Centrum Wiskunde & Informatica

REPORT *RAPPORT*

PNA

Probability, Networks and Algorithms

Probability, Networks and Algorithms

PNA

On the record process of time-reversible spectrally-negative Markov additive processes

J. Ivanovs, M.R.H. Mandjes

REPORT PNA-E0903 MARCH 2009

Centrum Wiskunde & Informatica (CWI) is the national research institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organisation for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Science Park 123, 1098 XG Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3711

On the record process of time-reversible spectrally-negative Markov additive processes

ABSTRACT
We study the record process of a spectrally-negative Markov additive process (MAP). Assuming time-reversibility, a number of key quantities can be given explicitly. It is shown how these key quantities can be used when analyzing the distribution of the all-time maximum attained by MAPs with negative drift, or, equivalently, the stationary workload distribution of the associated storage system; the focus is on Markov-modulated Brownian mo- tion, spectrally-negative and spectrally-positive MAPs. It is also argued how our results are of great help in the numerical analysis of systems in which the driving MAP is a superposition of multiple time-reversible MAPs.

2000 Mathematics Subject Classification: 60K25
Keywords and Phrases: Levy processes, fluctuation theory

ON THE RECORD PROCESS OF TIME-REVERSIBLE SPECTRALLY-NEGATIVE MARKOV ADDITIVE PROCESSES

JEVGENIJS IVANOVS AND MICHEL MANDJES

Abstract. We study the record process of a spectrally-negative Markov additive process (MAP). Assuming time-reversibility, a number of key quantities can be given explicitly. It is shown how these key quantities can be used when analyzing the distribution of the all-time maximum attained by MAPs with negative drift, or, equivalently, the stationary workload distribution of the associated storage system; the focus is on Markov-modulated Brownian motion, spectrally-negative and spectrally-positive MAPs. It is also argued how our results are of great help in the numerical analysis of systems in which the driving MAP is a superposition of multiple time-reversible MAPs.

1. Introduction

Continuous-time Markov additive processes (MAPs) have proven an important modelling tool in communications networking [17, Ch. 6-7] as well as finance [2, 11]. This has led to a vast body of literature; for an overview see for instance [1, Ch. XI]. A MAP is essentially a Lévy process whose Laplace exponent depends on the state of a (finite-state) Markovian background process. It is a non-trivial generalization of the standard Lévy process, with many analogous properties and characteristics, as well as new mathematical objects associated to it, posing new challenges.

Just as for standard Lévy processes, the class of MAPs with one-sided jumps is of high importance. On the one hand this class is rich, as it covers for instance Markov-modulated one-sided compound Poisson processes with drift, Markov-modulated Brownian motions, as well as 'Markov fluids' [1, Section XI.1b], but on the other hand it allows for fairly explicit results. In this paper we consider *spectrally-negative* MAPs, that is, processes which are only allowed to have negative jumps. We denote the MAP by $(X(\cdot), J(\cdot))$, with $X(\cdot)$ being the value of the MAP and $J(\cdot)$ the state of the Markovian background process. The focus of the present paper is on the Markov chain $J(\tau_x)$ associated with the record process $\tau_x := \inf\{t \geq 0 : X(t) > x\}$. Here it is noted that the (possibly defective) generator matrix Λ of the Markov chain $J(\tau_x)$ plays a crucial role in the fluctuation theory for one-sided MAPs,

Date: February 24, 2009.

see e.g. [5, 16]. In [18] a complete characterization of Λ is given for a special MAP, viz. a Markov fluid, but, to the best of our knowledge, no explicit expression for Λ has been developed in the literature for a general spectrally-negative MAP; it is only known that Λ solves a specific matrix integral equation, see [5].

The main assumption, which we impose in this paper and which makes the analysis tractable, is that we restrict ourselves to *time-reversible* MAPs. In this setting we succeed in identifying the Jordan normal form of the generator matrix Λ, which turns out to be similar to a real diagonal matrix, in the sense that $\Lambda = PDP^{-1}$ for a real diagonal matrix D and an invertible matrix P; see Section 2. We then show in Section 3 how this result and its ramifications can be used to determine the distribution of the all-time maximum attained by MAPs with negative drift, or, equivalently, the stationary workload distribution of the associated storage system; the focus is on Markov-modulated Brownian motion, on spectrally-negative and on spectrally-positive MAPs. In Section 4 we argue that our findings greatly reduce the computational efforts required to obtain numerical output, particularly for systems in which the driving MAP is a superposition of multiple time-reversible MAPs.

The rest of this Introduction is devoted to developing a set of prerequisites.

1.1. **Spectrally-negative MAP.** Before formally defining the class of MAPs, we first introduce some notation. Throughout this paper we use bold symbols to denote (column) vectors. For example, $\mathbf{1}$ and $\mathbf{0}$ denote vectors of 1-s and 0-s respectively, whereas $\boldsymbol{e}_i$ stands for a vector of 0-s but with the i-th element being 1. Moreover, $\boldsymbol{a} < \boldsymbol{b}$ means that $a_i < b_i$ for all indices i.

A MAP is a bivariate Markov process $(X(\cdot), J(\cdot)) \equiv (X(t), J(t))_t$ defined as follows. Let $J(\cdot)$ be an irreducible continuous-time Markov chain with finite state space $E := \{1, \dots, N\}$ and $N \times N$ generator matrix $Q = (q_{ij})$. For each state i of $J(\cdot)$ let $X_i(\cdot)$ be a Lévy process with Laplace exponent $\phi_i(\alpha) := \log(\mathbb{E}e^{\alpha X_i(1)})$. Letting T_n and T_{n+1} be two successive transition epochs of $J(\cdot)$, and given that $J(\cdot)$ jumps from state i to state j at T_n, we define the additive process $X(\cdot)$ in the time interval $[T_n, T_{n+1})$ through

$$X(t) := X(T_n-) + U_{ij}^n + [X_j(t) - X_j(T_n)],$$

where (U_{ij}^n) is a sequence of independent and identically distributed random variables with Laplace-Stieltjes transform

$$\tilde{G}_{ij}(\alpha) := \mathbb{E}e^{\alpha U_{ij}^1}, \quad \text{where } U_{ii}^1 \equiv 0,$$

describing the jumps at transition epochs. To make the MAP spectrally-negative, it is required that $U_{ij}^1 \leq 0$ (for all $i, j \in \{1, \dots, N\}$) and that $X_i(\cdot)$ is allowed to have only negative jumps (for all $i \in \{1, \dots, N\}$).

We let $N^{\downarrow}$ be the number of Lévy processes $X_i(\cdot)$, $i \in \{1, \ldots, N\}$, which are *downward subordinators*, i.e., stochastic processes with non-increasing paths a.s.; without loss of generality we assume that such processes have indices $N^+ + 1$ through N, where $N^+ = N - N^{\downarrow}$. As will turn out, these downward subordinators play a special role in our analysis. We also define $E^+ := \{1, \ldots, N^+\}$ and $E^{\downarrow} := \{N^+ + 1, \ldots, N\}$, and use $\boldsymbol{v}^+$ and $\boldsymbol{v}^{\downarrow}$ to denote the restrictions of a vector $\boldsymbol{v}$ to the indices from E^+ and $E^{\downarrow}$ respectively. Finally, in order to exclude trivialities it is assumed that $N^+ > 0$.

A central object, which can be considered as the multi-dimensional analog of a Laplace exponent, defining the law of a MAP, is the associated *generator* $F(\alpha)$, given by

$$F(\alpha) := Q \circ \tilde{G}(\alpha) + \mathrm{diag}(\phi_1(\alpha), \ldots, \phi_N(\alpha)),$$

where $\tilde{G}(\alpha) := (\tilde{G}_{ij}(\alpha))$ and for matrices A and B of the same dimensions $A \circ B := (a_{ij} b_{ij})$. The generator is finite for all $\alpha \geq 0$, and in addition

$$\mathbb{E}_i[e^{\alpha X(t)} 1_{\{J(t)=j\}}] = (e^{F(\alpha)t})_{ij},$$

cf. [1, Prop. XI.2.2], where $\mathbb{E}_i[\cdot]$ denotes expectation given that $J(0) = i$. Let $k(\alpha)$ be the eigenvalue of $F(\alpha)$ with maximal real part, which is, by Perron-Frobenius theory [1, Thm. I.6.5], well-defined, simple and real.

An important quantity associated to a MAP is the *asymptotic drift* $\lim_{t\to\infty} \mathbb{E}_i X(t)/t$, which does not depend on the initial state i of $J(t)$ and is given by

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}_i X(t) = k'(0) = \sum_i \pi_i \left(\phi_i'(0) + \sum_{j \neq i} q_{ij} \tilde{G}_{ij}'(0) \right),$$

where $\boldsymbol{\pi}$ is a unique stationary distribution of $J(\cdot)$ [1, Cor. XI.2.7].

In this paper we assume that the MAP $(X(\cdot), J(\cdot))$ under consideration is *time-reversible*, which is equivalent to saying that the Markov chain $J(\cdot)$ is time-reversible (that is, $\pi_i q_{ij} = \pi_j q_{ji}$) and U_{ij}^1 has the same law as U_{ji}^1 for all $i, j \in \{1, \ldots, N\}$, see [1, Section XI.2e]. Yet another equivalent definition of time-reversibility, which we will use in the present paper, is, with $\Delta_{\boldsymbol{x}}$ being a diagonal matrix with the vector $\boldsymbol{x}$ on its diagonal,

$$\Delta_{\boldsymbol{\pi}} F(\alpha) = (\Delta_{\boldsymbol{\pi}} F(\alpha))^{\mathrm{T}}. \tag{1}$$

1.2. **The Record Process.** Define the first passage time over level $x > 0$ for the process $X(\cdot)$ as

$$\tau_x := \inf\{t \geq 0 : X(t) > x\}.$$

Note that due to the absence of positive jumps the time-changed process $J(\tau_x)$ is again a Markov chain but taking values in the set E^+ only (see also [16]). Denote the corresponding $N^+ \times N^+$ dimensional generator matrix by Λ, so that, with $\mathbb{P}_i(\cdot)$ denoting probability given $J(0) = i$,

$$\mathbb{P}_i(J(\tau_x) = j, \tau_x < \infty) = (e^{\Lambda x})_{ij}, \text{ where } i, j \in E^+. \tag{2}$$

Another matrix of interest is the $N^\downarrow \times N^+$ matrix Π defined as follows

$$\Pi_{ij} := \mathbb{P}_{i+N^+}(J(\tau_0) = j, \tau_0 < \infty), \text{ where } i = 1, \ldots, N^\downarrow, j = 1, \ldots, N^+. \tag{3}$$

In the following we have to distinguish between two cases:

- if $k'(0) \geq 0$, then Λ is a non-defective generator matrix: $\Lambda \mathbf{1}^+ = \mathbf{0}^+$;
- if $k'(0) < 0$, then Λ is a defective generator matrix: $\Lambda \mathbf{1}^+ \leq \mathbf{0}^+$, with at least one strict inequality.

This follows from [1, Prop. XI.2.10], which states that in the case of a non-negative asymptotic drift $\limsup_{t\to\infty} X(t) = +\infty$ $\mathbb{P}_i$-a.s. for all i, and thus $\mathbb{P}_i(\tau_x < \infty) = 1$. In the case of a negative asymptotic drift $\lim_{t\to\infty} X(t) = -\infty$ $\mathbb{P}_i$-a.s. for all i, and thus $\mathbb{P}_i(\tau_x < \infty) < 1$. This also means that $\Pi \mathbf{1}^+ = \mathbf{1}^\downarrow$ if $k'(0) \geq 0$, and $\Pi \mathbf{1}^+ < \mathbf{1}^\downarrow$ if $k'(0) < 0$.

2. Characterization of Λ and Π

In this section we prove that under the time-reversibility assumption the generator matrix Λ is *similar* to some real diagonal matrix D, in the sense that $\Lambda = PDP^{-1}$ for some invertible matrix P. Moreover, we provide a procedure to construct the matrices D and P.

Let $\alpha_1, \ldots, \alpha_k$ and $m_1, \ldots, m_k$ be the zeros of $\det(F(\alpha))$ in $(0, \infty)$ and their multiplicities. For all $i = 1, \ldots, k$ let p_i denote the dimension of the null space of $F(\alpha_i)$ (geometric multiplicity of the null-eigenvalue) and $\boldsymbol{v}_i^1, \ldots, \boldsymbol{v}_i^{p_i}$ be some basis of this null space. It is well known, see e.g. [9, Lemma 2.4], that $m_i \geq p_i$. In the special case when $m_i = p_i$ the zero α_i is called *semi-simple*.

Let Υ_i be a $p_i \times p_i$ diagonal matrix with α_i on the diagonal and $V_i := [\boldsymbol{v}_i^1, \ldots, \boldsymbol{v}_i^{p_i}]$. Define

$$\Upsilon := \operatorname{diag}(\Upsilon_1, \ldots, \Upsilon_k) \text{ and } V := [V_1, \ldots, V_k], \qquad \text{if } k'(0) < 0$$

$$\Upsilon := \operatorname{diag}(0, \Upsilon_1, \ldots, \Upsilon_k) \text{ and } V := [\mathbf{1}, V_1, \ldots, V_k], \quad \text{if } k'(0) \geq 0 \tag{4}$$

and let the matrices V^+ and $V^\downarrow$ be the restrictions of the matrix V to the rows corresponding to E^+ and $E^\downarrow$ respectively:

$$V = \begin{pmatrix} V^+ \\ V^\downarrow \end{pmatrix}, \text{ where } V^+ \text{ has exactly } N^+ \text{ rows.}$$

It is the difficult part of the proof of our main result, Thm. 1, to show that V is composed of N^+ columns, which implies that Υ and V^+ are square $N^+ \times N^+$-dimensional matrices.

Theorem 1. *Let $(X(\cdot), J(\cdot))$ be a time-reversible spectrally-negative MAP. Then Υ and V^+ are $N^+ \times N^+$-dimensional matrices, V^+ is invertible, and*

$$\Lambda = -V^+\Upsilon(V^+)^{-1} \text{ and } \Pi = V^{\downarrow}(V^+)^{-1}.$$

We start the proof of Thm. 1 by establishing a lemma, which can be considered as a weak analog of this theorem.

Lemma 2. *If $\alpha > 0$ and $\boldsymbol{v}$ are such that $F(\alpha)\boldsymbol{v} = \boldsymbol{0}$ then*

$$\Lambda \boldsymbol{v}^+ = -\alpha \boldsymbol{v}^+ \text{ and } \boldsymbol{v}^{\downarrow} = \Pi \boldsymbol{v}^+. \tag{5}$$

Proof. By choosing $\lambda(\alpha) = 0$ and $\boldsymbol{h}(\alpha) = \boldsymbol{v}$ in [3, Lemma 2.1], we obtain that for any distribution of $J(0)$

$$M(t) := e^{\alpha X(t)} v_{J(t)}$$

is a martingale. Apply the optional sampling theorem to see that, for any $t > 0$ and any $x \geq 0$,

$$v_i = \mathbb{E}_i[e^{\alpha X(\tau_x \wedge t)} v_{J(\tau_x \wedge t)}].$$

Note that $M(t)$ is bounded in absolute value on $[0, \tau_x]$, due to the facts that $\alpha > 0$ and $X(t) \leq x$ on $[0, \tau_x]$. It moreover always holds that either $\mathbb{P}_i(\tau_x = \infty) = 0$ or $\lim_{t\to\infty} X(t) = -\infty$ a.s. (where the former case corresponds to $k'(0) \geq 0$, and the latter case to $k'(0) < 0$), so by applying 'dominated convergence' we have

$$v_i = \mathbb{E}_i[1_{\{\tau_x < \infty\}} e^{\alpha x} v_{J(\tau_x)}] = e^{\alpha x} \sum_{j \in E^+} \mathbb{P}_i(J(\tau_x) = j, \tau_x < \infty) v_j.$$

Choosing $x = 0$ and considering $i \in E^{\downarrow}$ we obtain

$$\boldsymbol{v}^{\downarrow} = \Pi \boldsymbol{v}^+,$$

see also (3). On the other hand, considering $i \in E^+$ we get

$$\boldsymbol{v}^+ = e^{\alpha x} e^{\Lambda x} \boldsymbol{v}^+,$$

see also (2). The first equality appearing in (5) is obtained by differentiating the above equality with respect to x and setting $x = 0$. □

Recall that if $k'(0) \geq 0$ then $\mathbb{P}_i(\tau_x < \infty) = 1$, so $\Lambda \boldsymbol{1}^+ = \boldsymbol{0}^+$ and $\Pi \boldsymbol{1}^+ = \boldsymbol{1}^{\downarrow}$. Using Lemma 2 one can see now that $\Lambda V^+ = -V^+\Upsilon$ and $V^{\downarrow} = \Pi V^+$. Note that the columns of the matrix V^+ are vectors from the bases of the eigenspaces of the matrix Λ, so they are linearly independent. But then the matrix V^+ (and so also V) cannot have more than N^+ columns. Thus, in order to prove Thm. 1, it remains to show that

(6) V is composed of at least N^+ column vectors.

We devote the rest of this section to proving this claim.

Lemma 3. *The eigenvalues of $F(\alpha), \alpha \geq 0$ are real with algebraic and geometric multiplicities being the same.*

Proof. Recall that time-reversibility of $(X(\cdot), J(\cdot))$ implies that the matrix $\Delta_{\boldsymbol{\pi}} F(\alpha)$, $\alpha \geq 0$ is real symmetric, see (1), and hence the same applies to $\Delta_{\boldsymbol{\pi}}^{1/2} F(\alpha) \Delta_{\boldsymbol{\pi}}^{-1/2}$, $\alpha \geq 0$. It is well known that a real symmetric matrix has real eigenvalues with algebraic and geometric multiplicities coinciding, see [8, Thms. 2.5.4 and 4.1.3]. The claim is now immediate in view of [8, Thm. 1.4.8]. □

Remark 2.1. There are three types of multiplicities mentioned in this section: the algebraic and geometric multiplicities of eigenvalues of $F(\alpha)$ for some fixed α, and the multiplicities of zeros of $\det(F(\alpha))$. Hence to every zero $\alpha_i > 0$ of $\det(F(\alpha))$, having a multiplicity that we denoted by m_i, we can associate a null-eigenvalue of the matrix $F(\alpha_i)$, which has the same algebraic and geometric multiplicities according to Lemma 3; we recall that this multiplicity is denoted by p_i.

Let $g_i(\alpha)$ be the i-th largest eigenvalue of $F(\alpha), \alpha \geq 0$ (so that $g_1(\alpha) = k(\alpha)$, the Perron-Frobenius eigenvalue defined earlier). Then $g_i : [0, \infty) \mapsto \mathbb{R}$ is a continuous function. The next lemma presents some properties of the functions $g_i(\cdot)$.

Lemma 4. *It holds that*

- $g_1(0) = 0$ *and* $g_i(0) < 0$ *for* $i = 2, \ldots, N$,
- $g_i(\alpha) \to \infty$ *as* $\alpha \to \infty$ *for* $i = 1, \ldots, N^+$.

Proof. The first statement follows immediately by noting that $F(0) = Q$ is an irreducible generator matrix; see also [1, Cor. I.4.9]. Consider the second statement. It is well known that $\lim_{\alpha\to\infty} \phi_i(\alpha) = \infty$ if $X_i(\cdot)$ is not a downward subordinator. Let $f(\alpha)$ be the N^+-th largest number out of $\phi_i(\alpha)$ for $i \in \{1, \ldots, N\}$, so $\lim_{\alpha\to\infty} f(\alpha) = \infty$ and $F(\alpha)/f(\alpha)$ goes to a diagonal matrix with at least N^+ positive (possibly infinite) elements. Take an arbitrary sequence $\alpha_n \to \infty$ and apply Lemma 7 of the Appendix to $F(\alpha_n)/f(\alpha_n)$ to obtain the result. □

Proof of Theorem 1. Recall that we are left to prove that the matrix V is composed of at least N^+ columns, see (6). Lemma 4 shows that the functions $g_2(\alpha), \ldots, g_{N^+}(\alpha)$, and in addition $g_1(\alpha)$ provided that $k'(0) < 0$, hit 0 in the interval $(0, \infty)$ at least once (recall that $k(\alpha) = g_1(\alpha)$). If these functions hit 0 for distinct α, then the claim is immediate, see the definition of matrix V given by (4). Suppose now that n of these functions hit 0 for some $\alpha = \alpha^*$. Then the algebraic multiplicity of the null-eigenvector of $F(\alpha^*)$ is n. But the algebraic and geometric multiplicities of all the eigenvalues of $F(\alpha), \alpha > 0$ are the same according to Lemma 3, so the null space of $F(\alpha^*)$ is of dimension n, and the claim follows. □

We conclude with the following immediate corollary of the above proof, which turns out to be important for practical reasons discussed in Section 4.

Corollary 5. *The functions $g_2(\alpha), \dots, g_{N^+}(\alpha)$, and in addition $g_1(\alpha)$ provided that $k'(0) < 0$, hit 0 exactly once on the interval $(0, \infty)$. Moreover, these are the only functions $g_i(\alpha)$ hitting 0 for some $\alpha \in (0, \infty)$.*

Proof. If there are more hits of 0 than stated in the Corollary, then the same arguments as in the proof of Thm. 1 show that V and thus also V^+ are composed of more than N^+ columns, which is impossible. □

Remark 2.2. It has been shown in [10] (for the case when $k'(0)$ is non-zero and finite) that the total number of zeros (taken according to their multiplicities) of $\det(F(\alpha))$ in $\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$ is $N^+ - 1_{\{k'(0) \geq 0\}}$. But $m_i \geq p_i$, so according to Thm. 1 it should hold that $m_i = p_i$ and $\alpha_1, \dots, \alpha_k$ are the only zeros of $\det(F(\alpha))$ in $\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$ (i.e., all the zeros are real and semi-simple). Finally, we note that $\alpha = 0$ is not necessarily semi-simple, nor it is true for a general spectrally-negative MAP that the zeros of $\det(F(\alpha))$ are real or semi-simple.

3. Applications

In order to illustrate the applicability of our results we first consider in Section 3.1 a Markov-modulated Brownian motion (MMBM) reflected at zero. This model has been extensively studied in [12], where the stationary distribution under the stability condition $k'(0) < 0$ is given. This distribution involves a set of unknown constants, which is shown to satisfy a system of linear equations. It was, however, not proved in [12] that these equations are *linearly independent*, and hence it is not guaranteed that they provide a unique solution. Below we show that under our time-reversibility assumption, the equations are indeed linearly independent. Moreover, the stationary distribution appears to be of a particularly simple form. We also provide some comments and further identities.

We proceed in Section 3.2 with time-reversible spectrally-negative MAP reflected at zero. A simple observation allows us to retrieve the end result obtained for MMBM in a trivial way. Making repeated use of Thm. 1 one can easily see that actually all the results discussed in the case of MMBM hold true.

Finally in Section 3.3 we make a brief comment on a *spectrally-positive* MAP reflected at zero, where we show that the result of our paper makes it possible to resolve an open issue posed in [3] for the special case of a time-reversible spectrally-positive MAP.

We recall that for a given MAP $(X(\cdot), J(\cdot))$ the *reflected process* (at 0) $(W(\cdot), J(\cdot))$ is defined through

$$W(t) := X(t) + \max\left\{-\inf_{s\in[0,t]} X(s), 0\right\}.$$

It is well known that this reflected process has a unique stationary distribution if the asymptotic drift of $(X(\cdot), J(\cdot))$ is negative (i.e., the stability condition $k'(0) < 0$ holds), which we assume in the sequel. Let a pair of random variables (W, J) have the stationary distribution of $(W(\cdot), J(\cdot))$, and denote the all-time maximum attained by $X(\cdot)$ through $\overline{X} := \sup_{t\geq 0} X(t)$. It is an immediate consequence of [1, Prop. XI.2.11] and time-reversibility that

$$(W|J = i) \text{ and } \big(\overline{X}|J(0) = i\big) \text{ have the same distribution.} \tag{7}$$

3.1. **Reflected MMBM.** Let $(X(\cdot), J(\cdot))$ be a MMBM, that is, every Lévy process $X_i(\cdot)$ is a Brownian motion with drift parameter $\mu_i \in \mathbb{R}$ and variance parameter $\sigma_i^2 \geq 0$, and the jumps at transition epochs U_{ij} are 0 with probability 1. It is immediate that

$$F(\alpha) = \alpha^2 \Delta_{\boldsymbol{\sigma}^2} + \alpha \Delta_{\boldsymbol{\mu}} + Q.$$

Note that $k'(0) = \sum_i \pi_i \mu_i < 0$ is the stability condition. Realize that $N^{\downarrow}$ is the number of indices i such that $\sigma_i^2 = 0$ and $\mu_i \leq 0$. We finally note that a time-reversible MMBM also appears in [13], where the authors succeed in identifying the stationary distribution of this process reflected at two barriers. This is however done under a rather restrictive assumption, namely all the pairs (μ_i, σ_i^2) are assumed to be proportional to (μ, σ^2), which e.g. implies that all drifts μ_i have the same sign.

The following results can be found in [12]. Firstly, for any index $i \in E$ the distribution of $W1_{\{J=i\}}$ has a mass at zero given by c_i and a density $p_i(x)$ for $x > 0$. Secondly, there are N^+ zeros λ_i (taken according to their multiplicities) of $\det(F(\alpha))$ with $\mathrm{Re}(\lambda_i) > 0$. Moreover, if these zeros are distinct and the vectors $\boldsymbol{\phi}_i$ are such that $\boldsymbol{\phi}_i^{\mathrm{T}} F(\lambda_i) = \mathbf{0}$, then the column vector of densities at $x > 0$ is given by

$$\boldsymbol{p}(x) := (p_1(x), \dots, p_N(x))^{\mathrm{T}} = \sum_{i=1}^{N^+} a_i e^{-\lambda_i x} \boldsymbol{\phi}_i, \tag{8}$$

and vectors $\boldsymbol{c}$ and $\boldsymbol{a}$ satisfy the normalizing condition

$$\boldsymbol{c} + \sum_{i=1}^{N^+} \frac{a_i}{\lambda_i} \boldsymbol{\phi}_i = \boldsymbol{\pi} \text{ and } \boldsymbol{c}^+ = \mathbf{0}. \tag{9}$$

From the theory of systems of second order homogenous differential equations and the results in [12] it can be seen that (8) actually holds under a weaker assumption: it is only required that the zeros λ_i are semi-simple [7, Thm. 8.1]. In this case the vectors $\boldsymbol{\phi}_i^{\mathrm{T}}$ corresponding to $\lambda_i = \lambda^*$ should form the basis of the left null space of $F(\lambda^*)$. As mentioned above, it was an open question if the above linear equations uniquely determine the unknown constants. In the following we give a positive answer for the case when $J(\cdot)$ is time-reversible.

Thm. 1 states that there are exactly N^+ zeros of $\det(F(\alpha))$ in $(0, \infty)$ (taken according to their geometric multiplicities), because $k'(0) < 0$. But then these zeros are all semi-simple and there are no other zeros in $\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$, because $m_i \geq p_i$ (see also Remark 2.2). Thus in the time-reversible case the identity (8) is always true with λ_i being real positive numbers.

It follows from (1) that

$$\boldsymbol{\phi}_i^{\mathrm{T}} F(\lambda_i) = \mathbf{0} \quad \Leftrightarrow \quad F(\lambda_i)\Delta_{\boldsymbol{\pi}}^{-1}\boldsymbol{\phi}_i = \mathbf{0},$$

so λ_i can be taken as the i-th diagonal entry of the matrix Υ and $\boldsymbol{\phi}_i = \Delta_\pi \boldsymbol{v}_i$, where $\boldsymbol{v}_i$ is the i-th column of the matrix V. Next note that the equations in (9) corresponding to the indices in E^+ can be rewritten as

$$\sum_{i=1}^{N^+} \frac{a_i}{\lambda_i} \boldsymbol{v}_i^+ = \mathbf{1}^+.$$

But now Thm. 1 states that the N^+ vectors $\boldsymbol{v}_i^+$ are linearly independent, and thus $\boldsymbol{a}$ can be uniquely determined through

$$\boldsymbol{a} = \Upsilon(V^+)^{-1}\mathbf{1}^+ = -(V^+)^{-1}\Lambda\mathbf{1}^+.$$

After a number of elementary manipulations one obtains

$$\boldsymbol{c}^{\downarrow} = \Delta_{\boldsymbol{\pi}^{\downarrow}}(\mathbf{1}^{\downarrow} - \Pi\mathbf{1}^+)$$

and finally using $V^+ e^{-\Upsilon x}(V^+)^{-1} = e^{\Lambda x}$ we get

$$\boldsymbol{p}(x) = \Delta_{\boldsymbol{\pi}} V e^{-\Upsilon x}\boldsymbol{a} = -\Delta_{\boldsymbol{\pi}} \begin{pmatrix} I^+ \\ \Pi \end{pmatrix} e^{\Lambda x}\Lambda\mathbf{1}^+, \tag{10}$$

where I^+ is the $N^+ \times N^+$ identity matrix.

We can provide a further interpretation of this result. Identity (10) means that for every index i the density $p_i(x)/\pi_i$ (of the stationary workload distribution W given $J = i$) is of phase-type with the *phase generator* Λ, the *exit vector* $-\Lambda\mathbf{1}^+$, the *initial distribution* $\boldsymbol{e}_i^+$ or the corresponding row of the matrix Π depending if $i \in E^+$ or $i \in E^{\downarrow}$ respectively; see [1, Section III.4] for an exposition about phase-type distributions.

3.2. **Reflected Spectrally-negative MAP.** It is claimed in (7) that $(W|J = i)$ and $\big(\overline{X}|J(0) = i\big)$ are equal in distribution. The crucial observation is that $\overline{X}$ is the life-time of $J(\tau_x)$, thus it is of phase type [1, Section III.4]. In particular, for $i \in E^+$ the distribution of $\overline{X}$ is of phase-type with generator matrix Λ, exit vector $-\Lambda\mathbf{1}^+$ and initial distribution $\boldsymbol{e}_i^+$. More care should be taken if $i \in E^{\downarrow}$. In this case we initiate the phase-type distribution with the vector $(\mathbb{P}_i(J(\tau_0) = 1, \tau_0 < \infty), \ldots \mathbb{P}_i(J(\tau_0) = N^+, \tau_0 < \infty))^{\mathrm{T}}$, which is the row of Π corresponding to the index i, and with probability $\mathbb{P}_i(\tau_0 = \infty)$ the random variable $\overline{X}$ has mass at zero as $X(\cdot)$ never hits $(0, \infty)$. This shows that identity (10) holds for a general time-reversible spectrally-negative MAP. Furthermore, using the ideas from Section 3.1 one can easily show that the representation (8) is still valid.

3.3. **Reflected Spectrally-positive MAP.** In this section we consider a time-reversible spectrally-positive MAP $(\hat{X}(\cdot), J(\cdot))$ with negative asymptotic drift and let $(W(\cdot), J(\cdot))$ be the corresponding reflected process. If $X(t) := -\hat{X}(t)$, then $(X(\cdot), J(\cdot))$ is a time-reversible spectrally-negative MAP. In the following we work only with the latter process. For instance, the state space E is split into E^+ and $E^\downarrow$ according to $(X(\cdot), J(\cdot))$. Note also that in this case $k'(0) > 0$ is the stability condition, which implies that matrix Λ is a non-defective generator matrix. Let $\boldsymbol{\pi}_\Lambda$ be its unique stationary distribution.

In [3] the Laplace-Stieltjes transform of (W, J) is obtained:

$$\mathbb{E}[e^{-\alpha W}(1_{\{J=1\}}, \ldots, 1_{\{J=N\}})] = \alpha \boldsymbol{\ell}^{\mathrm{T}} F(\alpha)^{-1}, \tag{11}$$

where $\boldsymbol{\ell}$ is a vector of unknown constants with $\boldsymbol{\ell}^\downarrow = \mathbf{0}^\downarrow$ and $\boldsymbol{\ell}^{\mathrm{T}}\mathbf{1} = k'(0)$. It was mentioned in [3] that it was an open problem how to identify vector $\boldsymbol{\ell}^+$.

Right-multiply both sides of equation (11) by $F(\alpha)$. Pick $\alpha = \lambda_i, i \neq 1$ (so that $\lambda_i \neq 0$) and right-multiply both sides of the equation by $\boldsymbol{v}_i$ to see that $\boldsymbol{\ell}^{\mathrm{T}}\boldsymbol{v}_i = 0$. Using $\boldsymbol{\ell}^{\mathrm{T}}\mathbf{1} = k'(0)$ we obtain $(\boldsymbol{\ell}^+)^{\mathrm{T}}V^+ = \boldsymbol{\ell}^{\mathrm{T}}V = k'(0)\boldsymbol{e}_1^{\mathrm{T}}$. Thm. 1 states that V^+ is an invertible matrix, so that

$$(\boldsymbol{\ell}^+)^{\mathrm{T}} = k'(0)\boldsymbol{e}_1^{\mathrm{T}}(V^+)^{-1}.$$

It is easy to check now that $\boldsymbol{\ell}^+ = k'(0)\boldsymbol{\pi}_\Lambda$.

4. Practical Issues

In the previous section we demonstrated that the possibility of determining the matrices Υ and V (and thus Λ, as well as the vector $\boldsymbol{\pi}_\Lambda$) is essential for finding the distribution of the stationary workload W (in conjunction with J, the stationary distribution of the state of the background process). In this section we discuss computational aspects that facilitate the numerical computation of the zeros of $\det(F(\alpha))$ and the corresponding null spaces, which are the building blocks of the matrices Υ and V as given by (4).

4.1. **A Few Structural Results.** Note that in the time-reversible case the zeros of $\det(F(\alpha))$ of interest are real positive numbers, whereas in general the right half of the complex plane is to be considered. Moreover, none of the zeros of $\det(F(\alpha)), \alpha > 0$ can be larger than $\max_i\{\psi_i(2q_i) : i \in E^+\}$, where $q_i := -Q_{ii}$ and $\psi_i(\cdot)$ is the right-inverse of $\phi_i(\cdot)$, see [15] for a definition. The above claim is true, because for larger α the matrix $F(\alpha)$ is diagonally dominant and thus non-singular (use basic properties of the functions $\phi_i(\cdot)$, or see [10]).

We now present a procedure with attractive numerical properties to find $\alpha > 0$ and $\boldsymbol{v}$ such that $F(\alpha)\boldsymbol{v} = \mathbf{0}$. Trivially, we can equivalently consider $\bar{F}(\alpha)\boldsymbol{v} = \mathbf{0}$ with $\bar{F}(\alpha) = F(\alpha)/\alpha$. Define $h_i(\alpha) := g_i(\alpha)/\alpha$ for

$\alpha > 0$ to be the i-th largest eigenvalue of $\bar{F}(\alpha)$. The following lemma shows that the functions $h_i(\alpha)$ are strictly increasing.

Define d_j to be the deterministic drift of Lévy process $X_j(\cdot)$ if this process has paths of bounded variation, and ∞ otherwise.

Lemma 6. *The functions $h_i(\alpha)$ are strictly increasing with*

(i) $\lim_{\alpha\downarrow 0} h_1(\alpha) = k'(0)$ *and* $\lim_{\alpha\downarrow 0} h_i(\alpha) = -\infty$ *for* $i > 1$,

(ii) $\lim_{\alpha\to\infty} h_i(\alpha) = c_i$, *where* c_i *is the* i*-th largest number among the* d_i*-s.*

Proof. Fix a $c \in \mathbb{R}$, and define the time-reversible generator $\tilde{F}(\alpha) := F(\alpha) - c\alpha$. Trivially $\tilde{g}_i(\alpha) = g_i(\alpha) - c\alpha$ and $\tilde{h}_i(\alpha) := h_i(\alpha) - c$. Lemma 5 applied to $\tilde{F}(\alpha)$ implies that the functions $\tilde{h}_i(\alpha)$ hit 0 in the interval $(0, \infty)$ at most once, which shows that $h_i(\alpha)$ are strictly increasing, because c is arbitrary.

Claim (i) now follows immediately from Lemma 4. Finally, note that $\tilde{N}^+$ (in self-evident notation) is decreasing in c. More precisely, $\tilde{N}^+$ decreases when $c = d_j$ for some j, because then $X_j(t) - ct$ is non-increasing in t, i.e., corresponds to a downward subordinator. Now the second claim follows directly from Lemma 5. $\square$

We now explain how the above lemma can be used to identify V. Recall that we wish to find $\alpha > 0$ and $\boldsymbol{v}$ such that $\bar{F}(\alpha)\boldsymbol{v} = \mathbf{0}$, i.e., we want to find $\alpha > 0$ such that $\bar{F}(\alpha)$ has eigenvalue 0. As the eigenvalues $h_i(\alpha)$ increase in α, for instance a simple bisection procedure can be used.

4.2. **Aggregates of Multiple MAPs.** We now consider the situation in which the MAP $X(\cdot)$ consists of the superposition of multiple independent MAPs $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$ [14]. Then $F(\alpha)$ can be written as $F^{(1)}(\alpha) \oplus \ldots \oplus F^{(M)}(\alpha)$, with $\oplus$ denoting the Kronecker sum [4], and $F^{(1)}(\alpha), \ldots, F^{(M)}(\alpha)$ being generators. If $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$ are spectrally-negative and time-reversible, clearly $X(\cdot)$ inherits these properties. Following the procedure outlined above, one can identify V by equating the eigenvalue functions $h_i(\cdot)$ of $\bar{F}(\alpha)$ to 0. If N_m is the dimension of generator $F^{(m)}(\alpha)$, this would require solving eigensystems of dimension $\prod_{m=1}^{M} N_m$ (as this is the dimension of $F(\alpha)$).

One can, however, find V in a considerably more efficient manner, following the approach presented in [19, Section 4]. There it is explained how to convert the $\sum_{m=1}^{M} N_m$ eigenvalue functions

$$h_i^{(m)}(\alpha),\;\; i = 1, \ldots, N_m,\;\; m = 1, \ldots, M$$

of the (low-dimensional) MAPs $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$ into the $\prod_{m=1}^{M} N_m$ eigenvalue functions of the (high-dimensional) MAP $X(\cdot)$. It essentially entails that the bulk of the computations can be performed at the level of individual MAPs $X^{(1)}(\cdot), \ldots, X^{(M)}(\cdot)$. The corresponding eigenvector (solving $\bar{F}(\alpha)\boldsymbol{v} = \mathbf{0}$) is then the Kronecker product of the eigenvectors associated to the lower dimensional MAPs, cf. [19, Eqn. (60)]. This procedure may lead to reducing the computational burden by several orders of magnitude. The function $h_1^{(m)}(\alpha) = k^{(m)}(\alpha)/\alpha$ is often referred to as the *effective bandwidth function* [6].

Acknowledgment

We thank Offer Kella for suggesting this problem and express our gratitude to Onno Boxma for insightful remarks and motivating discussions.

Appendix A

Lemma 7. *If a sequence of $N \times N$ matrices A_n goes to a diagonal matrix A with elements in $[-\infty, \infty]$, then the N eigenvalues of A_n go to the different (with regard to index) diagonal elements of A.*

Proof. It trivially follows from Gershgorin's theorem [8, Thm. 6.1.1] that for any $\delta > 0$ there exists n_0 such that the eigenvalues of A_n for $n > n_0$ belong to the discs $D_i \coloneqq \{z \in \mathbb{C} : |z - (A_n)_{ii}| < \delta\}$. Moreover, if a union of k of these discs is disjoint from all the remaining $N - k$ discs then there are exactly k eigenvalues of A_n in this union. Clearly, for sufficiently small δ and large enough n_0 the discs D_i and D_j are disjoint if $(A_n)_{ii}$ and $(A_n)_{jj}$ do not have the same limit. This concludes the proof. □

References

[1] S. Asmussen. *Applied Probability and Queues*. Applications of Mathematics. Springer-Verlag New York, Inc., 2nd edition, 2003.

[2] S. Asmussen, F. Avram, and M. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and their Applications*, 109:79–111, 2004.

[3] S. Asmussen and O. Kella. A multi-dimensional martingale for Markov additive processes and its applications. *Advances in Applied Probability*, 32(2):376–393, 2000.

[4] R. Bellman. *Introduction to Matrix Analysis*. McGraw-Hill, New York, 1960.

[5] A. B. Dieker and M. Mandjes. Extremes of Markov-additive processes with one-sided jumps, with queueing applications. 2008. (submitted). `http://www2.isye.gatech.edu/~adieker3/publications/modulatedfluid.pdf`.

[6] A.I. Elwalid and D. Mitra. Effective bandwidth of general Markovian traffic sources and admission control of high speed networks. *IEEE/ACM Transactions on Networking*, 1:329–343, 1993.

[7] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Computer Science and Applied Mathematics. Academic Press Inc., 1982.

[8] R. A. Horn and C. A. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[9] R. Hryniv and P. Lancaster. On the perturbation of analytic matrix functions. *Integral Equations Operator Theory*, 34(3):325–338, 1999.

[10] J. Ivanovs, O. Boxma, and M. Mandjes. Singularities of the generator of a Markov additive process with one-sided jumps. Technical Report 2008-037, EURANDOM, 2008. `http://www.eurandom.tue.nl/reports/2008/037-JI-report.pdf`.

[11] A. Jobert and L. C. G. Rogers. Option pricing with Markov-modulated dynamics. *SIAM Journal on Control and Optimization*, 44:2063–2078, 2006.

[12] R. L. Karandikar and V. G. Kulkarni. Second–order fluid flow models: Reflected Brownian motion in a random environment. *Operations Research*, 43(1):77–88, 1995.

[13] O. Kella and W. Stadje. A Brownian motion with two reflecting barriers and Markov-modulated speed. *Journal of Applied Probability*, 41(4):1237–1242, 2004.

[14] L. Kosten. *Stochastic theory of data-handling systems with groups of multiple sources*, pages 321–331. Performance of Computer-Communication Systems. Elsevier, Amsterdam, 1984.

[15] A. E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Universitext. Springer-Verlag Berlin Heidelberg, 2006.

[16] A. E. Kyprianou and Z. Palmowski. Fluctuations of spectrally negative Markov additive processes. *Séminaire de Probabilité*, XLI:121–135, 2008.

[17] N. U. Prabhu. *Stochastic Storage Processes*, volume 15 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998. Queues, insurance risk, dams, and data communication.

[18] L. C. G. Rogers. Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Annals of Applied Probability*, 4(2):390–413, 1994.

[19] T. E. Stern and A. I. Elwalid. Analysis of separable Markov-modulated rate models for information-handling systems. *Advances in Applied Probability*, 23(1):105–139, 1991.

Eurandom, Eindhoven University of Technology, and Korteweg-de Vries Institute for Mathematics, University of Amsterdam