ is a graded contraction-closed subalgebra of $T(W)$ that contains $T(W)^{S_n}$.*

*Proof.* First note that $J, B \in p_{a,B}(\mathcal{Q})$ as they are the image of $K_1^{\bullet} \cdot K_1^{\bullet}$ and $K_2^{\bullet\bullet}$ respectively. So by Lemma 6.13, $A^{-1}, I \in p_{a,B}(\mathcal{Q})$. Note note that for $w \in W^{\otimes l}$, the contraction, $C_{i,j}^l(w)$, of $w$ can be obtained from $A^{-1} \otimes w$, by contracting it two times with respect to the bilinear form $(\cdot,\cdot)_a$. Since contractions with respect to $(\cdot,\cdot)_a$), can be obtained by composing $U_i^{l-1}$ with $K_{i,j}^l$, it follows by (6.36) and (6.38) that $p_{a,B}(\mathcal{Q})$ is contraction closed.

Next, define for $k \in \mathbb{N}$, $h_k := \sum_{i=1}^n e_i^{\otimes k}$. By applying $K_{2,3}^4$ to $h_2 \otimes h_2$ we find that $\sum_{i=1}^n e_i^{\otimes 3} \in p_{a,B}(\mathcal{Q})$, as $h_2 = I \in p_{a,B}(\mathcal{Q})$. Similarly, $h_k \in p_{a,B}(\mathcal{Q})$ for any $k > 2$. For $k = 1$, we have $h_1 = p_{a,B}(K_1^{\bullet})$. From this we will deduce that $T(W)^{S_n} \subseteq p_{a,B}(\mathcal{Q})$.

First we need a definition. A tensor $u$ is called *mutation* of a tensor $v \in W^{\otimes l}$ if it is obtained from $v$ by permuting tensor factors. Note that $p_{a,B}(\mathcal{Q})$ is closed under mutations. Indeed, any mutation of $v \in W^{\otimes l}$ can be obtained by applying $l$ contractions to $v \otimes I^{\otimes l}$.

For $l \in \mathbb{N}$ and a partition $\lambda$ of $[l]$ define elements of $(W^{\otimes l})^{S_n}$ by

$$m_\lambda := \sum_{\substack{i_1,\ldots,i_l \in [n] : i_j = i_k \Leftrightarrow j,k \\ \text{are in the same block of } \lambda}} e_{i_1} \otimes \cdots \otimes e_{i_l},$$

$$p_\lambda := \sum_{\substack{i_1,\ldots,i_l \in [n] : i_j \neq i_k \Leftrightarrow j,k \\ \text{are in different blocks of } \lambda}} e_{i_1} \otimes \cdots \otimes e_{i_l}. \tag{6.42}$$

Observe that the $m_\lambda$ span $(W^{\otimes l})^{S_n}$. For a partition $\lambda$ of $[l]$, $p_\lambda = \sum_{\mu \trianglerighteq \lambda} m_\mu$, where for partitions $\mu$ and $\lambda$ of $[l]$ we set $\mu \trianglerighteq \lambda$ if each block of $\lambda$ is contained in some block of $\mu$. This defines a partial order on the set of partitions of $[l]$. By Möbius inversion (cf. [56, Theorem 3.3]), it follows that the $p_\lambda$ also span $(W^{\otimes l})^{S_n}$. Finally, observe that each $p_\lambda \in p_{a,B}(\mathcal{Q})$, as it can be obtained from a mutation of $h_{l_1} \otimes \cdots \otimes h_{l_t}$, where the $l_i$ denote the block sizes of $\lambda$. So we conclude that $T(W)^{S_n} \subseteq p_{a,B}(\mathcal{Q})$. □

### 6.3.3 Proof of Theorem 6.1

Theorem 6.1 is special case of the following result.

**Theorem 6.15.** *Let $(a, B)$ be a twin-free $n$-color vertex-coloring model over $\mathbb{F}$, such that $\sum_{i \in I} a_i \neq 0$ for all $I \subseteq [n]$. Then*

$$p_{a,B}(\mathcal{Q}) = T(W)^{\mathrm{Aut}(a,B)}. \tag{6.43}$$

*Proof of Theorem 6.1.* By Lemma 6.12 and Theorem 6.15, we only need to show that $(a, B)$ is nondegenerate. Suppose that $0 \neq v \in T(W)^{\mathrm{Aut}(a,B)}$. Then there exists $w \in T(W)$ such that $(v,w)_a \neq 0$. Then, as $v$ is $\mathrm{Aut}(a,B)$-invariant, we have

$$(v,w)_a = \frac{1}{|\mathrm{Aut}(a,B)|} \sum_{\pi \in \mathrm{Aut}(a,B)} (v, \pi w)_a. \tag{6.44}$$

As $\sum_{\pi \in \mathrm{Aut}(a,B)} \pi w \in T(W)^{\mathrm{Aut}(a,B)}$, Theorem 6.15 implies that $p_{a,B}(\mathcal{Q})$ is nondegenerate. □

To prove Theorem 6.15 we use a characterization of subalgebras of $T(W)$ that are algebras of $G$-invariants for subgroups $G$ of $S_n$.

**Theorem 6.16.** *Let $A \subseteq T(W)$. Then $A = T(W)^G$ for some subgroup $G \subseteq S_n$ if and only if $A$ is a graded contraction-closed subalgebra of $T(W)$ that contains $T(W)^{S_n}$.*

We will prove this theorem in Section 6.4. Now we will use it to prove Theorem 6.15.

*Proof of Theorem 6.15.* By Proposition 6.14 we can apply Theorem 6.16, to find that we have $p_{a,B}(\mathcal{Q}) = T(W)^G$, for some subgroup $G \subseteq S_n$.

We finish the proof by showing that $G = \mathrm{Aut}(a,B)$. First note that $a = U_1^2(h_2)$ and that $B \in p_{a,B}(\mathcal{Q})$ hence $G \subseteq \mathrm{Aut}(a,B)$. To see the converse, just observe that $T(W)^{\mathrm{Aut}(a,B)} \subseteq p_{a,B}(\mathcal{Q}) = T(W)^G$, as for each $l$-labeled graph $H$, each $\phi : [l] \to [n]$ and each $\pi \in \mathrm{Aut}(a,B)$, we have that $\hom_{\pi\cdot\phi}(H, G(a,B)) = \hom_{\phi}(H, G(a,B)))$ implying that $p_{a,B}(H)$ is invariant under $\mathrm{Aut}(a,B)$. □

*Remark.* Our proof of Theorem 6.1 is probably more involved than the proof of Lovász [41], but it has the advantage that it also works for $(a,B)$ where not all $a$ are positive, as long as the condition $\sum_{i \in S} a_i \neq 0$ for each $S \subseteq [n]$ is satisfied. In fact, the method by Lovász only requires that $(a,B)$ is nondegenerate, which is immediate if all $a_i$ are positive. It follows from our results that, if $\sum_{i \in S} a_i \neq 0$ for each $S \subseteq [n]$, then $(a,B)$ is nondegenerate. We do not know whether this can be shown directly, neither do we know whether we can remove this condition.

## 6.4 Proofs of Theorem 6.11 and Theorem 6.16

Both proofs are based on Schrijver's proof of Theorem 6.9 and have a similar structure. We will first prove Theorem 6.16 since proving Theorem 6.11 requires more advanced machinery.

**Theorem 6.16.** *Let $A \subseteq T(W)$. Then $A = T(W)^G$ for some subgroup $G \subseteq S_n$ if and only if $A$ is a graded contraction-closed subalgebra of $T(W)$ that contains $T(W)^{S_n}$.*

*Proof.* The 'only if' part is clear. To see the 'if' part, let $A \subseteq T(W)$ be a graded contraction-closed algebra containing $T(W)^{S_n}$.

Let $G := \{\pi \in S_n \mid \pi a = a \text{ for all } a \in A\}$. We will show that $A = T(W)^G$, where the inclusion $A \subseteq T(W)^G$ is direct. To see the opposite inclusion, let $X := S_n/G$ be the set of left $G$-cosets and define functions $f_{v,w} : X \to \mathbb{F}$ by $f_{v,w}(\pi G) := (\pi v, w)$, for $\pi \in S_n$, $v \in A \cap W^{\otimes k}$ and $w \in W^{\otimes k}$, for any $k$. This is well defined since if $\pi \in G$, then $\pi v = v$. Note that $f_{v,w} f_{v',w'} = f_{v \otimes v', w \otimes w'}$. By nondegeneracy, $f_{h_2,w}$ is the constant one function for some $w \in W^{\otimes 2}$.

Let $F$ be the algebra spanned by the functions $f_{v,w}$, for $v \in A \cap W^{\otimes k}$ and $w \in W^{\otimes k}$ and $k \in \mathbb{N}$. If $\pi G \neq \pi' G$, then by definition of $G$ there exists $v \in A \cap W^{\otimes k}$ for some $k$ such that $\pi^{-1}\pi' v \neq v$. So there exists $w \in W^{\otimes k}$ such that $(\pi v, w) \neq (\pi' v, w)$. Hence $f_{v,w}(\pi G) \neq f_{v,w}(\pi' G)$. So for each $\pi G \neq \pi' G \in X$, $F$ contains a function $f$ such that $f(\pi G) = 1$ and $f(\pi' G) = 0$, as $F$ contains the all-ones function. Since $F$ is an algebra it follows that $F = \mathbb{F}^X$. (This is actually the Stone-Weierstrass theorem for continuous functions on finite sets.)

Now let $x \in (W^{\otimes k})^G$ for some $k$. Then for any $\pi \in S_n$ we can write

$$\pi x = \sum_{\phi:[k]\to[n]} f_\phi(\pi) e_\phi, \tag{6.45}$$

for certain functions $f_\phi : S_n \to \mathbb{F}$. Since $x$ is $G$-invariant, the $f_\phi$ are actually functions on $X$. So we can write (6.45) as

$$\pi x = \sum_{\phi,i} f_{v_{\phi,i}, w_{\phi,i}}(\pi G) e_\phi, \tag{6.46}$$

for certain $v_{\phi,i} \in A \cap W^{\otimes k}$ and $w_{\phi,i} \in W^{\otimes k}$. Multiplying (6.46) by $\pi^{-1}$ we obtain, (as $(\cdot,\cdot)$ is $S_n$-invariant),

$$\text{for all } \pi \in S_n: \; x = \sum_{\phi,i} (\pi v_{\phi,i}, w_{\phi,i}) \pi^{-1} e_\phi = \sum_{\phi,i} (v_{\phi,i}, \pi^{-1} w_{\phi,i}) \pi^{-1} e_\phi. \tag{6.47}$$

Now note that $(v_{\phi,i}, \pi^{-1} w_{\phi,i}) \pi^{-1} e_{\phi,i}$ is equal to a series of contractions $K_{\phi,i}$ applied to $v_{\phi,i} \otimes \pi^{-1}(w_{\phi,i} \otimes e_\phi)$. Hence

$$x = \sum_{\phi,i} K_{\phi,i}(v_{\phi,i} \otimes \left( \frac{1}{n!} \sum_{\pi \in S_n} \pi^{-1}(w_{\phi,i} \otimes e_\phi) \right)), \tag{6.48}$$

implying that $x \in A$, as $A$ contains $T(W)^{S_n}$ and is a graded subalgebra of $T(W)$ that is closed under contractions. This finishes the proof of the theorem. □

The proof of Theorem 6.11 has the same structure as the proof of Theorem 6.16, but since the orthogonal group is a non-compact group, certain details require more advanced algebraic techniques.

**Theorem 6.11.** *Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $A \subseteq T(V)$ be a graded contraction closed subalgebra containing $T(V)^{\mathrm{O}_k}$. If $\mathrm{Stab}(A) = \mathrm{Stab}(w)$ for some $w \in A$ whose $O_k$-orbit is closed in the Zariski topology, then $A = T(V)^{\mathrm{Stab}(A)}$ and moreover $\mathrm{Stab}(A)$ is a reductive group.*

*Proof.* Let $w \in A$ be such that $G := \mathrm{Stab}(w)$ equals $\mathrm{Stab}(A)$. Write $w = w_1 + \ldots + w_t$ with $w_j \in W_j := V^{\otimes n_j}$ the homogeneous components of $w$, and assume that that $\mathrm{O}_k w \subseteq W := \bigoplus_{j=1}^t W_j$ is closed. The map $\mathrm{O}_k \to W$ given by $g \mapsto gw$ induces an isomorphism $\mathrm{O}_k/G \to \mathrm{O}_k w$ of quasi affine varieties (cf. [30, Section 12] or [9, Theorem 1.16]). As $\mathrm{O}_k w$ is closed, both varieties are affine and moreover regular functions on $\mathrm{O}_k w$ extend to regular functions (polynomials) on $W$. So they are generated by $W_j^*$ for $j = 1, \ldots, t$. This means that any regular function on $\mathrm{O}_k/G$ is a linear combination of functions of the form

$$\begin{aligned} gG &\mapsto (gw_1, u_1)^{d_1} \cdots (gw_t, u_t)^{d_t} \\ &= (w_1^{\otimes d_1} \otimes \cdots \otimes w_t^{\otimes d_t}, g^{-1}(u_1^{\otimes d_1} \otimes \cdots \otimes u_t^{\otimes d_t})), \end{aligned} \tag{6.49}$$

where $d_1, \ldots, d_t$ are natural numbers and $u_j \in W_j$ for all $j$. Since $A$ is a graded algebra, the tensor products of the $w_j$ are contained in $A$. So we find that every regular function on $\mathrm{O}_k/G$ is a linear combination of functions of the form $gG \mapsto (gq, u) = (q, g^{-1}u)$ with $u \in T(V)$ and $q \in A$ in the same graded component of $T(V)$.

Clearly, $A$ is contained in $T(V)^G$. To prove the converse, let $a \in (V^{\otimes k})^G$. Let $z_1, \ldots, z_s$ be a basis of $V^{\otimes k}$. Then we can write,

$$ga = \sum_{i=1}^{s} f_i(g) z_i, \tag{6.50}$$

for all $g \in \mathrm{O}_k$, where the $f_i$ are regular functions on $\mathrm{O}_k$. Since $gha = ga$ for all $h \in G$ it follows that the $f_i$ induce regular functions on $\mathrm{O}_k/G$. By the above, for each $i = 1, \ldots, s$, we can write

$$f_i(g) = \sum_j (q_{i,j}, g^{-1}u_{i,j}), \tag{6.51}$$

for certain $q_{i,j} \in A$ and $u_{i,j} \in T(V)$. Multiplying both sides of (6.50) by $g^{-1}$ we obtain

$$\text{for all } g \in \mathrm{O}_k: \; a = \sum_{i,j} (q_{i,j}, g^{-1}u_{i,j}) g^{-1} z_i = \sum_{i,j} K_{i,j}(q_{i,j} \otimes g^{-1}(u_{i,j} \otimes z_i)), \tag{6.52}$$

where $K_{i,j}$ denotes a certain series of contractions. Let $\rho_{\mathrm{O}_k}$ be the Reynolds operator of $\mathrm{O}_k$. Then we have

$$a = \sum_{i,j} K_{i,j}(q_{i,j} \otimes \rho_{\mathrm{O}_k}(u_{i,j} \otimes z_i)). \tag{6.53}$$

In the case where $\mathbb{F} = \mathbb{C}$, this follows immediately by integrating (6.52) over $g$ in the compact real orthogonal group (with respect to the Haar measure). In the general case this follows, by reductiveness of $\mathrm{O}_k$, from Lemma 4.2.

To complete the proof note that $q_{i,j} \in A$ and $\rho_{\mathrm{O}_k}(u_{i,j} \otimes z_i) \in T(V)^{\mathrm{O}_k} \subseteq A$. As $A$ is a contraction closed subalgebra of $T(V)$ it follows that $a \in A$.

Finally, since $\mathrm{O}_k/G$ is affine, Matsushima's Criterion (see [1] for an elementary proof) implies that $\mathrm{Stab}(A) = G$ is a reductive group. □

# Chapter 7

# Edge-reflection positive partition functions of vertex-coloring models

Recall from Section 5.1 that the partition function of a vertex-coloring model is also the partition function of an edge-coloring model. In this chapter we characterize, using some fundamental results from geometric invariant theory, for which vertex-coloring models their partition functions are edge-reflection positive, i.e., for which vertex-coloring models their partition functions are partition functions of real edge-coloring models.

This chapter is based on [54] except for Section 7.2, which is based on joint work with Jan Draisma [20, Section 6].

## 7.1 Introduction

In his paper [66] (see also [67]) on the characterization of partition functions of real edge-coloring models, Szegedy gave an explicit way to construct from a vertex-coloring model $(a, B)$ over $\mathbb{C}$ an edge-coloring model $h$ over $\mathbb{C}$ such that $p_{a,B}(H) = p_h(H)$ for every $H \in \mathcal{G}$. We will now describe this construction.

Let $(a, B)$ be an $n$-color vertex-coloring model over $\mathbb{C}$. As $B$ is symmetric we can write $B = U^T U$ for some $n \times k$ (complex) matrix $U$, for some $k$. Let $u_1, \ldots, u_n \in \mathbb{C}^k$ be the columns of $U$. Define the edge-coloring model $h$ by $h := \sum_{i=1}^n a_i \text{ev}_{u_i}$, where for $u \in \mathbb{C}^k$, $\text{ev}_u \in R(\mathbb{C})^*$ is the linear map defined by

$p \mapsto p(u)$ for $p \in R(\mathbb{C})$. (Recall that $R(\mathbb{C}) = \mathbb{C}[x_1, \ldots, x_k]$.)

**Lemma 7.1** (Szegedy [66]). *Let $(a, B)$ and $h$ be as above. Then $p_{a,B} = p_h$.*

*Proof.* Let $G = (V, E) \in \mathcal{G}$. Then $p_h(G)$ is equal to

$$\sum_{\phi:E\to[k]} \prod_{v\in V} h\Big(\prod_{e\in\delta(v)} x_{\phi(e)}\Big) = \sum_{\phi:E\to[k]} \prod_{v\in V} \left(\sum_{i=1}^{n} a_i \prod_{e\in\delta(v)} u_i(\phi(e))\right) \tag{7.1}$$

$$= \sum_{\phi:E\to[k]} \sum_{\psi:V\to[n]} \prod_{v\in V} \Big(a_{\psi(v)} \prod_{e\in\delta(v)} u_{\psi(v)}(\phi(e))\Big)$$

$$= \sum_{\psi:V\to[n]} \prod_{v\in V} a_{\psi(v)} \cdot \sum_{\phi:E\to[k]} \prod_{v\in V} \prod_{e\in\delta(v)} u_{\psi(v)}(\phi(e))$$

$$= \sum_{\psi:V\to[n]} \prod_{v\in V} a_{\psi(v)} \cdot \sum_{\phi:E\to[k]} \prod_{vw\in E} u_{\psi(v)}(\phi(vw)) u_{\psi(w)}(\phi(vw))$$

$$= \sum_{\psi:V\to[n]} \prod_{v\in V} a_{\psi(v)} \cdot \prod_{vw\in E} \sum_{i=1}^{k} u_{\psi(v)}(i) u_{\psi(w)}(i) = p_{a,B}(G).$$

Where the last line follows from the fact that $U^T U = B$. This completes the proof. ∎

Note that the proof of Lemma 7.1 also shows that if an edge-coloring model $h$ is of the form $h = \sum_{i=1}^n a_i \mathrm{ev}_{u_i}$ for certain nonzero $a_i \in \mathbb{C}^k$ and certain vectors $u_i \in \mathbb{C}$, then the partition function of $h$ is equal to the partition function of $(a, B)$ (on $\mathcal{G}$), where $a = (a_1, \ldots, a_n)$ and $B_{i,j} = u_i^T u_j$. We will sometimes abuse notation and call $h$ a vertex-coloring model.

Let $(a, B)$ be a vertex-coloring model. If $B$ is positive semidefinite, then $h$ can be taken to be real valued, that is, in view of Theorem 5.2, $p_{a,B}$ is edge-reflection positive. Szegedy [66] moreover observed that for $B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, with $a, b \geq 0$, $p_{a,B}$ is also edge-reflection positive. Clearly, for $a = 0$ and $b = 1$, this matrix is not positive semidefinite. This made him ask the question, which partition of vertex-coloring models are edge-reflection positive (cf. [66, Question 3.2]).

In this chapter we give a complete characterization of edge-reflection positive partition functions of vertex-coloring models over $\mathbb{C}$. Let $h = \sum_{i=1}^n a_i \mathrm{ev}_{u_i}$ for nonzero $a_i$ and distinct vectors $u_i \in \mathbb{C}^k$. We start by giving a simple characterization in terms of the $u_i$ and $a_i$ for $p_h$ to be edge-reflection positive.

**Lemma 7.2.** *Let $u_1, \ldots, u_n \in \mathbb{C}^k$ be distinct vectors, let $a \in (\mathbb{C}^*)^n$ and let $h := \sum_{i=1}^n a_i \mathrm{ev}_{u_i}$. Then $h$ is an edge-coloring model over $\mathbb{R}$ if and only if the set $\{\begin{pmatrix} u_i \\ a_i \end{pmatrix} \mid i = 1, \ldots, n\}$ is closed under complex conjugation.*

*Proof.* Suppose first that the set $\{\begin{pmatrix} u_i \\ a_i \end{pmatrix} \mid i = 1, \ldots, n\}$ is closed under complex conjugation. Then for $p \in R(\mathbb{R})$, $h(p) = \sum_{i=1}^n a_i p(u_i) = \sum_{i=1}^n \overline{a_i p(u_i)} = \overline{h(p)}$. Hence, $h(p) \in \mathbb{R}$. So $h$ is real valued.

Now the 'only if' part. By possibly adding some vectors to $\{u_1, \ldots, u_n\}$ and extending the vector $a$ with zero's, we may assume that $\{u_1, \ldots, u_n\}$ is closed under complex conjugation. We must show that $u_i = \overline{u_j}$ implies $a_i = \overline{a_j}$. We may assume that $u_1 = \overline{u_2}$. Using interpolating polynomials (cf. [17, Lemma 2.9]) we find $p \in R(\mathbb{C})$ such that $p(u_j) = 1$ if $j = 1, 2$ and 0 otherwise. Let $p' := 1/2(p + \overline{p})$. Then $p' \in R(\mathbb{R})$ and consequently, $h(p') = \sum_{i=1}^n a_i p(u_i) = a_1 + a_2 \in \mathbb{R}$. Similarly, there exists $q \in R(\mathbb{C})$ such that $q(u_1) = \mathrm{i}$, $q(u_2) = -\mathrm{i}$ and $q(u_j) = 0$ if $j > 2$. Setting $q' := 1/2(q + \overline{q})$ and applying $h$ to it, we find that $\mathrm{i}(a_1 - a_2) \in \mathbb{R}$. So we conclude that $a_1 = \overline{a_2}$. Continuing this way proves the lemma. $\square$

Lemma 7.2 clearly explains why for $B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, with $a, b \geq 0$, we have that $p_{\mathbb{1},B}$ is edge-reflection positive. Here is another example.

**Example 7.1.**

$$\text{Let } B = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 2 & 0 & 4 & 0 \end{pmatrix} \text{ and let } U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathrm{i} & -\mathrm{i} & \mathrm{i} & -\mathrm{i} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \mathrm{i} & -\mathrm{i} \end{pmatrix}. \tag{7.2}$$

Then $U^T U = B$, and so by Lemma 7.2, $p_{(\mathbb{1},B)}$ is equal to the partition function of a real edge-coloring model.

One might think that Lemma 7.2 already gives the answer to Szegedy's question, but the only thing it says is that if $h := \sum_{i=1}^n a_i \mathrm{ev}_{u_i}$ for certain $a \in (\mathbb{C}^*)^n$ and distinct $u_1, \ldots, u_n \in \mathbb{C}^k$, then it is easy to check whether $h$ is real. In case $h$ is not real valued, it does not rule out the possibility that there is another real-valued edge-coloring model $h'$ (with possible more than $k$ colors) such that $p_h(H) = p_{h'}(H)$ for all graphs $H$. Yet, surprisingly, a certain converse to Lemma 7.2 holds. We need however a few more definitions to state it.

For a $k \times n$ matrix $U$ we denote its columns by $u_1, \ldots, u_n$. By $U^*$ we denote the conjugate transpose of $U$. Let, for any $k$, $(\cdot, \cdot)$ denote the standard bilinear form on $\mathbb{C}^k$. We call the matrix $U$ *nondegenerate* if the span of $u_1, \ldots, u_n$ is nondegenerate with respect to $(\cdot, \cdot)$. In other words, if $\mathrm{rk}\,(U^T U) = \mathrm{rk}\,(U)$. We think of vectors in $\mathbb{C}^k$ as vectors in $\mathbb{C}^l$ for any $l \geq k$. We can now state the main result of this chapter. The proof will be given in Section 7.3.

**Theorem 7.3.** *Let $(a, B)$ be a twin-free $n$-color vertex-coloring model. Let $U$ be a nondegenerate $k \times n$ matrix such that $U^T U = B$. Then the following are equivalent:*

*(i)* $p_{a,B} = p_y$ *for some real edge-coloring model* $y$*,*

*(ii) there exist* $l \geq k$ *and* $g \in O_l(\mathbb{C})$ *such that the set* $\{\begin{pmatrix} gu_i \\ a_i \end{pmatrix} \mid i = 1, \ldots, n\}$ *is closed under complex conjugation,*

*(iii) there exist* $l \geq k$ *and* $g \in O_l(\mathbb{C})$ *such that* $\sum_{i=1}^n a_i \mathrm{ev}_{gu_i}$ *is real.*

*If moreover,* $UU^* \in \mathbb{R}^{k\times k}$*, then we can take* $g$ *equal to the identity in (ii) and (iii).*

Observe that if the set of columns of $gU$ is closed under complex conjugation, then $gU(gU)^*$ is real. So the existence of a nondegenerate matrix $U$ such that $U^T U = B$ and $UU^*$ is real, is a necessary condition for $p_{a,B}$ to be the partition function of an edge-coloring model over $\mathbb{R}$.

In case $B$ is real, there is an easy way to obtain a $k \times n$ rank $k$ matrix $U$, where $k = \mathrm{rk}\,(B)$, such that $UU^* \in \mathbb{R}^{k\times k}$ and $U^T U = B$, using the spectral decomposition of $B$. So by Theorem 7.3, we get the following characterization of partition functions of real vertex-colorings that are partition functions of real edge-coloring models. We will state it as a corollary.

**Corollary 7.4.** *Let $(a, B)$ be a twin-free $n$-color vertex-coloring model over $\mathbb{R}$. Then $p_{a,B} = p_h$ for some real edge-coloring model $h$ if and only if for each $i \in [n]$ there exists $j \in [n]$ such that*

**(i)** $a_i = a_j$,

**(ii)** *for each eigenvector $v$ of $B$ with eigenvalue $\lambda$ :* $\begin{cases} \lambda > 0 & \Rightarrow \quad v_i = v_j, \\ \lambda < 0 & \Rightarrow \quad v_i = -v_j. \end{cases}$

We will now give some examples to illustrate Theorem 7.3 and Corollary 7.4.

**Example 7.2.** Let $G$ be the graph on two nodes $x_1$ and $x_2$ with node weights equal to 1; the loop at $x_1$ has weight 1; the loop at $x_2$ has weight 0 and the edge $x_1x_2$ has weight 1. Then $\hom(H, G)$ is equal to the number of independent sets of $H$. Using Theorem 7.3, it is easy to see that the partition function of any real edge-coloring model can not be equal to $\hom(\cdot, G)$. This can also be easily seen using Theorem 5.2.

**Example 7.3.** For any $n \in \mathbb{N}$ with $n \geq 2$ consider $K_n$, the complete graph on $n$ vertices. Then $\hom(H, K_n)$ is equal to the number of proper $n$-colorings of

$H$. The corresponding vertex-coloring model is $(\mathbb{1}, J - I)$, where $\mathbb{1}$ denotes the all-ones vector, $J$ the all-ones matrix and $I$ the identity matrix. The eigenvalue $-1$ of $J - I$ has multiplicity $n - 1$. Using that the eigenspace corresponding to $-1$ is equal to $\mathbb{1}^{\perp}$, it is easy to see, using Corollary 7.4, that $\hom(\cdot, K_n)$ is equal to the partition function of a real edge-coloring model if and only if $n = 2$. We do not know whether it is easy to deduce this from Theorem 5.2.

In view of Theorem 5.2, Example 7.3 shows that for each $n \geq 3$ there exists $k, t \in \mathbb{N}$, $k$-fragments $F_1, \ldots, F_t$ and $\lambda \in \mathbb{R}^t$ such that $\sum_{i,j=1}^{t} \lambda_i \lambda_j \hom(F_i * F_j, K_n) < 0$. It would be interesting to characterize for which (twin-free) graphs $G$ the invariant $\hom(\cdot, G)$ is edge-reflection positive. By Corollary 7.4, this depends on spectral properties of $G$.

The remainder of this chapter is devoted to proving Theorem 7.3. The proof is based on a well-known generalization of the Hilbert-Mumford criterion, a fundamental result in geometric invariant theory. In the next section we use this criterion to characterize when the $\mathrm{O}_k(\mathbb{C})$-orbit of a vertex-coloring model is closed. In Section 7.3 we then use this result combined with some ideas of Kempf and Ness to give a proof of Theorem 7.3.

## 7.2 Orbits of vertex-coloring models

In this section we will work with a general algebraically closed field $\mathbb{F}$ of characteristic zero. Let $k \in \mathbb{N}$ and let $V$ be a $k$-dimensional vectorspace over $\mathbb{F}$ equipped with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$. Identify $V$ with $\mathbb{F}^k$ through the bilinear form. Let $u_1, \ldots, u_n \in V$ be distinct, and let $a \in (\mathbb{F}^*)^n$. Define the edge-coloring model $h$ by $h := \sum_{i=1}^{n} a_i \mathrm{ev}_{u_i}$. In this section we will consider the $\mathrm{O}_k$-orbit $\mathrm{O}_k h_{\leq e} \subset \mathbb{F}^{\mathbb{N}^k_{\leq e}}$ for $e \in \mathbb{N}$ (recall that $h_{\leq e}$ denotes the restriction of $h$ to the space of polynomials of degree at most $e$); we will characterize in terms of the $u_i$ when this orbit is closed. Our main tool will be a well-known generalization of the Hilbert-Mumford criterion.

### 7.2.1 The one-parameter subgroup criterion

There is a beautiful criterion for closedness of orbits involving *one-parameter subgroups* of $\mathrm{O}_k$, i.e., homomorphisms $\lambda : \mathbb{F}^* \to \mathrm{O}_k$ of algebraic groups. We call a basis $v_1, \ldots, v_k$ of $V$ such that $(v_i, v_j) = \delta_{k+1,i+j}$ for all $i, j$, (i.e. so that the Gram matrix of the basis has zeroes everywhere except ones on the longest anti-diagonal) a *canonical basis*. Let $\lambda : \mathbb{F}^* \to \mathrm{O}_k$ be a one-parameter subgroup. Then there exists a canonical basis $v_1, \ldots, v_k$ of $V$ such that $\lambda(t) v_i = t^{d_i} v_i$ for each $t \in \mathbb{F}^*$, for some integral *weights* $d_1 \geq \cdots \geq d_k$ satisfying $d_i = -d_{k+1-i}$ for all $i$. This follows, for instance, from [25, Section 2.1.2] or [4, §23.4] (ignoring the subtle rationality issues there as $\mathbb{F}$ is algebraically closed) and the fact that all maximal tori are conjugate [4, §11.3]. Conversely, given a canonical basis $v_1, \ldots, v_k$ and such a sequence of $d_i$'s, the homomorphism $\lambda : \mathbb{F}^* \to \mathrm{O}_k$ defined by $\lambda(t)v_i = t^{d_i}v_i$ is a one-parameter subgroup of $\mathrm{O}_k$.

The one-parameter subgroup criterion says the following: let $W$ be a finite-dimensional $\mathrm{O}_k$-module, and let $w \in W$. Consider the orbit $\mathrm{O}_k w \subseteq W$. By Theorem 4.7, the Zariski closure of this orbit contains a unique closed orbit $C$. Then there exists a one-parameter subgroup $\lambda$ such that $\lim_{t\to 0} \lambda(t)w$ exists and is contained in $C$ (the Hilbert-Mumford criterion considers the special case where $C = \{0\}$). Here the existence of the limit by definition means that the morphism $\mathbb{F}^* \to W$, $t \mapsto \lambda(t)w$ extends to $\mathbb{F}$. It then does so in a unique manner, and the value at 0 is declared the limit. Put differently, just like $V$, the $\lambda$-module $W$ decomposes into a direct sum of weight spaces (cf. [25, Lemma 1.6.4]), and the condition is that all components of $w$ in $\lambda$-weight spaces corresponding to negative weights are zero, and the component of $w$ in the zero weight space is the limit. We record the one-parameter subgroup criterion as a theorem.

**Theorem 7.5.** *Let $W$ be a finite dimensional $O_k$-module, let $w \in W$ and let $C$ be the unique closed orbit contained in $\overline{O_k w}$. Then there exists a one-parameter subgroup $\lambda : \mathbb{F}^* \to O_k$ such that the limit $\lim_{t\to 0} \lambda(t)w$ exists and is contained in $C$.*

For a proof of Theorem 7.5 see e.g. [3, Theorem 4.2] or [32, Theorem 1.4].

**Example 7.4.** Recall the edge-coloring model $h$ from Example 6.1 in Section 6.1, $h \in k[x_1, x_2]^*$ is zero on all polynomials of degree at least 2. The restriction of $h$ to the space of polynomials of degree at most 1 is an element of $(V^*)^* = V$, namely, equal to $v_1 := e_1 + \mathrm{i}e_2$. This is an isotropic vector relative to the bilinear form, and so is its complex conjugate $v_2 := e_1 - \mathrm{i}e_2$. So the sequence $1/\sqrt{2}v_1, 1/\sqrt{2}v_2$ forms a canonical basis of $V$. The linear map $V \to V$ scaling $v_1$ with $t \in \mathbb{F}$ and $v_2$ with $t^{-1}$ is an element of the orthogonal group. Explicitly, this gives the one-parameter subgroup

$$\lambda(t) = \frac{1}{2t} \begin{bmatrix} 1+t^2 & \mathrm{i}-\mathrm{i}t^2 \\ -\mathrm{i}+\mathrm{i}t^2 & 1+t^2 \end{bmatrix} \in \mathrm{O}_2 \tag{7.3}$$

with the property that $\lim_{t\to 0} \lambda(t)h_{\leq e} = 0$ for all $e$.

### 7.2.2 Application to vertex-coloring models

Here we will use the one-parameter subgroup criterion to characterize when the $O_k$-orbit of $h_{\leq e}$ is closed.

We will need the following well-known result.

**Proposition 7.6.** *Let $u_1, \ldots, u_n \in V$ be nonzero. If $w_1, \ldots, w_n \in V$ are nonzero vectors such that*

$$(u_i, u_j) = (w_i, w_j) \text{ for all } i, j = 1, \ldots, n, \tag{7.4}$$

*then there exists $g \in O_k$ such that $gu_i = v_i$ for all $i \in [n]$.*

For completeness we will sketch the proof.

*Proof.* Let $U$ denote the span of the $u_i$ and $W$ the span of the $w_i$. If $U = V$, we can just define a linear map $g : V \to V$ by $u_i \mapsto w_i$ for each $i$. It is easy to see that $g$ is well defined and that $g$ preserves the bilinear form, that is $g \in O_k$.

Next, if the bilinear form restricted to $U$ is nondegenerate, then we can reduce to the previous case by adding an ortonormal basis for $U^\perp$ to $\{u_1, \ldots, u_n\}$ and an orthonormal basis for $W^\perp$ to $\{w_1, \ldots, w_n\}$.

Finally, if $U$ is degenerate we can find $i \in [n]$ such that $(u_i, u_j) = 0$ for all $j \in [n]$. Let $U' \subset U$ and $W' \subset W$ be complements to $u_i$ and $w_i$ respectively. Then we may choose $u \in U'^\perp$ such that $(u_i, u) = 1$ and such that $(u, u) = 0$ (cf. [36, XV, §9]). Similarly, we may choose $w \in W'^\perp$ such that $(w_i, w) = 1$ and such that $(w, w) = 0$. Now add $u$ to $\{u_1, \ldots, u_n\}$ and $w$ to $\{w_1, \ldots, w_n\}$ and note that the dimension of $U$ (and of $W$) increases by one. Now just proceed until $U$ becomes nondegenerate so that we can reduce to the previous case. □

**Theorem 7.7.** *Let $\mathbb{F} = \overline{\mathbb{F}}$, let $u_1, \ldots, u_n \in V$ be distinct and let $a \in (\mathbb{F}^*)^n$. Let $h := \sum_{i=1}^n a_i \mathrm{ev}_{u_i}$ and let $e \geq 3n$. Then the orbit $O_k h_{\leq e}$ is closed if and only if the restriction of the bilinear form to the span of the $u_i$ is nondegenerate.*

*Proof.* Let $U \subset V$ denote the space spanned by the $u_i$. Suppose first that the bilinear form restricted to $U$ is degenerate. Then we may assume that $(u_1, u_i)$ is 0 for all $i \in [n]$. Define $h' = \sum_{i=2}^n a_i \mathrm{ev}_{u_i}$. By Proposition 7.6, there exists for each $\varepsilon > 0$ , $g \in O_k$ such that $gu_1 = \varepsilon u_1$ and $gu_i = u_i$ for $i \geq 2$. This implies that $h'_{\leq e}$ is contained in the closure of the orbit of $h_{\leq e}$. We will now show that $h'_{\leq e}$ is not contained in the orbit $h_{\leq e}$.

Let $I(h) \subset R$ be the set of polynomials $p$ of degree at most $n$ such that $h(pq) = 0$ for all polynomials $q$ of degree at most $n - 1$. Then

$$I(h) = \{p \in R \mid \deg(p) \leq n, p(u_i) = 0 \text{ for } i = 1, \ldots, n\}. \tag{7.5}$$

The inclusion '$\supseteq$' is clear. To see the other inclusion, let $p_1, \ldots, p_n$ be interpolating polynomials at the $u_i$, i.e., the $p_i$ are polynomials of degree $n-1$ such that $p_i(u_j) = \delta_{i,j}$ for all $i, j = 1, \ldots, n$ (cf. [17, Lemma 2.9]). Then for a polynomial $p$ of degree at most $n$ we have that $\deg(pp_i) \leq 2n-1 \leq e$ and $h(pp_i) = 0$ if and only if $p(u_i) = 0$. This shows (7.5), which in turn implies

$$\{u \in V \mid p(u) = 0 \text{ for all } p \in I(h)\} = \{u_1, \ldots, u_n\}. \tag{7.6}$$

But since the $u_i$ are distinct, (7.6) applied to $h'$ implies that $gh_{\leq e} \neq h'_{\leq e}$ for any $g \in \mathrm{O}_k$, showing that the orbit of $h_{\leq e}$ is not closed.

For the converse, assume that the bilinear form restricted to $U$ is nondegenerate. We will prove that the $\mathrm{O}_k$-orbit of $h_{\leq e}$ is closed. Let $\lambda : \mathbb{F}^* \to \mathrm{O}_k$ be a one-parameter subgroup such that $\lim_{t\to 0} \lambda(t)h_{\leq e}$ exists. We will show that it lies in the orbit of $h_{\leq e}$. Let $v_1, \ldots, v_k$ be a canonical basis of $V$ with $\lambda(t)v_j = t^{d_j}v_j$ for weights $d_1 \geq \cdots \geq d_k$. Let $x_1, \ldots, x_k$ be the basis of $V^*$ dual to $v_1, \ldots, v_k$. For any monomial $x^\alpha, \alpha \in \mathbb{N}^k$, we have

$$(\lambda(t)h)(x^\alpha) = h(\lambda(t)^{-1}x^\alpha) = h(t^{\alpha_1 d_1 + \ldots + \alpha_k d_k}x^\alpha) = t^{\alpha\cdot d}\sum_{i=1}^{n} a_i x^\alpha(u_i), \tag{7.7}$$

where $\alpha \cdot d := \alpha_1 d_1 + \ldots + \alpha_k d_k$. By assumption, if $x^\alpha$ is a monomial of degree at most $e$, the limit for $t \to 0$ in (7.7) exists. Note that this implies for $|\alpha| \leq e$:

$$\alpha \cdot d < 0 \Rightarrow h(x^\alpha) = \sum_{i=1}^{n} a_i x^\alpha(u_i) = 0. \tag{7.8}$$

In what follows, we exclude the trivial cases where $k = 0$ and where $k = 1$ and $u_1$ is the zero vector; in both of these cases the orbit of $h$ is just a single point.

Next let $b \in \{1, \ldots, k\}$ be the maximal index with $x_b(U) \neq \{0\}$, and order the $u_i$ such that $x_b(u_1), \ldots, x_b(u_l) \neq 0$ $(l > 0)$ and $x_b(u_{l+1}), \ldots, x_b(u_n) = 0$. By maximality of $b$, $U$ is contained in the span of $v_1, \ldots, v_b$. So if $d_b$ is nonnegative, then $\lim_{t\to 0} \lambda(t)(u_1, \ldots, u_n)$ exists, and is by Proposition 7.6 contained in the orbit of $(u_1, \ldots, u_n)$. (Since $U$ is nondegenerate, the equations describing the orbit are given by (7.4).) Then also $h_{\leq e}$ and $\lim_{t\to 0} \lambda(t)h_{\leq e}$ lie in the same orbit. So we may assume that $d_b < 0$. (In particular, $b > k/2$.)

Since the coordinates $x_{b+1}, \ldots, x_k$ vanish identically on $U$, it follows that $U$ is contained in the subspace of $V$ perpendicular to $v_1, \ldots, v_{k-b}$. As $U$ is nondegenerate, it does not contain a nonzero linear combination of $v_1, \ldots, v_{k-b}$. This means, in particular, that the coordinates $x_{k-b+1}, \ldots, x_b$ together separate the points $u_1, \ldots, u_l$. Then so do the monomials $x_{k-b+1}x_b^2, \ldots, x_{b-1}x_b^2, x_b^3$. Note that the dot product $\alpha \cdot d$ is negative for each of these (e.g., for the first, it equals $d_{k-b+1} + 2d_b = d_b < 0$ and from there the dot product decreases weakly to the right). It follows that there exists a linear combination $p$ of those cubic monomials for which $p(u_1), \ldots, p(u_l)$ are distinct and nonzero. Then, by (7.8) and the fact that $p(u_{l+1}) = \cdots = p(u_n) = 0$, the vector $(a_1, \ldots, a_l)^T$ is in the kernel of the Vandermonde matrix

$$\begin{bmatrix} p(u_1) & \ldots & p(u_l) \\ p(u_1)^2 & \ldots & p(u_l)^2 \\ \vdots & & \vdots \\ p(u_1)^l & \ldots & p(u_l)^l \end{bmatrix}, \tag{7.9}$$

since the degree of $p^l$ is $3l \le e$. This implies that $a_1, \ldots, a_l$ are all zero, contrary to the assumption that all $a_i$ are nonzero. This proves that the orbit of $h_{\le e}$ is closed for $e \ge 3n$. $\square$

## 7.3 Proof of Theorem 7.3

In this section we give a proof of Theorem 7.3 using Theorem 7.7. We first need some preparations.

Let $W \in \mathbb{C}^{l \times n}$ be any matrix and consider the function $f_W : \mathrm{O}_l(\mathbb{C}) \to \mathbb{R}$ defined by

$$g \mapsto \mathrm{tr}(W^* g^* g W) = \mathrm{tr}((gW)^* gW), \tag{7.10}$$

for $g \in \mathrm{O}_l(\mathbb{C})$, where $\mathrm{tr}(M)$ denotes the trace of a matrix $M$ and $M^*$ the conjugate transpose of $M$. This function was introduced by Kempf and Ness [33] in the context of connected reductive linear algebraic groups acting on finite dimensional vector spaces. Note that $f_W$ is left-invariant under $\mathrm{O}_l(\mathbb{R})$ and right-invariant under $\mathrm{Stab}(W) := \{g \in \mathrm{O}_l(\mathbb{C}) \mid gW = W\}$. Let $e \in \mathrm{O}_l(\mathbb{C})$ denote the identity. We are interested in the situation that the infimum of $f_W$ over $\mathrm{O}_l(\mathbb{C})$ is equal to $f_W(e)$.

**Lemma 7.8.** *The function $f_W$ has the following properties:*

**(i)** $\inf_{g \in O_l(\mathbb{C})} f_W(g) = f_W(e)$ *if and only if* $WW^* \in \mathbb{R}^{l \times l}$,

**(ii)** *If* $WW^* \in \mathbb{R}^{l \times l}$, *then* $f_W(e) = f_W(g)$ *if and only if* $g \in O_l(\mathbb{R}) \cdot \mathrm{Stab}(W)$.

*Proof.* We start by showing that

$$f_W \text{ has a critical point at } e \text{ if and only if } WW^* \in \mathbb{R}^{l \times l}. \tag{7.11}$$

By definition, a critical point of $f_W$ is a point $g \in \mathrm{O}_l(\mathbb{C})$ such that $(Df_W)_g(X) = 0$ for all $X \in T_g(\mathrm{O}_l(\mathbb{C}))$, where $T_g(\mathrm{O}_l(\mathbb{C}))$ is the tangent space of $\mathrm{O}_l(\mathbb{C})$ at $g$ and where $(Df_W)_g$ is the derivative of $f_W$ at $g$. It is well known that the tangent space of $\mathrm{O}_l(\mathbb{C})$ at $e$ is the space of skew-symmetric matrices, i.e., $T_e(\mathrm{O}_l(\mathbb{C})) = \{X \in \mathbb{C}^{l\times l} \mid X^T + X = 0\}$. It is easy to see that the derivative of $f_W$ at $e$ is the $\mathbb{R}$-linear map $(Df_W)_e \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^{l\times l}, \mathbb{R})$ defined by $Z \mapsto \mathrm{tr}(W^*(Z + Z^*)W)$. Now let $Z$ be skew-symmetric and write $Z = X + \mathrm{i}Y$, with $X, Y \in \mathbb{R}^{l\times l}$. Note that $Z$ is skew-symmetric if and only if both $X$ and $Y$ are skew-symmetric. Write $W = V + \mathrm{i}T$ with $V, T \in \mathbb{R}^{l\times l}$. Then $(Df_W)_e(Z)$ is equal to

$$\begin{aligned} & \mathrm{tr}((V^T - \mathrm{i}T^T)(X + \mathrm{i}Y + X^T - \mathrm{i}Y^T)(V + \mathrm{i}T)) \\ = \; & 2\mathrm{tr}((V^T - \mathrm{i}T^T)\mathrm{i}Y(V + \mathrm{i}T)) \\ = \; & 2\mathrm{tr}(T^TYV) - 2\mathrm{tr}(V^TYT) = 4\mathrm{tr}(T^TYV), \end{aligned} \tag{7.12}$$

where we use that $X$ and $Y$ are skew symmetric, and standard properties of the trace. So $Df_e(Z) = 0$ for all skew symmetric $Z$ if and only if $VT^T = TV^T$. That is, if and only if $WW^* \in \mathbb{R}^{l\times l}$. This shows (7.11).

By a result of Kempf and Ness (cf. [33, Theorem 0.1]) we can now conclude that (i) and (ii) hold. However, we will give an independent and elementary proof.

First the proof of (i). Note that (7.11) immediately implies that $f_W$ does not attain a minimum at $e$ if $WW^* \notin \mathbb{R}^{l\times l}$. (This follows easily from the method of Lagrange multipliers.) Conversely, suppose $WW^* \in \mathbb{R}^{l\times l}$. Since $WW^*$ is real and positive semidefinite there exists a real matrix $V$ such that $WW^* = VV^T$. Now note that, by the cyclic property of the trace, $f_W(g) = \mathrm{tr}(g^*gWW^*)$. So we have $f_W = f_V$. Let $I$ denote the identity matrix. Take any $g = X + \mathrm{i}Y \in \mathrm{O}_l(\mathbb{C})$, where $X, Y \in \mathbb{R}^{l\times l}$. Using that $X^TX - Y^TY = I$, and the fact that $f_W$ is real valued, we find that

$$\begin{aligned} f_W(g) &= \mathrm{tr}((X^TX + Y^TY)VV^T) = \mathrm{tr}(VV^T) + 2\mathrm{tr}(Y^TYVV^T) \\ &= 2\mathrm{tr}(VV^T) + \mathrm{tr}(YV(YV)^T) \geq \mathrm{tr}(VV^T) = f_W(e). \end{aligned} \tag{7.13}$$

This proves (i).

Next, suppose that $f_W(g) = f_W(e)$ for some $g \in \mathrm{O}_l(\mathbb{C})$. Again, since $WW^*$ is real and positive semidefinite there exists a real matrix $V$ such that $WW^* = VV^T$. Moreover, the span of the columns of $V$ is equal to the span of the columns of $W$. This implies that $\mathrm{Stab}(V) = \mathrm{Stab}(W)$. Now write $g = X + \mathrm{i}Y$, with $X, Y \in \mathbb{R}^{l\times l}$. As, by (7.13), $f_W(g) = f_W(e)$ if and only if $YV = 0$, it follows that $gV = XV + \mathrm{i}YV = XV$ is a real matrix. Let $v_1, \ldots, v_n$ be the columns of $V$. Then, since by definition of the orthogonal group, $(gv_i, gv_j) = (v_i, v_j)$ for all $i, j$, and since the $gv_i$ and the $v_i$ are real, there exists $g_1 \in \mathrm{O}_l(\mathbb{R})$ such that $g_1 g V = V$. This implies that $g \in \mathrm{O}_l(\mathbb{R}) \cdot \mathrm{Stab}(V)$. This finishes the proof of (ii). ☐

For any $e$, let $\langle \cdot, \cdot \rangle$ denote the Hermitian inner product on $\mathbb{C}^{\mathbb{N}^l_{\leq e}}$ induced by the standard Hermitian inner product on $\bigoplus_{i=1}^{e} (\mathbb{C}^l)^{\otimes i}$, by viewing elements of $\mathbb{C}^{\mathbb{N}^l_{\leq e}}$ as symmetric tensors. The next proposition has as conclusion a special case of Theorem 0.2 in [33].

**Proposition 7.9.** *Let $h$ be any $l$-color edge-coloring model. Let $C_e$ be the unique closed orbit in $\overline{O_l(\mathbb{C})h_{\leq e}}$. Then there exists $h'_{\leq e} \in C_e$ such that*

$$\inf_{g \in O_l(\mathbb{C})} \langle g h_{\leq e}, g h_{\leq e} \rangle \geq \langle h'_{\leq e}, h'_{\leq} e \rangle. \tag{7.14}$$

*Moreover, the infimum is attained if and only if $h_{\leq e} \in C_e$.*

*Proof.* Clearly, the infimum is attained at some $g \in \mathrm{O}_l(\mathbb{C})$ if $h_{\leq e} \in C_e$. So we can take $h' = gh$.

Now assume that $h_{\leq e} \notin C_e$. Fix any $g \in \mathrm{O}_l(\mathbb{C})$, write $y := gh$ and, as in the proof of Theorem 7.7, let $\lambda : \mathbb{C}^* \to \mathrm{O}_l(\mathbb{C})$ be a one-parameter subgroup such that $\lim_{t \to 0} \lambda(t) y_{\leq e} = y'_{\leq e} \in C_e$. Let $v_1, \ldots, v_l$ be a canonical basis of $\mathbb{C}^l$ with $\lambda(t) v_j = t^{d_j} v_j$ for weights $d_1 \geq \cdots \geq d_l$. Let $x_1, \ldots, x_l$ be the basis of $(\mathbb{C}^l)^*$ dual to $v_1, \ldots, v_l$. Recall from (7.7) that for any monomial $x^\alpha, \alpha \in \mathbb{N}^l$, we have $(\lambda(t)y)(x^\alpha) = t^{\alpha \cdot d} y(x^\alpha)$. Since, by assumption, the limit $\lim_{t \to 0} t^{\alpha \cdot d} y(x^\alpha)$ exists for $|\alpha| \leq e$, this implies:

$$y'_{\leq e}(x^\alpha) = \begin{cases} 0 (= y(x^\alpha)) & \text{if } \alpha \cdot d < 0, \\ y(x^\alpha) & \text{if } \alpha \cdot d = 0, \\ 0 & \text{if } \alpha \cdot d > 0. \end{cases} \tag{7.15}$$

For $e' \leq e$ and $\phi : [e'] \to [l]$ let $\phi \cdot d := \alpha \cdot d$, for $\alpha \in \mathbb{N}^l$ such that $x^\phi = x_{\phi(1)} \cdots x_{\phi(l)} = x^\alpha$. Then, as $y_{\leq e} \neq y'_{\leq e}$, by (7.15),

$$\langle y_{\leq e}, y_{\leq e} \rangle = \sum_{\substack{e'=0,\ldots,e \\ \phi:[e'] \to [l] \\ \phi \cdot d \geq 0}} y(x^\phi) \overline{y(x^\phi)} > \sum_{\substack{e'=0,\ldots,e \\ \phi:[e'] \to [l] \\ \phi \cdot d = 0}} y(x^\phi) \overline{y(x^\phi)} = \langle y'_{\leq e}, y'_{\leq e} \rangle. \tag{7.16}$$

So for each $g \in \mathrm{O}_l(\mathbb{C})$ we can find $y'_{\leq e} \in C_e$ such that $\langle g h_{\leq e}, g h_{\leq e} \rangle > \langle y'_{\leq e}, y'_{\leq e} \rangle$, proving the first statement. This moreover implies that the infimum is not attained if $h_{\leq e} \notin C_e$, finishing the proof. ☐

We need one more lemma before we can prove Theorem 7.3.

**Lemma 7.10.** *Let $h := \sum_{i=1}^{n} a_i \mathrm{ev}_{u_i} \in R(\mathbb{C})^*$, with $a \in (\mathbb{C}^*)^n$ and $u_1, \ldots, u_n \in \mathbb{C}^k$ distinct. Suppose the bilinear form restricted to the span of the $u_i$ is nondegenerate. If $y$ is a real $l$-color edge-coloring model such that $p_h(H) = p_y(H)$ for all $H \in \mathcal{G}$, then there exists $g \in O_l(\mathbb{C})$ such that $gh = y$.*

*Proof.* We may assume that $l \geq k$. Recall that in case $l > k$ we add colors to $h$. This is done by appending the $u_i$'s with zero's. Note that the bilinear form restricted to the span of the $u_i$ remains nondegenerate. Then, by Theorem 7.7, for each $d \geq 3n$, the orbit $\mathrm{O}_l h_{\leq d}$ is equal to the unique closed orbit $C_d$. We will now show that the orbit of $y_{\leq d}$ is also equal to $C_d$ for any $d$.

For any $e \leq d$, $\mathrm{O}_l(\mathbb{C})$ embeds naturally into $\mathrm{O}_{l^e}(\mathbb{C})$. Let $g \in \mathrm{O}_{l^e}(\mathbb{C})$, and write $g = X + \mathrm{i}Y$, with $X, Y \in \mathbb{R}^{l^e \times l^e}$. Then, using that $X^T X - Y^T Y = I$,

$$\begin{aligned} \langle g y_e, g y_e \rangle &= \langle X y_e, X y_e \rangle + \langle Y y_e, Y y_e \rangle \\ &= \langle y_e, y_e \rangle + 2\langle Y y_e, Y y_e \rangle \geq \langle y_e, y_e \rangle. \end{aligned} \tag{7.17}$$

As this holds for any $e \leq d$, we can now conclude by Proposition 7.9 that the orbit of $y_{\leq d}$ is closed.

We now claim that this implies that there exists $g \in \mathrm{O}_l(\mathbb{C})$ such that $gh = y$. Indeed, since $\mathrm{Stab}(y_{\leq d}) = \cap_{d' \leq d} \mathrm{Stab}(y_{\leq d'})$ and since $\mathrm{O}_l(\mathbb{C})$ is Noetherian, there exists $d_1 \geq 3n$ such that $\mathrm{Stab}(y_{\leq d_1}) = \cap_{d \in \mathbb{N}} \mathrm{Stab}(y_{\leq d})$. Recall that we have a canonical bijection from $\mathrm{O}_l(\mathbb{C})/\mathrm{Stab}(y_{\leq d})$ to $C_d$ given by

$$g\mathrm{Stab}(y_{\leq d}) \mapsto g y_{\leq d} \tag{7.18}$$

(cf. the proof of Theorem 6.11). This implies that for any $d \geq d_1$, if $g \in \mathrm{O}_l(\mathbb{C})$ is such that $g y_{\leq d} = h_{\leq d}$, then also $gy = h$. This proves the lemma. □

Now we can give a proof of Theorem 7.3.

**Theorem 7.3.** *Let $(a, B)$ be a twin-free $n$-color vertex-coloring model. Let $U$ be a nondegenerate $k \times n$ matrix such that $U^T U = B$. Then the following are equivalent:*

- *(i) $p_{a,B} = p_y$ for some real edge-coloring model $y$,*
- *(ii) there exist $l \geq k$ and $g \in O_l(\mathbb{C})$ such that the set $\{\begin{pmatrix} g u_i \\ a_i \end{pmatrix} \mid i = 1, \ldots, n\}$ is closed under complex conjugation,*
- *(iii) there exist $l \geq k$ and $g \in O_l(\mathbb{C})$ such that $\sum_{i=1}^{n} a_i \mathrm{ev}_{g u_i}$ is real.*

*If moreover, $UU^* \in \mathbb{R}^{k \times k}$, then we can take $g$ equal to the identity in (ii) and (iii).*

*Proof.* Observe that since $(a, B)$ is twin free, the columns of $U$ are distinct. Lemma 7.2 implies the equivalence of (ii) and (iii) for the same $g$ and $l$ in (ii) and (iii). Moreover, since $(gU)^T gU = U^T g^T gU = U^T U = B$, for any $g \in \mathrm{O}_l(\mathbb{C})$, Lemma 7.1 shows that (iii) implies (i).

Let $u_1, \ldots, u_n$ be the columns of $U$ and let $h := \sum_{i=1}^n a_i \mathrm{ev}_{u_i}$. We will now prove that (i) implies (iii). Let $y$ be a real $l$-color edge-coloring model such that $p_{a,B} = p_y$. Since $U$ is nondegenerate, we may assume, by Lemma 7.10, that $y = gh$ for some $g \in O_l(\mathbb{C})$. Now note that $gh = \sum_{i=1}^n a_i \mathrm{ev}_{gu_i}$. This shows that (i) implies (iii).

Now assume that $UU^* \in \mathbb{R}^{k \times k}$. We will show that (i) implies (iii) with $g = e$. Let $y$ be a real $l$-color edge-coloring model such that $p_{a,B} = p_y$. Just as above, we may assume that $y = \sum_{i=1}^n a_i \mathrm{ev}_{gu_i}$, for some $g \in \mathrm{O}_l(\mathbb{C})$. Lemma 7.2 implies that the set $\{gu_i\}$ is closed under complex conjugation. This implies that $gU(gU)^* \in \mathbb{R}^{l \times l}$. So by Lemma 7.8 (i) the infimum of $f_{gU}$ is attained at $e$. Equivalently, the infimum of $f_U$ is attained at $g$. Since $UU^* \in \mathbb{R}^{k \times k}$, this implies, by Lemma 7.8 (ii), that $g \in \mathrm{O}_l(\mathbb{R}) \cdot \mathrm{Stab}(U)$. Hence $g = g_1 \cdot s$ for some $g_1 \in \mathrm{O}_l(\mathbb{R})$ and $s \in \mathrm{Stab}(U)$. Now note that since $sh = h$ we have that $h = g_1^{-1} y$ and hence $h$ is real. $\square$

# Chapter 8

# Compact orbit spaces in Hilbert spaces and limits of edge-coloring models

We prove an abstract theorem about compact orbit spaces in Hilbert spaces. As a consequence we derive the existence of limits of certain sequences of edge-coloring models.

This chapter is based on joint work with Lex Schrijver [55].

## 8.1 Introduction

In [45] (which was awarded the Fulkerson prize in 2012) Lovász and Szegedy develop a theory of limits of dense graphs (here dense means that the number of edges is proportional to the number of vertices squared). The theory of graph limits has many connections to other areas of discrete mathematics, computer science and statistical mechanics. We refer to the book by Lovász [40] for details and references.

We shall now describe one of the main results from [45], but first we need to introduce a few definitions. For two simple graphs $H$ and $G$, we define the *homomorphism density* of $H$ in $G$ by

$$t(H,G) := p_{\mathbb{1}/n,B}(H) = \frac{1}{n^{|V(H)|}} \hom(H,G), \tag{8.1}$$

where $n$ is the number of nodes of $G$, $B$ is the adjacency matrix of $G$ and $\mathbb{1}/n$

denotes the vector with all entries equal to $1/n$. Then $t(H, G)$ is the probability that a random map from $V(H)$ to $V(G)$ is homomorphism. Central in the theory of graph limits is the following definition. A sequence $(G_n)$ of simple graphs is called *convergent* if for each simple graph $H$, $(t(H, G_n))$ is a convergent sequence of real numbers.

The main result in [45] is the discovery of a natural limit object for a convergent sequence of graphs, which we will now describe. A *graphon* is a symmetric Lebesgue measurable function $W : [0,1]^2 \to [0,1]$. For a graphon $W$ and a graph $H = ([k], E)$ define $t(H, W)$ by

$$t(H, W) := \int_{[0,1]^k} \prod_{ij \in E} W(x_i, x_j) dx_1 \cdots dx_k. \tag{8.2}$$

In the context of de la Harpe and Jones [28], we may view $t(H, W)$ as the *partition function* of $W$.

We can view a simple graph $G = ([n], E)$ as a $\{0,1\}$-valued graphon $W_G$ by scaling its adjacency matrix, i.e.,

$$W_H(x, y) := \begin{cases} 1 \text{ if } (\lceil nx \rceil, \lceil ny \rceil) \in E, \\ 0 \text{ otherwise.} \end{cases} \tag{8.3}$$

Then $t(H, G) = t(H, W_G)$ for each simple graph $H$. So (8.2) generalizes (8.1). Lovász and Szegedy [45] showed that graphons are natural limit objects of convergent graph sequences in the following sense.

**Theorem 8.1** (Lovász and Szegedy [45]). *Let $(G_n)$ be a convergent sequence of simple graphs. Then there exists a graphon $W$ such that* $\lim_{n\to\infty} t(H, G_n) = t(H, W)$ *for each simple graph $H$.*

We can view Theorem 8.1 as describing limit objects for certain convergent sequences of vertex-coloring models. From that perspective, the following definition is natural. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A sequence $(h^n)$ of edge-coloring models over $\mathbb{F}$ is called *convergent* if for each simple graph $H$, $(p_{h^n}(H))$ is a convergent sequence in $\mathbb{F}$.

If we would allow all graphs in this definition, and if the number of colors of each $h^n$ is bounded, by $k$ say, then its is easy to see by Theorem 5.2 in the real case, and by Theorem 5.3 in the complex case, that there exists a $k$-color edge coloring model $h$ such that $\lim_{n\to\infty} p_{h^n}(H) = p_h(H)$ for all graphs $H$. However, if the number of colors grows we can not represent the limit parameter as the partition function of an ordinary edge-coloring model, as the following example shows.

**Example 8.1.** Consider for $n \in \mathbb{N}$ the edge-coloring model: $h^n \in \mathbb{R}[x_1, \ldots, x_n]^*$ defined by $x^\alpha \mapsto 1$ if $x^\alpha = x_i^i$ for $i \in [n]$ and $x^\alpha \mapsto 0$ otherwise. Then $(h^n)$ is convergent. Indeed, $p_{h^n}(H) = 1$ if $H$ is the disjoint union of regular graphs of degree at most $n$ and 0 otherwise, implying that $\lim_{n\to\infty} p_{h^n}(H) = 1$ if $H$ is the disjoint union of regular graphs and 0 otherwise. Let $f$ denote the limit graph parameter. Then $f$ is not the partition function of any $k$-color edge-coloring model, for any $k \in \mathbb{N}$.

Indeed, let $k \in \mathbb{N}$ and let for $i = 1, \ldots k+1$, $H_i$ be an $i$-regular graph. Fix for each $i$ an edge $u_i v_i$ from $H_i$ and let $H_i'$ be the graph where this edge is removed. Let $H$ be the disjoint union of the $H_i'$. Define $s : \{u_1, \ldots, u_{k+1}\} \to V(H)$ by $s(u_i) = v_i$ for $i = 1, \ldots, k+1$. Now note that

$$\sum_{\pi \in S_{k+1}} \operatorname{sgn}(\pi) f(H_{s\circ\pi}) = f(H_s) = 1. \tag{8.4}$$

So by Theorem 5.3, it follows that $f$ is not the partition function of any $k$-color edge-coloring model over $\mathbb{C}$ (neither over any algebraically closed field of characteristic zero).

The limit graph parameter $f$ can be described as the partition function of $h \in \mathbb{R}[x_1, x_2 \ldots]^* \to \mathbb{R}$ defined by $h(x^\alpha) = 1$ if $x^\alpha = x_i^i$ for $i \in \mathbb{N}$ and $h(x^\alpha) = 0$ otherwise.

We shall show that under some boundedness conditions there exists a natural limit object for each convergent sequence of edge-coloring models $(h^n)$, which, as in the example above, is an infinite color edge-coloring model, just as a graphon can be considered as a vertex-coloring model with an (uncountably) infinite number of states. This answers a question posed by Lovász [39] and also, in a slightly different form, by Kannan [31].

To to do so, we state in the next section an abstract theorem about compact orbit spaces in Hilbert spaces (cf. Theorem 8.2), which generalizes a result from Lovász and Szegedy [46] and as such it allows to show Theorem 8.1. Moreover, it allows to construct limit objects for certain convergent sequences of edge-coloring models.

## 8.2 Compact orbit spaces in Hilbert spaces and applications

We state a theorem on compact orbit space in Hilbert spaces. In [55] this is done for real Hilbert spaces only. It is straightforward to extend the results to complex Hilbert spaces, which we will do here. We moreover show how the theorem applies to limits of both graphs and edge-coloring models. Throughout this section $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

### 8.2.1 Compact orbit spaces in Hilbert spaces

We start with a few definitions. Let $\mathcal{H}$ be a (complex or real) Hilbert space, i.e., $\mathcal{H}$ is a linear space equipped with an inner product $\langle \cdot, \cdot \rangle$, (which is linear in the first argument and conjugate linear in the second argument) such that $\mathcal{H}$ is complete with respect to the norm topology induced by the inner product. We denote the 2-norm of $x \in \mathcal{H}$ by $\|x\|$, where $\|x\| := \sqrt{\langle x, x \rangle}$, and the Hilbert metric is denoted by $d_2$, where $d_2(x,y) := \|x - y\|$ for $x, y \in \mathcal{H}$. By $B(\mathcal{H})$ we denote the closed unit ball in $\mathcal{H}$.

For a bounded subset $R \subset \mathcal{H}$ we define a seminorm $\| \cdot \|_R$ and a pseudometric[1] $d_R$ on $\mathcal{H}$ by for $x, y \in \mathcal{H}$:

$$\|x\|_R := \sup_{r \in R} |\langle x, r \rangle| \quad \text{and} \quad d_R(x,y) := \|x - y\|_R. \tag{8.5}$$

We use the topology induced by this pseudometric only if we explicitly mention it, otherwise we use the topology induced by the ordinary Hilbert norm. Note that if $R \subseteq B(\mathcal{H})$, then, by Cauchy-Schwarz, $d_R(x,y) \leq d_2(x,y)$ for any $x, y \in \mathcal{H}$.

A subset $\mathcal{W}$ of $\mathcal{H}$ is called *weakly compact* if it is compact in the weak topology on $\mathcal{H}$. (A set $U$ is open in the weak topology if for each $u \in U$, there exist $n \in \mathbb{N}$, $y_i \in \mathcal{H}$ and $\varepsilon_i > 0$ for $i = 1, \ldots, n$ such that $U$ contains $\bigcap_{i=1}^n \{x \in \mathcal{H} \mid |\langle u - x, y_i \rangle| < \epsilon_i\}$.) By the Banach-Alaoglu Theorem (cf. [15, Theorem V.3.1] and the Principle of Uniform Boundedness (cf. [15, Theorem III.14.1]), for any $\mathcal{W} \subseteq \mathcal{H}$:

$$\begin{aligned} \mathcal{W} \text{ closed, bounded and convex} &\Rightarrow \mathcal{W} \text{ weakly compact} \\ \mathcal{W} \text{ weakly compact} &\Rightarrow \mathcal{W} \text{ bounded.} \end{aligned} \tag{8.6}$$

Let $G$ be a group acting on a topological space $X$. The *orbit space* $X/G$ is the quotient space of $X$ taking the orbits of $G$ as classes. We can now state our result on compact orbit space in Hilbert spaces.

[1] A *seminom* is a norm except that nonzero elements may have norm 0. A pseudometric is a metric except that distinct points may have distance 0. One can turn a pseudometric space into a metric space by identifying points at distance 0, but for our purposes it is notationally easier and sufficient to maintain the original space. Notions like convergence pass easily over to pseudometric spaces, but limits need not be unique.

**Theorem 8.2.** *Let $\mathcal{H}$ be a Hilbert space and let G be a group of unitary transformations of $\mathcal{H}$. Let $\mathcal{W}$ and R be G-stable subsets of $\mathcal{H}$, with $\mathcal{W}$ weakly compact and $R^k/G$ compact for each $k \in \mathbb{N}$. Then $(\mathcal{W}, d_R)/G$ is compact.*

We postpone the proof of Theorem 8.2 to Section 8.3. Schrijver [63] found a nice application of it to low-rank approximation of polynomials. We will not describe this here. We will now show some applications of it to limits of both graphs and edge-coloring models.

### 8.2.2 Application of Theorem 8.2 to graph limits

Here we will show how Theorem 8.2 can be used to prove Theorem 8.1. In this subsection, measures are Lebesgue measure.

Let $\mathcal{H} := L^2([0,1]^2)$, the Hilbert space of all square integrable functions $[0,1]^2 \to \mathbb{R}$. Let $R$ be the collection of functions $\chi^A \times \chi^B$, where $A, B$ are measurable subsets of $[0,1]$ and where $\chi^A$ and $\chi^B$ denote the incidence functions of $A$ and $B$ respectively. Let $S_{[0,1]}$ be the group of measure space automorphisms of $[0,1]$. The group $S_{[0,1]}$ act naturally on $\mathcal{H}$ by $\pi W(x,y) = W(\pi^{-1}x, \pi^{-1}y)$ for $W \in \mathcal{H}$ and $\pi \in S_{[0,1]}$. Moreover, $R^k/S_{[0,1]}$ is compact for each $k$. (This can be derived from the fact that for each measurable $A \subseteq [0,1]$ there exists $\pi \in S_{[0,1]}$ such that $\pi(A)$ is an interval up to a set of measure 0 (cf. [49]).)

Let $\mathcal{W}_0 \subseteq \mathcal{H}$ be the set defined by all $[0,1]$-valued functions $W$ such that $W(x,y) = W(y,x)$ for all $x, y \in [0,1]$, that is, $\mathcal{W}_0$ is the set of all graphons. Then $\mathcal{W}_0$ is a closed bounded and convex $S_{[0,1]}$-stable subset of $\mathcal{H}$. So by (8.6) and by Theorem 8.2, we recover Theorem 5.1 from Lovász and Szegedy [46]:

$$(\mathcal{W}_0, d_R)/S_{[0,1]} \text{ is compact.} \tag{8.7}$$

Note that $t(H,W) = t(H,\pi W)$ for each $\pi \in S_{[0,1]}$, simple graph $H$ and graphon $W$. Two graphons $W, W' \in \mathcal{W}_0$ are considered to be the same if there exists $\pi \in S_{[0,1]}$ such that $\pi W = W'$. So one might say that the graphon space is compact with respect to $d_R$.

By $\mathcal{G}_{\text{sim}}$ we denote the set of all simple graphs. In [45], Lovász and Szegedy showed that the map $\tau : (\mathcal{W}_0, d_R) \to \mathbb{R}^{\mathcal{G}_{\text{sim}}}$ defined by $\tau(W)(H) := t(H,W)$ is continuous (here the restriction to simple graphs is really necessary). Since $(\mathcal{W}_0, d_R)/S_{[0,1]}$ is compact, and since $\tau$ is $S_{[0,1]}$-invariant, the image of $\tau$ in $\mathbb{R}^{\mathcal{G}_{\text{sim}}}$ is compact. Hence each sequence $\tau(W_1), \tau(W_2) \ldots \in \mathbb{R}^{\mathcal{G}_{\text{sim}}}$ of partition functions of graphons such that $t(H, W_i)$ converges for each simple graph $H$ converges to the partition function $\tau(W) \in \mathbb{R}^{\mathcal{G}_{\text{sim}}}$ of some graphon $W$. So, as simple graphs can be viewed as graphons, this gives a limit behavior of simple graphs, that is, it implies Theorem 8.1.

### 8.2.3 Application of Theorem 8.2 to edge-coloring models

We will now show how Theorem 8.2 can be applied to (limits of) edge coloring models. We will again extend the results of [55] to the complex setting. First we need to extend our definition of an edge-coloring model to a Hilbert space setting. After that we will state our main results about limits of edge-coloring models, postponing the proofs to Section 8.4.

We will use a different, but universal, model of Hilbert space. Let $C$ be a finite or infinite set, and consider for $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, the Hilbert space $l^2(C) := l^2(C, \mathbb{F})$, the set of all functions $f : C \to \mathbb{F}$ with $\sum_{c \in C} |f(c)|^2 < \infty$, having norm $\|f\| := (\sum_{c \in C} |f(c)|^2)^{1/2}$. The inner product on $l^2(C)$ is defined by $\langle f, h \rangle := \sum_{c \in C} f(c)\overline{h(c)}$ for $f, h \in l^2(C)$.

Define for each $k = 0, 1, \ldots$:

$$\mathcal{H}_k := l^2(C^k). \tag{8.8}$$

As usual, $\mathcal{H}_k^{S_k}$ denotes the set of elements of $\mathcal{H}_k$ that are invariant under the natural action $S_k$ on $\mathcal{H}_k$. We call an element $h = (h_k)_{k \in \mathbb{N}}$ of $\prod_{k=0}^{\infty} \mathcal{H}_k^{S_k}$ a *C-color edge-coloring model*. Note that for finite $C$ this agrees with our original definition of a $|C|$-color edge-coloring model, because we can view $h \in \prod_{k=0}^{\infty} \mathcal{H}_k^{S_k}$ as a linear map on $\mathbb{C}[x_1, \ldots, x_{|C|}]$ via the identification of symmetric tensors in $\mathcal{H}_k$ with homogeneous polynomials of degree $k$. Let $\mathcal{G}_0 \subset \mathcal{G}$ be the set of all graphs without loops. The *partition function* of $h$ is the graph parameter $p_h : \mathcal{G}_0 :\to \mathbb{F}$ defined by,

$$p_h(H) := \sum_{\phi : E \to C} \prod_{v \in V} h_{d(v)}(\phi(\delta(v))) \tag{8.9}$$

for a loopless graph $H = (V, E)$. Recall that $d(v)$ denotes the degree of the vertex $v$. Moreover, if $\delta(v)$ consists of the edges $e_1, \ldots, e_k$ (in some arbitrary order), then $\phi(\delta(v)) = (\phi(e_1), \ldots, \phi(e_k)) \in C^k$. As $h_k$ is $S_k$-invariant the order is irrelevant. We will show below (cf. (8.22)) that the sum (8.9) is absolutely convergent. Hence $p_h$ is well-defined. The next example shows that it is necessary for $H$ to not have loops.

**Example 8.2.** Define $h \in \mathcal{H}_2^{S_2}$ by

$$h(i,j) = \begin{cases} \frac{1}{i} \text{ if } i = j, \\ 0 \text{ otherwise,} \end{cases} \tag{8.10}$$

and let $H = C_1$. Then $\|h\|^2 < \infty$, but $p_h(H) = \sum_{k=1}^{\infty} 1/k$ and this series does not converge.

Define

$$\pi : \prod_{k=0}^{\infty} \mathcal{H}_k^{S_k} \to \mathbb{F}^{\mathcal{G}_{\text{sim}}} \quad \text{by} \quad \pi(h)(H) = p_h(H) \tag{8.11}$$

for $H \in \mathcal{G}_{\text{sim}}$. It is not difficult to show that $\pi$ is continuous on $\prod_{k=0}^{\infty} \mathcal{H}_k^{S_k}$, even if we replace $\mathcal{G}_{\text{sim}}$ by $\mathcal{G}_0$.

Let $\mathrm{O}(\mathcal{H})$ denote the group of invertible linear transformations of the **real** Hilbert space $l^2(C,\mathbb{R})$ that preserve the inner product. We call $\mathrm{O}(\mathcal{H})$ the *orthogonal group*. Note that $\mathrm{O}(\mathcal{H})$ is a subgroup of the group of unitary transformations of $l^2(C,\mathbb{C})$.

The tensor power $l^2(C)^{\otimes k}$ embeds naturally in $l^2(C^k)$. In fact, $l^2(C^k)$ is the completion of $l^2(C)^{\otimes k}$. Hence the group $\mathrm{O}(\mathcal{H})$ acts naturally on $\mathcal{H}_k$ for each $k$. Just as in the finite case, the partition functions of edge-coloring models are invariant under the orthogonal group. This follows directly from the case where $|C|$ is finite (cf. the proof of Proposition 6.7), as soon as we show that we can extend the definition of $p_h$ to fragments, which we will do in Section 8.4.

The standard orthonormal basis for $\mathcal{H}^k$ is given by the set $\{e_\phi \mid \phi : [k] \to C\}$, where for $\phi : [k] \to C$, $e_\phi := e_{\phi(1)} \otimes \ldots \otimes e_{\phi(k)}$, and where $e_c$ for $c \in C$ is the orthonormal basis for $l^2(C)$ given by $e_c(c') = \delta_{c,c'}$. Define

$$R_k := \{r_1 \otimes \ldots \otimes r_k \mid r_1, \ldots, r_k \in B(\mathcal{H}_1)\} \subset \mathcal{H}_k. \tag{8.12}$$

We will show that $\pi$ is continuous on $\prod_{k=0}^{\infty} B_k$, when $B_k := B(\mathcal{H}_k)^{S_k}$ is equipped with the metric $d_{R_k}$.

**Theorem 8.3.** *The map $\pi$ is continuous on $\prod_{k=0}^{\infty}(B_k, d_{R_k})$.*

From Theorem 8.2 we will derive:

**Theorem 8.4.** *The space $(\prod_{k=0}^{\infty}(B_k, d_{R_k}))/O(\mathcal{H})$ is compact.*

The proofs of Theorem 8.3 and 8.4 will be given in Section 8.4. Note that since $\pi$ is $\mathrm{O}(\mathcal{H})$-invariant, Theorem 8.3 and 8.4 imply:

**Corollary 8.5.** *The image $\pi(\prod_{k=0}^{\infty} B_k))$ of $\pi$ is compact.*

This implies:

**Corollary 8.6.** *Let $h^1, h^2 \ldots \in \prod_{k=0}^{\infty} B_k$ be a convergent sequence of edge-coloring models. Then there exists $h \in \prod_{k=0}^{\infty} B_k$ such that for each simple graph $H$,*

$$\lim_{n\to\infty} p_{h^n}(H) = p_h(H). \tag{8.13}$$

The corollary holds more generally for sequences in $\prod_{k=0}^{\infty} \lambda_k B_k$ for any fixed sequence $\lambda_0, \lambda_1 \ldots \in \mathbb{F}$.

Since $l^2(C)$ embeds naturally in $l^2(C')$ for $C \subseteq C'$, all edge-coloring models with a any finite number of states embed into $\prod_{k=0}^{\infty}(l^2(\mathbb{N}^k))^{S_k}$. So just as Theorem 8.1 describes a limit behavior of finite graphs, Corollary 8.6 describes a limit behavior of finite-state edge-coloring models, answering a question of Lovász [39]. Since we can think symmetric $k$-tensors as edge-coloring models, Corollary 8.6 also describes a limit behavior of symmetric tensors, providing an answer to a question of Kannan [31].

We end this section with two questions. In [7], Borg, Chayes, Lovász, Sós and Vesztergombi show that the map $\tau : \mathcal{W}_0/S_{[0,1]} \to [0,1]^{\mathcal{G}_{\text{sim}}}$ satisfies that if $\tau(W) = \tau(W')$, then $W'$ is contained in the closure of the $S_{[0,1]}$-orbit of $W$.

*Question* 1. Is it true that if $\pi(h) = \pi(h')$ for any $h, h' \in \prod_{k=0}^{\infty} B_k$, then $h'$ is contained in the closure of the $\mathrm{O}(\mathcal{H})$-orbit of $h$?

The image of the map $\tau$ was characterized by Lovász and Szegedy [45] in terms of some form of reflection positivity.

*Question* 2. Can one give a characterization of the image of $\pi$ for $\mathbb{F} = \mathbb{R}$ in terms of some form of edge-reflection positivity?

## 8.3 Proof of Theorem 8.2

We start by proving that a weakly compact space equipped with the $d_R$ metric (with $R$ bounded) is complete.

**Proposition 8.7.** *Let $\mathcal{H}$ be a Hilbert space and let $R, \mathcal{W} \subseteq \mathcal{H}$ with $R$ bounded and $\mathcal{W}$ weakly compact. Then $(\mathcal{W}, d_R)$ is complete.*

*Proof.* Let $a_1, a_2, \ldots \in \mathcal{W}$ be a Cauchy sequence with respect to $d_R$. We must show that it has a limit in $\mathcal{W}$ with respect to $d_R$. We may assume that $\mathcal{H}$ is separable, otherwise we can replace $\mathcal{H}$ by the closure of the linear span of the $a_i$.

Then, as $\mathcal{W}$ is weakly compact, the sequence has a weakly convergent subsequence (cf. [15, Theorem V.5.1]), say with limit $a \in \mathcal{W}$. Then $a$ is the required limit, that is, $\lim_{n\to\infty} d_R(a_n, a) = 0$. For choose $\varepsilon > 0$. As $a_1, a_2, \ldots$ is Cauchy with respect to $d_R$, there is a $p$ such that $d_R(a_n, a_m) < 1/2\varepsilon$ for $n, m \geq p$. Since $a$ is the weak limit of a subsequence of the $a_i$, there is for each $r \in R$ an $m \geq p$ such that $|\langle a_m - a, r\rangle| < 1/2\varepsilon$. This implies, by the triangle inequality, that for each $n \geq p$,

$$|\langle a_n - a, r\rangle| \leq |\langle a_n - a_m, r\rangle| + |\langle a_m - a, r\rangle| < \varepsilon. \tag{8.14}$$

So $d_R(a_n, a) \leq \varepsilon$ if $n \geq p$. □

Let $G$ be a group acting on a pseudometric space $(X, d)$ that leaves $d$ invariant. Define a pesudometric $d/G$ on $X$ by, for $x, y \in X$:

$$d/G(x,y) := \inf_{g \in G} d(x, gy). \tag{8.15}$$

Since $d$ is $G$-invariant, $(d/G)(x,y)$ is equal to the distance of the $G$-orbits $Gx$ and $Gy$. Any two points $x, y$ on the same $G$-orbit have $(d/G)(x,y) = 0$. If we identify points of $(X, d/G)$ that are on the same orbit, the topological space obtained is homeomorphic to the orbit space $(X, d)/G$ of the topological space $(X, d)$.

**Proposition 8.8.** *Let $(X, d)$ be a complete metric space and let $G$ be a group that acts on $(X, d)$, leaving $d$ invariant. Then $(X, d/G)$ is complete.*

*Proof.* Let $a_1, a_2, \ldots \in X$ be a Cauchy sequence with respect to $d/G$. Then it has a subsequence $b_1, b_2, \ldots$ such that $(d/G)(b_k, b_{k+1}) < 2^{-k}$ for all $k$.

Let $g_1 = 1 \in G$. If $g_k \in G$ has been chosen, let $g_{k+1} \in G$ such that $d(g_k b_k, g_{k+1} b_{k+1}) < 2^{-k}$. Then $g_1 b_1, g_2 b_2, \ldots$ is a Cauchy sequence with respect to $d$. Hence it has a limit $b$ say. Then $\lim_{k\to\infty}(d/G)(b_k, b) = 0$, implying $\lim_{n\to\infty}(d/G)(a_n, b) = 0$. □

Let $\mathcal{H}$ be a Hilbert space and let $R \subseteq \mathcal{H}$. For any $k \geq 0$, define

$$Q_k := \{\lambda_1 r_1 + \ldots + \lambda_k r_k \mid r_i \in R, |\lambda_i| \leq 1 \text{ for } i = 1, \ldots, k\}. \tag{8.16}$$

For any pseudometric $d$, let $B_d(Z, \varepsilon)$ denote the set of points at most distance $\varepsilon$ from $Z$. The following is a form of 'weak Szemerédi regularity'. (cf. Lemma 4.1 of Lovász and Szegedy [46], extending a result of Fernandez de la Vega, Kannan, Karpinski and Vempala [23].)

**Proposition 8.9.** *If $R \subseteq B(\mathcal{H})$, then for each $k \geq 1$:*

$$B(\mathcal{H}) \subseteq B_{d_R}(Q_k, 1/\sqrt{k}). \tag{8.17}$$

*Proof.* Choose $a \in B(\mathcal{H})$ and set $a_0 := a$. If, for some $i \geq 0$, $a_i$ has been found, and $\|a_i\|_R > 1/\sqrt{k}$, choose $r \in R$ with $|\langle a_i, r\rangle| > 1/\sqrt{k}$. Define $a_{i+1} := a_i - \langle a_i, r\rangle r$. Then, with induction,

$$\begin{aligned} \|a_{i+1}\|^2 &= \|a_i\|^2 - 2|\langle a_i, r\rangle|^2 + |\langle a_i, r\rangle|^2\|r\|^2 = \|a_i\|^2 - |\langle a_i, r\rangle|^2(2 - \|r\|^2) \\ &\leq \|a_i\|^2 - |\langle a_i, r\rangle|^2 \leq \|a_i\|^2 - 1/k \leq 1 - (i+1)/k. \end{aligned} \tag{8.18}$$

Moreover, since $|\langle a_i, r\rangle| \leq 1$, we know by induction that $a - a_i \in Q_i$.

By (8.18), as $\|a_{i+1}\|^2 \geq 0$, the process terminates for some $i \leq k$. For this $i$ one has $\|a_i\|_R \leq 1/\sqrt{k}$. Hence, since $Q_i \subseteq Q_k$,

$$d_R(a, Q_k) \leq d_R(a, Q_i) \leq d_R(a, a - a_i) = \|a_i\|_R \leq 1/\sqrt{k}. \tag{8.19}$$

□

We can now give a proof of Theorem 8.2.

**Theorem 8.2.** *Let $\mathcal{H}$ be a Hilbert space and let $G$ be a group of unitary transformations of $\mathcal{H}$. Let $\mathcal{W}$ and $R$ be $G$-stable subsets of $\mathcal{H}$, with $\mathcal{W}$ weakly compact and $R^k/G$ compact for each $k \in \mathbb{N}$. Then $(\mathcal{W}, d_R)/G$ is compact.*

*Proof.* As $R/G$ is compact, $R$ is bounded. So by (8.6), we may assume that both $R$ and $\mathcal{W}$ are contained in $B(\mathcal{H})$.

By Propositions 8.7 and 8.8, $(\mathcal{W}, d_R/G)$ is complete. So it suffices to show that $(\mathcal{W}, d_R/G)$ is *totally bounded*; that is for each $\varepsilon > 0$, $\mathcal{W}$ can be covered by finitely many $d_R/G$-balls of radius $\varepsilon$. For suppose $a_1, a_2, \ldots$ is some sequence in $\mathcal{W}$. Then there exists a ball $B_1$ of $d_R/G$-radius $2^{-1}$ containing infinitely many of the $a_i$. Let $N_1 := \{n \in \mathbb{N} \mid a_n \in B_1\}$. If $B_k$ and $N_k$ have been chosen, choose a ball $B_{k+1}$, of $d_R/G$-radius $2^{-k-1}$, such that $N_{k+1} := \{n \in N_k \mid a_n \in B_{k+1}\}$ is infinite. Now choose for $k \geq 1$, $n_k \in N_k$ with $n_k > n_{k-1}$ and set $b_k := a_{n_k}$. Then $(d_R/G)(b_k, b_{k+1}) \leq 2^{-k+1}$. Hence $b_1, b_2, \ldots$ forms a Cauchy sequence in $(\mathcal{W}, d_R/G)$ and thus has a limit $b \in \mathcal{W}$, proving compactness of $(\mathcal{W}, d_R/G)$.

Now we will show that $(\mathcal{W}, d_R/G)$ is totally bounded. Let $\varepsilon > 0$ and set $k := \lceil 4/\varepsilon^2 \rceil$. As $R^k/G$ is compact, $Q_k/G$ is compact (since the function $R^k \times \{\lambda \mid |\lambda| \leq 1\}^k \to \mathcal{Q}_k$ mapping $(r_1, \ldots, r_k, \lambda_1, \ldots, \lambda_k)$ to $\lambda_1 r_1 + \ldots + \lambda_k r_k$ is continuous, surjective and $G$-equivariant.) Hence (as $d_R \leq d_2$) $(Q_k, d_R)/G$ is compact, equivalently, $(Q_k, d_R/G)$ is compact. Therefore, there exists some finite set $F$ such that $Q_k \subseteq B_{d_R/G}(F, 1/\sqrt{k})$. Then by Proposition 8.9 and the triangle inequality,

$$\begin{aligned} \mathcal{W} &\subseteq B(\mathcal{H}) \subseteq B_{d_R}(Q_k, 1/\sqrt{k}) \subseteq B_{d_R/G}(Q_k, 1/\sqrt{k}) \\ &\subseteq B_{d_R/G}(F, 2/\sqrt{k}) \subseteq B_{d_R/G}(F, \varepsilon). \end{aligned} \tag{8.20}$$

□

## 8.4 Proofs of Theorem 8.3 and 8.4.

Throughout this section, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

### 8.4.1 Properties of the map $\pi$

We start by showing some properties of the map $\pi$, after which we will prove Theorem 8.3.

For an $l$-fragment $F = ([n], E)$ without loops nor open edges and $h = (h_v)_{v\in[n]} \in \prod_{v\in[n]} B_{d(v)}$, define $p_h(F) \in \mathcal{H}_l$ by

$$p_h(F)(c_1,\dots,c_l) = \sum_{\substack{\phi:E\to C\\ \phi(i)=c_i \text{ for all } i\in[l]}} \prod_{v\in V} h_v(\phi(\delta(v))). \tag{8.21}$$

Then

$$\|p_h(F)\| \le \prod_{v\in[n]} \|h_v\|. \tag{8.22}$$

This in particular shows that the sum (8.9) is absolutely convergent and that (8.21) is well-defined. We prove (8.22) by induction on $|E\setminus[l]|$. The case $E=[l]$ being trivial. Let $|E\setminus[l]|\ge 1$ and choose an edge $ab\in E\setminus[l]$. Set $E' = E\setminus\{ab\}$, $\delta'(v) := \delta(v)\setminus\{ab\}$ and $d'(v) = |\delta'(v)|$ for each $v\in[n]$. Let $F'$ be the fragment obtained from $F$ by deleting the edge $ab$. For $c_1,\dots,c_m \in C$ and $h\in\mathcal{H}_k^{S_k}$ $h(c_1,\dots,c_m)$ is the element of $\mathcal{H}_{k-m}^{S_{k-m}}$ defined by $h(c_1,\dots,c_m)(c_{m+1},\dots,c_k) = h(c_1,\dots,c_k)$. Since

$$|p_h(F)(c_1,\dots,c_l)| \le \sum_{\substack{\phi:E\to C\\ \phi(i)=c_i \text{ for all } i\in[l]}} \prod_{v\in[n]} |h_v(\phi(\delta(v)))|, \tag{8.23}$$

we may assume that $h$ takes values in $\mathbb{R}_{\ge0}$. Then

$$\begin{aligned} &p_h(F)(c_1,\dots,c_l) = \\ &\quad\sum_{\substack{\phi:E'\to C\\ \phi(i)=c_i \text{ for all } i\in[l]}} \sum_{c\in C} h_a(\phi(\delta'(a)),c)h_b(\phi(\delta'(b)),c)\cdot \prod_{v\in[n]\setminus\{a,b\}} h_v(\phi(\delta(v))) \\ \le&\quad \sum_{\substack{\phi:E'\to C\\ \phi(i)=c_i \text{ for all } i\in[l]}} \|h_a(\phi(\delta'(a)))\|\,\|h_b(\phi(\delta'(b)))\|\cdot \prod_{v\in[n]\setminus\{a,b\}} h_v(\phi(\delta(v))), \end{aligned} \tag{8.24}$$

by Cauchy-Schwarz. Now define $h'_v = h_v$ for $v\notin\{a,b\}$ and for $v\in\{a,b\}$, $h'_v\in\mathcal{H}_{d'(v)}$ is defined by

$$h'_v(c_1,\dots,c_{d'(v)}) := \|h_v(c_1,\dots,c_{d'(v)})\|. \tag{8.25}$$

Then the last line of (8.24) is equal to $p_{h'}(F')(c_1,\dots,c_l)$. Since $\|h'_v\| = \|h_v\|$ for all $v\in V$, (8.24) implies with induction that

$$\|p_h(F)\| \le \|p_{h'}(F')\| \le \prod_{v\in[n]} \|h_v\|. \tag{8.26}$$

This proves (8.22).

Next, for a graph without loops $H = ([n], E)$ define a function

$$\pi_F : \prod_{v\in[n]} \mathcal{H}_{d(v)}^{S_{d(v)}} \to \mathbb{F} \quad \text{by} \quad \pi_H(h) := \sum_{\phi:E\to C} \prod_{v\in[n]} h_v(\phi(\delta(v))) \tag{8.27}$$

for $h = (h_v)_{v\in[n]} \in \prod_{v\in[n]} \mathcal{H}_{d(v)}^{S_{d(v)}}$.

**Proposition 8.10.** *For a simple graph $H = (V, E)$, the map $\pi_H$ is continuous on $\prod_{v\in V}(B_{d(v)}, d_{R_{d(v)}})$.*

*Proof.* We start by showing that for each $u \in V$,

$$|\pi_H(h)| \leq \|h_u\|_{R_{d(u)}} \prod_{v\in V\setminus\{u\}} \|h_v\|. \tag{8.28}$$

To see this, let $N(u)$ be the set of neighbors of $u$, $H' = H - u$, $\delta'(v) := \delta(v) \setminus \delta(u)$ for $v \in V \setminus \{u\}$ and $d'(v) = |\delta'(v)|$. As above, define for $v \neq u$, $h'_v \in \mathcal{H}_{d'(v)}^{S_{d'(v)}}$ by $h'_v = h_v$ if $v \notin N(u)$ and $h'_v(c_1, \ldots, c_{d'(v)}) = \|h_v(c_1, \ldots, c_{d'(v)})\|$ if $v \in N(u)$. Again, $\|h'_v\| = \|h_v\|$ for all $v$. Then

$$\begin{aligned} |\pi_H(h)| = \Big|\sum_{\phi:E\to C} \prod_{v\in V} h_v(\phi(\delta(v)))\Big| \leq \\ \sum_{\phi:E\to C} \Big|\langle h_u, \bigotimes_{v\in N(u)} \overline{h_v(\phi(\delta'(v)))}\rangle\Big| \cdot \prod_{v\in V(H')\setminus N(u)} |h_v(\phi(\delta(v)))| \leq \\ \sum_{\phi:E(H')\to C} \|h_u\|_{R_{d(u)}} \prod_{v\in V(H')} |h'_v(\phi(\delta'(v)))| \leq \|h_u\|_{R_{d(u)}} \prod_{v\in V(H')} \|h_v\|, \end{aligned} \tag{8.29}$$

where the inequalities follow from the definition of $\|\cdot\|_{R_{d(u)}}$ and from (8.22) (applied to $H'$). This proves (8.28).

Next, identify $V$ with $[n]$ and let $g, h \in \prod_{v\in[n]} B_{d(v)}$. For $u = 1, \ldots, n$ define $p^u \in \prod_{i\in[n]} B_{d(i)}$ by $p^u_i := g_i$ if $i < u$, $p^u_u := g_u - h_u$, and $p^u_i := h_i$ if $i > u$. Moreover, for $u = 0, \ldots, n$ define $q^u \in \prod_{i\in[n]} B_{d(i)}$ by $q^u_i := g_i$ if $i \leq u$ and $q^u_i := h_i$ if $i > u$. So $q^n = g$ and $q_0 = h$. By the multilinearity of $\pi_H$ we have $\pi_H(q^u) - \pi_H(q^{u-1}) = \pi_H(p^u)$. Hence by (8.28) we have the following, proving the proposition,

$$\begin{aligned} |\pi_H(g) - \pi_H(h)| = \Big|\sum_{u=1}^n (\pi_H(q^u) - \pi_H(q^{u-1})\Big| = \Big|\sum_{u=1}^n \pi_H(p^u)\Big| \\ \leq \sum_{u=1}^n \|p^u_u\|_{R_{d(u)}} = \sum_{u=1}^n \|g_u - h_u\|_{R_{d(u)}}. \end{aligned} \tag{8.30}$$

□

We can use Proposition 8.10 to prove Theorem 8.3.

**Theorem 8.3.** *The map $\pi$ is continuous on $\prod_{k=0}^{\infty}(B_k, d_{R_k})$.*

*Proof.* For each simple graph $H$, the function $\psi : \prod_{k=0}^{\infty} B_k \to \prod_{v\in v(H)} B_{d(v)}$ mapping $(h_k)_{k=0}^{\infty}$ to $(h_{d(v)})_{v\in V(H)}$ is continuous. As $\pi(\cdot)(H) = \pi_H(\psi(\cdot))$, the theorem follows from Proposition 8.10. □

Note that we really need simple graphs in Theorem 8.3, as the following example shows.

**Example 8.3.** Let $H = C_2 :=$  and let $h^n \in B_2$ be defined by

$$h^n(i,j) := \begin{cases} n^{-1/2} \text{ if } i = j \leq n, \\ 0 \text{ otherwise.} \end{cases} \tag{8.31}$$

Then $\|h^n\|^2 = p_H(h^n) = 1$ for all $n$, but $\lim_{n\to\infty} \|h^n\|_{d_{R_2}} = 0$. So $\pi_H$ is not continuous with respect to $d_{R_2}$.

It is easy to see that Theorem 8.3 remains true if we replace $B_{d(i)}$ by $\lambda_i B_{d(i)}$ for any $\lambda_0, \lambda_1, \ldots \in \mathbb{F}$. (As it only affects the bound in (8.30) by a factor of $(\max_{v\in V} |\lambda_{d(v)}|)^{n-1}$.) But $\pi_H$ is not continuous on $\prod_{i\in[n]}(\mathcal{H}_{d(i)}^{S_{d(i)}}, d_{R_{d(i)}})$, as the following example shows.

**Example 8.4.** Define $h^n \in \mathcal{H}_2^{S_2}$ by

$$h^n(i,j) := \begin{cases} n^{-1/3} \text{ if } i = j \leq n, \\ 0 \text{ otherwise.} \end{cases} \tag{8.32}$$

Then for $H = C_3$, we have $p_H(h^n) = 1$ for all $n$, but $\lim_{n\to\infty} \|h^n\|_{d_{R_2}} = 0$.

However, with respect to the Hilbert metric we have continuity (and even differentiability) on $\prod_{k=0}^{\infty} \mathcal{H}_k^{S_k}$. Indeed, let $H = ([n], E)$ be a graph without loops, and let $h, x \in \prod_{i\in[n]} \mathcal{H}_{d(i)}^{S_{d(i)}}$. Let for $i = 1, \ldots, n$, $y^i \in \prod_{i\in[n]} \mathcal{H}_{d(i)}^{S_{d(i)}}$ be defined by $y_i^i := x_i$ and $y_j^i := h_j$ if $i \neq j$. Then by (8.22) and by the multilinearity of $\pi_H$,

$$\pi_H(h + x) = \pi_H(h) + \pi_H(y^1) + \ldots + \pi_H(y^n) + o(x). \tag{8.33}$$

This implies that the derivative of $\pi_H(\cdot)$ at $h$ is the linear map $D(\pi_H, h) : \prod_{i\in[n]} \mathcal{H}_{d(i)}^{S_{d(i)}} \to \mathbb{F}$ given by $x = (x_i)_{i\in[n]} \mapsto \pi_H(y^1) + \ldots + \pi_H(y^n)$. One can similarly find that $\pi_H(\cdot)$ is $k$ times differentiable for any $k$.

We can realize the derivative $D(\pi_H, h)$ as the image of a quantum fragment (assuming for simplicity that $\mathbb{F} = \mathbb{R}$). For a graph $H = (V, E)$, let for $v \in V$, $F_v$ be the quantum $d(v)$-fragment obtained from $H$ by deleting vertex $v$, but keeping all the edges adjacent to $v$ as open ends, and taking the sum over all possible labelings of the open ends. Then

$$D(\pi_H, h) = \big(\frac{1}{d(v)!} p_h(F_v)\big)_{v \in V} \in \prod_{v \in V} \mathcal{H}_{d(v)}^{S_{d(v)}}, \tag{8.34}$$

where we identify a Hilbert space with its dual space.

*Remark.* In [59] Schrijver characterizes partition functions of (finite color) edge-coloring models over $\mathbb{R}$ using these derivatives. Perhaps they can also be used to characterize the image of the map $\pi$.

### 8.4.2 Proof of Theorem 8.4

Here we give a proof of Theorem 8.4. But first we show:

**Proposition 8.11.** *Let $(X_1, \delta_1), (X_2, \delta_2), \ldots$ be complete metric spaces and let $G$ be a group acting on each $X_k$, leaving $\delta_k$ invariant ($k = 1, 2, \ldots$). Then $(\prod_{k=1}^{\infty} X_k)/G$ is compact if and only $(\prod_{k=1}^{t} X_k)/G$ is compact for each $t$.*

*Proof.* Necessity being direct, we show sufficiency. We may assume that space $X_k$ has diameter at most $1/k$. Let $A := \prod_{k=1}^{\infty} X_k$, and let $d$ be the supremum metric on $A$ (i.e. $d(a, b) := \sup_k \delta_k(a_k, b_k)$ for $a = (a_k)$ and $b = (b_k)$). Then $d$ is $G$-invariant and $\prod_{k=1}^{\infty}(X_k, \delta_k)$ is $G$-homeomorphic with $(A, d)$. Indeed, a set $B_d(x, \varepsilon)$ is open in $\prod_{k=1}^{\infty}(X_k, \delta_k)$, as it only gives open conditions for $k < 1/\varepsilon$. Conversely, a basic open set $\{x \in \prod_{i=1}^{\infty} X_i \mid \delta_k(x_k, z_k) < \varepsilon\}$ is open in $(A, d)$, as it is equal to the union of $B_d(y, \varepsilon)$ over all $y \in \prod_{i=1}^{\infty} X_i$ with $y_k = z_k$. So it suffices to show that $(A, d)/G$ is compact.

As each $(X, \delta_k)$ is complete, $(A, d)$ is complete. (The limit of a Cauchy sequence $(x^n)$ is the point $x \in A$, where $x_k$ is equal to the pointwise limit of the sequence $(x_k^n)$ in $X_k$, which exists since $(x_k^n)$ is a Cauchy sequence in $X_k$.) By Proposition 8.8 $(A, d)/G$ is complete. So it suffices to show that $(A, d/G)$ is totally bounded. Let $\varepsilon > 0$. Set $t := \lceil \varepsilon^{-1} \rceil$. Let $B := \prod_{k=1}^{t} X_k$ and $C := \prod_{k=t+1}^{\infty} X_k$, with supremum metrics $d_B$ and $d_C$ respectively. As $B/G$ is compact (by assumption), it can be covered by finitely many $d_B/G$-balls of radius $\varepsilon$. As $C$ has diameter at most $1/(t+1) \leq \varepsilon$, $A = B \times C$ can be covered by finitely many $d/G$-balls of radius $\varepsilon$. □

This proposition allows us to prove Theorem 8.4.

**Theorem 8.4.** *The space* $(\prod_{k=0}^{\infty}(B_k, d_{R_k}))/O(\mathcal{H})$ *is compact.*

*Proof.* As each $(B_k, d_{R_k})$ is complete by Proposition 8.7, it suffices by Proposition 8.11 to show that for each $t$, $(\prod_{k=0}^{t}(B_k, d_{R_k}))/\mathrm{O}(\mathcal{H})$ is compact. Consider the Hilbert space $\prod_{k=0}^{t}\mathcal{H}_k$, and let $\mathcal{W} := \prod_{k=0}^{t} B_k$ and $R := \prod_{k=0}^{t} R_k$. Then the identity function is a homeomorphism from $(\mathcal{W}, d_R)$ to $\prod_{k=0}^{t}(B_k, d_{Rk})$. So it suffices to show that $(\mathcal{W}, d_R)/\mathrm{O}(\mathcal{H})$ is compact. Now for each $n$, $R^n/\mathrm{O}(\mathcal{H})$ is compact, as it is the continuous image of $B(\mathcal{H}_1)^m/\mathrm{O}(\mathcal{H})$, with $m := n(1 + 2 + \ldots + t)$. The latter space is compact, as it is the continuous image of the compact space $B(\mathbb{R}^m)^m$ in case $\mathbb{F} = \mathbb{R}$. Since $B(l^2(C, \mathbb{C}))^m$ can be seen as a closed subset of $B(l^2(C, \mathbb{R}))^{2m}$, the previous argument implies that also for $\mathbb{F} = \mathbb{C}$, $B(\mathcal{H}_1)^m/\mathrm{O}(\mathcal{H})$ is compact. (Assuming $|C| = \infty$ in both cases, otherwise $B(\mathcal{H}_1)$ is itself compact). So by Theorem 8.2, $(\mathcal{W}, d_R)/O(\mathcal{H})$ is compact. □

Note that the proof also shows that for any fixed $\lambda_0, \lambda_1, \ldots \in \mathbb{F}$ the space $(\prod_{k=0}^{\infty}(\lambda_k B_k, d_{R_k}))/\mathrm{O}(\mathcal{H})$ is compact.

# Summary

This thesis is concerned with links between certain graph parameters and the invariant theory of the orthogonal group and some of its subgroups. These links are given through so-called *partition functions of edge-coloring models*. These partition functions can be seen as graph parameters as well as polynomials that are invariant under a natural action of the orthogonal group.

Partition functions of edge-coloring models were introduced as graph parameters by de la Harpe and Jones [28] in 1993. For $k \in \mathbb{N}$, a *k-color edge-coloring model* (actually called *vertex model* in [28]) is a statistical physics model. Given a graph $G$, we can think of the edges of $G$ as particles, the vertices as interactions between particles and the colors as states. Given a coloring of the edges of $G$ with $k$ colors (i.e. an assignment of states to the particles), at each vertex we see a multiset of colors to which the edge-coloring model assigns a number. The *weight* of the coloring is the product over the vertices of $G$ of these numbers; in statistical mechanics it is called the *Boltzmann weight*. The *partition function* of the model is the sum, over all possible colorings of the edges of $G$ with $k$ colors, of the weights associated to these colorings.

Many interesting graph parameters are partition functions of edge-coloring models. For example, the number of perfect matchings, the number of proper $k$-edge-colorings for fixed $k \in \mathbb{N}$, but also the number of homomorphisms into a fixed graph.

In this thesis we characterize when a graph parameter $f$ is the partition function of a complex-valued $k$-color edge-coloring model, for a fixed $k \in \mathbb{N}$, in terms of an infinite number of equations of the form $\sum_{i=1}^{n} \lambda_i f(G_i) = 0$, for certain $\lambda_i \in \{\pm 1\}^n$, graphs $G_1, \ldots, G_n$ and $n \in \mathbb{N}$. These equations can be thought of as describing an ideal in a polynomial ring $R$ with infinitely many variables. The proof of the characterization is based on a combinatorial interpretation of these polynomials in $R$ that are invariant under the orthogonal group, which in turn is proved using the First and Second Fundamental Theorem of invariant theory for the orthogonal group, and on Hilbert's Nullstellensatz.

An important tool are certain labeled graphs, called *fragments*. One can construct, for any edge-coloring model $h$, a natural map from the space of formal linear combinations of fragments to the tensor algebra. If $h$ is real valued, then the image of this map turns out to be the algebra of those tensors that are invariant under the subgroup of the orthogonal group consisting of the elements leaving $h$ invariant. This is proved using a theorem of Schrijver [58]. If $h$ is complex valued the situation is more complicated, but a similar statement can be proved. The connection between fragments and invariant tensors allows us to answer a question posed by Szegedy [66].

Besides introducing the edge-coloring model, de la Harpe and Jones also introduced the *vertex-coloring model* (which is called spin model in statistical mechanics). Given a graph $G$, we can also think of the vertices of $G$ as particles, the edges as interactions between particles and again the colors as states. Given a coloring of the vertices of $G$ with $n$ colors (i.e. an assignment of states to the particles), at every edge one sees a a pair of colors; the vertex-coloring model assigns a number to each such a pair. The *weight* of the coloring is the product over the edges of $G$ of the numbers associated to these pairs. This is called the *Boltzmann weight* in statistical mechanics. The *partition function* of a vertex-coloring model is the sum over all possible colorings of the vertices of the graph with $n$ colors of the associated weights. Partition functions of vertex coloring models generalize counting graph homomorphisms.

Szegedy [66] showed that any partition function of a vertex-coloring model can also be obtained as the partition function of a complex edge-coloring model. Using advanced methods from geometric invariant theory we are able to characterize in this thesis for which vertex-coloring models the edge-coloring model can be taken to be real valued.

In [45], Lovász and Szegedy introduce vertex-coloring models with infinitely many colors and show how they can be seen as limits of certain sequences of simple graphs when the set of simple graphs is equipped with a topology based on homomorphism densities. Motivated by this work, we introduce in this thesis edge-coloring models with infinitely many colors and show how they can be seen as limit objects of certain sequences of edge-coloring models with finitely many colors if the latter set is equipped with a particular topology.

# Samenvatting

Dit proefschrift gaat over verbanden tussen bepaalde graafparameters en de invariantentheorie van de orthogonale groep en enkele van zijn ondergroepen. De verbanden worden gelegd door zogenaamde *partitiefuncties van lijnkleuring modellen*. Deze partitiefuncties kunnen zowel beschouwd worden als graafparameters alsmede als polynomen die invariant zijn onder een natuurlijke actie van de orthogonale groep.

Partitiefuncties van lijnkleuring modellen werden geïntroduceerd als graafparameters door de la Harpe en Jones [28] in 1993. Voor $k \in \mathbb{N}$, is een *k-kleur lijnkleuring model* een statistisch mechanisch model. Voor een gegeven graaf $G$, kunnen we de lijnen van $G$ beschouwen als deeltjes, de punten als interacties tussen de deeltjes en de kleuren als toestanden. Gegeven een kleuring van de lijnen van $G$ met $k$ kleuren (dat wil zeggen, een toewijzing van toestanden aan de deeltjes), zien we bij elk punt van de graaf een multiverzameling van kleuren, waaraan het lijnkleuring model een waarde toekent. Het *gewicht* van de kleuring is het product over de punten van $G$ van deze waarden; in de statistische mechanica wordt dit het *Boltzmann gewicht* genoemd. De *partitiefunctie* van het model is de som, over alle mogelijke kleuringen van de lijnen van $G$ met $k$ kleuren, van de gewichten behorend bij deze kleuringen.

Veel interessante graafparameters zijn partitiefuncties van lijnkleuring modellen. Bijvoorbeeld, het aantal perfecte matchings, het aantal propere lijnkleuringen met $k$ kleuren voor vaste $k \in \mathbb{N}$, maar ook het aantal homomorfismen in een vaste graaf.

In dit proefschrift geven we een karakterisatie van graafparameters $f$, die partitiefuncties zijn van complexwaardige $k$-kleur lijnkleuring modellen, voor vaste $k \in \mathbb{N}$, door middel van een oneindig aantal vergelijkingen van de vorm $\sum_{i=1}^{n} \lambda_i f(G_i) = 0$, voor zekere $\lambda_i \in \{\pm 1\}^n$, grafen $G_1, \ldots, G_n$ en $n \in \mathbb{N}$. We kunnen deze vergelijkingen beschouwen als de beschrijving van een ideaal in een polynoomring $R$ met een oneindig aantal variabelen. Het bewijs van de karakterisatie is gebaseed op een combinatorische interpretatie van de polynomen in $R$ die invariant zijn onder de orthogonale groep, welke op zijn beurt bewezen wordt gebruikmakende van de Eerste en Tweede Hoofdstelling van de invariantentheorie van de orthogonale groep en op Hilbert's Nullstellensatz.

Een belangrijk instrument zijn zekere gemarkeerde grafen, welke *fragmenten* genoemd worden. Voor een lijnkleuring model $h$, kan men een natuurlijke afbeelding definiëren van de ruimte van formele lineare combinaties van fragmenten naar de tensor algebra. Wanneer $h$ reëelwaardig is, dan blijkt het beeld van deze afbeelding de algebra van tensoren te zijn die invariant zijn onder de ondergroep van de orthogonale groep bestaande uit de elementen die $h$ stabilizeren. Dit wordt bewezen gebruikmakende van een stelling van Schrijver [58]. De situatie is ingewikkelder waneer $h$ complexwaardig is, maar een zelfde soort resultaat kan bewezen worden. Het verband tussen fragementen en invariante tensoren stelt ons in staat om een vraag van Szegedy [66] te beantwoorden.

Naast de introductie van het lijnkleuring model, introduceerden de la Harpe en Jones ook het *puntkleuring model*. Voor een gegeven graaf $G$, kunnen we ook de punten van $G$ beschouwen als deeltjes, de lijnen als interacties tussen de deeltjes en de kleuren wederom als toestanden. Gegeven een kleuring van de punten van $G$ met $n$ kleuren (dat wil zeggen, een toewijzing van toestanden aan de deeltjes), zien we bij elk lijn van de graaf een paar kleuren, waaraan het puntkleuring model een waarde aan toekent. Het *gewicht* van de kleuring is het product over de lijnen van $G$ van deze waarden; in de statistische mechanica wordt dit het *Boltzmann gewicht* genoemd. De *partitiefunctie* van het model is de som, over alle mogelijke kleuring van de punten van $G$ met $n$ kleuren, van de gewichten behorend bij deze kleuringen. Partitiefuncties van puntkleuring modellen zijn generalisaties van het tellen van graaf homomorfismen.

Szegedy [66] heeft laten zien dat elke partitiefunctie van een puntkleuring model gelijk is aan de partitiefunctie van een complexwaardig lijnkleuring model. Door gebruik te maken van gevanceerde technieken uit de geometrische invariantentheorie, is het ons gelukt om te karakteriseren voor welke puntkleuring modellen de bijbehorende lijnkleuring modellen reëelwaardig zijn.

In [45], introduceerden Lovász en Szegedy een puntkleuring model met een oneindig aantal kleuren en lieten ze zien hoe deze modellen gezien kunnen worden als limieten van bepaalde rijtjes enkelvoudige grafen, in het geval de verzameling van grafen voorzien wordt van een toplogie gebaseerd op het tellen van homomorfismen. Geïnspireerd door deze resultaten, introduceren wij lijnkleuring modellen met een oneindig aantal kleuren en laten we zien hoe zij fungeren als limiet objecten voor bepaalde rijtjes lijnkleuring modellen, wanneer de verzameling van deze modellen voorzien is van een zekere topologie.

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# List of symbols

$|\alpha|$ sum of the $\alpha_i$, 47
$\mathrm{Aut}(a, B)$ automorphism group of the weighed graph $G(a, B)$, 51
$\mathcal{A}$ algebra of all fragments, 54

$(\cdot,\cdot)_w$ bilinear form: $(e_i, e_j)_w := w_i \delta_{i,j}$, 61
$B(\mathcal{H})$ closed unit ball in $\mathcal{H}$, 86

$C^l_{i,j}$ contraction operator for tensors, 20
$C^{\bullet}_1$ labeled loop, 9
$C_n$ $n$-th Catalan number, 41
$\mathbb{C}$ field of complex numbers, 7
$\bigcirc$ circle; the graph with one edge and no vertices, 8
$\mathcal{C}^l_{i,j}$ contraction operator for fragments, 56

$F_1 \cdot F_2$ gluing product of $2l$-fragments $F_1$ and $F_2$, 10
$\delta(v)$ set of edges incident with the vertex $v$, 8
$\delta_{s_1,s_2}$ the delta function (equal to 1 if $s_1 = s_2$ and 0 otherwise), 7
$d(V)$ degree of the vertex $v$, 8

$E(F)$ edge set of the fragment $F$, 55
$E(H)$ edge set of the graph $H$, 8
$E_s$ edges associated to the map $s$, 36
$\mathrm{ev}_u$ evaluation map, 69
$\mathrm{End}(V)$ linear maps from $V$ to itself, 8
$e_\phi$ $e_{\phi(1)} \otimes \cdots \otimes e_{\phi(n)}$, 54

$\mathcal{F}_l$ set of all $l$-fragments, 10
$\mathbb{F}\mathcal{F}_l$ space of $l$-quantum fragments, 54

$\mathbb{F}$ field of characteristic zero, 7
$\mathbb{F}^*$ nonzero elements of the field $\mathbb{F}$, 8
$\overline{\mathbb{F}}$ algebraic closure of $\mathbb{F}$, 7
FFT First Fundamental Theorem, 27

$G(a, B)$ weighted graph with vertex weights $a$ and edge weights $B$, 18
$\mathcal{G}'$ set of all graphs including $\bigcirc$, 8
$\mathcal{G}$ set of all graphs, 8
$\mathcal{G}_l$ set of all $l$-labeled graphs, 9
$\mathcal{G}_n$ set of graphs with vertex set $[n]$, 43
$\mathcal{G}_{\text{sim}}$ set of all simple graphs, 87
$\mathbb{F}\mathcal{G}_l$ semigroup algebra of $\mathcal{G}_l$, 12
$\mathrm{GL}(W)$ group of invertible linear maps from $W$ to itself, 25

$H/s$ graph obtained from $H_s$ by contracting the edges in $E_s$, 36
$H_1H_2$ product of the labeled graphs $H_1$ and $H_2$, 9
$H_s$ graph obtained from $H$ by adding the edges in $E_s$, 36
$\mathcal{H}$ Hilbert space, 86
$\mathcal{H}_k^{S_k}$ space of $S_k$-invariants in $\mathcal{H}_k$, 88
$\mathcal{H}_k$ the Hilbert space $l^2(C^k)$, 88
$\hom(H, G)$ number of homomorphisms from $H$ to $G$, 18
$h_l$ restriction of $h$ to the space of homogenous polynomials of degree $l$, 57

$I_V$ $(I)$ identity map in $\mathrm{End}(V)$, 8
$\mathcal{I}_l(f)$ ideal in $\mathbb{F}\mathcal{G}_l$ generated by the kernel of $f$, 13
$\mathcal{I}_l(h)$ kernel of $M_{p_h,l}$, 54

$K_{i,j}^l$ labeled contraction operator for tensors, 62
$K_1^\bullet$ labeled vertex, 9
$K_2^{\bullet\bullet}$ 2-labeled edge, 9
$\mathcal{K}_{i,j}^l$ labeled contraction operator for labeled graphs, 62

$M_h$ moment matrix of $h$, 36
$M_{f,l}$ $l$-th edge connection matrix of $f$, 11
$\mathcal{M}_m$ set of perfect matchings on $[2m]$, 27

$N_{f,l}$ $l$-th vertex connection matrix of $f$, 11
$[n]$ the set $\{0, 1, \ldots, n\}$, 7
$\mathbb{N}$ the natural numbers including 0, 7
$\mathbb{N}_{\leq d}^k$ set of those $\alpha \in \mathbb{N}^k$ with $\mid \alpha \mid \leq d$, 47
$\| x \|_R$ seminorm associated to $R$, 86

$\mathcal{O}(V)$ algebra generated by the dual of $V$, 22
$\overline{h(p)}$ complex conjugate of $h(p)$, 71
$\overline{A}$ Zariski closure of $A$, 29
$\mathrm{O}(\mathcal{H})$ orthogonal group of the real Hilbert space $l^2(C, \mathbb{R})$, 89
$\mathrm{O}_k(\mathbb{F})$ orthogonal group over $\mathbb{F}$, 21

$\mathrm{pr}_d$ projection from $\mathbb{N}^k_{\leq d'}$ onto $\mathbb{N}^k_{\leq d}$, 47
$p$ map from $\mathcal{G}$ to $T$, 43
$p_h(\mathcal{A})$ image of $\mathcal{A}$ in the tensor algebra under the map $p_h$, 57
$p_n$ restriction of $p$ to the set of graphs with $n$ vertices, 43
$p_{a,B}$ partition function of $(a, B)$, 18

$\mathcal{Q}_l(f)$ quotient algebra $\mathbb{F}\mathcal{G}_l/\mathcal{I}_l(f)$, 13

$R(\mathbb{F})$ polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$, 18
$R$ polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$, 18
$R_k$ $\{r_1 \otimes \ldots \otimes r_k \mid r_1, \ldots, r_k \in B(\mathcal{H}_1)\}$, 89
$\mathbb{R}$ field of real numbers, 7
$\mathrm{rk}\,(M)$ rank of the matrix $M$, 8

$(C \circ D)$ Schur product of $C$ and $D$, 62
$C * D$ operation on 2-tensors, 62
$F_1 * F_2$ gluing operation of $F_1$ and $F_2$, 10
$S\mathbb{F}^{n\times n}$ space of symmetric $n \times n$ matrices in $\mathbb{F}^{n\times n}$, 28
$S_n$ symmetric group, 30
$\mathrm{Stab}(A)$ pointwise stabilizer of $A$, 57
$\mathrm{Stab}(h)$ stabilizer of the edge-coloring model $h$, 52
SFT Second Fundamental Theorem, 27

$F_1 \otimes F_2$ tensor product of the fragments $F_1$ and $F_2$, 54
$M^*$ conjugate transpose of the matrix $M$, 8
$M^T$ transpose of the matrix $M$, 8
$T(V)^{\mathrm{Stab}(h)}$ algebra of tensors invariant under the stabilizer of $h$, 58
$T$ polynomial ring in the variables $y_\alpha$, $\alpha \in \mathbb{N}^k$, 42
$T_n$ homogeneous polynomials in $T$ of degree $n$, 43
tr trace, 77
$t_M$ tensor associated to the perfect matching $M$, 27

$U_i^l$ unlabeling operator for tensors, 62
$[\![H]\!]$ underlying graph of the labeled graph $H$, 9
$\mathcal{U}_i^l$ unlabeling operator for labeled graphs, 62

$(V^{\otimes 2m})^{\mathrm{O}_k}$ space of $\mathrm{O}_k$-invariant $2m$-tensors, 27
$V(F)$ vertex set of the fragment $F$, 55
$V(H)$ vertex set of the graph $H$, 8
$V^*$ dual vectorspace of the vectorspace $V$, 8

$W^G$ subspace of $G$-invariants in $W$, 26

$X/G$ orbit space of $G$ acting on $X$, 86

$Y_d$ the common zeros of the polynomials $p(H) - f(H)$, with $H \in \mathcal{G}$ of max. degree $d$, 47