TORIC IDEALS AND DIAGONAL 2-MINORS

ANARGYROS KATSABEKIS

Abstract. Let G be a simple graph on the vertex set $\{1,\dots,n\}$. An algebraic object attached to G is the ideal P_G generated by diagonal 2-minors of an $n \times n$ matrix of variables. In this paper we first provide some general results concerning the ideal P_G. It is also proved that if G is bipartite, then every initial ideal of P_G is generated by squarefree monomials. Furthermore, we completely characterize all graphs G for which P_G is the toric ideal associated to a finite simple graph. As a byproduct we obtain classes of toric ideals associated to non-bipartite graphs which have quadratic Gröbner bases. Finally, we provide information in certain cases about the universal Gröbner basis of P_G.

1. Introduction

Let $X = (x_{ij})$ be an $n \times n$ matrix of variables and $R = K[x_{ij} | 1 \leq i, j \leq n]$ be the polynomial ring in n^2 variables over a field K. For $1 \leq i < j \leq n$ we denote by f_{ij} the diagonal 2-minor of X given by the elements that stand at the intersection of the rows i, j and the columns i, j. Thus $f_{ij} := x_{ii}x_{jj} - x_{ij}x_{ji}$ is a binomial in R, namely a difference of two monomials. This paper deals with ideals in R generated by collections of diagonal 2-minors of X. Given a simple graph G on the vertex set $\{1,\dots,n\}$, we shall denote by P_G the ideal of R generated by the binomials f_{ij} such that $i < j$ and $\{i, j\}$ is an edge of G. By a simple graph G we mean an undirected graph without loops or multiple edges.

The ideal P_G was considered for the first time in [6]. By Proposition 1.1 in [6] the ideal P_G is complete intersection of height $\mathrm{ht}(P_G) = |E(G)|$, where $E(G)$ is the set of edges of G. Furthermore, the authors notice that the set of generators of P_G is the reduced Gröbner basis of P_G with respect to the reverse lexicographical order induced by the natural ordering of variables

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > x_{22} > \cdots > x_{2n} > \cdots > x_{n1} > \cdots > x_{nn}.$$

Thus the initial ideal of P_G with respect to this order is generated by squarefree monomials. They also proved that P_G is a prime ideal and the ring R/P_G is a normal domain. Since P_G is a prime ideal generated by binomials, we

1991 *Mathematics Subject Classification.* 13F20, 13P10, 05C25.

Key words and phrases. Toric ideals, Ideals of diagonal 2-minors, Gröbner bases.

This work was carried out during the tenure of an ERCIM "Alain Bensoussan" Fellowship Programme. The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement no 246016.

have, from Theorem 5.5 in [5], that it is also a toric ideal (see section 2 for the definition of such an ideal). This is the starting point of the present paper.

In section 2 we study in detail the fact that P_G is a toric ideal. Additionally, we show that if G is bipartite, then every initial ideal of P_G is generated by squarefree monomials (see Theorem 2.8). Given a bipartite graph G, we also provide an upper bound for the maximum degree in the universal Gröbner basis of P_G, i.e. the union of all reduced Gröbner bases of P_G. Our bound is sharp, see Remark 3.12 and Remark 3.19.

An interesting problem is to determine when an ideal generated by binomials is the toric ideal I_H associated to a finite graph H. In [12] H. Ohsugi and T. Hibi consider it for the class of ideals generated by adjacent 2-minors. In this paper, we study the above problem for ideals generated by diagonal 2-minors. More precisely we prove the following:

Theorem 3.7. Let G be a simple graph on the vertex set $\{1, \dots, n\}$. Then there exists a finite simple graph H such that P_G is the toric ideal associated to H, i.e. $P_G = I_H$, if and only if every connected component of G has at most one cycle.

The proof of Theorem 3.7 is constructive, namely we explicitly determine a graph H with the above property.

The problem of finding homogeneous ideals which possess quadratic Gröbner bases has been studied by many authors on commutative algebra, see for example [8], [11], [13]. It is well known that if a homogeneous ideal $I \subset S$, where S is a polynomial ring, has a quadratic Gröbner basis, then the ring S/I is Koszul. A case of particular interest is that of toric ideals associated to graphs. The class of bipartite graphs was studied in [9] where they determine all toric ideals which have quadratic Gröbner bases. However, if the graph is not bipartite, it is generally unknown when the toric ideal has a quadratic Gröbner basis. Our approach produces new examples of nonbipartite graphs H such that the toric ideal I_H has a quadratic Gröbner basis.

As another application of Theorem 3.7, we characterize in graph theoretical terms the elements of the universal Gröbner basis of P_G, when G is a connected graph with at most one cycle. Moreover we explicitly calculate the number of elements and the maximum degree in the universal Gröbner basis of P_G in 2 cases:

(1) G is a star graph,
(2) G is a path graph.

2. General results for ideals generated by diagonal 2-minors

In this section first we recall some basic facts about toric ideals associated to vector configurations. Next we associate to a simple graph G on the vertex set $\{1,\dots,n\}$ the vector configuration $\mathcal{A}_G$ and the matrix N_G with columns the vectors of $\mathcal{A}_G$. It turns out (see Proposition 2.2) that P_G is the toric ideal associated to $\mathcal{A}_G$. Furthermore, we show that the rational polyhedral cone $pos_{\mathbb{Q}}(\mathcal{A}_G)$ has exactly $2m+n$ extreme rays, where m is the number of edges of G. Also, Theorem 2.7 provides a necessary and sufficient condition for the matrix N_G to be totally unimodular. This implies that every initial ideal of P_G is generated by squarefree monomials, when G is a bipartite graph. Moreover, for a bipartite graph G, the universal Gröbner basis of P_G is described by the circuits of the toric ideal associated to $\mathcal{A}_G$.

2.1. **Basics on toric ideals.**

Let $\mathcal{A}=\{\mathbf{a}_1,\dots,\mathbf{a}_s\}\subset\mathbb{Z}^r$ be a vector configuration and let $\mathbb{N}\mathcal{A}=\{l_1\mathbf{a}_1+\cdots+l_s\mathbf{a}_s|l_i\in\mathbb{N}\}$ be the corresponding affine semigroup. We introduce one variable y_i for each vector $\mathbf{a}_i$ and form the polynomial ring $K[y_1,\dots,y_s]$. For every $\mathbf{a}_i=(a_{i,1},\dots,a_{i,s})$, $1\le i\le s$, we let $\mathbf{t}^{\mathbf{a}_i}:=t_1^{a_{i,1}}\cdots t_r^{a_{i,r}}$. The *toric ideal* $I_{\mathcal{A}}$ associated to $\mathcal{A}$ is the kernel of the K-algebra homomorphism

$$\phi: K[y_1,\dots,y_s]\to K[t_1^{\pm1},\dots,t_r^{\pm1}]$$

given by $\phi(y_i)=\mathbf{t}^{\mathbf{a}_i}$ for all $i=1,\dots,s$. We grade the polynomial ring $K[y_1,\dots,y_s]$ by the semigroup $\mathbb{N}\mathcal{A}$ setting $\deg_{\mathcal{A}}(y_i)=\mathbf{a}_i$ for $i=1,\dots,s$. The $\mathcal{A}$-degree of a monomial $K[y_1,\dots,y_s]\ni\mathbf{y}^{\mathbf{u}}=y_1^{u_1}\cdots y_s^{u_s}$ is defined by

$$\deg_{\mathcal{A}}(\mathbf{y}^{\mathbf{u}})=u_1\mathbf{a}_1+\cdots+u_s\mathbf{a}_s\in\mathbb{N}\mathcal{A}.$$

The ideal $I_{\mathcal{A}}$ is generated by all the binomials $\mathbf{y}^{\mathbf{u}}-\mathbf{y}^{\mathbf{v}}$ such that $\deg_{\mathcal{A}}(\mathbf{y}^{\mathbf{u}})=\deg_{\mathcal{A}}(\mathbf{y}^{\mathbf{v}})$, see [15]. Every binomial $\mathbf{y}^{\mathbf{u}}-\mathbf{y}^{\mathbf{v}}$ in $I_{\mathcal{A}}$ is $\mathcal{A}$-homogeneous, i.e. $\deg_{\mathcal{A}}(\mathbf{y}^{\mathbf{u}})=\deg_{\mathcal{A}}(\mathbf{y}^{\mathbf{v}})$. For such binomials, we define $\deg_{\mathcal{A}}(\mathbf{y}^{\mathbf{u}}-\mathbf{y}^{\mathbf{v}}):=\deg_{\mathcal{A}}(\mathbf{y}^{\mathbf{u}})$. By Lemma 4.2 in [15], the height of $I_{\mathcal{A}}$ is equal to $s-\mathrm{rank}(D)$, where $\mathrm{rank}(D)$ is the rank of the matrix D with columns the vectors of $\mathcal{A}$.

For a vector $\mathbf{u}=(u_1,\dots,u_s)\in\mathbb{Z}^s$ we let $\mathrm{supp}(\mathbf{u})=\{i\in\{1,\dots,s\}|u_i\neq 0\}$ be the support of $\mathbf{u}$. The support of a binomial $B=\mathbf{y}^{\mathbf{u}}-\mathbf{y}^{\mathbf{v}}$ is $\mathrm{supp}(B)=\mathrm{supp}(\mathbf{u})\cup\mathrm{supp}(\mathbf{v})$. An irreducible binomial B belonging to $I_{\mathcal{A}}$ is called a *circuit* of $I_{\mathcal{A}}$ if there exists no binomial $B'\in I_{\mathcal{A}}$ such that $\mathrm{supp}(B')\subsetneq\mathrm{supp}(B)$. We shall denote by $\mathcal{C}(I_{\mathcal{A}})$ the set of circuits of $I_{\mathcal{A}}$. The set $\mathcal{C}(I_{\mathcal{A}})$ can be computed easily, see [3] or [4].

An irreducible binomial $\mathbf{y}^{\mathbf{u}}-\mathbf{y}^{\mathbf{v}}$ belonging to $I_{\mathcal{A}}$ is called *primitive* if there is no other binomial $\mathbf{y}^{\mathbf{c}}-\mathbf{y}^{\mathbf{d}}\in I_{\mathcal{A}}$ such that $\mathbf{y}^{\mathbf{c}}$ divides $\mathbf{y}^{\mathbf{u}}$ and $\mathbf{y}^{\mathbf{d}}$ divides $\mathbf{y}^{\mathbf{v}}$. The set of all primitive binomials is the Graver basis of $I_{\mathcal{A}}$, denoted by $\mathrm{Gr}(I_{\mathcal{A}})$.

The universal Gröbner basis of $I_{\mathcal{A}}$, denoted by $U(I_{\mathcal{A}})$, is the union of all its reduced Gröbner bases. It is well known (see [15]) that $U(I_{\mathcal{A}})$ is a finite set and also a Gröbner basis of $I_{\mathcal{A}}$ with respect to every term order. Any toric ideal $I_{\mathcal{A}}$ has a universal Gröbner basis. Furthermore, for every toric ideal $I_{\mathcal{A}}$ we have that $\mathcal{C}(I_{\mathcal{A}})\subseteq U(I_{\mathcal{A}})\subseteq\mathrm{Gr}(I_{\mathcal{A}})$, see [15].

Toric ideals associated to graphs serve as interesting examples of toric ideals. Let H be a finite simple graph with vertices $V(H)=\{v_1,\dots,v_r\}$

and edges $E(H) = \{z_1, \ldots, z_s\}$. The *incidence matrix* of H is the $r \times s$ matrix $M_H := (b_{i,j})$ defined by

$$b_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is one of the vertices in } z_j \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{B}_H = \{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$ be the set of vectors in $\mathbb{Z}^r$, where $\mathbf{b}_i = (b_{i,1}, \ldots, b_{i,r})$ for $1 \leq i \leq s$. With I_H we denote the toric ideal $I_{\mathcal{B}_H}$ in $K[y_1, \ldots, y_s]$. This ideal is commonly known as the toric ideal associated to H. By Lemma 8.3.2 in [18], the rank of the matrix M_H equals $r - b(H)$, where $b(H)$ is the number of connected components of H which are bipartite. Thus the height of I_H equals $s - r + b(H)$.

A *walk of length* q from vertex $v_{i_1} \in V(H)$ to vertex $v_{i_{q+1}} \in V(H)$ is a finite sequence of the form

$$w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_{q+1}}\})$$

such that each $\{v_{i_k}, v_{i_{k+1}}\}$, $1 \leq k \leq q$, is an edge of H. In some cases we may also denote a walk only by vertices $(v_{i_1}, v_{i_2}, \ldots, v_{i_{q+1}})$. The walk w is *closed* if $v_{i_1} = v_{i_{q+1}}$. An *even* (respectively *odd*) closed walk is a closed walk of even (respectively odd) length. A *cycle* is a closed walk

$$(\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_q}, v_{i_1}\})$$

in which $v_{i_j} \neq v_{i_k}$, for every $1 \leq j < k \leq q$. For an edge $z_i = \{v_{i_k}, v_{i_l}\}$ of H, it is clear that $\phi(y_i) = t_{i_k} t_{i_l}$. Given an even closed walk $w = (z_{i_1}, \ldots, z_{i_{2q}})$ of H with every $z_k \in E(H)$, we have that

$$\phi(\prod_{k=1}^{q} y_{i_{2k-1}}) = \phi(\prod_{k=1}^{q} y_{i_{2k}})$$

and therefore the binomial

$$B_w := \prod_{k=1}^{q} y_{i_{2k-1}} - \prod_{k=1}^{q} y_{i_{2k}}$$

belongs to I_H. Proposition 3.1 in [17] asserts that the toric ideal I_H is generated by binomials of this form.

2.2. **Ideals generated by diagonal 2-minors.**

Let G be a simple graph on the vertex set $\{1, \ldots, n\}$ with edges $\{z_1, \ldots, z_m\}$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_{n+1}, \ldots, \mathbf{e}_{n+m}\}$ be the canonical basis of $\mathbb{Z}^{n+m}$. For every edge $z_i = \{i_k, i_l\}$ of G we consider 2 vectors, namely $\mathbf{a}_{i_k i_l} = \mathbf{e}_{i_k} + \mathbf{e}_{i_l} - \mathbf{e}_{n+i}$ and $\mathbf{a}_{i_l i_k} = \mathbf{e}_{n+i}$. Also we consider the vectors $\mathbf{a}_{jj} = \mathbf{e}_j$, $1 \leq j \leq n$. In this way we form the set $\mathcal{A}_G$ consisting of the above $2m + n$ vectors. Actually $I_{\mathcal{A}_G}$ is an ideal in the polynomial ring

$$S := K[\{x_{ij}, x_{ji} | \{i,j\} \text{ is an edge of } G\} \cup \{x_{ii} | 1 \leq i \leq n\}].$$

From now on we will denote by N_G the $(n+m) \times (2m+n)$-matrix with columns all the vectors of $\mathcal{A}_G$.

Example 2.1. Let G be the graph on the vertex set $\{1,\dots,5\}$ with edges $z_1=\{1,2\}$, $z_2=\{2,3\}$, $z_3=\{3,4\}$, $z_4=\{1,4\}$, $z_5=\{1,5\}$ and $z_6=\{3,5\}$. The set $\mathcal{A}_G$ consists of the following 17 vectors:

$$\mathbf{a}_{12}=(1,1,0,0,0,-1,0,0,0,0,0),\mathbf{a}_{21}=(0,0,0,0,0,1,0,0,0,0,0),$$

$$\mathbf{a}_{23}=(0,1,1,0,0,0,-1,0,0,0,0),\mathbf{a}_{32}=(0,0,0,0,0,0,1,0,0,0,0),$$

$$\mathbf{a}_{34}=(0,0,1,1,0,0,0,-1,0,0,0),\mathbf{a}_{43}=(0,0,0,0,0,0,0,1,0,0,0),$$

$$\mathbf{a}_{14}=(1,0,0,1,0,0,0,0,-1,0,0),\mathbf{a}_{41}=(0,0,0,0,0,0,0,0,1,0,0),$$

$$\mathbf{a}_{15}=(1,0,0,0,1,0,0,0,0,-1,0),\mathbf{a}_{51}=(0,0,0,0,0,0,0,0,0,1,0),$$

$$\mathbf{a}_{35}=(0,0,1,0,1,0,0,0,0,0,-1),\mathbf{a}_{53}=(0,0,0,0,0,0,0,0,0,0,1),$$

$$\mathbf{a}_{11}=(1,0,0,0,0,0,0,0,0,0,0),\mathbf{a}_{22}=(0,1,0,0,0,0,0,0,0,0,0),$$

$$\mathbf{a}_{33}=(0,0,1,0,0,0,0,0,0,0,0),\mathbf{a}_{44}=(0,0,0,1,0,0,0,0,0,0,0),$$

$$\mathbf{a}_{55}=(0,0,0,0,1,0,0,0,0,0,0).$$

The toric ideal $I_{\mathcal{A}_G}$ is the kernel of the K-algebra homomorphism

$$\phi: S\to K[t_1^{\pm1},\dots,t_{11}^{\pm1}]$$

given by $\phi(x_{12})=t_1t_2t_6^{-1}$, $\phi(x_{21})=t_6$, $\phi(x_{23})=t_2t_3t_7^{-1}$, $\phi(x_{32})=t_7$, $\phi(x_{34})=t_3t_4t_8^{-1}$, $\phi(x_{43})=t_8$, $\phi(x_{14})=t_1t_4t_9^{-1}$, $\phi(x_{43})=t_9$, $\phi(x_{15})=t_1t_5t_{10}^{-1}$, $\phi(x_{51})=t_{10}$, $\phi(x_{35})=t_3t_5t_{11}^{-1}$, $\phi(x_{53})=t_{11}$, $\phi(x_{11})=t_1$, $\phi(x_{22})=t_2$, $\phi(x_{33})=t_3$, $\phi(x_{44})=t_4$, $\phi(x_{55})=t_5$. It is easy to see that $I_{\mathcal{A}_G}$ is generated by the following 6 binomials: f_{12}, f_{23}, f_{34}, f_{14}, f_{15} and f_{35}. Thus $P_G=I_{\mathcal{A}_G}$.

Proposition 2.2. *Let G be a simple graph on the vertex set $\{1,\dots,n\}$. Then the ideal P_G coincides with the toric ideal $I_{\mathcal{A}_G}$.*

Proof. Given an edge $z_i=\{i_k,i_l\}$ of G, we have that

$$\mathbf{a}_{i_ki_l}+\mathbf{a}_{i_li_k}=(\mathbf{e}_{i_k}+\mathbf{e}_{i_l}-\mathbf{e}_{n+i})+\mathbf{e}_{n+i}=\mathbf{e}_{i_k}+\mathbf{e}_{i_l}=\mathbf{a}_{i_ki_k}+\mathbf{a}_{i_li_l},$$

so the binomial $f_{i_ki_l}$ belongs to $I_{\mathcal{A}_G}$ and therefore $P_G\subseteq I_{\mathcal{A}_G}$. Now the rank of N_G is equal to $n+m$, since the vectors $\mathbf{e}_1,\dots,\mathbf{e}_n,\mathbf{e}_{n+1},\dots,\mathbf{e}_{n+m}$ are columns of N_G, and therefore the height of $I_{\mathcal{A}_G}$ is equal to $2m+n-(n+m)=m$. Since P_G is prime and has height m, we deduce that $P_G=I_{\mathcal{A}_G}$. □

Remark 2.3. (1) Given two edges $\{i_k,i_l\}$ and $\{j_r,j_s\}$ of G, we have that $\deg_{\mathcal{A}_G}(f_{i_ki_l})\neq\deg_{\mathcal{A}_G}(f_{j_rj_s})$ since $\deg_{\mathcal{A}_G}(f_{i_ki_l})=\mathbf{e}_{i_k}+\mathbf{e}_{i_l}$ and $\deg_{\mathcal{A}_G}(f_{j_rj_s})=\mathbf{e}_{j_r}+\mathbf{e}_{j_s}$.
(2) Let G' be a subgraph of G on the vertex set $\{i_1,\dots,i_k\}$ with $i_1\leq\dots\leq i_k$. By Proposition 4.13 in [15] we have that

$$P_{G'}=P_G\cap K[\{x_{rs},x_{sr}|\ \{r,s\}\text{ is an edge of }G'\}\cup\{x_{jj}|i_1\leq j\leq i_k\}].$$

(3) There is a term order such that the initial ideal of $I_{\mathcal{A}_G}$ is generated by squarefree quadratic monomials. Thus we have, from Corollary 8.9 in [15], that the vector configuration $\mathcal{A}_G$ has a unimodular regular triangulation.

We associate to the toric ideal $I_{\mathcal{A}_G}$ the rational convex polyhedral cone $\sigma = pos_{\mathbb{Q}}(\mathcal{A}_G)$ consisting of all non-negative linear combinations of the vectors in $\mathcal{A}_G$. The dimension of σ is equal to the rank of N_G and therefore equals $m+n$. A *face* $\mathcal{F}$ of σ is any set of the form

$$\mathcal{F} = \sigma \cap \{\mathbf{x} \in \mathbb{Q}^{n+m} : \mathbf{cx} = \mathbf{0}\},$$

where $\mathbf{c} \in \mathbb{Q}^{n+m}$ and $\mathbf{cx} \geq 0$ for all $\mathbf{x} \in \sigma$. Faces of dimension one are called *extreme rays*. Notice that σ is strongly convex, i.e. $\mathbf{0}$ is a face of σ with defining vector $\mathbf{c}$ having coordinates $c_i = 1$, for every $i = 1, \ldots, n+m$. If G has at least 2 vertices, then from Corollary 3.4 in [7] the cone σ has at most $2(m+n)-2 = 2m+2n-2$ extreme rays. We will prove that the number of extreme rays of σ is equal to $2m+n$.

Proposition 2.4. *The cone $\sigma = pos_{\mathbb{Q}}(\mathcal{A}_G)$ has exactly $2m+n$ extreme rays.*

Proof. Given an edge $z_i = \{v_{i_k}, v_{i_l}\}$, $1 \leq i \leq m$, of G, we have that $pos_{\mathbb{Q}}(\mathbf{a}_{i_k i_l})$ is a face of σ with defining vector $\mathbf{c} = (c_1, \ldots, c_{n+m}) \in \mathbb{Z}^{n+m}$ having coordinates

$$c_r = \begin{cases} 2, & \text{if } \ r = n+i, \\ 1, & \text{otherwise.} \end{cases}$$

Furthermore, $pos_{\mathbb{Q}}(\mathbf{a}_{i_l i_k})$ is a face of σ with defining vector $\mathbf{c} = (c_1, \ldots, c_{n+m}) \in \mathbb{Z}^{n+m}$ having coordinates

$$c_r = \begin{cases} 0, & \text{if } \ r = n+i, \\ 1, & \text{otherwise.} \end{cases}$$

Finally we have that $pos_{\mathbb{Q}}(\mathbf{a}_{jj})$, $1 \leq j \leq n$, is a face of σ with defining vector $\mathbf{c} = (c_1, \ldots, c_n, c_{n+1}, \ldots, c_{n+m}) \in \mathbb{Z}^{n+m}$ having coordinates

$$c_r = \begin{cases} 0, & \text{if } \ r = j, \\ 2, & \text{if } \ r \in \{1, 2, \ldots, n\} \setminus \{j\}, \\ 1, & \text{if } \ r \in \{n+1, \ldots, n+m\}. \end{cases} \qquad \square$$

A *binomial* $B \in P_G$ is called *indispensable* if every system of binomial generators of P_G contains B or $-B$, while a *monomial* M is called *indispensable* if every system of binomial generators of P_G contains a binomial B such that M is a binomial of B. Let $\mathcal{M}_G$ be the ideal generated by all monomials $\mathbf{x}^{\mathbf{u}}$ for which there exists a nonzero $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in P_G$. By Proposition 3.1 in [2], the set of indispensable monomials of P_G is the unique minimal generating set of $\mathcal{M}_G$.

Remark 2.5. If $\{B_1 = M_1 - N_1, \ldots, B_s = M_s - N_s\}$ is a generating set of P_G, then $\mathcal{M}_G = (M_1, N_1, M_2, N_2, \ldots, M_s, N_s)$.

We will prove that P_G is generated by its indispensable.

Proposition 2.6. *Let G be a simple graph on the vertex set $\{1, \ldots, n\}$ with m edges, then the ideal P_G has a unique minimal system of binomial generators.*

Proof. By Remark 2.5 the set $\{x_{ii}x_{jj}, x_{ij}x_{ji} | \ \{i,j\} \text{ is an edge of } G\}$ generates the monomial ideal $\mathcal{M}_G$. In fact it is a minimal generating set of $\mathcal{M}_G$, so for every edge $\{i,j\}$ of G the monomials $x_{ii}x_{jj}$ and $x_{ij}x_{ji}$ are indispensable of P_G. Thus P_G has exactly $2m$ indispensable monomials. Now the cone σ is strongly convex, so, from the graded Nakayama's Lemma, every minimal binomial generating set of P_G has exactly m binomials. By Remark 2.3 (1), every binomial f_{ij} is indispensable of P_G and therefore P_G has a unique minimal system of binomial generators. □

A matrix M with $\text{rank}(M) = d$ is called *unimodular* if all non-zero $d \times d$-minors of M have the same absolute value. The matrix M is called *totally unimodular* when every minor of M is 0 or ± 1. It is well known (see for example [14, Chapter 19]) that the graph G is bipartite if and only if its incidence matrix M_G is totally unimodular.

Theorem 2.7. *Let G be a simple graph on the vertex set $\{1, \ldots, n\}$ with m edges, then the matrix N_G is totally unimodular if and only if G is bipartite.*

Proof. ($\Rightarrow$) Suppose that N_G is totally unimodular. Then also M_G is totally unimodular, since it is a submatrix of N_G, and therefore G is bipartite.
($\Leftarrow$) Suppose that G is bipartite. Since total unimodularity is preserved under the unit vectors, it is enough to consider the matrix Q with columns $\mathbf{a}_{i_k i_l} = \mathbf{e}_{i_k} + \mathbf{e}_{i_l} - \mathbf{e}_{n+i}$, where $z_i = \{i_k, i_l\}$ is an edge of G. It suffices to prove that Q is totally unimodular. The incidence matrix M_G is totally unimodular, since G is bipartite. So the $m \times (n+m)$-matrix $(M_G^t | R)$ is totally unimodular, where R is the matrix with rows $-\mathbf{e}_1, \ldots, -\mathbf{e}_m$, and therefore the transpose Q of this matrix is totally unimodular. □

Theorem 2.8. *If G is a bipartite graph, then the initial ideal of P_G is generated by squarefree monomials with respect to any term order. Furthermore, the equality $U(P_G) = \mathcal{C}(I_{\mathcal{A}_G})$ holds.*

Proof. We have, from Theorem 2.7, that the matrix N_G is totally unimodular and hence also unimodular. By Corollary 8.9 in [15] every initial ideal of $P_G = I_{\mathcal{A}_G}$ is generated by squarefree monomials. Now Proposition 8.11 in [15] asserts that $\mathcal{C}(I_{\mathcal{A}_G}) = \text{Gr}(I_{\mathcal{A}_G})$. Thus the equality $U(P_G) = \mathcal{C}(I_{\mathcal{A}_G})$ holds. □

Remark 2.9. Let G be a bipartite graph. By the proof of Proposition 8.11 in [15], for every circuit $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{\mathcal{A}_G}$ the monomials $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ are squarefree.

Corollary 2.10. *Let G be a bipartite graph on the vertex set $\{1, \ldots, n\}$ with m edges, then the maximum degree of a binomial in the universal Gröbner basis of P_G is least than or equal to $\lfloor \frac{m+n+1}{2} \rfloor$.*

Proof. By Lemma 4.8 in [15], the cardinality of the support of a circuit $B \in I_{\mathcal{A}_G}$ is least than or equal to $m + n + 1$. Since every binomial in

$I_{\mathcal{A}_G}$ is homogeneous with respect to the standard grading, we have that the degree of any circuit is less than or equal to $\lfloor \frac{m+n+1}{2} \rfloor$. By Theorem 2.8, the maximum degree of a binomial in the universal Gröbner basis of $P_G = I_{\mathcal{A}_G}$ is least than or equal to $\lfloor \frac{m+n+1}{2} \rfloor$. □

Remark 2.11. If G is bipartite, then we have, from Corollary 8.9 in [15], that every regular triangulation of $\mathcal{A}_G$ is unimodular.

Example 2.12. We come back to Example 2.1. The circuits of $I_{\mathcal{A}_G}$ are the following:

$$\mathcal{C}(I_{\mathcal{A}_G}) = \{f_{12}, f_{23}, f_{34}, f_{14}, f_{15}, f_{35}, x_{44}x_{35}x_{53} - x_{55}x_{34}x_{43},$$
$$x_{11}x_{35}x_{53} - x_{33}x_{15}x_{51}, x_{44}x_{15}x_{51} - x_{55}x_{14}x_{41}, x_{22}x_{15}x_{51} - x_{55}x_{12}x_{21},$$
$$x_{33}x_{14}x_{41} - x_{11}x_{34}x_{43}, x_{22}x_{14}x_{41} - x_{44}x_{12}x_{21}, x_{22}x_{34}x_{43} - x_{44}x_{23}x_{32},$$
$$x_{11}x_{23}x_{32} - x_{33}x_{12}x_{21}, x_{22}x_{35}x_{53} - x_{55}x_{23}x_{32}, x_{14}x_{41}x_{35}x_{53} - x_{34}x_{43}x_{15}x_{51},$$
$$x_{14}x_{41}x_{35}x_{53} - x_{33}x_{44}x_{15}x_{51}, x_{14}x_{41}x_{35}x_{53} - x_{11}x_{55}x_{34}x_{43},$$
$$x_{11}x_{44}x_{35}x_{53} - x_{34}x_{43}x_{15}x_{51}, x_{12}x_{21}x_{35}x_{53} - x_{23}x_{32}x_{15}x_{51},$$
$$x_{11}x_{22}x_{35}x_{53} - x_{23}x_{32}x_{15}x_{51}, x_{12}x_{21}x_{35}x_{53} - x_{22}x_{33}x_{15}x_{51},$$
$$x_{12}x_{21}x_{35}x_{53} - x_{11}x_{55}x_{23}x_{32}, x_{34}x_{43}x_{15}x_{51} - x_{33}x_{55}x_{14}x_{41},$$
$$x_{23}x_{32}x_{15}x_{51} - x_{33}x_{55}x_{12}x_{21}, x_{23}x_{32}x_{14}x_{41} - x_{12}x_{21}x_{34}x_{43},$$
$$x_{23}x_{32}x_{14}x_{41} - x_{11}x_{22}x_{34}x_{43}, x_{22}x_{33}x_{14}x_{41} - x_{12}x_{21}x_{34}x_{43},$$
$$x_{23}x_{32}x_{14}x_{41} - x_{33}x_{44}x_{12}x_{21}, x_{12}x_{21}x_{34}x_{43} - x_{11}x_{44}x_{23}x_{32},$$
$$x_{22}x_{14}x_{41}x_{35}x_{53} - x_{44}x_{23}x_{32}x_{15}x_{51}, x_{22}x_{14}x_{41}x_{35}x_{53} - x_{55}x_{12}x_{21}x_{34}x_{43},$$
$$x_{44}x_{12}x_{21}x_{35}x_{53} - x_{55}x_{23}x_{32}x_{14}x_{41}, x_{44}x_{12}x_{21}x_{35}x_{53} - x_{22}x_{34}x_{43}x_{15}x_{51},$$
$$x_{22}x_{34}x_{43}x_{15}x_{51} - x_{55}x_{23}x_{32}x_{14}x_{41}, x_{44}x_{23}x_{32}x_{15}x_{51} - x_{55}x_{12}x_{21}x_{34}x_{43}\}.$$

The graph G is bipartite and a partition of its vertices is $\{1,3\} \cup \{2,4,5\}$. Notice that G has at least 2 even cycles, for instance $(\{1,2\},\{2,3\},\{3,4\},\{4,1\})$ and $(\{1,5\},\{5,3\},\{3,2\},\{2,1\})$. By Theorem 2.8, the universal Gröbner basis of P_G consists of the above 36 binomials.

3. Classification of all graphs G such that the equality $P_G = I_H$ holds

In this section we completely characterize all simple graphs G for which there exists a finite simple graph H such that P_G is the toric ideal associated to H. We start with Proposition 3.1 which gives a sufficient condition for the equality $P_G = I_H$.

Proposition 3.1. *Let G be a simple graph on the vertex set $\{1, \ldots, n\}$ with m edges. If there exists a finite simple graph H such that $P_G = I_H$, then every connected component of G has at most one cycle.*

Proof. Let $G_i, 1 \leq i \leq s$, be the connected components of G. It is easy to see that any two binomials $B \in P_{G_i}$ and $B' \in P_{G_j}$, $i \neq j$, have no common variable. Thus there exists a finite simple graph H such that $P_G = I_H$ if and only if for every $1 \leq i \leq s$ there exists a finite simple graph H_i such that $P_{G_i} = I_{H_i}$. Therefore we may assume that the graph G is connected. Recall that P_G is an ideal of the polynomial ring

$$S = K[\{x_{ij}, x_{ji} | \{i,j\} \text{ is an edge of } G\} \cup \{x_{ii} | 1 \leq i \leq n\}].$$

Let H be a simple graph such that $P_G = I_H$, then H has $2m+n$ edges. Without loss of generality we can assume that H has no isolated vertices. Let $\mathcal{B}_H = \{\mathbf{b}_{ij}, \mathbf{b}_{ji} | \{i,j\} \text{ is an edge of } G\} \cup \{\mathbf{b}_{11}, \ldots, \mathbf{b}_{nn}\}$ be the set of columns of the incidence matrix M_H of H. Every column of M_H has exactly two non-zero entries, which are equal to 1. In particular every $\mathbf{b}_{ii}$, $1 \leq i \leq n$, has exactly two non-zero entries, so the cardinality of the set $\{\text{supp}(\mathbf{b}_{11}), \ldots, \text{supp}(\mathbf{b}_{nn})\}$ is at most $2n$. Given an edge $\{i,j\}$ of G we have that $f_{ij} = x_{ii}x_{jj} - x_{ij}x_{ji} \in P_G$, so $f_{ij} \in I_H$ and therefore $\mathbf{b}_{ii} + \mathbf{b}_{jj} = \mathbf{b}_{ij} + \mathbf{b}_{ji}$. Thus $\mathbf{b}_{ij} = \mathbf{b}_{ii} + \mathbf{b}_{jj} - \mathbf{b}_{ji}$. If $\text{supp}(\mathbf{b}_{ji}) \cap \text{supp}(\mathbf{b}_{ii}) = \emptyset$ or $\text{supp}(\mathbf{b}_{ji}) \cap \text{supp}(\mathbf{b}_{jj}) = \emptyset$, then the vector $\mathbf{b}_{ij}$ has at least one entry which is equal to -1, a contradiction. Thus the cardinality of the set formed by the supports of all vectors of $\mathcal{B}_H$ is at most $2n$. Let d be the number of vertices of H, then d is least than or equal to $2n$. Let $b(H)$ be the number of connected components of H which are bipartite. Using the equality $\text{ht}(P_G) = \text{ht}(I_H)$ we have that $m = 2m + n - d + b(H)$ and therefore $m + n + b(H) = d \leq 2n$. So $m \leq m + b(H) \leq n$, while $n \leq m + 1$ since G is connected. Thus $n \in \{m, m+1\}$ and therefore G has at most one cycle. Note that G is a tree when $n = m+1$, while G has exactly one cycle in the case that $n = m$. $\square$

Let G be a simple connected graph with k vertices and l edges. We will associate to G the prism over G, i.e. a new graph $G^\star$ with $2k$ vertices and $k + 2l$ edges. Consider two graphs G_1 and G_2, which are isomorphic to G, with vertices $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$, correspondingly, such that $\{p_1, \ldots, p_k\} \cap \{q_1, \ldots, q_k\} = \emptyset$. Given an edge $\{i,j\}$ of G, we let $z_{ij} = \{p_i, p_j\}$ and $z_{ji} = \{q_i, q_j\}$ be edges of G_1 and G_2, respectively. We define the graph $G^\star$ as follows. The vertex set of $G^\star$ is $\{p_1, \ldots, p_k\} \cup \{q_1, \ldots, q_k\}$. Also both G_1 and G_2 are subgraphs of $G^\star$. Thus each one of the edges of G_1 and G_2 is also an edge of $G^\star$. Finally we let $z_{ii} = \{p_i, q_i\}$ be an edge of $G^\star$, for every vertex i of G. It holds that $P_G \subseteq I_{G^\star}$, since $w = (z_{ii}, z_{ji}, z_{jj}, z_{ij})$ is an even cycle of $G^\star$ and therefore $x_{ii}x_{jj} - x_{ij}x_{ji} = B_w \in I_{G^\star}$.

Example 3.2. Let G be the graph on the vertex set $\{1, \ldots, 4\}$ with edges $z_{12} = \{1,2\}$, $z_{13} = \{1,3\}$, $z_{14} = \{1,4\}$ and $z_{34} = \{3,4\}$. Let $G_1 = G$ and G_2 be the graph with vertices $\{5,6,7,8\}$ and edges $z_{21} = \{5,6\}$, $z_{31} = \{5,7\}$, $z_{41} = \{5,8\}$, $z_{43} = \{7,8\}$. The graph $G^\star$ has 8 vertices, namely $\{1, \ldots, 8\}$, and 12 edges, namely the 4 edges of G_1, the 4 edges of G_2 and also the edges $z_{11} = \{1,5\}$, $z_{22} = \{2,6\}$, $z_{33} = \{3,7\}$ and $z_{44} = \{4,8\}$.

Lemma 3.3. *Let W be a connected subgraph of a simple graph G. If W is either a tree or a non-bipartite graph with exactly one cycle, then there exists a connected graph $W^\star$ such that $P_W = I_{W^\star}$.*

Proof. Let W be a tree with k vertices and l edges. Since W is a tree, we have that $k = l + 1$. If $W^\star$ is not bipartite, then

$$\mathrm{ht}(I_{W^\star}) = (k + 2l) - 2k = 2l - k = 2l - (l + 1) = l - 1 < l = \mathrm{ht}(P_W),$$

a contradiction to the fact that $P_W \subseteq I_{W^\star}$. Thus $W^\star$ is bipartite, so

$$\mathrm{ht}(I_{W^\star}) = (k + 2l) - 2k + 1 = l$$

and therefore $P_W = I_{W^\star}$.
Let W be a non-bipartite graph with exactly one cycle. Let r, s be the number of vertices and edges, respectively, of W, then $r = s$. We have that $W^\star$ is not bipartite and therefore

$$\mathrm{ht}(I_{W^\star}) = (2s + r) - 2r = 2s - r = s = \mathrm{ht}(P_W).$$

Since $P_W \subseteq I_{W^\star}$, we take the equality $P_W = I_{W^\star}$. □

Given an even cycle $C = (\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_k, i_1\})$ of G of length $k \geq 4$, we will introduce a connected graph $\overline{C}$ with $2k$ vertices and $3k$ edges. Consider two graphs C' and C'', which are isomorphic to the path $Y = (\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{k-1}, i_k\})$, with disjoint vertex sets $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$, correspondingly. Given an edge $\{r, s\}$ of Y we let $z_{rs} = \{p_r, p_s\}$ and $z_{sr} = \{q_r, q_s\}$ be edges of C' and C'', respectively. We define the graph $\overline{C}$ as follows. The vertex set of $\overline{C}$ is $\{p_1, \ldots, p_k\} \cup \{q_1, \ldots, q_k\}$. Also both C' and C'' are subgraphs of $\overline{C}$. Additionally, for $r = i_1$, $s = i_k$ we let $z_{rs} = \{p_r, q_s\}$ and $z_{sr} = \{p_s, q_1\}$ be edges of $\overline{C}$. Finally we let $z_{rr} = \{p_r, q_r\}$ be an edge of $\overline{C}$, for every vertex r of C. The graph $\overline{C}$ is a Möbius-band, see [1] for more information about such graphs. When $k = 4$ and $C = (\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\})$ the graph $\overline{C}$ is drawn in Figure 1.

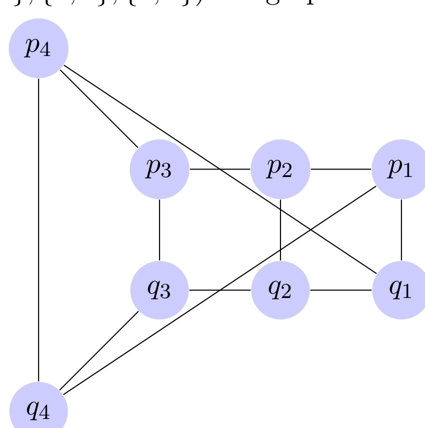

Figure 1.

The next Lemma asserts that $P_C = I_{\overline{C}}$. This result can be seen as a special case of Theorem 4.4 in [1]. However, our argument seems to be more appropriate in the context of this paper.

Lemma 3.4. *Let C be an even cycle of a simple graph G of length $k \geq 4$. Then there is a connected graph $\overline{C}$ such that $P_C = I_{\overline{C}}$.*

Proof. The graph $\overline{C}$ is not bipartite, since $w = (p_{i_1}, q_{i_k}, p_{i_k}, p_{i_{k-1}}, \ldots, p_{i_2}, p_{i_1})$ is an odd cycle of length $k+1$. Thus $\mathrm{ht}(I_{\overline{C}}) = 3k - 2k = k$. For an edge $\{r, s\}$ of Y we have that $w = (z_{rr}, z_{sr}, z_{ss}, z_{rs})$ is an even cycle of $\overline{C}$, so $x_{rr}x_{ss} - x_{rs}x_{sr} = B_w \in I_{\overline{C}}$. Moreover, for $r = i_1$ and $s = i_k$ we have that also $\gamma = (z_{rr}, z_{sr}, z_{ss}, z_{rs})$ is an even cycle of $\overline{C}$, so $x_{rr}x_{ss} - x_{rs}x_{sr} \in I_{\overline{C}}$ and therefore $P_C \subseteq I_{\overline{C}}$. Thus the equality $P_C = I_{\overline{C}}$ holds, since $\mathrm{ht}(P_C) = k$. $\square$

Remark 3.5. Let $C = (\{1,2\}, \{2,3\}, \ldots, \{k,1\})$ be an even cycle of a simple graph G.
(1) We have that $P_C \neq I_{C^\star}$. The graph $C^\star$ is bipartite, since

$$\{p_1, p_3, p_5, \ldots, p_{k-1}, q_2, q_4, \ldots, q_k\} \cup \{p_2, p_4, \ldots, p_k, q_1, q_3, q_5, \ldots, q_{k-1}\}$$

is a partition of its vertices. Then $\mathrm{ht}(I_{C^\star}) = 3k - 2k + 1 = k + 1 \neq k = \mathrm{ht}(P_C)$.
(2) The graph $\overline{C}$ has an even cycle of length $2k$, namely

$$(p_1, p_2, \ldots, p_{k-1}, p_k, q_k, q_{k-1}, \ldots, q_1, p_1).$$

(3) The subgraph of $\overline{C}$ consisting of all edges, except from z_{1k} and z_{k1}, is bipartite and a partition of its vertices is

$$\{p_1, p_3, \ldots, p_{k-1}, q_2, q_4, \ldots, q_k\} \cup \{p_2, p_4, \ldots, p_k, q_1, q_3, \ldots, q_{k-1}\}.$$

Let $G_1 = (V(G_1), E(G_1))$, $G_2 = (V(G_2), E(G_2))$ be graphs such that $G_1 \cap G_2$ is a complete graph. The new graph $G = G_1 \bigoplus G_2$ with the vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$ is called the *clique sum* of G_1 and G_2 in $G_1 \cap G_2$. If the cardinality of $V(G_1) \cup V(G_2)$ is $k+1$, then this operation is called a k-sum of the graphs. We write $G = G_1 \bigoplus_{\widehat{v}} G_2$ to indicate that G is the clique sum of G_1 and G_2 and that $V(G_1) \cap V(G_2) = \widehat{v}$.

Example 3.6. Let G be the graph on the vertex set $\{1, \ldots, 6\}$ consisting of the even cycle $C = (\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\})$, as well as the tree T with edges $\{1,5\}$ and $\{1,6\}$. Let $\overline{C}$ be the graph on the vertex set $\{7,8,9,10\} \cup \{12,13,15,17\}$ consisting of the edges $z_{12} = \{7,8\}$, $z_{23} = \{8,9\}$, $z_{34} = \{9,10\}$, $z_{14} = \{7,17\}$, $z_{21} = \{12,13\}$, $z_{32} = \{13,15\}$, $z_{43} = \{15,17\}$, $z_{41} = \{10,12\}$, $z_{11} = \{7,12\}$, $z_{22} = \{8,13\}$, $z_{33} = \{9,15\}$, $z_{44} = \{10,17\}$.
Also consider the graph $T^\star$ on the vertex set $\{7,18,19\} \cup \{12,20,21\}$ consisting of the edges $z_{15} = \{7,18\}$, $z_{51} = \{12,20\}$, $z_{16} = \{7,19\}$, $z_{61} = \{12,21\}$, z_{11}, $z_{55} = \{18,20\}$, $z_{66} = \{19,21\}$. Notice that $\overline{C} \cap T^\star$ is the graph on the vertex set $\widehat{v} = \{7,12\}$ consisting of the edge z_{11}. The 1-clique sum H of the graphs $\overline{C}$ and $T^\star$ is drawn in Figure 2.

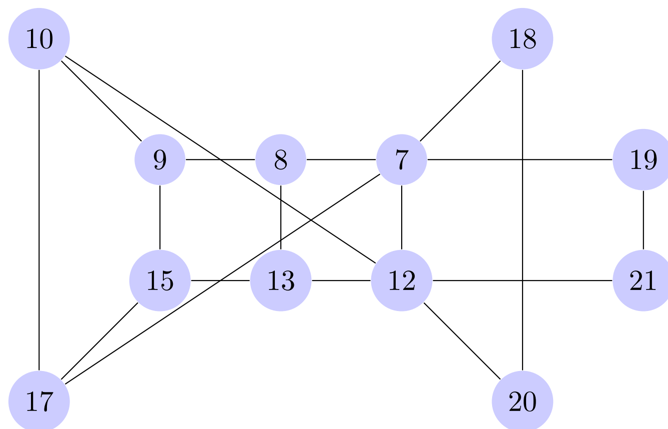

Figure 2.

It is easy to see that $P_G \subseteq I_H$. Moreover $\text{ht}(P_G) = 6$ and $\text{ht}(I_H) = 18 - 12 = 6$, since H is not bipartite.
Thus $P_G = I_H$, so $\{f_{12}, f_{23}, f_{34}, f_{14}, f_{15}, f_{16}\}$ is a quadratic Gröbner basis for I_H with respect to the reverse lexicographic term order induced by the ordering of the variables

$$x_{11} > x_{12} > x_{14} > x_{15} > x_{16} > x_{21} > x_{22} > x_{23} > x_{32} > x_{33} > x_{34} >$$

$$x_{41} > x_{43} > x_{44} > x_{51} > x_{55} > x_{61} > x_{66}.$$

The following Theorem determines all graphs G such that the ideal P_G is of the form I_H, for a finite simple graph H.

Theorem 3.7. *Let G be a simple graph on the vertex set $\{1, \ldots, n\}$. Then there exists a finite simple graph H such that P_G is the toric ideal associated to H, i.e. $P_G = I_H$, if and only if every connected component of G has at most one cycle.*

Proof. We may assume that the graph G is connected. If there exists a finite simple graph H such that $P_G = I_H$, then we have, from Proposition 3.1, that G has at most one cycle. Conversely if either G has no cycle, i.e. G is a tree, or it is non-bipartite with exactly one cycle, then Lemma 3.3 asserts that P_G is the toric ideal associated to $G^\star$. Thus it is enough to consider the case that G is bipartite and has exactly one cycle. Let, say, that $C = (\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_k, i_1\})$ is the unique cycle of G, where $k \geq 4$ is even. The graph G can be written as the 0-sum of the cycle C and some trees. More precisely we have that

$$G = C \bigoplus_{i_1} T_1 \bigoplus_{i_2} T_2 \bigoplus_{i_3} \cdots \bigoplus_{i_s} T_s,$$

for some vertices $i_1, \ldots, i_s$ of C. An example of a graph G is drawn in Figure 3.

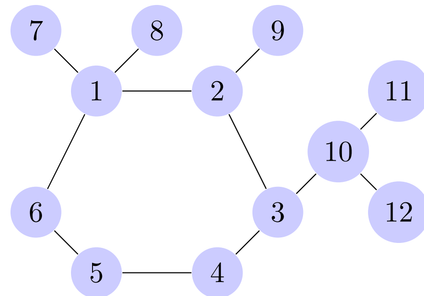

Figure 3.

By Lemma 3.4, there is a connected graph $\overline{C}$ such that $P_C = I_{\overline{C}}$. Moreover, $\overline{C}$ has exactly $3k$ edges and $2k$ vertices. Denote by $\widehat{v_j}$ the set of vertices of the edge $z_{i_j i_j}$, $1 \leq j \leq s$. By Lemma 3.3, there exists a connected graph $T_j^\star$ such that $P_{T_j} = I_{T_j^\star}$, for every $1 \leq j \leq s$. If the tree T_j, $1 \leq j \leq s$, has g_j edges, then $T_j^\star$ has $2g_j + 2$ vertices and $3g_j + 1$ edges. Without loss of generality we can assume that $V(T_j^\star) \cap V(\overline{C}) = \widehat{v_j}$, for every $1 \leq j \leq s$, and also $V(T_i^\star) \cap V(T_j^\star) = \emptyset$, for every $i \neq j$. Let $H = \overline{C} \bigoplus_{\widehat{v_1}} T_1^\star \bigoplus_{\widehat{v_2}} T_2^\star \bigoplus_{\widehat{v_3}} \cdots \bigoplus_{\widehat{v_s}} T_s^\star$ be the 1-clique sum of the graphs $\overline{C}$ and $T_i^\star$, $1 \leq i \leq s$. We have that $P_G = P_C + P_{T_1} + \cdots + P_{T_s}$, so $P_G = I_{\overline{C}} + I_{T_1^\star} + \cdots + I_{T_s^\star}$ and therefore $P_G \subseteq I_H$. Notice that $\mathrm{ht}(P_G) = k + g_1 + \cdots + g_s$ and also

$$\mathrm{ht}(I_H) = (3k + 3g_1 + \cdots + 3g_s) - (2k + 2g_1 + \cdots + 2g_s) =$$
$$= k + g_1 + \cdots + g_s.$$

Consequently $P_G = I_H$, since $\mathrm{ht}(P_G) = \mathrm{ht}(I_H)$. □

Remark 3.8. Let G be a connected graph with exactly one cycle. Denote by H either the graph constructed in the proof of Theorem 3.7, when G is bipartite, or the graph $G^\star$ when G is not bipartite. The toric ideal I_H associated to the non-bipartite graph H has a quadratic Gröbner basis, with respect to the reverse lexicographical order induced by the natural ordering of variables

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > x_{22} > \cdots > x_{2n} > \cdots > x_{n1} > \cdots > x_{nn}.$$

Example 3.9. Consider the non-bipartite graph G on the vertex set $\{1, \ldots, 4\}$ with edges $z_{12} = \{1, 2\}$, $z_{23} = \{2, 3\}$, $z_{13} = \{1, 3\}$ and $z_{14} = \{1, 4\}$. The graph G has exactly one odd cycle, namely (z_{12}, z_{23}, z_{13}). We let $G^\star$ be the graph with vertices $\{1, \ldots, 4\} \cup \{5, \ldots, 8\}$ and 12 edges, namely the 4 edges of G and the edges $z_{21} = \{5, 6\}$, $z_{32} = \{6, 7\}$, $z_{31} = \{5, 7\}$, $z_{41} = \{5, 8\}$, $z_{11} = \{1, 5\}$, $z_{22} = \{2, 6\}$, $z_{33} = \{3, 7\}$, $z_{44} = \{4, 8\}$. From Lemma 3.3 the equality $P_G = I_{G^\star}$ holds. By using Algorithm 7.2 in [15] we determine the Graver basis of $I_{G^\star}$ which consists of the following 16 binomials:

$$f_{12}, f_{23}, f_{13}, f_{14}, x_{33}x_{14}x_{41} - x_{44}x_{13}x_{31}, x_{22}x_{14}x_{41} - x_{44}x_{12}x_{21},$$
$$x_{22}x_{13}x_{31} - x_{11}x_{23}x_{32}, x_{22}x_{13}x_{31} - x_{33}x_{12}x_{21}, x_{11}x_{23}x_{32} - x_{33}x_{12}x_{21},$$
$$x_{23}x_{32}x_{14}x_{41} - x_{22}x_{44}x_{13}x_{31}, x_{23}x_{32}x_{14}x_{41} - x_{33}x_{44}x_{12}x_{21}, x_{33}^2 x_{12}x_{21} - x_{23}x_{32}x_{13}x_{31},$$

$$x_{11}^2x_{23}x_{32} - x_{12}x_{21}x_{13}x_{31}, x_{22}^2x_{13}x_{31} - x_{12}x_{21}x_{23}x_{32},$$
$$x_{11}x_{23}x_{32}x_{14}x_{41} - x_{44}x_{12}x_{21}x_{13}x_{31}, x_{12}x_{21}x_{13}x_{31}x_{44}^2 - x_{23}x_{32}x_{14}^2x_{41}^2.$$

Notice that $\mathrm{Gr}(I_{G^\star}) \neq \mathcal{C}(I_{G^\star})$, since the binomial $B = x_{11}x_{23}x_{32}x_{14}x_{41} - x_{44}x_{12}x_{21}x_{13}x_{31}$ is primitive and not a circuit of $I_{G^\star}$. The set $\{f_{12}, f_{23}, f_{13}, f_{14}\}$ constitutes a quadratic Gröbner basis for the toric ideal $I_{G^\star}$ with respect to the reverse lexicographical order induced by the ordering of variables

$$x_{11} > x_{12} > x_{13} > x_{14} > x_{21} > x_{22} > x_{23} > x_{31} > x_{32} > x_{33} > x_{41} > x_{44}.$$

Let G be a connected bipartite graph with exactly one cycle C. Consider the 1-clique sum H of the graphs $\overline{C}$ and $T_i^\star$, $1 \leq i \leq s$, which appeared in the proof of Theorem 3.7. Combining Theorem 2.8 and Theorem 3.7 we take the equality $U(P_G) = \mathcal{C}(I_H)$. Theorem 3.11 detects all circuits of I_H. In order to prove this Theorem, we will use the following result:

Theorem 3.10. *([17]) Let H be a finite, connected and simple graph. Then a binomial B is a circuit of I_H if and only if $B = B_w$ where w is an even closed walk of H which has one of the following forms*

(1) *w is an even cycle.*
(2) *w consists of two odd cycles intersecting in exactly one vertex.*
(3) *w consists of two vertex disjoint odd cycles joined by a path.*

Theorem 3.11. *Let G be a simple connected graph on the vertex set $\{1, \dots, n\}$. Assume that G is bipartite with exactly one cycle. Then a binomial $B \in P_G$ belongs to the universal Gröbner basis of P_G if and only if $B = B_w$ where w is an even cycle of H.*

Proof. From the assumption G has a unique cycle, say C, and suppose that it has even length $k \geq 4$. Let $\{p_1, \dots, p_k, q_1, \dots, q_k\}$ be the vertices of $\overline{C}$, where $\{p_1, \dots, p_k\} \cap \{q_1, \dots, q_k\} = \emptyset$. Figure 4 shows an example of a graph

$$H = \overline{C} \bigoplus_{\widehat{v_1}} T_1^\star \bigoplus_{\widehat{v_2}} T_2^\star \bigoplus_{\widehat{v_3}} \cdots \bigoplus_{\widehat{v_s}} T_s^\star.$$

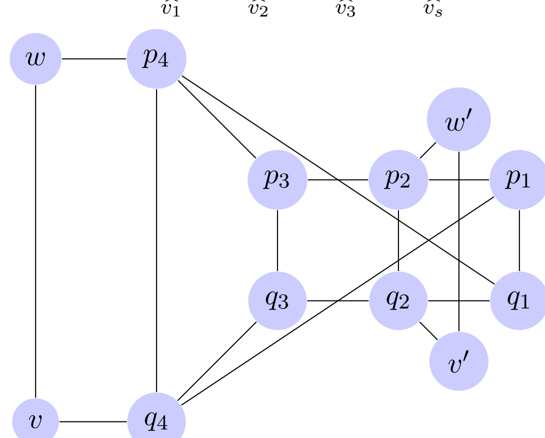

Figure 4.

Every odd cycle in H contains at least one of the edges $z_{1k} = \{p_1, q_k\}$ and $z_{k1} = \{q_1, p_k\}$, since the subgraph F of H consisting of all the edges of H, except from z_{1k} and z_{k1}, is bipartite. We will prove that any pair of two odd cycles in H share at least 2 vertices. Suppose that there exist two odd cycles w_1 and w_2 in H which share at most one vertex. Let, say, that w_1 contains z_{1k} and w_2 contains z_{k1}. We can take the cycles w_1 and w_2 to start from the vertices p_1 and q_1, respectively. Moreover, we can assume that z_{1k} and z_{k1} are the first edges of w_1 and w_2, respectively. We claim that the second edge of w_1 is $\{q_k, q_{k-1}\}$. If $\{q_k, q_{k-1}\}$ is not the second edge, then either $\{q_k, p_k\}$ is the second edge of w_1 or there exists a vertex $v \in T_i^\star$, such that $\{q_k, v\}$ is the second edge of w_1. In the latter case there exists a path in w_1 of length > 2 connecting q_k with p_k. Since w_1 is a cycle, we have that in both cases $\{p_k, p_{k-1}\}$ is an edge of w_1. Now $\{q_1, p_k\}$ is the first edge of w_2, so either $\{p_k, p_{k-1}\}$ is the second edge of w_2 or there exists a path in w_2 of length ≥ 1 connecting p_k with q_k. In both cases we arrive at a contradiction, since w_1, w_2 have at most one common vertex. Consequently $\{q_k, q_{k-1}\}$ is the second edge of w_1 and analogously we have that $\{p_k, p_{k-1}\}$ is the second edge of w_2. Using similar arguments we conclude that $\{p_1, q_k\}$, $\{q_k, q_{k-1}\}, \ldots, \{q_3, q_2\}$ are all edges of w_1, while $\{q_1, p_k\}$, $\{p_k, p_{k-1}\}, \ldots, \{p_3, p_2\}$ are all edges of w_2. We claim that $\{q_2, q_1\}$ is the next edge of w_1. Suppose not, then either $\{q_2, p_2\}$ is an edge of w_1 or there exists a vertex $v' \in T_j^\star$ such that $\{q_2, v'\}$ is an edge of w_1. In both cases there exists a path in w_1 of length ≥ 1 connecting q_2 with p_2. Thus $\{p_2, p_1\}$ is an edge of w_1. But $\{p_3, p_2\}$ is an edge of w_2, so either $\{p_2, p_1\}$ is the next edge of w_2 or there exists a path in w_2 of length ≥ 1 connecting p_2 with q_2. Since w_1, w_2 have at most one vertex in common, we arrive at a contradiction. Thus $\{q_2, q_1\}$ is an edge of w_1 and analogously $\{p_2, p_1\}$ is an edge of w_2. But then w_1, w_2 have 2 vertices in common, namely p_1 and q_1, contradicting our assumption. As a consequence any pair of two odd cycles in H share at least 2 vertices. From Theorem 3.10 we have that the universal Gröbner basis of P_G consists of all binomials of the form B_w, where w is an even cycle of H. □

Remark 3.12. Let C be an even cycle of G of length $k \geq 4$, then, from Remark 3.5 (2), the maximum degree of a binomial in the universal Gröbner basis of P_C is equal to k. Notice that $\lfloor \frac{k+k+1}{2} \rfloor = k$.

Let $C = (\{p_1, p_2\}, \{p_2, p_3\}, \ldots, \{p_k, p_1\})$ be an odd cycle of G of length $k \geq 3$. We consider the graph $C^\star$ on the vertex set $\{p_1, \ldots, p_k, q_1, \ldots, q_k\}$ and let C' be the odd cycle $(\{q_1, q_2\}, \ldots, \{q_k, q_1\})$.

Proposition 3.13. *A binomial $B \in P_C$ belongs to the universal Gröbner basis of P_C if and only if $B = B_w$ where w is an even closed walk of $C^\star$ which has one of the following forms*

(1) *w is an even cycle.*
(2) *$w = (C, z_{ii}, C')$ where $z_{ii} = \{p_i, q_i\}$, $1 \leq i \leq k$.*

Proof. By Proposition 6.1 in [1], the graph $C^\star$ has exactly two vertex disjoint odd cycles, namely C and C'. We have, from Lemma 3.2 in [10], that every primitive binomial $B \in I_{C^\star}$ is of the form $B = B_w$, where

(1) w is an even cycle of $C^\star$ or
(2) w consists of two odd cycles of $C^\star$ intersecting in exactly one vertex or
(3) $w = (C, z_{ii}, C')$, i.e. it consists of the vertex disjoint odd cycles C, C' joined by the edge z_{ii}.

Since $C^\star$ has no vertex of degree greater than three, we deduce that there are no two odd cycles intersecting in exactly one vertex. Consequently the universal Gröbner basis of $P_C = I_{C^\star}$ consists of all binomials of the form B_w, where w is either an even cycle of $C^\star$ or $w = (C, z_{ii}, C')$. □

Remark 3.14. Let G be a connected non-bipartite graph with exactly one cycle. From Theorem 3.4 in [16] we have that a binomial B belongs to the universal Gröbner basis of P_G if and only if $B = B_w$, where w is a mixed even closed walk of $G^\star$. For the definition of a mixed walk see [16].

Another interesting case of a bipartite graph is that of a tree G. We will prove that every element in the universal Gröbner basis of P_G corresponds to an even cycle of $G^\star$.

Theorem 3.15. *If G is a tree, then the universal Gröbner basis of P_G consists of all the binomials B_w where w is an even cycle of $G^\star$.*

Proof. We have, from Lemma 3.3, that the graph $G^\star$ is bipartite and also $P_G = I_{G^\star}$. Now Theorem 3.10 asserts that the universal Gröbner basis of $I_{G^\star}$ consists of all the binomials B_w, where w is an even cycle of $G^\star$. □

For the rest of this section we are going to study the universal Gröbner bases of two special classes of trees, namely star graphs and path graphs.

Proposition 3.16. *Let G be a star graph with at least $n \geq 3$ vertices, then the universal Gröbner basis of P_G consists of $2n-2$ binomials. Futhermore, the maximum degree of a binomial in the universal Gröbner basis of P_G is equal to three.*

Proof. Let G be a star graph with vertices $\{p_1, \ldots, p_n\}$ and edges $\{p_1, p_2\}$, $\{p_1, p_3\}, \ldots$, $\{p_1, p_n\}$. Consider the graph $G^\star$ on the vertex set $\{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_n\}$, where $\{p_1, \ldots, p_n\} \cap \{q_1, \ldots, q_n\} = \emptyset$, with $3n-2$ edges. Every edge of G is also an edge of $G^\star$. Moreover $\{q_1, q_k\}$, $1 \leq k \leq n$, is an edge of $G^\star$. Finally for every $1 \leq k \leq n$ we let $\{p_k, q_k\}$ be an edge of $G^\star$. For $n = 3$ the graph $G^\star$ is drawn in Figure 5.

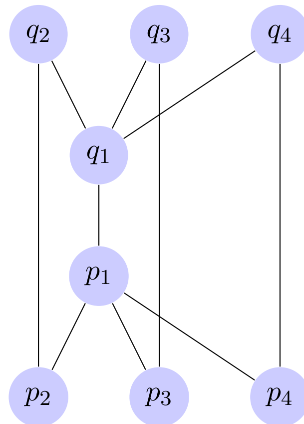

Figure 5.

The graph $G^\star$ is bipartite. Furthermore, it has at least one even cycle of length 6. For example $(\{p_1,p_2\},\{p_2,q_2\},\{q_2,q_1\},\{q_1,q_3\},\{q_3,p_3\},\{p_3,p_1\})$ is an even cycle of length 6 in $G^\star$. We claim that $G^\star$ has no even cycle of length greater than 6. Every even cycle w in $G^\star$ contains both p_1 and q_1. In fact we can take w to start from the vertex p_1. There are three cases for w:

(1) $w = (\{p_1,q_1\},\{q_1,q_k\},\{q_k,p_k\},\{p_k,p_1\})$ where $2 \le k \le n$.
(2) $w = (\{p_1,p_l\},\{p_l,q_l\},\{q_l,q_1\},\{q_1,p_1\})$ where $2 \le l \le n$.
(3) $w = (\{p_1,p_r\},\{p_r,q_r\},\{q_r,q_1\},\{q_1,q_s\},\{q_s,p_s\},\{p_s,p_1\})$ where $2 \le r,s \le n$ and $r \ne s$.

Thus $G^\star$ has no even cycles of length greater than 6. By Theorem 3.15 the maximum degree of a binomial in the universal Gröbner basis of P_G is equal to three. Furthermore, the graph $G^\star$ has exactly $(n-1)$ even cycles of length 6. Since the number of even cycles of length 4 equals also $(n-1)$, we have that the universal Gröbner Basis of P_G consists of $(n-1)+(n-1) = 2n-2$ binomials. □

Example 3.17. Let G be the tree on the vertex set $\{1,\dots,4\}$ with edges $z_{12} = \{1,2\}$, $z_{13} = \{1,3\}$ and $z_{14} = \{1,4\}$. Consider the graph $G^\star$ on the vertex set $\{1,\dots,8\}$ which has 10 edges, namely z_{12}, z_{13}, z_{14}, $z_{21} = \{5,6\}$, $z_{31} = \{5,7\}$, $z_{41} = \{5,8\}$, $z_{11} = \{1,5\}$, $z_{22} = \{2,6\}$, $z_{33} = \{3,7\}$, $z_{44} = \{4,8\}$. The graph $G^\star$ has three even cycles of length 4 and three even cycles of length 6, namely

$$w_1 = (z_{12}, z_{22}, z_{21}, z_{31}, z_{33}, z_{13}), w_2 = (z_{12}, z_{22}, z_{21}, z_{41}, z_{44}, z_{14})$$

and $w_3 = (z_{13}, z_{33}, z_{31}, z_{41}, z_{44}, z_{14})$. The universal Gröbner basis of P_G consists of the following 6 binomials:

$$x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{33} - x_{13}x_{31}, x_{11}x_{44} - x_{14}x_{41},$$

$$B_{w_1} = x_{12}x_{21}x_{33} - x_{22}x_{31}x_{13}, B_{w_2} = x_{12}x_{21}x_{44} - x_{22}x_{41}x_{14},$$

$$B_{w_3} = x_{13}x_{31}x_{44} - x_{33}x_{41}x_{14}.$$

Proposition 3.18. *Let G be a path graph with n vertices where $n \geq 2$. Then the universal Gröbner basis of P_G consists of exactly $\frac{n(n-1)}{2}$ binomials. Furthermore, the maximum degree of a binomial in the universal Gröbner basis of P_G is equal to n.*

Proof. Let G be a path graph with vertices $\{p_1, \ldots, p_n\}$ and edges

$$\{p_1, p_2\}, \{p_2, p_3\}, \ldots, \{p_{n-1}, p_n\}.$$

Consider the graph $G^\star$ on the vertex set $\{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_n\}$, where $\{p_1, \ldots, p_n\} \cap \{q_1, \ldots, q_n\} = \emptyset$, with $3n-2$ edges. Every edge of G is also an edge of $G^\star$. Also $\{q_k, q_{k+1}\}$, $1 \leq k \leq n-1$, is an edge of $G^\star$. Finally for every $1 \leq k \leq n$ we let $\{p_k, q_k\}$ be an edge of $G^\star$. For $n = 5$ the graph $G^\star$ is drawn in Figure 6.

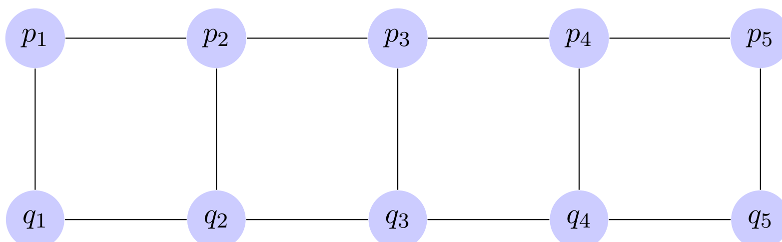

Figure 6.

The graph $G^\star$ is bipartite and the maximum length of an even cycle in $G^\star$ is $2n$. Actually there is only one cycle with this length, namely

$$(\{p_1, q_1\}, \{q_1, q_2\}, \{q_2, q_3\}, \ldots, \{q_{n-1}, q_n\}, \{q_n, p_n\}, \{p_n, p_{n-1}\}, \ldots, \{p_2, p_1\}).$$

Since the universal Gröbner basis consists of all the binomials of the form B_w, where w is an even cycle of $G^\star$, we have that the maximum degree of a binomial in the universal Gröbner basis of P_G is equal to n. Let s_k be the number of even cycles in $G^\star$ of length k. It is easy to see that $s_4 = n-1$, $s_6 = n-2$, $s_8 = n-3$, $s_{10} = n-4, \ldots$, $s_{2n} = 1$. Consequently $G^\star$ has exactly

$$1 + 2 + \cdots + (n-3) + (n-2) + (n-1) = \frac{n(n-1)}{2}$$

even cycles and therefore the universal Gröbner basis of P_G consists of $\frac{n(n-1)}{2}$ binomials. □

Remark 3.19. For a path graph G with n vertices, we have that the maximum degree n of a binomial in the universal Gröbner basis of P_G equals $\left\lfloor \frac{n+(n-1)+1}{2} \right\rfloor$.

Example 3.20. Let G be the path graph on the vertex set $\{1, \ldots, 5\}$ with edges $\{i, i+1\}$, $1 \leq i \leq 4$. Consider the graph $G^\star$ on the vertex set $\{1, \ldots, 10\}$ which has 13 edges, namely $z_{12} = \{1, 2\}$, $z_{23} = \{2, 3\}$, $z_{34} = \{3, 4\}$, $z_{45} = \{4, 5\}$, $z_{21} = \{6, 7\}$, $z_{32} = \{7, 8\}$, $z_{43} = \{8, 9\}$, $z_{54} = \{9, 10\}$, $z_{11} = \{1, 6\}$, $z_{22} = \{2, 7\}$, $z_{33} = \{3, 8\}$, $z_{44} = \{4, 9\}$, $z_{55} = \{5, 10\}$. The graph $G^\star$ has 4 even cycles of length 4, 3 even cycles of length 6, 2 even cycles of length 8 and 1 even cycle of length 10. The universal Gröbner basis of P_G consists of the following 10 binomials:

$$x_{11}x_{22}-x_{12}x_{21}, x_{22}x_{33}-x_{23}x_{32}, x_{33}x_{44}-x_{34}x_{43}, x_{44}x_{55}-x_{45}x_{54},$$
$$x_{12}x_{33}x_{21}-x_{23}x_{32}x_{11}, x_{23}x_{44}x_{32}-x_{34}x_{43}x_{22}, x_{34}x_{55}x_{43}-x_{45}x_{54}x_{33},$$
$$x_{12}x_{34}x_{43}x_{21}-x_{23}x_{44}x_{32}x_{11}, x_{23}x_{45}x_{54}x_{32}-x_{34}x_{55}x_{43}x_{22},$$
$$x_{12}x_{34}x_{55}x_{43}x_{21}-x_{23}x_{45}x_{54}x_{32}x_{11}.$$

References

[1] I. Bermejo, I. Garcia-Marco, E. Reyes, Complete intersection graphs, arXiv: 1210.1950.

[2] H. Charalambous, A. Katsabekis, A. Thoma, Minimal systems of binomial generators and the indispensable complex of a toric ideal, Proc. Amer. Math. Soc. **135** (2007) 3443-3451.

[3] J. A. De Loera, Triangulations of polytopes and computational algebra, PhD thesis, Cornell University, 1995.

[4] P. Diaconis, D. Eisenbud, B. Sturmfels, Lattice walks and primary decomposition, in: B. E. Sagan, R. P. Stanley (Eds.), Mathematical Essays in Honor of Gian-Carlo Rota, Birkhäuser, Boston, 1998, pp. 173-193.

[5] V. Ene, J. Herzog, Gröbner bases in Commutative Algebra, Graduate Studies in Mathematics **130**, American Mathematical Society, Providence, RI, 2012.

[6] V. Ene, A. A. Qureshi, Ideals generated by diagonal 2-minors, Communications in Algebra (in press).

[7] K. Fischer, W. Morris, J. Shapiro, Affine semigroup rings that are complete intersections, Proc. Amer. Math. Soc. **125** (1997) 3137-3145.

[8] C. Haase, A. Paffenholz, Quadratic Gröbner bases for smooth 3×3 transportation polytopes, J. Algebraic Combin. **30** (2009) 477-489.

[9] H. Ohsugi, T. Hibi, Koszul bipartite graphs, Adv. in Appl. Math. **22** (1999) 25-28.

[10] H. Ohsugi, T. Hibi, Toric ideals generated by quadratic binomials, J. Algebra **218** (1999) 509-527.

[11] H. Ohsugi, T. Hibi, Two way subtable sum problems and quadratic Gröbner bases, Proc. Amer. Math. Soc. **137** (2009) 1539-1542.

[12] H. Ohsugi, T. Hibi, Toric ideals of finite graphs and adjacent 2-minors, Mathematica Scandinavica (in press).

[13] I. Peeva, V. Reiner, B. Sturmfels, How to shell a monoid, Math. Ann. **310** (1998) 379-393.

[14] A. Schrijver, Theory of Linear and Integer Programming, John Wiley and Sons, New York, 1986.

[15] B. Sturmfels, Gröbner Bases and Convex Polytopes, University Lecture Series, No. 8 American Mathematical Society Providence, R.I. 1995.

[16] C. Tatakis, A. Thoma, On the universal Gröbner bases of toric ideals of graphs, J. Combin. Theory Ser. A **118** (2011) 1540-1548.

[17] R. H. Villarreal, Rees algebras of edge ideals, Comm. Algebra **23** (1995) 3513-3524.

[18] R. H. Villarreal, Monomial Algebras, Marcel Dekker, New York, 2001.

Centrum Wiskunde & Informatica (CWI), Postbus 94079, 1090 GB Amsterdam, The Netherlands

E-mail address: katsabek@aegean.gr