



SOME APPROXIMATIONS TO THE PERCENTAGE POINTS OF THE NON-CENTRAL t -DISTRIBUTION¹

by

Constance van Eeden

Mathematical Centre, Amsterdam

MATHEMATISCH CENTRUM
Statistische Afdeling

O. SUMMARY

In this paper the following approximations to the percentage points of the non-central t -distribution are considered.

- I. Three closely related approximations based on a normal approximation to the distribution of $\underline{x} + k\sqrt{\underline{\chi}^2}$,² where $\underline{x}$ possesses a normal distribution with zero mean and variance 1 and where $\underline{\chi}^2$ possesses a χ^2 -distribution with f degrees of freedom ($\underline{x}$ and $\underline{\chi}^2$ are distributed independently). Two of these approximations were derived by W. J. Jennett and B. L. Welch (1939) and by N. L. Johnson and B. L. Welch (1940) respectively. The third one was derived by W. A. Hendricks (1936) for the percentage points of the central t -distribution (Student's distribution) and could be generalized for the non-central t -distribution.
- II. Four closely related approximations based on a method used by H. Goldberg and H. Levine (1946) for the percentage points of the central t -distribution. In this case the percentage-points are expressed in a power-series in $\frac{1}{f}$, the coefficients being functions of a normal variable with zero mean and variance 1.
- III. The approximation of M. Merrington and E. S. Pearson (1958) based on a Pearson type IV-approximation to the non-central t -distribution.
- IV. The approximation of B. I. Harley (1957) based on an approximation of the distribution of a function of t by the distribution of the correlation coefficient in samples from a bivariate normal distribution.
- V. An approximation mentioned in D. B. Owen's table of factors for one-sided tolerance limits for a normal distribution (1958). This approximation is based on an approximate relation between one- and two-sided tolerance limits.
- VI. An approximation based on a normal approximation to the non-central t -distribution.

These approximations are compared with the exact percentage points for several values of the non-centrality parameter δ , the probability α and the number of degrees of freedom f . For the central t -distribution the approximations are especially important for large values of f , where no tables are available. Then one of the approximations II is the best one: it is identical with the exact percentage points in three decimal places for $0.50 \leq \alpha \leq 0.99$. For the non-central t -distribution we used $f = 2(1)9$. For these values of f the approximations I and II are the most important ones. In some

¹ Shortened version of report S 242 of the Statistical Department of the Mathematical Centre, Amsterdam (1958).

² Random variables are distinguished from numbers (e.g. from the values they take in an experiment) by underlining their symbols.

cases III or IV may be better, but for these approximations more computational work is needed. The approximations V and VI are not recommended; VI only gives reasonable results for the central t -distribution.

Further the paper contains a description of the exact tables of the non-central t -distribution.

1. INTRODUCTION

This paper contains some approximations to the percentage points of the non-central t -distribution. Most of these approximations are known in literature; some of them however were only known for the special case of the central t -distribution (Student's distribution), but could be generalized for the non-central t -distribution.

In section 2 a description will be given of the existing tables of the non-central t -distribution. The approximations to the percentage points will be given in section 3; in section 4 the approximations will be compared with the exact percentage points.

2. DESCRIPTION OF THE TABLES OF THE NON-CENTRAL t -DISTRIBUTION

Consider two independent random variables $\underline{x}$ and $\underline{\chi^2}$, where $\underline{x}$ possesses a $N(0,1)$ -distribution³ and $\underline{\chi^2}$ a χ^2 -distribution with f degrees of freedom. Then the random variable

$$(2; 1) \quad \underline{t} \stackrel{\text{def}}{=} \frac{\underline{x} + \delta}{\underline{\chi}} \sqrt{f}$$

possesses a non-central t -distribution with f degrees of freedom and non-centrality parameter δ .

Let $t(f, \delta, \alpha)$ be defined by

$$(2; 2) \quad P[\underline{t} \geq t(f, \delta, \alpha) | f, \delta] = \alpha.$$

Now we have

$$(2; 3) \quad P[\underline{t} \geq t | f, \delta] = P[\underline{t} \leq -t | f, -\delta],$$

consequently

$$(2; 4) \quad t(f, \delta, \alpha) = -t(f, -\delta, 1 - \alpha).$$

Tables of the non-central t -distribution are

I. G. J. Resnikoff and G. J. Lieberman (1957).

This table contains

a. A table of $\frac{t(f, \delta, \alpha)}{\sqrt{f}}$ as a function of f , α and

$$(2; 5) \quad p \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} e^{-1/2 x^2} dx$$

$$\frac{\sqrt{f+1}}{\sqrt{f+1}}$$

³ " $\underline{x}$ possesses a $N(\mu, \sigma)$ -distribution" is an abbreviation for: $\underline{x}$ possesses a normal distribution with mean μ and variance σ^2 .

for $f = 2(1)24(5)49$,

$$\alpha = \begin{cases} 0.995; 0.99; 0.95; 0.90; 0.75; 0.50; 0.25; 0.10; 0.05 & \text{for } f = 2(1)4, \\ 0.995; 0.99; 0.95; 0.90; 0.75; 0.50; 0.25; 0.10; 0.05; 0.01; 0.005 & \\ \text{for } f = 5(1)24(5)49 \end{cases}$$

and $p = 0.25; 0.15; 0.10; 0.065; 0.04; 0.025; 0.01; 0.004; 0.0025; 0.001$.

b. A table of

$$(2; 6) \quad P\left[\frac{t}{\sqrt{f}} \leq x \mid f, \delta\right]$$

as a function of x, f and p , for the above mentioned values of f and p .

This table only contains values of $p < 0.5$ (so values of $\delta > 0$). For negative values of δ (2; 3) and (2; 4) may be used. For $p = 0.5$ ($\delta = 0$) the non-central t -distribution reduces to Student's distribution. In this case the exact percentage points may e.g. be found in the tables of E. S. Pearson and H. O. Hartley (1954) (p. 138). This table contains $t(f, 0, \alpha)$ for

$$f = 1(1)30, 40, 60, 120$$

and $\alpha = 0.40; 0.25; 0.10; 0.05; 0.025; 0.01; 0.005; 0.0025; 0.001; 0.0005$.

The table of Resnikoff and Lieberman will now be illustrated by means of an example⁴. Let $\underline{y}$ possess a $N(\mu, \sigma)$ -distribution (μ and σ unknown); let $y_1, \dots, y_n$ be n independent observations of $\underline{y}$. Using these observations the hypothesis

$$(2; 7) \quad H_0: P[\underline{y} \geq a] = p_0,$$

will be tested, where a and p_0 ($0 < p_0 < 1$) are given numbers. Then, if H_0 is true, we have

$$(2; 8) \quad \frac{a - \mu}{\sigma} = u_{p_0},$$

where u_{ε} is defined by

$$(2; 9) \quad P[\underline{u} \geq u_{\varepsilon}] = \varepsilon \quad (\underline{u} \text{ possesses a } N(0, 1)\text{-distribution}).$$

The test statistic is

$$(2; 10) \quad t = \frac{\bar{y} - a}{s} \sqrt{n},$$

where

$$(2; 11) \quad \begin{cases} \bar{y} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n y_i, \\ s^2 \stackrel{\text{def}}{=} \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2. \end{cases}$$

Now we have

$$(2; 12) \quad t = \frac{\frac{\bar{y} - \mu}{\sigma} \sqrt{n} - \frac{a - \mu}{\sigma} \sqrt{n}}{\frac{s \sqrt{n-1}}{\sigma}} \sqrt{n-1},$$

⁴ Further examples of the use of the non-central t -distribution are given by N. L. Johnson and B. L. Welch (1940) and by G. J. Resnikoff and G. J. Lieberman (1957).

where

$$(2; 13) \quad \left\{ \begin{array}{l} \underline{x} \stackrel{\text{def}}{=} \frac{\bar{y} - \mu}{\sigma} \sqrt{n} \text{ possesses a } N(0,1)\text{-distribution,} \\ \underline{\chi^2} \stackrel{\text{def}}{=} \left\{ \frac{s \sqrt{n-1}}{\sigma} \right\}^2 \text{ possesses a } \chi^2\text{-distribution} \\ \text{with } n-1 \text{ degrees of freedom.} \end{array} \right.$$

Further $\underline{x}$ and $\underline{\chi^2}$ are distributed independently, consequently $\underline{t}$ possesses a non-central t -distribution with parameters $f = n - 1$ and $\delta = -\frac{\bar{a} - \mu}{\sigma} \sqrt{n}$. If the hypothesis H_o is true the non-centrality parameter is (cf. (2; 8))

$$(2; 14) \quad \delta = -u_{p_o} \sqrt{n}.$$

If an upper critical region is chosen the tailprobability is

$$(2; 15) \quad \begin{aligned} P[\underline{t} \geq t \mid f = n - 1, \delta = -u_{p_o} \sqrt{n}] &= \\ &= \begin{cases} P[\underline{t} \leq -t \mid f = n - 1, \delta = u_{p_o} \sqrt{n}] \text{ if } p_o < 0.5 \\ 1 - P[\underline{t} \leq t \mid f = n - 1, \delta = u_{1-p_o} \sqrt{n}] \text{ if } p_o > 0.5. \end{cases} \end{aligned}$$

These probabilities may be found in table b mentioned above for $f = n - 1$ and (cf. (2; 5))

$$(2; 16) \quad \begin{cases} p = p_o & \text{if } p_o < 0.5 \\ p = 1 - p_o & \text{if } p_o > 0.5. \end{cases}$$

The critical values with level of significance α may be found in table a .

If e.g.

$$n = 10, p_o = 0.25, \alpha = 0.05$$

then we have (cf. (2; 4))

$$(2; 17) \quad \begin{aligned} t(9; -u_{0.25} \sqrt{10}; 0.05) &= -t(9; u_{0.25} \sqrt{10}; 0.95) = \\ &= -0.164 \sqrt{9} = -0.492. \end{aligned}$$

With $\alpha = 0.10$ and the same values of n and p_o we have

$$(2; 18) \quad \begin{aligned} t(9; -u_{0.25} \sqrt{10}; 0.10) &= -t(9; u_{0.25} \sqrt{10}; 0.90) = \\ &= -0.283 \sqrt{9} = -0.849. \end{aligned}$$

If, however, $0.25 < p_o < 0.5$ or $0.5 < p_o < 0.75$ then the percentage points and tailprobabilities cannot be found in the table of Resnikoff and Lieberman. In many cases the percentage points may then be found in the table of

II N. L. Johnson and B. L. Welch (1940).

This table contains

a . A table (table V) of a quantity $\lambda(f, \delta, \alpha)$ as a function of f and of

$$(2; 19) \quad \eta' \stackrel{\text{def}}{=} \frac{\delta}{\sqrt{2f + \delta^2}}$$

for $\alpha = 0.05$

and $f = 4(1)9, 16, 36, 144, \infty$.

For other values of f , $\lambda(f, \delta, \alpha)$ may be found by linear interpolation with respect to $\frac{12}{\sqrt{f}}$.

From f, δ and $\lambda(f, \delta, \alpha)$ then follows

$$(2; 20) \quad t(f, \delta, \alpha) = \frac{\delta + \lambda(f, \delta, \alpha) \sqrt{1 + \frac{1}{2f} [\delta^2 - \{\lambda(f, \delta, \alpha)\}^2]}}{1 - \frac{1}{2f} [\lambda(f, \delta, \alpha)]^2}.$$

By means of this table and the relation (2; 4) the percentage points $t(f, \delta, \alpha)$ may be found for $\alpha = 0.05$ and $\alpha = 0.95$. For the example mentioned above ($f = 9, \delta = -u_{0.25} \sqrt{10}$) we find

$$\delta = -2.1329, \quad \eta' = -0.4492.$$

Linear interpolation in table V gives $\lambda(f, \delta, \alpha) = 1.6312$ and from (2; 20) follows

$$(2; 21) \quad t(9; -u_{0.25} \sqrt{10}; 0.05) = -0.491.$$

b. For other values of α the following table (table IV) of Johnson and Welch may be used.

Let $\delta(f, t_o, \alpha)$ be defined by

$$(2; 22) \quad P[t \geq t_o | f, \delta(f, t_o, \alpha)] = \alpha.$$

By means of table IV $\delta(f, t_o, \alpha)$ may be found. This table contains a quantity $\lambda(f, t_o, \alpha)$ as a function of f, α and

$$(2; 23) \quad \begin{cases} y' = \frac{t_o}{\sqrt{2f + t_o^2}} & \text{if } |t_o| \leq 0.75 \sqrt{2f} \\ y = \sqrt{\frac{2f}{2f + t_o^2}} & \text{if } |t_o| \geq 0.75 \sqrt{2f} \end{cases}$$

for $\alpha = 0.50; 0.40; 0.30; 0.20; 0.10; 0.05; 0.025; 0.01; 0.005$
and $f = 4(1)9, 16, 36, 144, \infty$.

For other values of f , $\lambda(f, t_o, \alpha)$ may be found by linear interpolation with respect to $\frac{12}{\sqrt{f}}$.

From f, t_o and $\lambda(f, t_o, \alpha)$ then follows

$$(2; 24) \quad \delta(f, t_o, \alpha) = t_o - \lambda(f, t_o, \alpha) \sqrt{\frac{2f + t_o^2}{2f}}.$$

This table may also be used to calculate $t_i^*(f, \delta, \alpha)$. From (2; 20) we see that $t(f, \delta, \alpha)$ may be found if $\lambda(f, \delta, \alpha)$ is known. In table IV λ is given as a function of f, t_o and α ; consequently an iteration method must be used in order to find $t(f, \delta, \alpha)$. As a first approximation Johnson and Welch give

$$(2; 25) \quad t(f, \delta, \alpha) \approx \frac{\delta + u_\alpha \sqrt{1 + \frac{1}{2f} (\delta^2 - u_\alpha^2)}}{1 - \frac{1}{2f} u_\alpha^2}.$$

In the example mentioned above with $f = 9$, $\delta = -u_{0.25} \sqrt{10}$ and $\alpha = 0.10$ this approximation gives $t(f, \delta, \alpha) = -0.827$; the exact value is (cf. (2; 18)) -0.849 .

If e.g. $f = 9$, $\delta = -u_{0.30} \sqrt{10}$ and $\alpha = 0.10$ then $t(f, \delta, \alpha)$ cannot be found in the table of Resnikoff and Lieberman, p being equal to 0.30. Also table V of Johnson and Welch cannot be used, α being equal to 0.10. Consequently in this case $t(f, \delta, \alpha)$ can only be found by iteration, using table IV of Johnson and Welch.

Finally we mention the table of

III D. B. Owen (1958).

This table contains

$$(2; 26) \quad \frac{t(f, \delta, \alpha)}{\sqrt{f+1}}$$

as a function of n , γ and P with

$$(2; 27) \quad \begin{cases} n = f + 1 \\ \gamma = 1 - \alpha \\ \delta = u_{1-P} \sqrt{n}. \end{cases}$$

The table is constructed for determining tolerance limits for a normal distribution. Let $\underline{y}$ possess a $N(\mu, \sigma)$ -distribution (μ and σ unknown) and let $\bar{y}$ and s^2 be defined by (2; 11). Let further (cf. (2; 26) and (2; 27))

$$(2; 28) \quad k(n, P, \gamma) \stackrel{\text{def}}{=} \frac{t(f, \delta, \alpha)}{\sqrt{f+1}},$$

then $\bar{y} + k(n, P, \gamma) s$ is an upper tolerance limit with tolerance coefficient P and confidence coefficient γ , i.e. at least a proportion P of the normal distribution is less than $\bar{y} + k(n, P, \gamma) s$ with confidence γ . The quantity $\bar{y} - k(n, P, \gamma) s$ is a lower tolerance limit with tolerance coefficient P and confidence coefficient γ , i.e. at least a proportion P of the normal distribution is larger than $\bar{y} - k(n, P, \gamma) s$ with confidence γ .

The table consists of the following four parts:

- a. A table of $k(n, P, \gamma)$ calculated by means of table V of Johnson and Welch for the same values of f and α , consequently for

$$\begin{aligned} n = f + 1 &= 5(1)10, 17, 37, 145, \infty \\ \gamma &= 1 - \alpha = 0.95. \end{aligned}$$

For P the following values are chosen

$$P = 0.70(0.05)0.85; 0.875; 0.90; 0.935; 0.95; 0.96; 0.975; 0.99; 0.995; 0.996; 0.9975; 0.999; 0.9995.$$

- b. A table of $k(n, P, \gamma)$ calculated by means of the above mentioned table of Resnikoff and Lieberman (Ia). The table contains, for f and $p = 1 - P$, the same values as the table of Resnikoff and Lieberman; consequently

$$\begin{aligned} n = f + 1 &= 3(1)25(5)50 \\ P = 1 - p &= 0.75; 0.85; 0.90; 0.935; 0.96; 0.975; 0.99; 0.996; 0.9975; 0.999. \end{aligned}$$

Further it contains $n = \infty$. For α however the table only contains values $< 1/2$:

$$\gamma = 1 - \alpha = \begin{cases} 0.75; 0.90; 0.95 \text{ for } n = 3(1)5 \\ 0.75; 0.90; 0.95; 0.99; 0.995 \text{ for } n = 6(1)25(5)50, \infty. \end{cases}$$

- c. A table of the approximation to $k(n, P, \gamma)$ according to (2; 25). This approximation is mentioned in the form

$$(2; 29) \quad k(n, P, \gamma) \approx \frac{-u_p + \sqrt{u_p^2 - AB}}{A}^5$$

where

$$(2; 30) \quad \begin{cases} A = 1 - \frac{u_\gamma^2}{2(n-1)} \\ B = u_p^2 - \frac{u_\gamma^2}{n} \end{cases}$$

This table contains the following values of n , P and γ ,

$$n = 2(1)200(5)400(25)1000, \infty$$

$$P = 0.70; 0.80; 0.90; 0.95; 0.99; 0.999$$

$$\gamma = 0.70; 0.80; 0.90; 0.95; 0.99; 0.999.$$

- d. A table of another approximation to $k(n, P, \gamma)$. This approximation will be described in section 3.5. The values of n , P and γ used in this table are

$$n = 2(1)102(2)180(5)300(10)400(25)750(50)1000, \infty$$

$$P = 0.875; 0.95; 0.975; 0.995; 0.9995$$

$$\gamma = 0.75; 0.90; 0.95; 0.99.$$

By means of Owen's tables $t(f, \delta, \alpha)$ may thus be found for positive values of δ and for $\alpha < 1/2$ (cf. (2; 27) and (2; 28)).

From the foregoing we see

1. that the exact percentage points may easier be found from table *a* of Resnikoff and Lieberman and the tables *a* and *b* of Owen than from the table of Johnson and Welch. Thus it is advisable to use the table of Johnson and Welch only if $t(f, \delta, \alpha)$ cannot be found in the other tables,
2. that the table of Resnikoff and Lieberman does not only contain percentage points, but also tailprobabilities,
3. that an iteration method is necessary for the table of Johnson and Welch if $\alpha \neq 0.05$ and $\neq 0.95$. In this case an accurate approximation is very useful,
4. that for some cases (e.g. for $f = 2$ or $f = 3$ and $0 < p < 0.50$ or $0.50 < p < 0.75$) the exact percentage points are not available. Thus also in these cases an accurate approximation will be very useful.

3. THE APPROXIMATIONS TO $t(f, \delta, \alpha)$

3.1 Approximations based on a normal approximation to the distribution of $\underline{x}$

It follows from (2; 1) that the inequality $\underline{t} \geq t(f, \delta, \alpha)$ may be written in the form

$$(3. 1; 1) \quad \underline{x} - \frac{t(f, \delta, \alpha)}{\sqrt{f}} \underline{x} \geq -\delta.$$

⁵ In C. Eisenhart, M. W. Hastay and W. A. Wallis (1947) this approximation is mentioned in the form (2; 29) on p. 19-20 and in the form (2; 25) on p. 75.

Now let

$$(3.1; 2) \quad \beta_f \stackrel{\text{def}}{=} \frac{\sqrt{2} \Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f} \Gamma\left(\frac{f}{2}\right)}$$

then we have

$$(3.1; 3) \quad \varepsilon \underline{\chi} = \beta_f \sqrt{f}$$

and

$$(3.1; 4) \quad \sigma^2(\underline{\chi}) = f(1 - \beta_f^2).$$

Further we have (cf. C. van Eeden (1958), Appendix A. 1)

$$(3.1; 5) \quad \begin{cases} 1. \lim_{f \rightarrow \infty} \beta_f = 1, \\ 2. \lim_{f \rightarrow \infty} f(1 - \beta_f^2) = 1/2, \\ 3. \frac{2f-1}{2f} < \beta_f^2 < \frac{2f}{2f+1} \text{ for } f = 1, 2, \dots \end{cases}$$

Further the random variable

$$\frac{\underline{\chi} - \varepsilon \underline{\chi}}{\sigma(\underline{\chi})}$$

is, for $f \rightarrow \infty$, asymptotically $N(0, 1)$ -distributed.

Therefore the distribution of $\underline{\chi}$ is, for finite f , approximated by a $N(\varepsilon \underline{\chi}, \sigma(\underline{\chi}))$ -distribution. Thus the distribution of (cf. (3.1; 1))

$$\underline{x} - \frac{t(f, \delta, \alpha)}{\sqrt{f}} \underline{\chi}$$

is approximated by a normal distribution with mean

$$-t(f, \delta, \alpha) \beta_f$$

and variance

$$1 + \{t(f, \delta, \alpha)\}^2 (1 - \beta_f^2).$$

This gives the approximation

$$(3.1; 6) \quad P[\underline{t} \geq t(f, \delta, \alpha) \mid f, \delta] \approx P\left[\underline{u} \geq \frac{-\delta + t(f, \delta, \alpha) \beta_f}{\sqrt{1 + \{t(f, \delta, \alpha)\}^2 (1 - \beta_f^2)}}\right],$$

where $\underline{u}$ is $N(0, 1)$ -distributed.⁶ An approximation to $t(f, \delta, \alpha)$ thus follows from (cf. (2; 9))

$$(3.1; 7) \quad \frac{-\delta + t(f, \delta, \alpha) \beta_f}{\sqrt{1 + \{t(f, \delta, \alpha)\}^2 (1 - \beta_f^2)}} \approx u_\alpha$$

and from (3.1; 7) follows

$$\text{Ia.} \quad t(f, \delta, \alpha) \approx \frac{\delta \beta_f + u_\alpha \sqrt{\beta_f^2 + (1 - \beta_f^2)(\delta^2 - u_\alpha^2)}}{\beta_f^2 - u_\alpha^2 (1 - \beta_f^2)}$$

⁶ If $t(f, \delta, \alpha) = 0$ the equality sign holds in (3.1; 6).

This approximation was derived by W. J. Jennett and B. L. Welch (1939).

A table of β_f may e.g. be found in the tables of E. S. Pearson and H. O. Hartley (1954, p. 184) for $f = 1(1)20(5)50(10)100$. A table of β_f , β_f^2 and $1 - \beta_f^2$ for $f = 1(1)20$ may be found in C. van Eeden (1958, table A). The paper of Jennett and Welch (1939) contains a table of $\beta_f \sqrt{f}$ and $f \sqrt{2(1 - \beta_f^2)}$.

For Student's distribution ($\delta = 0$) Ia reduces to

$$(3.1; 8) \quad t(f, 0, \alpha) \approx \frac{u_\alpha}{\sqrt{\beta_f^2 - u_\alpha^2 (1 - \beta_f^2)}}.$$

In an analogous way W. A. Hendricks derived an approximation to the percentage points of Student's distribution. He approximates the distribution of χ by a $N(\epsilon\chi, \sqrt{1/2})$ -distribution (cf. (3.1; 4) and (3.1; 5.2)). If this approximation is applied for the non-central t -distribution, we obtain

$$\text{Ib.} \quad t(f, \delta, \alpha) \approx \frac{\delta \beta_f + u_\alpha \sqrt{\beta_f^2 + \frac{1}{2f}(\delta^2 - u_\alpha^2)}}{\beta_f^2 - \frac{1}{2f}u_\alpha^2}$$

For $\delta = 0$ Ib. reduces to (cf. Hendricks' paper (1936))

$$(3.1; 9) \quad t(f, 0, \alpha) \approx \frac{u_\alpha}{\sqrt{\beta_f^2 - \frac{1}{2f}u_\alpha^2}}$$

If we put $\beta_f = 1$ in Ib (cf. (3.1; 5.1)) we obtain

$$\text{Ic.} \quad t(f, \delta, \alpha) \approx \frac{\delta + u_\alpha \sqrt{1 + \frac{1}{2f}(\delta^2 - u_\alpha^2)}}{1 - \frac{1}{2f}u_\alpha^2}$$

This is the approximation mentioned by Johnson and Welch (1940) (cf. also (2; 25)). In this case the distribution of χ is approximated by a $N(\sqrt{f}, \sqrt{1/2})$ -distribution.

For Student's distribution Ic reduces to

$$(3.1; 10) \quad t(f, 0, \alpha) \approx \frac{u_\alpha}{\sqrt{1 - \frac{1}{2f}u_\alpha^2}}$$

The approximations Ia, b and c can only be applied if respectively

$$(3.1; 11) \quad u_\alpha^2 \leq \begin{cases} \delta^2 + \frac{\beta_f^2}{1 - \beta_f^2} \\ \delta^2 + 2f\beta_f^2 \\ \delta^2 + 2f \end{cases}$$

Now it follows from (3.1; 5.3) that

$$(3.1; 12) \quad 2f\beta_f^2 < \frac{\beta_f^2}{1 - \beta_f^2} < 2f.$$

Consequently, for fixed f and δ , the approximation Ic can be applied for more values of α than Ia and Ia for more values of α than Ib.

If these approximations are applied to the examples of section 2 we obtain (cf. also (2; 17) and (2; 18))

TABLE 3.1 ;1
COMPARISON OF THE APPROXIMATIONS Ia, b AND c WITH THE EXACT
PERCENTAGE POINTS FOR $f = 9$ AND $\delta = -u_{0.25} \sqrt{10}$.

α	$t(f, \delta, \alpha)$ exact	approximations, to $t(f, \delta, \alpha)$		
		Ia	Ib	Ic
0.05	— 0.492	— 0.491	— 0.490	— 0.478
0.10	— 0.849	— 0.850	— 0.849	— 0.827

From table 3.1; 1 we see that the approximations Ia and b are in this case better than Ic.

3.2 *The approximations of H. Goldberg and H. Levine (1946) and of A. M. Peiser (1943 and 1949).*

H. Goldberg and H. Levine (1946) derived the following approximation to the percentage points of Student's distribution

$$(3.2; 1) \quad t(f, 0, \alpha) \approx u_{\alpha} + \frac{u_{\alpha}^3 + u_{\alpha}}{4f} + \frac{5u_{\alpha}^5 + 16u_{\alpha}^3 + 3u_{\alpha}}{96f^2}.$$

In order to simplify the computations they give a table of u_{α} ,

$\frac{u_{\alpha}^3 + u_{\alpha}}{4}$ and $\frac{5u_{\alpha}^5 + 16u_{\alpha}^3 + 3u_{\alpha}}{96}$ for $\alpha = 0.25; 0.10; 0.05; 0.025; 0.01; 0.005; 0.0025; 0.001$.

The approximation (3.2; 1) is derived by means of a method given by E. A. Cornish and R. A. Fisher (1937). By means of this method a random variable, whose distribution depends on a parameter f and possesses, for $f \rightarrow \infty$, asymptotically a $N(0,1)$ -distribution, can be expressed as a function of a $N(0,1)$ -variable in powers of $\frac{1}{f}$.

If this method is applied to the non-central t -distribution we obtain (cf. C. van Eeden (1958), Appendix A. 2)

$$\begin{aligned} \text{IIa.} \quad t(f, \delta, \alpha) \approx & u_{\alpha} + \frac{u_{\alpha}^3 + u_{\alpha}}{4f} + \frac{5u_{\alpha}^5 + 16u_{\alpha}^3 + 3u_{\alpha}}{96f^2} + \\ & + \delta \left[1 + \frac{2u_{\alpha}^2 + 1}{4f} + \frac{u_{\alpha}}{4f} \delta + \frac{4u_{\alpha}^4 + 12u_{\alpha}^2 + 1}{32f^2} + \frac{u_{\alpha}^3 + 4u_{\alpha}}{16f^2} \delta - \frac{u_{\alpha}^2 - 1}{24f^2} \delta^2 - \frac{u_{\alpha}}{32f^2} \delta^3 \right] \end{aligned}$$

If $t(f, 0, \alpha)$ is substituted for the first three terms of IIa (cf. (3.2; 1)) we obtain

$$\begin{aligned} \text{IIb.} \quad t(f, \delta, \alpha) \approx & t(f, 0, \alpha) + \\ & + \delta \left[1 + \frac{2u_{\alpha}^2 + 1}{4f} + \frac{u_{\alpha}}{4f} \delta + \frac{4u_{\alpha}^4 + 12u_{\alpha}^2 + 1}{32f^2} + \frac{u_{\alpha}^3 + 4u_{\alpha}}{16f^2} \delta - \frac{u_{\alpha}^2 - 1}{24f^2} \delta^2 - \frac{u_{\alpha}}{32f^2} \delta^3 \right] \end{aligned}$$

A table of

$$\frac{2u_\alpha^2 + 1}{4}, \frac{u_\alpha}{4}, \frac{4u_\alpha^4 + 12u_\alpha^2 + 1}{32}, \frac{u_\alpha^3 + 4u_\alpha}{16}, \frac{u_\alpha^2 - 1}{24} \text{ and } \frac{u_\alpha}{32}$$

for $\alpha = 0.50; 0.30; 0.25; 0.10; 0.05; 0.025; 0.01; 0.005; 0.0025; 0.001$ may be found in C. van Eeden (1958, table B).

Omitting in the approximations the terms of degree greater than 1 in $\frac{1}{f}$ we obtain

$$\text{Iic.} \quad t(f, \delta, \alpha) \approx u_\alpha + \frac{u_\alpha^3 + u_\alpha}{4f} + \delta \left[1 + \frac{2u_\alpha^2 + 1}{4f} + \frac{u_\alpha}{4f} \delta \right]$$

and

$$\text{IId.} \quad t(f, \delta, \alpha) \approx t(f, 0, \alpha) + \delta \left[1 + \frac{2u_\alpha^2 + 1}{4f} + \frac{u_\alpha}{4f} \delta \right]$$

For $\delta = 0$ Iic reduces to

$$(3.2; 2) \quad t(f, 0, \alpha) \approx u_\alpha + \frac{u_\alpha^3 + u_\alpha}{4f}.$$

The approximation (3.2; 2) was derived by A. M. Peiser (1943 and 1949).

If these approximations are applied to the examples of section 2 ($f = 9$, $\delta = u_{0.25} \sqrt{10}$) we obtain (cf. also table 3.1; 1).

TABLE 3.2; 1
COMPARISON OF THE APPROXIMATIONS IIa, b, c AND d WITH THE EXACT
PERCENTAGE POINTS FOR $f = 9$ AND $\delta = -u_{0.25} \sqrt{10}$.

α	$t(f, \delta, \alpha)$ exact	approximations to $t(f, \delta, \alpha)$			
		IIa	IIb	Iic	IId
0.05	— 0.492	— 0.491	— 0.490	— 0.491	— 0.472
0.10	— 0.849	— 0.850	— 0.850	— 0.849	— 0.842

Thus in this case the approximations IIa, b and c are better than IId.

3.3 The approximation of M. Merrington and E. S. Pearson (1958)

M. Merrington and E. S. Pearson (1958) give an approximation to the percentage points of the non-central t -distribution based on an approximation of the distribution of t by a Pearson type IV distribution.

This approximation is calculated as follows. The first four moments of t are

$$(3.3; 1) \quad \begin{cases} \mu \stackrel{\text{def}}{=} E(t | f, \delta) = \frac{f}{f-1} \delta \beta_f, \\ \sigma^2 \stackrel{\text{def}}{=} \sigma^2(t | f, \delta) = \frac{f}{f-2} (1 + \delta^2) - \mu^2, \\ \mu_3 \stackrel{\text{def}}{=} E\{(t - \mu)^3 | f, \delta\} = \mu \left\{ \frac{f(2f-3+\delta^2)}{(f-2)(f-3)} - 2\sigma^2 \right\}, \\ \mu_4 \stackrel{\text{def}}{=} E\{(t - \mu)^4 | f, \delta\} = \\ = \frac{f^2}{(f-2)(f-4)} (3 + 6\delta^2 + \delta^4) - \mu^2 \left[\frac{f\{(f+1)\delta^2 + 3(3f-5)\}}{(f-2)(f-3)} - 3\sigma^2 \right]. \end{cases}$$

Now let

$$(3.3; 2) \quad b_1 \stackrel{\text{def}}{=} \frac{\tilde{\mu}_3^2}{\sigma^6} \quad b_2 \stackrel{\text{def}}{=} \frac{\tilde{\mu}_4}{\sigma^4}$$

and let $z(b_1, b_2, \alpha)$ be defined by

$$(3.3; 3) \quad P \left[\frac{\underline{z} - \varepsilon \underline{z}}{\sigma(\underline{z})} \geq z(b_1, b_2, \alpha) \right] = \alpha,$$

where $\underline{z}$ possesses a Pearson-distribution with parameters b_1 and b_2 . Then we have

$$\text{III} \quad t(f, \delta, \alpha) \approx \mu + \sigma z(b_1, b_2, \alpha)$$

A table of $z(b_1, b_2, \alpha)$ may e.g. be found in the tables E. S. Pearson and H. O. Hartley (1954) (p. 206) for

$$b_1 = 0.00; 0.01; 0.03; 0.05(0.05) 0.20(0.10) 1.00,$$

$$b_2 = 1.8(0.2) 5.0$$

$$\text{and } \alpha = 0.995; 0.99; 0.975; 0.95; 0.05; 0.025; 0.01; 0.005.$$

For Student's distribution we obtain (cf. (3.3; 1) and 3.3; 2))

$$(3.3; 4) \quad b_1 = 0, \quad b_2 = \frac{3(f-2)}{f-4}.$$

Consequently

$$(3.3; 5) \quad t(f, 0, \alpha) \approx \sqrt{\frac{f}{f-2}} z\left(0, \frac{3(f-2)}{f-4}, \alpha\right).$$

The approximation III cannot be applied for all values of f , δ and α because

1. the fourth moment of $\underline{t}$ does not exist for $f \leq 4$. Consequently the approximation can only be applied for $f \geq 5$.

2. the above mentioned table of $z(b_1, b_2, \alpha)$ does not contain all values of b_1 and b_2 of the non-central t -distribution. If e.g. $\delta = 0$ and $f = 6$ then (cf. (3.3; 4))

$$(3.3; 6) \quad b_1 = 0, \quad b_2 = 6,$$

but the table only contains values of $b_2 \leq 5$. For $f = 7$ and $\delta = 0$ we have

$$(3.3; 7) \quad b_1 = 0, \quad b_2 = 5.$$

Consequently for Student's distribution the approximation can only be used for $f \geq 7$.

For a more detailed description of the relation between f , δ , b_1 and b_2 we refer to the paper of Merrington and Pearson.

3. Finally the approximation can only be applied for the values of α in the table of $z(b_1, b_2, \alpha)$.

3.4 The approximation of B. I. Harley (1957).

Let $\underline{r}$ denote the estimate of the correlation coefficient ρ of a two-dimensional normal distribution, calculated from n independent pairs of observations. Then, if $\rho = 0$, the random variable

$$(3.4; 1) \quad \frac{\underline{r}}{\sqrt{1-r^2}} \sqrt{n-2}$$

possesses a Student-distribution with $f = n - 2$ degrees of freedom. Thus if $r(n, \rho, \alpha)$ is defined by

$$(3.4; 2) \quad P[\underline{r} \geq r(n, \rho, \alpha) \mid n, \rho] = \alpha,$$

we have

$$(3.4; 3) \quad t(f, 0, \alpha) = \frac{r(f+2, 0, \alpha)}{\sqrt{1 - \{r(f+2, 0, \alpha)\}^2}} \sqrt{f}.$$

B. I. Harley (1957) shows that, for $\rho \neq 0$, the random variable

$$(3.4; 4) \quad \frac{\underline{r}}{\sqrt{1 - \underline{r}^2}} \sqrt{n-2} \frac{\sqrt{2(1 - \rho^2)}}{2 - \rho^2}$$

is approximately distributed as $\underline{t}$ with

$$(3.4; 5) \quad f = n - 2, \quad \delta = \rho \sqrt{\frac{2n-3}{2 - \rho^2}}.$$

Consequently

$$\text{IV } t(f, \delta, \alpha) \approx \frac{r(f+2, \rho, \alpha)}{\sqrt{1 - \{r(f+2, \rho, \alpha)\}^2}} \sqrt{\frac{f(2f+1)}{2f+1 + \delta^2}} \text{ with } \rho = \delta \sqrt{\frac{2}{2f+1 + \delta^2}} \neq 0.$$

For $\rho \neq 0$ no tables of $r(n, \rho, \alpha)$ are available. The table of F. N. David (1938) contains the exact distribution of $\underline{r}$ for $n = 3(1)25, 50, 100, 200, 400$ and $\rho = 0.0(0.1)0.9$. From this table the percentage points $r(n, \rho, \alpha)$ may be obtained by interpolation with respect to ρ and r , but this is not very accurate. Therefore the approximation can better be considered as an approximation to $r(n, \rho, \alpha)$ on the basis of the exact tables of $t(f, \delta, \alpha)$. We then obtain

$$(3.4; 6) \quad r(n, \rho, \alpha) \approx t(n-2, \delta, \alpha) \sqrt{\frac{2 - \rho^2}{2(n-2)(1 - \rho^2) + (2 - \rho^2)\{t(n-2, \delta, \alpha)\}^2}}$$

with

$$(3.4; 7) \quad \delta = \rho \sqrt{\frac{2n-3}{2 - \rho^2}} \neq 0.$$

A comparison of the approximation (3.4; 6) with the exact percentage points $r(n, \rho, \alpha)$ is difficult, because of the lack of tables of $r(n, \rho, \alpha)$.

B. I. Harley gives a comparison of approximation IV with the exact values of $t(f, \delta, \alpha)$ for $f = 7$ and 16. He chooses δ in such a way that ρ is a value occurring in the table of F. N. David; then only an interpolation with respect to r is necessary to obtain $r(n, \rho, \alpha)$.

3.5 The approximation in the table of D. B. Owen (1958)

Apart from the one-sided tolerance limits $\bar{y} + k(n, P, \gamma)s$ and $\bar{y} - k(n, P, \gamma)s$ mentioned in section 2, also two-sided tolerance limits may be obtained. Then a quantity $\lambda(n, \beta, \gamma)$ is determined in such a way that the proportion of the normal distribution between $\bar{y} - \lambda(n, \beta, \gamma)s$ and $\bar{y} + \lambda(n, \beta, \gamma)s$ is at least β with confidence γ . A. Wald and J. Wolfowitz (1946) derived the following approximation to $\lambda(n, \beta, \gamma)$

$$(3.5; 1) \quad \lambda(n, \beta, \gamma) \approx r \sqrt{\frac{n-1}{\chi^2(n-1, \gamma)}},$$

where $\chi^2(n-1, \gamma)$ is defined by

$$(3.5; 2) \quad P[\chi^2 \geq \chi^2(n-1, \gamma) | n-1] = \gamma$$

and r by

$$(3.5; 3) \quad \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{n}} - r}^{\frac{1}{\sqrt{n}} + r} e^{-1/2 t^2} dt = \beta.$$

A table of $\lambda(n, \beta, \gamma)$ according to (3.5; 1) may be found in C. Eisenhart, N. W. Hastay and W. A. Wallis (1946, p. 102-107) for

$$n = 2(1) 102(2) 180(5) 300(10) 400(25) 750(50) 1000, \infty$$

$$\beta = 0.75; 0.90; 0.95; 0.99; 0.999,$$

$$\gamma = 0.75; 0.90; 0.95; 0.99.$$

According to McClung (1955) we have

$$(3.5; 4) \quad k(n, P, \gamma) \approx 1.05 \lambda(n, 2P-1, \gamma)$$

and from (3.5; 4) follows (cf. (2; 27) and (2; 28))

$$\boxed{V \quad t(f, \delta, \alpha) \approx 1.05 \sqrt{f+1} \lambda(f+1, 2P-1, 1-\alpha) \text{ with } u_{1-P} = \frac{\delta}{\sqrt{f+1}} > 0.}$$

The table of D. B. Owen (1958) contains this approximation to $\frac{t(f, \delta, \alpha)}{\sqrt{f+1}}$ for the abovementioned values of n, β and γ , thus for

$$f = n-1 = 1(1) 101(2) 179(5) 299(10) 399(25) 749(50) 999, \infty,$$

$$P = \frac{\beta+1}{2} = 0.875; 0.95; 0.975; 0.995; 0.9995,$$

$$\alpha = 1-\gamma = 0.25; 0.10; 0.05; 0.01.$$

Now it may easily be seen that

$$(3.5; 5) \quad \lim_{n \rightarrow \infty} k(n, P, \gamma) = \lim_{n \rightarrow \infty} \lambda(n, 2P-1, \gamma) = u_{1-P}.$$

Consequently

$$(3.5; 5) \quad \lim_{n \rightarrow \infty} \frac{k(n, P, \gamma)}{\lambda(n, 2P-1, \gamma)} = 1,$$

and from (3.5; 5) it follows that, although the approximation may be accurate for small values of f , for large values of f , the accuracy decreases if f increases. Therefore we do not here consider this approximation any further.

3.6 An approximation based on a normal approximation to the non-central t -distribution

The random variable

$$\frac{t - \epsilon t}{\sigma(t)},$$

where (cf. (3.3; 1))

$$(3.6; 1) \quad \begin{cases} \varepsilon \underline{t} = \frac{f}{f-1} \delta \beta_f \\ \sigma^2(\underline{t}) = \frac{f}{f-2} (1 + \delta^2) - \frac{f^2}{(f-1)^2} \delta^2 \beta_f^2 \end{cases}$$

is, for $f \rightarrow \infty$, asymptotically $N(0,1)$ -distributed. Consequently

$$\text{VI} \quad t(f, \delta, \alpha) \approx \frac{f}{f-1} \delta \beta_f + u_\alpha \sqrt{\frac{f}{f-2} (1 + \delta^2) - \frac{f^2}{(f-1)^2} \delta^2 \beta_f^2} \text{ for } f > 2.$$

For Student's distribution this approximation reduces to

$$(3.6; 2) \quad t(f, 0, \alpha) \approx u_\alpha \sqrt{\frac{f}{f-2}} \text{ for } f > 2.$$

The normal approximation to the distribution of $\frac{t - \varepsilon \underline{t}}{\sigma(\underline{t})}$ may only be used for very small values of δ and large values of f . Therefore we only applied approximation VI for Student's distribution.

4. COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS

The approximations were compared with the exact percentage points for

$$\begin{aligned} \alpha &= 0.99; 0.975; 0.95; 0.90; 0.70; 0.50; 0.30; 0.10; 0.05; 0.025; 0.01, \\ p &= 0.50; 0.45; 0.40; 0.35; 0.30; 0.25; 0.15; 0.125 \end{aligned}$$

and for all values of $f \leq 9$, where exact percentage points are available. For $p = 0.50$ we moreover used $f = 30, 40$ and 60 .

The results are given in the tables C–M in C. van Eeden (1958). The results for

$$\alpha = 0.99; 0.975; 0.95; 0.05; 0.025 \text{ and } 0.01$$

are summarized in the tables I–X at the end of this paper.

In all tables the most accurate approximation is indicated as follows. Approximation III being applicable only in a few cases, we first indicated the most accurate approximation of all except III by printing it in bold type. If III is as accurate as the most accurate of all other approximations, it is printed in bold type too; if III is the most accurate of all approximations it is printed in italics. The most accurate approximation is defined as the one with the smallest difference with the exact value according to Johnson and Welch; if this exact value is not available we compare the approximations with the exact value according to Resnikoff and Lieberman.

We first consider the approximations to *Student's distribution* ($p = 0.50$; tables I and II). In this case the approximations

$$\text{I } a, b, c, \quad \text{II } a, c, \quad \text{III and VI}$$

may be applied.

The approximations II b and d reduce to the equality $t(f, 0, \alpha) = t(f, 0, \alpha)$ and approximation IV gives, for $p = 0.50$, the exact percentage points on the basis of tables of $r(n, 0, \alpha)$ (cf. section 3; 4).

Student's distribution being symmetric we only applied the approximations for values of $\alpha < 1/2$.

From table I we see that, for small values of f , IIa is the most accurate approximation of all except III. In some cases approximation III is better. However an approximation to $t(f, 0, \alpha)$ for small values of f will only be necessary for values of α not occurring with existing tables. Then however III cannot be applied, the tables for this approximation only containing the values $\alpha = 0.05; 0.025; 0.01$ and 0.005 .

For large values of f (table II) IIa is the most accurate approximation; it is, in three decimal places, identical with the exact value.

We now consider the *non-central t-distribution*. The exact values were calculated as follows:

For all values of α and p mentioned in the beginning of this section the exact percentage points were calculated from the tables of Johnson and Welch for $f = 4(1)9$. A table of these exact values is given in C. van Eeden (1958, table N); for

$$\alpha = 0.99; 0.975; 0.95, 0.005; 0.025 \text{ and } 0.01$$

they are summarized in table XI at the end of this paper.

For $p = 0.25$ and 0.15 they were also calculated from the table of Resnikoff and Lieberman (cf. section 2, table Ia and Ib). For

$$\alpha = \begin{cases} 0.99; 0.95; 0.90; 0.50; 0.10; 0.05 \text{ and } f = 2(1)9 \\ 0.01 \text{ and } f = 5(1)9 \end{cases}$$

we used table Ia. For the other values of α , thus for

$$\alpha = \begin{cases} 0.975; 0.70; 0.30; 0.025 \text{ and } f = 2(1)9 \\ 0.01 \text{ and } f = 2(1)4 \end{cases}$$

the percentage points were found by linear interpolation in the table of the distribution function of t (cf. section 2, table Ib). The following approximations were applied

I a , b and c in all cases were (cf. section 3.1)

$$\beta_f^2 - (1 - \beta_f^2)(\delta_f^2 - u_\alpha^2) \geq 0 \quad \text{for Ia}$$

$$\beta_f^2 - \frac{\delta_f^2 - u_\alpha^2}{2f} \geq 0 \quad \text{for Ib}$$

$$1 - \frac{\delta_f^2 - u_\alpha^2}{2f} \geq 0 \quad \text{for Ic.}$$

II a , b , c and d in all cases

III in all cases where it can be applied (cf. section 3.3), i.e. for

$$p = 0.45 \text{ and } f = 8, 9$$

$$p = 0.40 \text{ and } f = 8, 9$$

$$p = 0.35 \text{ and } f = 9$$

and $\alpha = 0.99; 0.975; 0.95; 0.05; 0.025; 0.01$.

Approximation IV is not applied. As mentioned in section 3.4 it requires an interpolation in the tables of the distribution of r with respect to r and ρ ; this will not give a very accurate result. Therefore this approximation is only compared with the exact values and with the other approximations for the same values of f , δ and α as used by Harley (cf. table X).

Approximation VI is not applied for the non-central t -distribution.

The tables III–IX are summarized in fig. 1. In this figure we indicated, for all values of p and α used in the tables, the approximations which may be considered as good approximations to $t(f, \delta, \alpha)$ for all values of $f \leq 9$. However this figure may also be used for larger values of f ; the only difference for larger values of f will be that more approximations can be used than indicated in fig. 1.

From fig. 1 we see that, among the approximations I, II and III, I and II are the most important ones and that Ia, b and IIa, b are the best ones.

Finally we consider approximation IV. From table X we see that this is a very good approximation. However, as mentioned earlier, it is difficult to apply it for those values of δ for which ρ does not occur in the tables of F. N. David. But, of course, it may be used as an approximation to $r(\rho, n, \alpha)$ from the exact value $t(f, \delta, \alpha)$.

Fig. 1

$\alpha \backslash p$		0,50				0,45				0,40				0,35				0,30				0,25				0,15				0,125			
		a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d				
0,99	I																																
	II																																
0,975	I																																
	II																																
0,95	I																																
	II																																
0,905	I																																
	II																																
0,9025	I																																
	II																																
0,901	I																																
	II																																

FIG. 1

In this figure we indicated, for all values of α and p used in the tables III–IX, the approximations which may be considered as good approximations to $t(f, \delta, \alpha)$.

The approximations IIb and IIc cannot be used as an approximation for cases with $p = 0.50$; they lead in this case to the equality $t(f, 0, \alpha) = t(f, 0, \alpha)$. However they are important for values of p near 0.50; therefore we indicated these approximations in the column $p = 0.50$.

This figure may also be used for values of α and p , not used in the tables, with $0.01 < \alpha < 0.99$ and $0.125 < p < 0.50$. E.g. for $\alpha = 0.95$ and any p with $0.35 < p < 0.30$ Ia and IIa may be used as an approximation to $t(f, \delta, \alpha)$.

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RESUME

Quelques approximations des percentiles de la distribution non centrée de t . Dans cet article nous voulons considérer les approximations suivantes des percentiles de la distribution non centrée de t .

I. 3 approximations qui sont en relation étroite l'une avec l'autre et sont basées sur une approximation normale de la distribution de $\underline{x} + k\sqrt{\underline{X}^2}$, où $\underline{x}$ a une distribution normale avec moyenne zéro et avec variance 1 et où $\underline{X}^2$ a une distribution de χ^2 avec f degrés de liberté ($\underline{x}$ et $\underline{X}^2$ sont distribués indépendamment). Deux de ces approximations ont été établies par W. J. Jennett et B. L. Welch (1939) et par N. L. Johnson et B. L. Welch (1940) respectivement. La troisième a été établie par W. A. Hendricks (1936) pour les percentiles de la distribution centrée de t (distribution de Student) et peut être généralisée pour la distribution non centrée de t .

II. 4 approximations qui sont en relation étroite l'une avec l'autre et qui ont été trouvées au moyen du développement de Cornish-Fisher (1937). Ces approximations-ci sont une généralisation des approximations de H. Goldberg et H. Levine (1946) et de A. M. Peiser (1943) pour les percentiles de la distribution centrée de t .

III. L'approximation de M. Merrington et E. S. Pearson (1958) basée sur l'approximation de type IV de Pearson de la distribution non centrée de t .

IV. L'approximation de B. I. Harley (1957) basée sur une approximation de la distribution d'une fonction de t par la distribution du coefficient de corrélation dans des échantillons d'une distribution normale à deux variables.

V. Une approximation mentionnée dans la table de coefficients pour les limites de tolérance d'une distribution normale, dressée par D. B. Owen (1958). Cette approximation est basée sur une relation approximative entre des limites de tolérance unilatérales et bilatérales.

VI. Une approximation basée sur une approximation normale de la distribution non centrée de t .

Ces approximations sont comparées avec les percentiles exactes pour plusieurs valeurs du paramètre de non-centralité δ , la probabilité α et le nombre de degrés de liberté f . Pour la distribution centrée de t , les approximations sont spécialement importantes pour les valeurs larges de f , pour lesquelles on ne peut pas disposer de tables. Alors une des approximations II est la meilleure; elle est identique aux percentiles exactes jusqu'à 3 décimales pour $0,50 \leq \alpha \leq 0,99$. Pour la distribution non centrée de t nous avons employé $f = 2(1)9$. Pour ces valeurs-ci de f les approximations I et II sont les plus importantes. Dans quelques cas III et IV peuvent être meilleures, mais pour ces approximations il faut plus de calculs. Les approximations V et VI ne sont pas recommandables. L'approximation VI seulement donne des résultats raisonnables pour la distribution centrée de t .

Pour le reste l'article contient une description des tables exactes de la distribution non centrée de t

TABLE I
COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR STUDENT'S
DISTRIBUTION ($f = 2(1)9$)

α	f	$t(f, 0, \alpha)$ exact	approximations to $t(f, 0, \alpha)$						
			I			II		III	VI
			a	b	c	a	c		
0.05	2	2.920	3.635	4.983	2.892	2.762	2.407	—	—
	3	2.353	2.480	2.608	2.220	2.311	2.153	—	2.849
	4	2.132	2.181	2.227	2.022	2.115	2.026	—	2.326
	5	2.015	2.041	2.064	1.926	2.006	1.950	—	2.123
	6	1.943	1.959	1.973	1.869	1.938	1.899	—	2.015
	7	1.895	1.905	1.915	1.831	1.892	1.863	1.893	1.946
	8	1.860	1.867	1.874	1.805	1.858	1.835	1.865	1.899
	9	1.833	1.839	1.844	1.784	1.832	1.814	1.837	1.865
0.025	2	4.303	—	—	9.849	3.852	3.146	—	—
	3	3.182	3.785	4.293	3.268	3.064	2.751	—	3.395
	4	2.776	2.967	3.086	2.719	2.729	2.553	—	2.772
	5	2.571	2.662	2.715	2.498	2.547	2.434	—	2.530
	6	2.447	2.500	2.530	2.377	2.434	2.355	—	2.400
	7	2.365	2.399	2.418	2.301	2.356	2.299	2.366	2.319
	8	2.306	2.330	2.343	2.248	2.301	2.256	2.309	2.263
	9	2.262	2.280	2.290	2.210	2.258	2.224	2.268	2.222
0.01	2	6.965	—	—	—	5.621	4.191	—	—
	3	4.541	13.278	—	7.432	4.205	3.569	—	4.029
	4	3.747	4.620	5.112	4.090	3.616	3.259	—	3.290
	5	3.365	3.708	3.855	3.435	3.301	3.072	—	3.003
	6	3.143	3.325	3.395	3.140	3.107	2.948	—	2.849
	7	2.998	3.110	3.152	2.970	2.976	2.859	2.993	2.753
	8	2.896	2.973	3.000	2.860	2.882	2.792	2.893	2.686
	9	2.821	2.876	2.896	2.782	2.811	2.741	2.823	2.638

TABLE II
COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR STUDENT'S
DISTRIBUTION ($f = 30, 40, 60$)

α	f	$t(f, 0, \alpha)$ exact	approximations to $t(f, 0, \alpha)$						
			I			II		III	VI
			a	b	c	a	c		
0.05	30	1.697	1.698	1.698	1.683	1.697	1.696	1.698	1.703
	40	1.684	1.684	1.684	1.673	1.684	1.683	1.683	1.688
	60	1.671	1.671	1.671	1.664	1.671	1.670	1.668	1.673
0.025	30	2.042	2.043	2.044	2.026	2.042	2.039	2.041	2.029
	40	2.021	2.022	2.022	2.009	2.021	2.019	2.019	2.011
	60	2.000	2.001	2.001	1.992	2.000	1.999	2.000	1.994
0.01	30	2.457	2.460	2.461	2.439	2.457	2.451	2.458	2.048
	40	2.423	2.425	2.425	2.409	2.423	2.420	2.425	2.387
	60	2.390	2.391	2.391	2.381	2.390	2.388	2.392	2.366

TABLE III

COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR $p = 0.45$

α	f	$t(f, \delta, \alpha)$ exact	approximations to $t(f, \delta, \alpha)$							
			I			II				III
			a	b	c	a	b	c	d	
0.99	4	-3.13	-3.66	-3.96	-3.28	-3.05	-3.18	-2.78	-3.27	—
	5	-2.80	-3.01	-3.10	-2.80	-2.76	-2.82	-2.59	-2.89	—
	6	-2.59	-2.70	-2.75	-2.56	-2.57	-2.61	-2.46	-2.66	—
	7	-2.46	-2.52	-2.55	-2.41	-2.44	-2.47	-2.36	-2.50	—
	8	-2.35	-2.40	-2.41	-2.31	-2.35	-2.36	-2.29	-2.39	-2.35
	9	-2.27	-2.30	-2.32	-2.23	-2.27	-2.28	-2.22	-2.30	-2.28
0.975	4	-2.279	-2.393	-2.469	-2.204	-2.252	-2.299	-2.129	-2.352	—
	5	-2.092	-2.144	-2.177	-2.017	-2.078	-2.102	-2.002	-2.139	—
	6	-1.969	-1.999	-2.017	-1.904	-1.963	-1.976	-1.911	-2.003	—
	7	-1.882	-1.901	-1.912	-1.825	-1.878	-1.887	-1.842	-1.908	—
	8	-1.815	-1.828	-1.836	-1.765	-1.813	-1.818	-1.785	-1.835	<i>-1.816</i>
	9	-1.760	-1.770	-1.776	-1.717	-1.759	-1.763	-1.739	-1.777	-1.762
0.95	4	-1.707	-1.734	-1.763	-1.612	-1.698	-1.715	-1.640	-1.746	—
	5	-1.589	-1.604	-1.618	-1.516	-1.586	-1.595	-1.551	-1.616	—
	6	-1.510	-1.519	-1.527	-1.451	-1.508	-1.513	-1.485	-1.529	—
	7	-1.451	-1.456	-1.462	-1.401	-1.450	-1.453	-1.434	-1.466	—
	8	-1.403	-1.407	-1.410	-1.360	-1.403	-1.405	-1.390	-1.415	-1.408
	9	-1.363	-1.366	-1.368	-1.326	-1.363	-1.364	-1.353	-1.372	-1.363
0.05	4	2.579	2.664	2.732	2.462	2.555	2.572	2.428	2.534	—
	5	2.461	2.506	2.542	2.361	2.447	2.456	2.364	2.429	—
	6	2.396	2.424	2.445	2.310	2.387	2.392	2.328	2.372	—
	7	2.356	2.377	2.391	2.282	2.352	2.355	2.307	2.339	—
	8	2.333	2.348	2.359	2.268	2.331	2.333	2.295	2.320	2.332
	9	2.320	2.332	2.340	2.262	2.318	2.319	2.290	2.309	2.319
0.025	4	3.303	3.604	3.779	3.285	3.235	3.282	2.996	3.219	—
	5	3.077	3.225	3.302	3.017	3.042	3.066	2.885	3.022	—
	6	2.947	3.037	3.081	2.882	2.929	2.942	2.817	2.909	—
	7	2.870	2.929	2.957	2.805	2.857	2.866	2.774	2.840	—
	8	2.819	2.861	2.881	2.758	2.811	2.816	2.744	2.794	<i>2.817</i>
	9	2.783	2.816	2.831	2.728	2.778	2.782	2.726	2.764	<i>2.779</i>
0.01	4	4.41	5.75	6.51	5.02	4.22	4.35	3.76	4.25	—
	5	3.96	4.50	4.71	4.14	3.88	3.94	3.57	3.87	—
	6	3.73	4.01	4.11	3.77	3.67	3.71	3.46	3.65	—
	7	3.57	3.75	3.81	3.57	3.54	3.56	3.38	3.51	—
	8	3.46	3.59	3.63	3.45	3.45	3.46	3.32	3.42	3.46
	9	3.39	3.49	3.51	3.37	3.38	3.39	3.28	3.36	3.39

TABLE IV

COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR $p = 0.40$

α	f	$t(f, \delta, \alpha)$ exact	approximations to $t(f, \delta, \alpha)$							
			I			II				III
			a	b	c	a	b	c	d	
0.99	4	-2.56	-2.85	-3.01	-2.59	-2.50	-2.64	-2.32	-2.81	—
	5	-2.26	-2.38	-2.43	-2.22	-2.24	-2.30	-2.13	-2.42	—
	6	-2.08	-2.13	-2.16	-2.02	-2.06	-2.10	-1.99	-2.19	—
	7	-1.94	-1.97	-1.99	-1.89	-1.93	-1.95	-1.88	-2.02	—
	8	-1.83	-1.85	-1.86	-1.79	-1.83	-1.84	-1.79	-1.90	-1.83
	9	-1.75	-1.76	-1.77	-1.71	-1.74	-1.75	-1.72	-1.80	-1.74
0.975	4	-1.812	-1.872	-1.916	-1.732	-1.797	-1.884	-1.719	-1.942	—
	5	-1.635	-1.661	-1.681	-1.567	-1.628	-1.652	-1.582	-1.719	—
	6	-1.512	-1.526	-1.536	-1.456	-1.509	-1.522	-1.479	-1.571	—
	7	-1.417	-1.426	-1.432	-1.371	-1.415	-1.424	-1.396	-1.462	—
	8	-1.341	-1.346	-1.350	-1.301	-1.340	-1.345	-1.325	-1.375	-1.345
	9	-1.275	-1.279	-1.281	-1.241	-1.274	-1.278	-1.265	-1.303	-1.274
0.95	4	-1.303	-1.316	-1.332	-1.227	-1.299	-1.316	-1.266	-1.372	—
	5	-1.182	-1.188	-1.196	-1.125	-1.181	-1.190	-1.162	-1.227	—
	6	-1.092	-1.096	-1.100	-1.048	-1.092	-1.097	-1.081	-1.125	—
	7	-1.020	-1.022	-1.025	-0.985	-1.020	-1.023	-1.013	-1.045	—
	8	-0.958	-0.960	-0.961	-0.929	-0.959	-0.961	-0.953	-0.978	-0.958
	9	-0.903	-0.905	-0.906	-0.879	-0.904	-0.905	-0.900	-0.919	-0.910
0.05	4	3.062	3.188	3.284	2.938	3.023	3.040	2.852	2.958	—
	5	2.936	3.006	3.056	2.827	2.915	2.924	2.801	2.866	—
	6	2.874	2.920	2.951	2.779	2.862	2.867	2.779	2.823	—
	7	2.844	2.877	2.899	2.761	2.838	2.841	2.774	2.806	—
	8	2.833	2.858	2.873	2.758	2.829	2.831	2.777	2.802	2.833
	9	2.832	2.852	2.864	2.764	2.829	2.830	2.787	2.806	2.831
0.025	4	3.873	4.312	4.556	3.911	3.777	3.824	3.466	3.689	—
	5	3.620	3.839	3.946	3.582	3.570	3.594	3.362	3.499	—
	6	3.483	3.618	3.679	3.427	3.456	3.469	3.304	3.396	—
	7	3.406	3.498	3.537	3.346	3.388	3.397	3.273	3.339	—
	8	3.361	3.428	3.456	3.301	3.349	3.354	3.257	3.307	3.360
	9	3.336	3.387	3.408	3.278	3.326	3.330	3.253	3.291	3.336
0.01	4	5.12	7.05	8.16	6.09	4.88	5.01	4.29	4.78	—
	5	4.61	5.38	5.67	4.93	4.49	4.56	4.10	4.40	—
	6	4.34	4.76	4.90	4.47	4.27	4.31	3.99	4.19	—
	7	4.18	4.44	4.53	4.23	4.13	4.16	3.92	4.06	—
	8	4.07	4.26	4.31	4.08	4.04	4.06	3.88	3.98	4.08
	9	4.00	4.14	4.18	4.00	3.98	3.99	3.85	3.93	4.00

TABLE V

COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR $p = 0.35$

α	f	$t(f, \delta, \alpha)$ exact	approximations to $t(f, \delta, \alpha)$							
			I			II				III
			a	b	c	a	b	c	d	
0.99	4	-2.02	-2.16	-2.24	-1.98	-1.99	-2.12	-1.87	-2.36	—
	5	-1.76	-1.80	-1.83	-1.70	-1.74	-1.80	-1.67	-1.97	—
	6	-1.57	-1.60	-1.61	-1.52	-1.57	-1.60	-1.53	-1.72	—
	7	-1.44	-1.45	-1.46	-1.39	-1.43	-1.46	-1.41	-1.55	—
	8	-1.32	-1.33	-1.34	-1.29	-1.32	-1.34	-1.31	-1.41	—
	9	-1.23	-1.24	-1.24	-1.20	-1.23	-1.24	-1.22	-1.30	-1.23
0.975	4	-1.366	-1.392	-1.415	-1.294	-1.359	-1.406	-1.315	-1.538	—
	5	-1.195	-1.205	-1.215	-1.140	-1.192	-1.216	-1.168	-1.305	—
	6	-1.067	-1.072	-1.077	-1.025	-1.066	-1.079	-1.052	-1.144	—
	7	-0.963	-0.966	-0.969	-0.930	-0.962	-0.971	-0.954	-1.020	—
	8	-0.874	-0.876	-0.878	-0.848	-0.875	-0.880	-0.868	-0.918	—
	9	-0.795	-0.797	-0.798	-0.774	-0.795	-0.799	-0.793	-0.831	-0.801
0.95	4	-0.913	-0.917	-0.924	-0.857	-0.912	-0.929	-0.895	-1.001	—
	5	-0.785	-0.787	-0.790	-0.746	-0.784	-0.793	-0.777	-0.842	—
	6	-0.682	-0.683	-0.685	-0.655	-0.682	-0.687	-0.679	-0.723	—
	7	-0.595	-0.596	-0.597	-0.574	-0.596	-0.599	-0.593	-0.625	—
	8	-0.518	-0.518	-0.519	-0.502	-0.519	-0.521	-0.516	-0.541	—
	9	-0.447	-0.448	-0.448	-0.435	-0.448	-0.449	-0.446	-0.465	-0.451
0.05	4	3.586	3.766	3.894	3.461	3.531	3.548	3.309	3.415	—
	5	3.450	3.552	3.619	3.334	3.420	3.429	3.270	3.335	—
	6	3.390	3.458	3.500	3.287	3.373	3.378	3.262	3.306	—
	7	3.368	3.418	3.447	3.276	3.358	3.361	3.272	3.304	—
	8	3.368	3.405	3.427	3.284	3.361	3.363	3.291	3.316	—
	9	3.379	3.410	3.426	3.303	3.374	3.375	3.317	3.336	3.376
0.025	4	4.494	5.104	5.430	4.609	4.366	4.413	3.973	4.196	—
	5	4.208	4.519	4.661	4.205	4.141	4.165	3.875	4.012	—
	6	4.062	4.254	4.335	4.023	4.023	4.036	3.828	3.920	—
	7	3.985	4.118	4.171	3.934	3.959	3.968	3.810	3.876	—
	8	3.944	4.043	4.081	3.890	3.928	3.933	3.807	3.857	—
	9	3.928	4.004	4.033	3.872	3.914	3.918	3.817	3.855	3.925
0.01	4	5.89	8.54	10.06	7.30	5.58	5.72	4.87	5.35	—
	5	5.31	6.37	6.77	5.81	5.16	5.23	4.68	4.97	—
	6	5.01	5.59	5.78	5.23	4.92	4.96	4.57	4.77	—
	7	4.84	5.21	5.32	4.94	4.78	4.80	4.51	4.65	—
	8	4.73	4.99	5.06	4.78	4.69	4.70	4.47	4.58	—
	9	4.66	4.86	4.91	4.68	4.63	4.64	4.46	4.54	4.66

TABLE VI
COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR $p = 0.30$

α	f	$t(f, \delta, \alpha)$ exact	approximations to $t(f, \delta, \alpha)$						
			I			II			
			a	b	c	a	b	c	d
0.99	4	-1.50	-1.55	-1.59	-1.44	-1.49	-1.62	-1.42	-1.91
	5	-1.25	-1.28	-1.29	-1.21	-1.25	-1.32	-1.22	-1.51
	6	-1.09	-1.09	-1.10	-1.04	-1.08	-1.12	-1.06	-1.26
	7	-0.95	-0.95	-0.95	-0.91	-0.94	-0.97	-0.93	-1.07
	8	-0.83	-0.83	-0.83	-0.80	-0.82	-0.84	-0.82	-0.92
	9	-0.72	-0.72	-0.72	-0.70	-0.72	-0.73	-0.72	-0.80
0.975	4	-0.935	-0.943	-0.953	-0.880	-0.933	-0.980	-0.912	-1.135
	5	-0.763	-0.766	-0.770	-0.727	-0.763	-0.787	-0.754	-0.891
	6	-0.627	-0.629	-0.631	-0.602	-0.628	-0.641	-0.623	-0.715
	7	-0.512	-0.512	-0.513	-0.494	-0.511	-0.520	-0.510	-0.576
	8	-0.409	-0.409	-0.410	-0.396	-0.409	-0.414	-0.408	-0.458
	9	-0.315	-0.315	-0.316	-0.307	-0.315	-0.319	-0.315	-0.353
0.95	4	-0.529	-0.531	-0.533	-0.498	-0.531	-0.548	-0.525	-0.631
	5	-0.390	-0.392	-0.392	-0.372	-0.391	-0.400	-0.389	-0.454
	6	-0.272	-0.273	-0.274	-0.262	-0.273	-0.278	-0.273	-0.317
	7	-0.168	-0.169	-0.169	-0.163	-0.170	-0.173	-0.169	-0.201
	8	-0.074	-0.074	-0.074	-0.072	-0.075	-0.077	-0.074	-0.099
	9	0.014	0.014	0.014	0.013	0.014	0.013	0.014	-0.005
0.05	4	4.163	4.408	4.575	4.041	4.089	4.106	3.810	3.916
	5	4.012	4.155	4.243	3.894	3.973	3.982	3.782	3.847
	6	3.954	4.049	4.104	3.845	3.930	3.935	3.789	3.833
	7	3.940	4.010	4.048	3.840	3.925	3.928	3.815	3.847
	8	3.950	4.004	4.032	3.859	3.939	3.941	3.851	3.876
	9	3.973	4.018	4.040	3.890	3.966	3.967	3.893	3.912
0.025	4	5.181	5.995	6.418	5.392	5.013	5.060	4.530	4.753
	5	4.856	5.276	5.460	4.898	4.768	4.792	4.438	4.575
	6	4.698	4.960	5.065	4.683	4.645	4.658	4.402	4.494
	7	4.618	4.802	4.871	4.582	4.583	4.592	4.396	4.462
	8	4.581	4.721	4.770	4.537	4.558	4.563	4.408	4.458
	9	4.573	4.681	4.719	4.523	4.554	4.558	4.432	4.470
0.01	4	6.74	10.25	12.24	8.69	6.37	6.50	5.50	5.99
	5	6.09	7.49	8.00	6.80	5.90	5.96	5.31	5.60
	6	5.76	6.53	6.77	6.10	5.64	5.67	5.21	5.40
	7	5.56	6.06	6.21	5.75	5.48	5.51	5.15	5.29
	8	5.46	5.81	5.90	5.55	5.39	5.41	5.13	5.23
	9	5.38	5.65	5.72	5.44	5.34	5.35	5.12	5.20

TABLE VII

COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR $p = 0.25$

α	f	$t(f, \delta, \alpha)$ exact		approximations to $t(f, \delta, \alpha)$							
		I ¹⁾	II ²⁾	I			II				
				a	b	c	a	b	c	d	
0.99	2	-2.085	—	—	—	—	-1.958	-3.302	-1.693	-4.467	
	3	-1.341	—	-1.420	-1.493	-1.272	-1.322	-1.658	-1.244	-2.216	
	4	-1.000	-0.99	-1.014	-1.029	-0.944	-0.995	-1.126	-0.967	-1.455	
	5	-0.774	-0.77	-0.777	-0.782	-0.736	-0.772	-0.836	-0.761	-1.054	
	6	-0.600	-0.60	-0.599	-0.601	-0.573	-0.598	-0.634	-0.593	-0.788	
	7	-0.450	-0.45	-0.450	-0.451	-0.434	-0.450	-0.472	-0.448	-0.587	
	8	-0.320	-0.32	-0.320	-0.320	-0.310	-0.320	-0.334	-0.318	-0.422	
	9	-0.204	-0.20	-0.202	-0.201	-0.196	-0.201	-0.211	-0.201	-0.281	
	0.975	2	-1.146	—	-1.224	-1.350	-1.042	-1.128	-1.579	-1.044	-2.201
3		-0.747	—	-0.752	-0.764	-0.687	-0.743	-0.861	-0.723	-1.154	
4		-0.512	-0.511	-0.512	-0.515	-0.480	-0.511	-0.558	-0.505	-0.728	
5		-0.333	-0.334	-0.334	-0.335	-0.318	-0.334	-0.358	-0.332	-0.469	
6		-0.191	-0.186	-0.186	-0.186	-0.178	-0.186	-0.199	-0.185	-0.277	
7		-0.058	-0.054	-0.054	-0.053	-0.052	-0.054	-0.063	-0.054	-0.120	
8		0.060	0.065	0.065	0.065	0.063	0.065	0.060	0.066	0.016	
9		0.174	0.176	0.176	0.176	0.171	0.177	0.173	0.176	0.138	
0.95		2	-0.605	—	-0.611	-0.626	-0.534	-0.605	-0.763	-0.583	-1.096
	3	-0.336	—	-0.336	-0.338	-0.309	-0.337	-0.379	-0.333	-0.533	
	4	-0.148	-0.147	-0.148	-0.148	-0.139	-0.148	-0.165	-0.147	-0.253	
	5	0.007	0.008	0.008	0.008	0.007	0.008	-0.001	0.007	-0.058	
	6	0.145	0.145	0.144	0.144	0.138	0.144	0.139	0.144	0.100	
	7	0.267	0.268	0.268	0.268	0.259	0.268	0.265	0.268	0.236	
	8	0.382	0.383	0.383	0.383	0.371	0.383	0.381	0.383	0.358	
	9	0.492	0.491	0.491	0.490	0.478	0.491	0.490	0.491	0.472	
	0.05	2	6.589	—	10.724	19.628	7.755	5.903	6.061	4.792	5.305
3		5.238	—	5.988	6.585	5.223	5.025	5.067	4.472	4.672	
4		4.806	4.809	5.135	5.347	4.697	4.713	4.730	4.372	4.478	
5		4.642	4.642	4.833	4.945	4.523	4.589	4.598	4.356	4.421	
6		4.583	4.584	4.713	4.783	4.470	4.550	4.555	4.379	4.423	
7		4.575	4.576	4.672	4.721	4.471	4.554	4.557	4.422	4.454	
8		4.596	4.596	4.672	4.709	4.499	4.581	4.583	4.474	4.499	
9		4.632	4.633	4.695	4.724	4.543	4.621	4.622	4.535	4.554	
0.025		2	9.475	—	-51.848	-10.482	60.045	7.844	8.295	5.917	7.074
	3	6.798	—	10.024	12.683	8.187	6.323	6.441	5.373	5.804	
	4	5.950	5.954	7.011	7.544	6.283	5.739	5.786	5.158	5.381	
	5	5.579	5.584	6.135	6.367	5.683	5.469	5.493	5.071	5.208	
	6	5.408	5.408	5.757	5.890	5.428	5.338	5.351	5.045	5.137	
	7	5.329	5.325	5.572	5.660	5.311	5.277	5.286	5.053	5.119	
	8	5.295	5.293	5.480	5.543	5.263	5.259	5.264	5.079	5.129	
	9	5.293	5.290	5.440	5.487	5.252	5.265	5.269	5.119	5.157	
	0.01	2	15.132	—	—	—	—	10.944	12.288	7.483	10.257
3		9.440	—	82.384	-45.229	28.803	8.293	8.629	6.600	7.572	
4		7.725	7.73	12.199	14.719	10.269	7.241	7.372	6.213	6.701	
5		6.979	6.96	8.768	9.415	7.939	6.720	6.784	6.019	6.312	
6		6.587	6.59	7.593	7.895	7.074	6.435	6.471	5.921	6.116	
7		6.371	6.38	7.033	7.211	6.655	6.271	6.293	5.875	6.014	
8		6.254	6.26	6.725	6.843	6.425	6.176	6.190	5.861	5.965	
9		6.180	6.18	6.544	6.630	6.296	6.125	6.135	5.868	5.948	

¹⁾ Exact percentage points according to Resnikoff and Lieberman (1957).²⁾ Exact percentage points according to Johnson and Welch (1940).

TABLE VIII
COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR $p = 0.15$

α	f	$t(f, \delta, \alpha)$ exact		approximations to $t(f, \delta, \alpha)$							
		I ¹⁾	II ²⁾	I			II				
				a	b	c	a	b	c	d	
0.99	2	-0.728	—	-0.756	-0.803	-0.652	-0.722	-2.066	-0.680	-3.454	
	3	-0.291	—	-0.291	-0.293	-0.267	-0.291	-0.627	-0.287	-1.259	
	4	-0.008	-0.01	-0.009	-0.009	-0.009	-0.009	-0.140	-0.010	-0.498	
	5	0.217	0.22	0.218	0.218	0.207	0.218	0.154	0.218	-0.075	
	6	0.416	0.42	0.417	0.416	0.400	0.416	0.380	0.416	0.221	
	7	0.593	0.60	0.598	0.596	0.578	0.597	0.575	0.596	0.457	
	8	0.764	0.77	0.766	0.764	0.743	0.764	0.750	0.763	0.659	
	9	0.921	0.92	0.924	0.922	0.899	0.922	0.912	0.919	0.839	
0.975	2	-0.196	—	-0.195	-0.197	-0.169	-0.196	-0.647	-0.192	-1.349	
	3	0.118	—	0.120	0.120	0.111	0.120	0.002	0.120	-0.311	
	4	0.362	0.364	0.364	0.363	0.343	0.364	0.317	0.364	0.141	
	5	0.574	0.576	0.576	0.575	0.550	0.576	0.552	0.575	0.438	
	6	0.764	0.767	0.768	0.766	0.738	0.767	0.754	0.765	0.673	
	7	0.942	0.943	0.945	0.943	0.914	0.944	0.935	0.940	0.874	
	8	1.106	1.108	1.112	1.109	1.079	1.109	1.104	1.104	1.055	
	9	1.261	1.266	1.269	1.266	1.236	1.268	1.264	1.259	1.221	
0.95	2	0.164	—	0.165	0.164	0.146	0.164	0.006	0.164	-0.349	
	3	0.438	—	0.439	0.437	0.405	0.438	0.396	0.438	0.238	
	4	0.670	0.669	0.670	0.668	0.632	0.669	0.652	0.668	0.562	
	5	0.877	0.876	0.878	0.875	0.837	0.877	0.868	0.873	0.808	
	6	1.066	1.065	1.068	1.064	1.026	1.067	1.062	1.060	1.016	
	7	1.244	1.242	1.245	1.242	1.203	1.243	1.240	1.235	1.203	
	8	1.409	1.409	1.412	1.408	1.370	1.410	1.408	1.400	1.375	
	9	1.566	1.567	1.569	1.567	1.529	1.569	1.568	1.556	1.537	
0.05	2	8.962	—	15.372	29.027	10.949	7.857	8.015	6.303	6.816	
	3	7.039	—	8.245	9.162	7.145	6.685	6.727	5.922	6.122	
	4	6.452	6.458	6.992	7.321	6.372	6.281	6.298	5.824	5.930	
	5	6.239	6.238	6.561	6.736	6.124	6.134	6.142	5.832	5.897	
	6	6.173	6.173	6.395	6.507	6.054	6.100	6.105	5.889	5.933	
	7	6.178	6.176	6.346	6.425	6.064	6.123	6.126	5.970	6.002	
	8	6.220	6.220	6.355	6.415	6.114	6.178	6.180	6.064	6.089	
	9	6.285	6.286	6.398	6.445	6.185	6.251	6.252	6.166	6.185	
0.025	2	12.822	—	-81.418	-17.985	90.756	10.317	10.768	7.679	8.836	
	3	9.081	—	14.125	18.199	11.413	8.318	8.436	7.025	7.456	
	4	7.918	7.934	9.621	10.437	8.574	7.563	7.610	6.786	7.009	
	5	7.432	7.436	8.337	8.694	7.695	7.225	7.249	6.706	6.843	
	6	7.208	7.210	7.793	7.999	7.328	7.071	7.084	6.703	6.795	
	7	7.112	7.112	7.533	7.670	7.166	7.011	7.019	6.741	6.807	
	8	7.087	7.085	7.409	7.508	7.104	7.007	7.012	6.801	6.851	
	9	7.098	7.100	7.360	7.435	7.097	7.034	7.038	6.877	6.915	
0.01	2	20.435	—	-7.706	-4.803	-9.520	14.222	15.566	9.576	12.350	
	3	12.557	—	124.726	-71.500	42.569	10.793	11.129	8.517	9.489	
	4	10.200	10.25	17.196	21.047	14.336	9.441	9.572	8.070	8.558	
	5	9.210	9.22	12.060	13.048	10.859	8.783	8.847	7.861	8.154	
	6	8.713	8.70	10.330	10.793	9.589	8.431	8.467	7.770	7.965	
	7	8.435	8.43	9.516	9.790	8.980	8.238	8.260	7.742	7.881	
	8	8.287	8.29	9.075	9.260	8.653	8.135	8.149	7.753	7.857	
	9	8.211	8.21	8.822	8.957	8.473	8.089	8.098	7.789	7.869	

¹⁾ Exact percentage points according to Resnikoff and Lieberman (1957).

²⁾ Exact percentage points according to Johnson and Welch (1940).

TABLE IX

COMPARISON OF THE APPROXIMATIONS WITH THE EXACT PERCENTAGE POINTS FOR $p = 0.125$

α	f	$t(f, \delta, \alpha)$ exact	approximations to $t(f, \delta, \alpha)$						
			I			II			
			a	b	c	a	b	c	d
0.99	4	0.25	0.25	0.25	0.24	0.25	0.12	0.25	-0.24
	5	0.49	0.49	0.49	0.47	0.49	0.42	0.49	0.20
	6	0.70	0.70	0.70	0.67	0.70	0.66	0.70	0.50
	7	0.89	0.90	0.89	0.87	0.89	0.87	0.89	0.75
	8	1.08	1.08	1.07	1.05	1.08	1.06	1.07	0.96
	9	1.25	1.25	1.25	1.22	1.25	1.24	1.24	1.16
0.975	4	0.605	0.607	0.604	0.572	0.606	0.559	0.605	0.382
	5	0.832	0.835	0.831	0.796	0.833	0.809	0.829	0.692
	6	1.039	1.043	1.039	1.003	1.041	1.028	1.033	0.941
	7	1.232	1.236	1.231	1.196	1.235	1.226	1.223	1.157
	8	1.414	1.419	1.415	1.378	1.417	1.412	1.402	1.352
	9	1.587	1.592	1.588	1.551	1.591	1.587	1.572	1.533
0.95	4	0.902	0.905	0.900	0.854	0.903	0.886	0.897	0.791
	5	1.127	1.131	1.126	1.080	1.130	1.121	1.118	1.053
	6	1.335	1.339	1.335	1.289	1.339	1.334	1.322	1.278
	7	1.530	1.534	1.530	1.484	1.534	1.531	1.514	1.482
	8	1.714	1.718	1.714	1.669	1.718	1.716	1.695	1.670
	9	1.888	1.893	1.889	1.844	1.893	1.892	1.867	1.848
0.05	4	7.001	7.599	7.966	6.919	6.782	6.799	6.309	6.415
	5	6.758	7.125	7.320	6.646	6.627	6.636	6.324	6.389
	6	6.689	6.944	7.068	6.570	6.594	6.599	6.390	6.434
	7	6.696	6.890	6.979	6.582	6.624	6.627	6.483	6.515
	8	6.746	6.902	6.969	6.638	6.687	6.689	6.589	6.614
	9	6.821	6.951	7.003	6.718	6.769	6.770	6.704	6.723
0.025	4	8.579	10.476	11.383	9.324	8.145	8.192	7.331	7.554
	5	8.042	9.058	9.455	8.354	7.784	7.808	7.253	7.390
	6	7.797	8.459	8.688	7.950	7.624	7.637	7.256	7.348
	7	7.694	8.174	8.326	7.772	7.564	7.573	7.302	7.368
	8	7.669	8.038	8.149	7.705	7.564	7.569	7.373	7.423
	9	7.687	7.985	8.070	7.698	7.597	7.601	7.459	7.497
0.01	4	11.08	18.82	23.10	15.66	10.14	10.27	8.69	9.18
	5	9.95	13.14	14.24	11.82	9.44	9.50	8.48	8.77
	6	9.41	11.23	11.74	10.41	9.07	9.10	8.39	8.58
	7	9.10	10.33	10.63	9.74	8.86	8.89	8.37	8.50
	8	8.94	9.84	10.05	9.38	8.76	8.77	8.38	8.49
	9	8.87	9.57	9.72	9.19	8.71	8.72	8.43	8.51

TABLE X
COMPARISON OF APPROXIMATION IV WITH THE EXACT PERCENTAGE POINTS AND THE OTHER
APPROXIMATIONS
 $f = 7$

δ	p	α	$t(f, \delta, \alpha)$ exact	approximations to $t(f, B, B)$								III	IV
				I			II						
				a	b	c	a	b	c	d			
0.5533	0.42	0.4	0.843	0.842	0.842	0.813	0.843	0.843	0.841	0.841	—	0.843	
		0.2	1.512	1.511	1.514	1.457	1.512	1.512	1.503	1.506	—	1.513	
		0.05	2.621	2.648	2.667	2.542	2.616	2.619	2.561	2.593	—	2.623	
		0.01	3.90	4.12	4.19	3.93	3.86	3.88	3.67	3.81	—	3.90	
1.8146	0.26	0.4	2.190	2.183	2.184	2.105	2.190	2.190	2.180	2.180	—	2.188	
		0.2	2.992	2.988	2.998	2.876	2.992	2.992	2.963	2.966	—	2.992	
		0.05	4.434	4.524	4.572	4.331	4.414	4.417	4.286	4.318	—	4.436	
		0.01	6.20	6.82	6.99	6.45	6.10	6.12	5.71	5.85	—	6.20	
2.6568	0.17	0.4	3.105	3.091	3.094	2.981	3.110	3.110	3.091	3.091	—	3.100	
		0.2	4.030	4.020	4.036	3.867	4.027	4.027	3.991	3.994	—	4.026	
		0.05	5.739	5.887	5.958	5.628	5.697	5.701	5.542	5.574	—	5.741	
		0.01	7.87	8.83	9.08	8.34	7.70	7.73	7.22	7.36	—	7.89	
3.1953	0.13	0.4	3.699	3.675	3.679	3.544	3.706	3.706	3.679	3.679	—	3.688	
		0.2	4.708	4.694	4.714	4.512	4.701	4.701	4.671	4.674	—	4.701	
		0.05	6.603	6.791	6.878	6.488	6.533	6.536	6.389	6.421	—	6.608	
		0.01	8.99	10.18	10.48	9.60	8.75	8.77	8.25	8.39	—	9.01	

$f = 16$

0.8207	0.42	0.4	1.096	1.096	1.096	1.079	1.096	1.096	1.096	1.096	—	—	1.096
		0.2	1.727	1.727	1.727	1.699	1.727	1.727	1.725	1.726	—	—	1.727
		0.05	2.674	2.680	2.683	2.635	2.674	2.674	2.660	2.666	2.673	—	2.674
		0.01	3.60	3.64	3.64	3.57	3.60	3.60	3.56	3.58	3.60	—	3.60
2.6914	0.26	0.4	3.029	3.025	3.025	2.978	3.029	3.029	3.026	3.026	—	—	3.027
		0.2	3.760	3.757	3.760	3.697	3.760	3.760	3.752	3.753	—	—	3.758
		0.05	4.920	4.945	4.955	4.860	4.917	4.917	4.887	4.893	4.919	—	4.922
		0.01	6.12	6.23	6.26	6.12	6.10	6.10	6.01	6.03	6.11	—	6.12
3.9407	0.17	0.4	4.334	4.326	4.326	4.258	4.336	4.336	4.330	4.330	—	—	4.330
		0.2	5.159	5.155	5.159	5.071	5.157	5.157	5.158	5.159	—	—	5.156
		0.05	6.502	6.546	6.562	6.431	6.487	6.487	6.475	6.480	6.502	—	6.509
		0.01	7.91	8.11	8.14	7.95	7.87	7.87	7.79	7.82	7.92	—	7.94
4.7394	0.13	0.4	5.173	5.161	5.162	5.080	5.177	5.177	5.170	5.170	—	—	5.165
		0.2	6.068	6.061	6.067	5.962	6.060	6.060	6.078	6.079	—	—	6.063
		0.05	7.538	7.596	7.616	7.462	7.503	7.504	7.532	7.537	7.543	—	7.546
		0.01	9.10	9.35	9.39	9.17	9.02	9.02	8.99	9.01	9.10	—	9.12

TABLE XI
EXACT PERCENTAGE POINTS OF THE NON-CENTRAL t -DISTRIBUTION FOR
 $\alpha = 0.99; 0.975; 0.95; 0.05; 0.025; 0.01$
 $f = 4(1)9$

FOUND FROM THE TABLES OF JOHNSON AND WELCH

α	f	p						
		0.45	0.40	0.35	0.30	0.25	0.15	0.125
0.99	4	-3.13	-2.56	-2.02	-1.50	-0.99	-0.01	0.25
	5	-2.80	-2.26	-1.76	-1.25	-0.77	0.22	0.49
	6	-2.59	-2.08	-1.57	-1.09	-0.60	0.42	0.70
	7	-2.46	-1.94	-1.44	-0.95	-0.45	0.60	0.89
	8	-2.35	-1.83	-1.32	-0.83	-0.32	0.77	1.08
	9	-2.27	-1.75	-1.23	-0.72	-0.20	0.92	1.25
0.975	4	-2.279	-1.812	-1.366	-0.935	-0.511	0.364	0.605
	5	-2.092	-1.635	-1.195	-0.763	-0.334	0.576	0.832
	6	-1.969	-1.512	-1.067	-0.627	-0.186	0.767	1.039
	7	-1.882	-1.417	-0.963	-0.512	-0.054	0.943	1.232
	8	-1.815	-1.341	-0.874	-0.409	0.065	1.108	1.414
	9	-1.760	-1.275	-0.795	-0.315	0.176	1.266	1.587
0.95	4	-1.707	-1.303	-0.913	-0.529	-0.147	0.669	0.902
	5	-1.589	-1.182	-0.785	-0.390	0.008	0.876	1.127
	6	-1.510	-1.092	-0.682	-0.272	0.145	1.065	1.335
	7	-1.451	-1.020	-0.595	-0.168	0.268	1.242	1.530
	8	-1.403	-0.958	-0.518	-0.074	0.383	1.409	1.714
	9	-1.363	-0.903	-0.447	0.014	0.491	1.567	1.888
0.05	4	2.579	3.062	3.586	4.163	4.809	6.458	7.001
	5	2.461	2.936	3.450	4.012	4.642	6.238	6.758
	6	2.396	2.874	3.390	3.954	4.584	6.173	6.689
	7	2.356	2.844	3.368	3.940	4.576	6.176	6.696
	8	2.333	2.833	3.368	3.950	4.596	6.220	6.746
	9	2.320	2.832	3.379	3.973	4.633	6.286	6.821
0.025	4	3.303	3.873	4.494	5.181	5.954	7.934	8.579
	5	3.077	3.620	4.208	4.856	5.584	7.436	8.042
	6	2.947	3.483	4.062	4.698	5.408	7.210	7.797
	7	2.870	3.406	3.985	4.618	5.325	7.112	7.694
	8	2.819	3.361	3.944	4.581	5.293	7.085	7.669
	9	2.783	3.336	3.928	4.573	5.290	7.100	7.687
0.01	4	4.41	5.12	5.89	6.74	7.73	10.25	11.08
	5	3.96	4.61	5.31	6.09	6.96	9.22	9.95
	6	3.73	4.34	5.01	5.76	6.59	8.70	9.41
	7	3.57	4.18	4.84	5.56	6.38	8.43	9.10
	8	3.46	4.07	4.73	5.46	6.26	8.29	8.94
	9	3.39	4.00	4.66	5.38	6.18	8.21	8.87

