



Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

MAS

Modelling, Analysis and Simulation



Modelling, Analysis and Simulation

Realization theory for linear hybrid systems, part I:
Existence of realization

M. Petreczky, J.H. van Schuppen

REPORT MAS-R0802 FEBRUARY 2008

Centrum voor Wiskunde en Informatica (CWI) is the national research institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organisation for Scientific Research (NWO). CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

Copyright © 2008, Stichting Centrum voor Wiskunde en Informatica
P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3703

Realization theory for linear hybrid systems, part I: Existence of realization

ABSTRACT

The paper is the first part of a series of papers which deal with realization theory for linear hybrid systems. Linear hybrid systems are hybrid systems in continuous-time without guards whose continuous dynamics is determined by linear control systems and whose discrete dynamics is determined by a finite state automaton. In Part I of the current series of papers we will formulate necessary and sufficient conditions for the existence of a linear hybrid system realizing a specified set of input-output maps. We will also sketch a realization algorithm for computing a linear hybrid system from the input-output data. In Part II we will present conditions for observability and span-reachability of linear hybrid systems and we will show that minimality is equivalent to observability and span-reachability; we will also discuss algorithms for checking observability and span-reachability and for transforming a linear hybrid system to a minimal one.

2000 Mathematics Subject Classification: 93B15 93B20 93B25 93C99

Keywords and Phrases: hybrid systems; realization theory; Hankel-matrix; formal power series; automata

Realization Theory For Linear Hybrid Systems,

Part I: Existence of Realization

Mihály Petreczky

Department of Mechanical Engineering, Eindhoven University of Technology

PO Box 513, 5600 MB Eindhoven, The Netherlands

`M.Petreczky@cw.nl`

Jan H. van Schuppen

Centrum voor Wiskunde en Informatica (CWI)

P.O.Box 94079, 1090GB Amsterdam, The Netherlands

`J.H.van.Schuppen@cw.nl`

Abstract

The paper is the first part of a series of papers which deal with realization theory for linear hybrid systems. Linear hybrid systems are hybrid systems in continuous-time without guards whose continuous dynamics is determined by linear control systems and whose the discrete dynamics is determined by a finite state automaton. In Part I of the current series of papers we will formulate necessary and sufficient conditions for the existence of a linear hybrid system realizing a specified set of input-output maps. We will also sketch a realization algorithm for computing a linear hybrid system from the input-output data. In Part II we will present conditions for observability and span-reachability of linear hybrid systems and we will show that minimality is equivalent to observability and span-reachability; we will also discuss algorithms for checking observability and span-reachability and for transforming a linear hybrid system to a minimal one.

The presented work was carried out during the first author's stay at Centrum voor Wiskunde en Informatica (CWI) in Amsterdam, The Netherlands

I. INTRODUCTION

Realization theory is one of the central topics of system theory. It studies the relationship between control systems and their input-output behaviours. Realization theory helps to understand such important system theoretic properties as observability, controllability and minimality. In addition, realization theory provides the theoretical foundations for systems identification and filtering. In fact, the well-known subspace identification methods for linear systems are based on realization theory. There are several reasons for studying realization theory of linear hybrid systems. First of all, understanding realization theory for linear hybrid systems might help solving the realization problem for other classes of hybrid systems with linear continuous dynamics, but with autonomous switching. In turn, the latter subclass of systems has a wide range of applications. Second, there are applications where linear hybrid systems of the form described below are used.

This paper develops realization theory for a special class of hybrid systems called *linear hybrid systems*. Hybrid systems are systems which exhibit both discrete and continuous behaviour, for more on the topic see [1] and the references therein. A linear hybrid system is a hybrid system *without guards* of the form

$$H : \begin{cases} \frac{d}{dt}x(t) = A_{q(t)}x(t) + B_{q(t)}u(t), & y(t) = C_{q(t)}x(t) \\ q(t+) = \delta(q(t), \gamma(t)), & x(t+) = M_{q(t+), \gamma(t), q(t)}x(t-), \text{ and } o(t) = \lambda(q(t)) \end{cases} \quad (1)$$

Here $q(t) \in Q$ is the *discrete state* at time t , $x(t) \in \mathbb{R}^{n_q(t)} = \mathcal{X}_{q(t)}$ is the *continuous state* at time t , $y(t) \in \mathbb{R}^p$ is the *continuous output* at time t , and $o(t) \in O$ is the *discrete output* at time t . The behaviour of the system at time t is influenced by *continuous inputs* $u(t) \in \mathbb{R}^m$, and *discrete inputs* $\gamma(t) \in \Gamma$. Further, Q is the *finite set of discrete states* of H , $\mathcal{X}_q = \mathbb{R}^{n_q}$, $n_q > 0$ is the *continuous state-space associated with the discrete state* $q \in Q$, O is the *finite set of discrete outputs*, Γ is the *finite set of discrete inputs (events)*, \mathbb{R}^m is the *set of continuous input values*, and \mathbb{R}^p is the *set of continuous output values*. The *state-space of* H is the set of all pairs (q, x) where $q \in Q$ is a discrete state and $x \in \mathcal{X}_q$ is a continuous state. For two different discrete states $q_1, q_2 \in Q$, the dimensions of the corresponding components \mathcal{X}_{q_1} and \mathcal{X}_{q_2} are allowed to be different. The continuous state $x(t)$ lives in the continuous component $\mathcal{X}_{q(t)}$ which corresponds to the discrete state $q(t)$. For each discrete state $q \in Q$, the matrices $A_q \in \mathbb{R}^{n_q \times n_q}$, $B_q \in \mathbb{R}^{n_q \times m}$ and $C_q \in \mathbb{R}^{p \times n_q}$ define a continuous-time linear system (A_q, B_q, C_q) on $\mathcal{X}_q = \mathbb{R}^{n_q}$.

The map $\delta : Q \times \Gamma \rightarrow Q$ is called the *discrete state-transition map*, and the map $\lambda : Q \rightarrow O$ is called the *discrete readout map*. For each discrete state $q \in Q$ and discrete input $\gamma \in \Gamma$, the matrix $M_{\delta(q,\gamma),\gamma,q} \in \mathbb{R}^{n_{\delta(q,\gamma)} \times n_q}$ is referred to as *reset map*. The continuous dynamics of the linear hybrid system H is determined by the linear systems (A_q, B_q, C_q) and the reset maps. The discrete dynamics is determined by the finite Moore-automaton $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$. Informally, a Moore-automaton is just a finite-state deterministic automaton equipped with outputs. A formal definition will be presented in Section VI. Notice that the classical linear systems are a special subclass of linear hybrid systems.

The evolution of the system (1) takes place according to the classical definition [1]. Assume that we feed in a \mathbb{R}^m -valued input signal $u(t) \in \mathbb{R}^m$. We assume that the discrete inputs (events) are indeed inputs, that is, we can create any discrete input at any time. In other words, linear hybrid systems *have no guards; the discrete state-transition takes place independently of the continuous state*. As long as the value of the discrete state does not change, the continuous state and the continuous output change according to the linear system determined by $(A_{q(t)}, B_{q(t)}, C_{q(t)})$. The discrete state $q(t)$ changes at time t if a discrete input $\gamma(t)$ arrives at time t . Then the new discrete state is determined by the discrete state-transition rule as $q(t+) = \delta(q(t), \gamma(t))$. The new continuous state $x(t+) \in \mathbb{R}^{n_{q(t+)}}$ is obtained from the current continuous state $x(t-) \in \mathbb{R}^{n_{q(t)}}$ by applying the corresponding reset map, that is, $x(t+) = M_{q(t+),\gamma(t),q(t)}x(t-) \in \mathbb{R}^{n_{q(t+)}}$. The discrete output is obtained from the discrete state by applying the discrete readout map to the current discrete state, that is, $o(t) = \lambda(q(t))$. After that, the continuous state and output evolve according to the continuous-time linear system $(A_{q(t+)}, B_{q(t+)}, C_{q(t+)})$ associated with the new discrete state $q(t+)$ and the discrete state and the discrete output remain unchanged until the arrival of the next discrete input. A more formal definition of the semantics of linear hybrid systems will be presented in Section II.

The current paper is the first part of a series of papers devoted to realization theory of linear hybrid systems. In Part II of the series we shall address the problem of minimality, observability and reachability. In Part I (the current paper) we will present necessary and sufficient conditions for existence of a realization by a linear hybrid system of sets of input-output maps. Notice that unlike in the classical formulation of the realization problem, in this paper we are looking at realizability of a *set of input-output maps* rather than a *single input-output map*. By looking at families of input-output maps we hope to provide a first step towards a *behavioural approach*,

[2] for hybrid systems. Obviously, the case of a single input-output map follows from the results of the paper. The conditions for existence of a realization by a linear hybrid system involve the requirement that the rank of the generalized infinite Hankel-matrix computed from the input-output maps is finite. If applied to linear systems, these conditions yield the classical ones. It will be shown that a minimal linear hybrid system can be constructed from the columns of the generalized Hankel-matrix. The constructions presented in the paper can be implemented; a minimal linear hybrid system can be computed either from input-output maps or from an existing realization. In fact, it is possible to formulate a partial realization theory for linear hybrid systems, see [3], [4]. However, in this paper we will not present the realization algorithms in full detail. Instead, we will just sketch the main steps of the algorithm and we will refer to [3], [4] for details. We plan to devote Part III of the current series of papers to the algorithmic aspects of realization theory of linear hybrid systems.

The class of hybrid systems studied in this paper is completely different from linear hybrid automata defined in [5]. The class of hybrid systems studied in this paper bears a certain resemblance to linear switching systems [6]. However, in [6] the external discrete events are viewed as disturbances not as inputs and the finite state automaton is non-deterministic. To the best of our knowledge, the only results on realization theory of hybrid systems are the ones presented in [3] and the references therein. In [4], [7] some of the results of the current paper were stated, but most of the proofs were omitted. The results of the current paper were included into the first author's PhD thesis [3].

Theory of rational formal power series [8], [9], and classical automata theory [10], [11] are the main mathematical tools used in the paper. Formal power series were already used for realization theory of nonlinear systems, see [12], [13], [14] and the references therein.

The outline of the paper is the following. Section II defines linear hybrid systems and presents the basic notions and notations which will be used in the paper. Section III presents the main theorems of the paper formally. Section VI presents the necessary background on finite Moore-automata. Section V contains the necessary results on formal power series. Section IV presents certain properties of the input-output maps of linear hybrid systems which are needed for the proof of the main results. Section VII contains the proof of Theorem 1 which gives necessary and sufficient conditions for existence of a realization. Section VIII discusses the computational aspects of realization theory.

II. PROBLEM FORMULATION

We will start by fixing some notation and terminology. In Subsection II-A. we will present the definition of a Moore-automaton, and in Subsection II-B we will define linear hybrid systems and the related concepts.

Notation Denote by \mathbb{N} the set of natural numbers including 0. Denote by \mathbb{N}^k the set of k tuples of natural numbers. Let $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^p$ be a smooth map of k variables and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{N}^k$ be a k tuple of natural numbers. We denote by $D^\alpha \phi$ the following partial derivative of ϕ evaluated at $(0, 0, \dots, 0)$,

$$D^\alpha \phi = \frac{d^{\alpha_1}}{dt_1^{\alpha_1}} \frac{d^{\alpha_2}}{dt_2^{\alpha_2}} \cdots \frac{d^{\alpha_k}}{dt_k^{\alpha_k}} \phi(t_1, t_2, \dots, t_k) \Big|_{t_1=t_2=\dots=t_k=0}.$$

For each $n > 0$ and $j = 1, \dots, n$, denote by e_j the j th unit vector of \mathbb{R}^n , i.e. $e_j = (\sigma_{1,j}, \sigma_{2,j}, \dots, \sigma_{n,j})^T$, where $\sigma_{j,j} = 1$ and $\sigma_{i,j} = 0$ for $i \neq j$. Denote by T the *time-axis* $[0, +\infty) \subseteq \mathbb{R}$ formed by all non-negative reals. Denote by $PC(T, \mathbb{R}^n)$ the set of piecewise-continuous maps with values in \mathbb{R}^n . Let Σ be a finite set which will be referred to as *alphabet*. Denote by Σ^* the set of finite *strings or words* of elements of Σ , i.e. an element of Σ^* is a finite sequence of the form $w = a_1 a_2 \cdots a_k$, where $a_1, a_2, \dots, a_k \in \Sigma$, and $k \geq 0$; k is *the length of w* and it is denoted by $|w|$. The empty sequence (word) is denoted by ϵ , and its length is 0. The concatenation of two strings $v = v_1 \cdots v_k$, and $w = w_1 \cdots w_m \in \Sigma^*$ is the string $vw = v_1 \cdots v_k w_1 \cdots w_m$. The empty sequence ϵ is a unit element with respect to the concatenation, i.e. $\epsilon w = w \epsilon = w$ for all $w \in \Sigma^*$. We denote by w^k the string $\underbrace{w \cdots w}_{k\text{-times}}$. The word w^0 is just the empty word ϵ . By abuse of notation we will denote any constant function $f : T \rightarrow A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then f will be denoted by a . For any function f the range of f will be denoted by $\text{Im} f$, i.e. if $f : A \rightarrow B$, then $\text{Im} f = \{f(a) \mid a \in A\}$. For any set A we will denote by $\text{card}(A)$ the cardinality of A . For any two sets J and X , an *indexed subset* of X with the *index set* J is simply a map $Z : J \rightarrow X$, denoted by $Z = \{a_j \in X \mid j \in J\}$, where $a_j = Z(j), j \in J$. That is, Z is a collection of (not necessarily distinct) elements a_j of X indexed by $j \in J$. For any two sets A, B , denote by Π_A and Π_B the functions which map any pair $(a, b) \in A \times B$ to its A -valued (respectively B -valued) component, i.e. $\Pi_A((a, b)) = a$ and $\Pi_B((a, b)) = b$. For any family of vector spaces $V_i, i \in I$, denote by $\bigoplus_{i \in I} V_i$ the direct sum of the vector spaces $V_i, i \in I$. If $T : V \rightarrow W$ is a linear map between vector spaces V and W , then for each element $v \in V$, Tv stands for $T(v)$, i.e. the value of T at v . If $S : H \rightarrow V$ is another linear map, then

the composition $T \circ S : H \rightarrow W$ of the maps T and S is denoted by TS , i.e. $TSv = T(S(v))$ for all $v \in H$.

A. Definition of Moore-automaton

A *finite Moore-automaton* is a tuple $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ where **(1)** Q is a finite set, called the *state-space* of \mathcal{A} , **(2)** Γ is a finite set, called the *input space* of \mathcal{A} , **(3)** O is a (not necessarily finite) set, called the *output space* of \mathcal{A} , **(4)** $\delta : Q \times \Gamma \rightarrow Q$ is a map, called the *state-transition map* of \mathcal{A} , **(5)** $\lambda : Q \rightarrow O$ is a map, called the *readout map* of \mathcal{A} . The elements of the input space Γ will sometimes be referred to as input symbols. We will denote by $\text{card}(\mathcal{A})$ the cardinality of the state-space Q of \mathcal{A} , i.e. $\text{card}(\mathcal{A}) = \text{card}(Q)$. A Moore-automaton can be thought of as a machine or system, which can be in finitely many states. The machine has an input tape and an output tape. The machine repeats the following sequence of actions; it reads from the input tape, changes its internal state and it writes a symbol onto the output tape. If the machine is in the state q , and it reads the symbol $\gamma \in \Gamma$ from the input tape, then it changes its state to $\delta(q, \gamma)$ and writes the output symbol $\lambda(q)$ on its output tape, and positions itself to read the next symbol from the input tape.

We can extend the functions δ and λ to act on sequences of input symbols. More precisely, define the function $\tilde{\delta} : Q \times \Gamma^* \rightarrow Q$ recursively as follows; let $\tilde{\delta}(q, \epsilon) = q$, and for each word $w \in \Gamma^*$ and input symbol $\gamma \in \Gamma$ let $\tilde{\delta}(q, w\gamma) = \delta(\tilde{\delta}(q, w), \gamma)$. Define the map $\tilde{\lambda} : Q \times \Gamma^* \rightarrow O$ by $\tilde{\lambda}(q, w) = \lambda(\tilde{\delta}(q, w))$ for each input word $w \in \Gamma^*$ and discrete state $q \in Q$. By abuse of notation we will denote $\tilde{\delta}$ and $\tilde{\lambda}$ simply by δ and λ respectively. An automaton $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is called *reachable from* $Q_0 \subseteq Q$, if for all $q \in Q$ there exists a sequence of input symbols $w \in \Gamma^*$ and a state $q_0 \in Q_0$ such that $q = \delta(q_0, w)$. Two states $q_1, q_2 \in Q$ are called *indistinguishable* if for any input sequence w , the output produced by q_1 equals the output produced by q_2 , i.e. $\lambda(q_1, w) = \lambda(q_2, w)$. The automaton \mathcal{A} is called *observable* or *reduced*, if there are no two distinct states $q_1, q_2 \in Q$, $q_1 \neq q_2$, such that q_1 and q_2 are indistinguishable.

Natural candidates for input-outputs maps of a Moore-automaton \mathcal{A} are maps of the form $f : \Gamma^* \rightarrow O$ which map words over Γ to elements in O . Let $\mathcal{D} = \{\phi_j : \Gamma^* \rightarrow O \mid j \in J\}$ be an indexed set of such functions with some index set J . Consider a map $\zeta : J \rightarrow Q$. The pair (\mathcal{A}, ζ) will be called *an automaton realization*. We will say that the automaton realization (\mathcal{A}, ζ) is a

*realization*¹ of \mathcal{D} if for all $j \in J$ the input-output map induced by the state $\zeta(j)$ is identical to the element ψ_j of \mathcal{D} , more precisely, for each sequence of input symbols $w \in \Gamma$, $\lambda(\zeta(j), w) = \phi_j(w)$. The automaton \mathcal{A} is said to be a *realization* of \mathcal{D} if there exists a $\zeta : J \rightarrow Q$ such that (\mathcal{A}, ζ) is a realization of \mathcal{D} . Notice the function ζ is just used to specify those states of \mathcal{A} , which generate an input-output map identical to some element of \mathcal{D} . An automaton realization (\mathcal{A}, ζ) of \mathcal{D} is called *minimal* if (\mathcal{A}, ζ) has the smallest state-space cardinality among all realizations of \mathcal{D} , i.e. for each automaton realization (\mathcal{A}', ζ') of \mathcal{D} , $\text{card}(\mathcal{A}) \leq \text{card}(\mathcal{A}')$. A realization (\mathcal{A}, ζ) is called *reachable* if \mathcal{A} is reachable from the range of ζ , i.e if it is reachable from the set $\text{Im}\zeta = \{\zeta(j) \mid j \in J\}$; and (\mathcal{A}, ζ) is called *observable* if \mathcal{A} is observable. Let (\mathcal{A}, ζ) and (\mathcal{A}', ζ') be two automaton realizations. Assume that $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ and $\mathcal{A}' = (Q', \Gamma, O, \delta', \lambda')$. A map $\phi : Q \rightarrow Q'$ is said to be an *automaton morphism* from (\mathcal{A}, ζ) to (\mathcal{A}', ζ') , denoted by $\phi : (\mathcal{A}, \zeta) \rightarrow (\mathcal{A}', \zeta')$ if ϕ commutes with the state-transition and readout maps, that is, $\phi(\delta(q, \gamma)) = \delta'(\phi(q), \gamma), \forall q \in Q, \gamma \in \Gamma$, $\lambda(q) = \lambda'(\phi(q)), \forall q \in Q$, $\phi(\zeta(j)) = \zeta'(j), \forall j \in J$. The automaton morphism ϕ is called *injective* (*surjective*) if the map ϕ is injective (surjective). The automaton morphism ϕ is called an *isomorphism*, if it is bijective. Two Moore-automata realizations are called isomorphic, if there exists an isomorphism between them.

B. Linear Hybrid System

Notation 1 (Linear hybrid systems): A linear hybrid system of the form (1) is denoted by

$$H = (\mathcal{A}, \mathbb{R}^m, \mathbb{R}^p, (\mathcal{X}_q, A_q, B_q, C_q)_{q \in Q}, \{M_{\delta(q, \gamma), \gamma, q} \mid q \in Q, \gamma \in \Gamma\})$$

where $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$ is the Moore-automaton formed by the discrete-state transition and discrete readout map of the system H . The automaton \mathcal{A} is denoted by \mathcal{A}_H , and the *state space* of H will be denoted by $\mathcal{H}_H = \bigcup_{q \in Q} \{q\} \times \mathcal{X}_q$.

Below we will describe the dynamics of linear hybrid systems, which follows the classical definition [1]. Denote the set of timed sequences of discrete inputs by $(\Gamma \times T)^*$, i.e. a typical element of $(\Gamma \times T)^*$ is a finite sequence of the form $w = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k)$ where $k \geq 0$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $t_1, \dots, t_k \in T$. The interpretation of the sequence w is the following. The event γ_i took place *after* the event γ_{i-1} and t_{i-1} is the *elapsed time between the arrival*

¹Notice that here we define the concept of realization for *families* of input-output maps rather than for a single input-output map as it is done in the classical literature, see [11], [10].

of γ_{i-1} and the arrival of γ_i . That is, t_i is the difference of the arrival times of γ_i and γ_{i-1} . Consequently, $t_i \geq 0$ but we allow $t_i = 0$, that is, we allow γ_i to arrive instantly after γ_{i-1} . If $i = 1$, then t_1 is simply the time when the first event γ_1 arrived. The inputs of the linear hybrid system H are piecewise-continuous input functions $u \in PC(T, \mathbb{R}^m)$ and timed sequences of discrete inputs (events) $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$. For an arbitrary state $h_0 = (q_0, x_0)$ of H define the *continuous state* $x_H(h_0, u, w, t_{k+1}) \in \mathcal{X}_{q_k}$ reached from h_0 with inputs u and w at time $\sum_{j=1}^k t_j + t_{k+1}$ recursively on k as follows. For each $i = 0, \dots, k$, denote by q_i the discrete state $q_i = \delta(q_0, \gamma_1 \gamma_2 \cdots \gamma_i)$ reachable from the initial discrete state q_0 with the sequence $\gamma_1 \gamma_2 \cdots \gamma_i$. Let the map $x : T \ni t \rightarrow x_H(h_0, u, \epsilon, t) \in \mathcal{X}_{q_0}$ be the solution of the differential equation $\frac{d}{dt}x(t) = A_{q_0}x(t) + B_{q_0}u(t)$ with the initial state $x(0) = x_0$. Let $k > 0$, and assume that for $v = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_{k-1}, t_{k-1})$ the continuous state $x_H(h_0, u, v, t_k) \in \mathcal{X}_{q_{k-1}}$ is already defined. Define $x_H(h_0, u, w, t_{k+1}) \in \mathcal{X}_{q_k}$, where $q_k = \delta(q_{k-1}, \gamma_k)$ so that the map $x : [0, t_{k+1}] \ni t \mapsto x_H(h_0, u, w, t_{k+1})$ is the solution of the differential equation $\frac{d}{dt}x(t) = A_{q_k}x(t) + B_{q_k}u(t + \sum_{j=1}^k t_j)$ with the initial condition $x(0) = M_{q_k, \gamma_k, q_{k-1}}x_H(h_0, u, v, t_k)$. Define the *state* $\xi_H(h_0, u, w, t_{k+1})$ reached from h_0 under inputs u, w at time $\sum_{j=1}^{k+1} t_j$ by $\xi_H(h_0, u, w, t_{k+1}) = (\delta(q_0, \gamma_1 \cdots \gamma_k), x_H(h_0, u, w, t_{k+1}))$. In fact, with the notation above, using the well-known expression for trajectories of linear systems

$$\begin{aligned}
x_H(h_0, u, w, t_{k+1}) &= e^{A_{q_k} t_{k+1}} M_{q_k, \gamma_k, q_{k-1}} e^{A_{q_{k-1}} t_k} \cdots \\
&\cdots M_{q_1, \gamma_1, q_0} e^{A_{q_0} t_1} x_0 + \sum_{i=0}^k e^{A_{q_k} t_{k+1}} M_{q_k, \gamma_k, q_{k-1}} e^{A_{q_{k-1}} t_k} \cdots \\
&\cdots M_{q_{i+1}, \gamma_{i+1}, q_i} \int_0^{t_{i+1}} e^{A_{q_i}(t_{i+1}-s)} B_{q_i} u(s + \sum_{j=1}^i t_j) ds
\end{aligned} \tag{2}$$

Define the *output* $v_H(h_0, u, w, t_{k+1})$ induced by h_0 under inputs u, w at time $\sum_{j=1}^{k+1} t_j$ as

$$v_H(h_0, u, w, t_{k+1}) = (\lambda(q_0, w), C_{q_k} x_H(h_0, u, w, t_{k+1}))$$

Define the *input-output map of the system H induced by the state $h_0 \in \mathcal{H}_H$ of H* as the function

$$\begin{aligned}
v_H(h, \cdot) : PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T \ni (u, w, t) \mapsto \\
v_H(h, u, w, t) \in O \times \mathbb{R}^p
\end{aligned} \tag{3}$$

From (3) it follows that the input-output maps of interest are maps of the form $f : PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T \rightarrow O \times \mathbb{R}^p$. We will denote the class of all such functions by $F(PC(T, \mathbb{R}^m) \times$

$(\Gamma \times T)^* \times T, O \times \mathbb{R}^p$). Throughout the paper we will mostly be concerned with realization of a *set of input-output maps*. It means that we will have to look at systems which have not one, but several initial states. We will use the following formalism to deal with the issue. Let H be a linear hybrid system of the form (1) and let Φ be a subset of the set of input-output maps. Let $\mu : \Phi \rightarrow \mathcal{H}_H$ be any map. We will call the pair (H, μ) a *realization*. The map μ just specifies a way to associate an initial state to each element of Φ . The statement that (H, μ) is a realization *does not imply* that H is realized Φ from the set of initial states $\text{Im}\mu$. The set Φ is said to be *realized* by a hybrid realization (H, μ) where $\mu : \Phi \rightarrow \mathcal{H}_H$, if for each input-output map f from the set Φ , the map f and the input-output map $v_H(\mu(f), \cdot)$ induced by the initial state $\mu(f)$ are identical, that is

$$\forall f \in \Phi: \quad v_H(\mu(f), \cdot) = f$$

In other words, for each input $u \in PC(T, \mathbb{R}^m)$, for each timed sequence of discrete inputs $w \in (\Gamma \times T)^*$ and for each time $t \in T$,

$$v_H(\mu(f), u, w, t) = f(u, w, t)$$

We will say that H *realizes* Φ if there exists a map $\mu : \Phi \rightarrow \mathcal{H}_H$ such that (H, μ) realizes Φ . With slight abuse of terminology, sometimes we will call both H and (H, μ) a *realization* of Φ . Thus, H realizes Φ if and only if for each $f \in \Phi$ there exists a state $h \in \mathcal{H}$ such that $v_H(h, \cdot) = f$. We will denote by μ_D the Q -valued component of μ , and by μ_C the continuous valued component of μ , that is, for each $f \in \Phi$, $\mu(f) = (\mu_D(f), \mu_C(f))$. The map μ can be thought of as a map which assigns to each input-output map f an initial state of the system H ; it is just an alternative way to fix a set of initial states.

Notation 2 (Products of System Matrices): The following notational convention will be used throughout the rest of the paper. Consider a linear hybrid system H of the form (1). Let $k \geq 0$ and consider an arbitrary sequence of discrete inputs $\gamma_1, \dots, \gamma_k \in \Gamma$ of length k . Consider an arbitrary sequence of natural numbers $\alpha_1, \alpha_2, \dots, \alpha_{k+1} \geq 0$ of length $k+1$. Pick discrete states q_0, q_1, \dots, q_k such that for each $i = 1, \dots, k$, the state q_i is defined recursively by $q_i = \delta(q_{i-1}, \gamma_i)$. Consider the product of matrices

$$\begin{aligned} & A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} A_{q_{k-1}}^{\alpha_k} M_{q_{k-1}, \gamma_{k-1}, q_{k-2}} \cdots \\ & \cdots A_{q_1}^{\alpha_2} M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} \end{aligned} \tag{4}$$

Following the widespread convention, if $\alpha_i = 0$ for some i , then $A_{q_{i-1}}^{\alpha_i}$ is interpreted as the identity matrix. Notice that (4) is *uniquely defined by the choice of q_0 and $\gamma_1, \dots, \gamma_k$, and $\alpha_1, \dots, \alpha_{k+1}$* . In the rest of the paper, unless stated otherwise, if we use an expression of the form (4), we will always assume that $q_i = \delta(q_0, \gamma_1 \gamma_2 \dots \gamma_i)$ holds. We will also adopt the same assumption for expressions of the form

$$\begin{aligned} & C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} A_{q_{k-1}}^{\alpha_k} M_{q_{k-1}, \gamma_{k-1}, q_{k-2}} \dots \\ & \dots A_{q_1}^{\alpha_2} M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} \end{aligned} \quad (5)$$

That is, when writing (5), we will automatically assume that $q_i = \delta(q_0, \gamma_1 \gamma_2 \dots \gamma_i)$ holds for all $i = 1, \dots, k$. Let $l = 1, \dots, k+1$ and consider the expression

$$\begin{aligned} & A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} A_{q_{k-1}}^{\alpha_k} M_{q_{k-1}, \gamma_{k-1}, q_{k-2}} \dots \\ & \dots A_{q_l}^{\alpha_{l+1}} M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l} B_{q_{l-1}} \end{aligned} \quad (6)$$

Again, (6) makes sense only if $q_i = \delta(q_{i-1}, \gamma_i)$ for all $i = l, \dots, k$ and hence we will always assume that $q_i = \delta(q_{l-1}, \gamma_l \dots \gamma_i)$ when we use expression (6). If $k = 0$ then (4) is understood to be $A_{q_1}^{\alpha_1}$ and (4) is understood to be $C_{q_1} A_{q_1}^{\alpha_1}$. If $l = k+1$, then (6) is understood to be the matrix $A_{q_{k+1}}^{\alpha_{k+1}} B_{q_{k+1}}$. *When it does not create confusion, we will use $A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \dots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_0}$ instead of the full expression (4), we will use $C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \dots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_0}$ instead of (5), and we will use $A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \dots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l} B_{q_{l-1}}$ instead of (6).*

In order to illustrate the notions introduced above, we will consider the following example.

Example 1: Consider a linear hybrid system H of the form (1), with the following system parameters. Assume that the set of discrete event $\Gamma = \{a, b\}$ consists of two events a and b . Assume that there are 4 discrete states, i.e. $Q = \{q_1, q_2, q_3, q_4\}$. The discrete state transitions are of the form: $\delta(q_1, a) = q_1$, $\delta(q_2, a) = q_1$, $\delta(q_1, b) = q_2$, $\delta(q_2, b) = q_2$, $\delta(q_3, a) = q_4$, $\delta(q_3, b) = q_3$, $\delta(q_4, b) = q_3$, $\delta(q_4, a) = q_4$, and the corresponding discrete outputs are $\lambda(q_1) = o$, $\lambda(q_2) = o$, $\lambda(q_3) = d$ and $\lambda(q_4) = g$. Denote by I_n the $n \times n$ identity matrix. The linear systems and the reset maps are of the following form.

$$\begin{aligned} A_{q_1} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_{q_1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, M_{q_1, a, q_1} = I_3, M_{q_2, b, q_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \\ A_{q_2} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, B_{q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}, M_{q_2, b, q_2} = I_2, M_{q_1, a, q_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

$$A_{q_3} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_{q_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C_{q_3} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, M_{q_3,b,q_3} = I_3, M_{q_4,a,q_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$A_{q_4} = \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix}, B_{q_4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C_{q_4} = \begin{bmatrix} 0 & 0 \end{bmatrix}, M_{q_4,a,q_4} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}, M_{q_3,b,q_4} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \text{ Consider the}$$

state $h_1 = (q_2, \begin{bmatrix} 1 & 0 \end{bmatrix}^T)$ and consider the input-output map $v_H(h_1, \cdot)$ induced by the state h . The analytic expression for $v_H(h, \cdot)$ is rather complex, therefore we will show it only for the following switching scenario; $(a, t_1)(b, t_2)(b, t_3)(a, t_4)$. Then for arbitrary piecewise-continuous input u , the output induced by the state h under the sequence of discrete inputs $(a, t_1)(b, t_2)(b, t_2)(a, t_4), t_5)$ is of the form

$$v_H(h, u, (a, t_1)(b, t_2)(b, t_3)(a, t_4), t_5) = (o, e^{-2t_5} e^{-3t_4} e^{-3t_3} e^{-2t_2} e^{-3t_1} + \int_0^{t_1+t_2+\dots+t_5} e^{-t_1-\dots-t_5-s} u(s) ds)$$

Consider the states $h_1 = (q_2, \begin{bmatrix} 1 & 0 \end{bmatrix}^T)$ and $h_2 = (q_3, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T)$. Define the input-output maps $f_1 = v_H(h_1, \cdot)$ and $f_2 = v_H(h_2, \cdot)$, and consider the set $\Phi = \{f_1, f_2\}$. Define the map $\mu : \Phi \rightarrow \mathcal{H}_H$ by $\mu(f_1) = h_1$ and $\mu(f_2) = h_2$. Then it is immediate from the definition of Φ that (H, μ) is a realization of Φ .

III. MAIN RESULTS

The goal of the section is to present the main results of the paper in a formal way. That is, we will present necessary and sufficient conditions for existence of a realization by a linear hybrid system.

In order to formulate the necessary and sufficient conditions mentioned above we will introduce the notion of hybrid kernel representation. Let Φ be a set of input-output maps, i.e. let $\Phi \subseteq F(PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T, O \times \mathbb{R}^p)$. For each input-output map $f \in \Phi$, denote by f_C the \mathbb{R}^p -valued part of the map f ; and denote by f_D the O -valued part of the map f_D . That is, $f(u, w, t) = (f_D(u, w, t), f_C(u, w, t)) \in O \times \mathbb{R}^p$ for all $u \in PC(T, \mathbb{R}^m)$, $w \in (\Gamma \times T)^*$ and $t \in T$. Informally, Φ has a hybrid kernel representation if,

- (a) For each $f \in \Phi$, f_D depends only on the discrete inputs.
- (b) For each $f \in \Phi$, f_C continuous and affine in continuous inputs, moreover for constant continuous inputs, f_C is analytic in time.

More formally, the definition goes as follows.

Definition 1 (Hybrid kernel representation): Φ is said to have *hybrid kernel representation* if for each input-output map $f \in \Phi$

- 1) The function f_D depends only on Γ^* , i.e. f_D can be regarded as a function $f_D : \Gamma^* \rightarrow O$. That is, for any two timed sequences of discrete inputs $w_1 = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k)$ and $w_2 = (\gamma_1, \tau_1)(\gamma_2, \tau_2) \cdots (\gamma_k, \tau_k)$ which differ only in the switching times, and for any two $t_{k+1}, \tau_{k+1} \in T$ and piecewise-continuous inputs u_1, u_2 ; $f_D(u_1, w_1, t_{k+1}) = f_D(u_2, w_2, \tau_{k+1})$.
- 2) For each input-output map $f \in \Phi$ and for each sequence of discrete inputs $v = \gamma_1 \gamma_2 \cdots \gamma_k$, $\gamma_1, \dots, \gamma_k \in \Gamma$ there exist analytic functions $K_v^{f, \Phi} : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^p$ and $G_{v,j}^{f, \Phi} : \mathbb{R}^j \rightarrow \mathbb{R}^{p \times m}$ where $j = 1, 2, \dots, k+1$, such that for all $t_1, \dots, t_{k+1} \in T$

$$f_C(u, w, t_{k+1}) = K_v^{f, \Phi}(t_1, \dots, t_k, t_{k+1}) + \sum_{i=0}^k \int_0^{t_{i+1}} G_{v, k+1-i}^{f, \Phi}(t_{i+1} - s, t_{i+2}, \dots, t_{k+1}) \sigma_{k+1-i} u(s) ds \quad (7)$$

where $\sigma_j u(s) = u(s + \sum_{i=1}^{j-1} t_i)$ and $w = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$.

A formal theorem relating hybrid kernel representations with conditions (a) and (b) is presented in [3] Section 7.1, Theorem 30, page 205. The role of the maps $K_v^{f, \Phi}$ and $G_{v,j}^{f, \Phi}$ is best understood by analogy with the theory of linear systems. Consider a map of the form $y : PC(T, \mathbb{R}^m) \times T \rightarrow \mathbb{R}^p$ and recall from [15] that a necessary condition for existence of a linear system realization of y is that there exists analytic functions $G : T \rightarrow \mathbb{R}^{p \times m}$ and $K : T \rightarrow \mathbb{R}^p$ such that $y(u, t) = K(t) + \int_0^t G(t-s)u(s)ds$. If the linear system (A, B, C) with the initial state x_0 is a realization of y , then $G(t) = Ce^{At}B$ and $K(t) = Ce^{At}x_0$. The requirement that Φ has a hybrid kernel representation is analogous to requiring that $y(u, t) = K(t) + \int_0^t G(t-s)u(s)ds$ with analytic K and G .

Remark 1: Similarly to linear systems, it is impossible to check computationally if a set of input-output maps has a hybrid kernel representation or not. One has to treat it as an assumption which has to be validated. To this end, the alternative characterization of hybrid kernel representations presented in [3] might be useful. Notice that the knowledge of the functions $K_w^{f, \Phi}$ and $G_{w,l}^{f, \Phi}$ is not at all needed for constructing a realization of Φ .

Remark 2: Let H be a hybrid system of the form (1) and let $\mu : \Phi \rightarrow \mathcal{H}_H$ be a map assigning initial states. It is easy to see that (H, μ) is a realization of Φ if and only if Φ has a hybrid

kernel representation of the form

$$\begin{aligned}
K_w^{f,\Phi}(t_1, \dots, t_{k+1}) &= C_{q_k} e^{A_{q_k} t_{k+1}} M_{q_k, \gamma_k, q_{k+1}} \cdots \\
&\cdots M_{q_1, \gamma_1, q_0} e^{A_{q_0} t_0} \mu_C(f) \\
G_{w, k+2-l}^{f,\Phi}(t_l, \dots, t_{k+1}) &= C_{q_k} e^{A_{q_k} t_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots \\
&\cdots e^{A_{q_l} t_{l+1}} M_{q_l, \gamma_l, q_{l-1}} e^{A_{q_{l-1}} t_l} B_{q_{l-1}}
\end{aligned} \tag{8}$$

and $f_D(w) = \lambda(\mu_D(f), w)$ for each $w = \gamma_1 \cdots \gamma_k$, $\gamma_1, \dots, \gamma_k \in \Gamma$. Recall that the maps μ_D and μ_C are defined by $\mu(f) = (\mu_C(f), \mu_D(f))$.

Using the notation above, define for each $f \in \Phi$ the function the map $y_0^{f,\Phi} : PC(T, \mathbb{R}^m) \times (\Gamma \times T)^* \times T \rightarrow \mathbb{R}^p$ as

$$\begin{aligned}
y_0^{f,\Phi}(u, w, t_{k+1}) &= \\
&\sum_{i=0}^k \int_0^{t_{i+1}} G_{v, k+1-i}^{f,\Phi}(t_{i+1} - s, t_{i+2}, \dots, t_{k+1}) \sigma_{k+i-1} u(s) ds
\end{aligned}$$

for each $u \in PC(T, \mathbb{R}^m)$, $w = (\gamma_1, t_1) \cdots (\gamma_k, t_k) \in (\Gamma \times T)^*$, $k \geq 0$, $t_{k+1} \in T$. It follows that $y_0^{f,\Phi}(u, w, t_{k+1}) = f_C(u, w, t_{k+1}) - f_C(0, w, t_{k+1})$. The intuition behind the definition of $y_0^{f,\Phi}$ is the following. If (H, μ) is a realization of Φ , then $y_0^{f,\Phi} = \Pi_{\mathbb{R}^p} \circ v_H((\mu_D(f), 0), \cdot)$, i.e. $y_0^{f,\Phi}$ can be thought of as the continuous-valued part of the input-output map induced by a hybrid state whose continuous-state component is zero.

As the next step, we will define the notion of the Hankel-matrix of a family of input-output maps admitting a hybrid kernel representation. Consider the following finite set, $\tilde{\Gamma} = \Gamma \cup \{e\}$, where e is chosen such that $e \notin \Gamma$, i.e. e is not a discrete input event. Every word $w \in \tilde{\Gamma}^*$ can be uniquely written as $w = e^{\alpha_1} \gamma_1 e^{\alpha_2} \gamma_2 \cdots \gamma_k e^{\alpha_{k+1}}$ for some $\gamma_1, \dots, \gamma_k \in \Gamma$, $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{N}$. Recall that e^k denotes the k letter word $\underbrace{ee \cdots e}_{k\text{-times}}$.

For each input-output maps $f \in \Phi$, for each continuous input $u \in PC(T, \mathbb{R}^m)$, and for each sequence of discrete inputs $w = \gamma_1 \cdots \gamma_k \in \Gamma^*$, $\gamma_1, \dots, \gamma_k \in \Gamma$ define the maps

$$\begin{aligned}
f_C(u, w, \cdot) : T^{k+1} \ni (t_1, \dots, t_{k+1}) &\mapsto f_C(u, (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k), t_{k+1}) \\
y_0^{f,\Phi}(u, w, \cdot) : T^{k+1} \ni (t_1, \dots, t_{k+1}) &\mapsto y_0^{f,\Phi}(u, (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k), t_{k+1})
\end{aligned} \tag{9}$$

It is easy to see that if u is constant, then the maps $f_C(u, w, \cdot)$ and $y_0^{f,\Phi}(u, w, \cdot)$ are analytic.

For each $f \in \Phi$ define the maps $Z_f : \tilde{\Gamma}^* \rightarrow \mathbb{R}^p$ and $Z_{f,j} : \tilde{\Gamma}^* \rightarrow \mathbb{R}^p$ as

$$\begin{aligned} Z_f(e^{\alpha_1} \gamma_1 e^{\alpha_2} \dots \gamma_k e^{\alpha_{k+1}}) &= D^\alpha f_C(0, w, \cdot) \text{ and} \\ Z_{f,j}(e^{\alpha_1} \gamma_1 e^{\alpha_2} \dots \gamma_k e^{\alpha_{k+1}}) &= D^\alpha y_0^{f,\Phi}(e_j, w, \cdot) \end{aligned} \quad (10)$$

where $w = \gamma_1 \dots \gamma_k$ and $\alpha = (\alpha_1, \dots, \alpha_{k+1})$. Recall that e_j is the j th unit vector in \mathbb{R}^m and that we agreed to identify e_j with the constant input function $T \ni t \mapsto e_j \in \mathbb{R}^m$, whose value is the j th unit vector. Similarly, 0 denotes the constant function $T \ni t \mapsto 0 \in \mathbb{R}^m$. Notice that the maps $Z_{f,j}$ and Z_f can be viewed as sequences of high-order time derivatives of the maps f_C and $y_0^{f,\Phi}$. Notice that $Z_{f,j}(v) = 0$ for all sequences of discrete inputs $v \in \Gamma^*$. Notice that the exact knowledge of the functions $K_w^{f,\Phi}$ and $G_{w,l}^{f,\Phi}$ is not needed in order to construct the functions $Z_f, Z_{f,j}$. In fact, one can think of Z_f as an object containing all the information on the behaviour of f with the zero continuous input. The functions $Z_{f,j}$, $j = 1, \dots, m$ contains all the information on the behaviour of $y_0^{f,\Phi}(e_j, \cdot)$. The maps $Z_{f,j}$, Z_f can be thought of as generalizations of Markov parameters for linear systems. In fact, later on we will show that if (H, μ) is realization of Φ and H is of the form (1), then $Z_{f,j}$ and Z_f are of the form

$$\begin{aligned} Z_{f,j}(e^{\alpha_1} \gamma_1 \dots \gamma_k e^{\alpha_{k+1}}) &= \\ C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \dots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l-1} B_{q_{l-1}} e_j \\ Z_f(e^{\alpha_1} \gamma_1 e^{\alpha_2} \dots \gamma_k e^{\alpha_{k+1}}) &= \\ C_{q_k} A_{q_k}^{\alpha_k} M_{q_k, \gamma_k, q_{k-1}} \dots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} \mu_C(f) \end{aligned} \quad (11)$$

We define the *Hankel matrix* of Φ , denoted by H_Φ , as the following infinite matrix formed by values of Z_f and $Z_{f,j}$. Consider the index set $I_\Phi = \Phi \cup (\Phi \times \{1, \dots, m\})$ formed by elements of Φ and pairs of the form (f, j) where f is an input-output map from Φ and $j = 1, \dots, m$. The columns of the matrix H_Φ are indexed by pairs $(w, l) \in \tilde{\Gamma}^* \times I_\Phi$ whose first component is a word over $\tilde{\Gamma}$ and whose second component is an index from the set I_Φ . The rows of the matrix H_Φ are indexed by pairs of the form (v, i) , where $v \in \tilde{\Gamma}^*$ is a word over $\tilde{\Gamma}$ and $i = 1, \dots, p$. That is, H_Φ has an infinite number of columns and rows. The element of H_Φ lying on the intersection of the row indexed by (v, i) and of the column indexed by (w, l) equals the i th row of the column vector $Z_l(wv) \in \mathbb{R}^p$ if $l \in \Phi$ or the i th row of the column vector $Z_{f,j}(wv) \in \mathbb{R}^p$ if $l = (f, j)$, $f \in \Phi$, $j = 1, \dots, m$, that is,

$$(H_\Phi)_{(v,i),(w,l)} = (Z_l(wv))_i \in \mathbb{R} \quad (12)$$

for all $w, v \in \tilde{\Gamma}^*, l \in I_\Phi, i = 1, \dots, p$. The *rank of H_Φ* (denoted by $\text{rank } H_\Phi$) is understood to be the dimension of the vector space spanned by the columns of H_Φ . Notice that the classical Hankel matrix of linear systems is a special case of the Hankel matrix defined above.

Denote by $H_{\Phi,O}$ the set of those columns of the matrix H_Φ which are indexed by (w, l) where $w \in \Gamma^*$ and $l = (f, j)$ for some $f \in \Phi$ and $j \in \{1, \dots, m\}$, i.e.

$$H_{\Phi,O} = \{(H_\Phi)_{(w,l)} \mid w \in \Gamma^*, l \in \Phi \times \{1, \dots, m\}\} \quad (13)$$

where $(H_\Phi)_{(w,l)}$ is the column of H_Φ indexed by (w, l) . For each sequence of discrete inputs $w \in \Gamma^*$ and for each input-output map $f \in \Phi$, define the *shift of f_D by w* as $w \circ f_D : \Gamma^* \ni v \mapsto f_D(wv) \in O$. Denote by W_{Φ_D} is the set of all maps of the form $w \circ f_D$, i.e.

$$W_{\Phi_D} = \{w \circ f_D : \Gamma^* \rightarrow O \mid w \in \Gamma^*, f \in \Phi\} \quad (14)$$

Notice that the value of $w \circ f_D$ at v is the value of f_D for the sequence wv , where v is preceded by w , hence the use of the word shift. This definition is standard in automata theory [11], [10], [8].

The intuition behind the definitions above is the following. The Hankel-matrix H_Φ contains all the information on the continuous-valued components of the input-output maps. The set W_{Φ_D} contains all the information on the discrete-valued components of the input-output maps. Finally, $H_{\Phi,O}$ contains information on those continuous-valued components, which should be interpreted as discrete outputs.

Now we are ready to state the main results of the paper.

Theorem 1 (Realization By Linear Hybrid Systems): Φ has a realization by a linear hybrid system if and only if

- (1) Φ has a hybrid kernel representation, and
- (2) the rank of the Hankel-matrix H_Φ is finite, and the sets $H_{\Phi,O}$ and W_{Φ_D} are finite, i.e. $\text{rank } H_\Phi < +\infty$, $\text{card}(W_{\Phi_D}) < +\infty$ and $\text{card}(H_{\Phi,O}) < +\infty$.

The proof of the above theorem can be found in Section VII. As we already mentioned, we can compute a realization of Φ from finite data, see the discussion in Section VIII or [3], [4].

IV. INPUT-OUTPUT MAPS OF LINEAR HYBRID SYSTEMS

The goal of this section is to present some technical results on input-output maps of linear hybrid systems. These results will play an important role in the proof of the realization theorem.

Let Φ be a set of input-output maps. Assume that Φ has a hybrid kernel representation and recall from Section III the definition of the map $y_0^{f,\Phi}$ for each $f \in \Phi$. Recall that for each $j = 1, \dots, m$, e_j denotes both the j th unit vector of \mathbb{R}^m and the constant map whose value is the j th unit vector of \mathbb{R}^m . Similarly, 0 denotes the constant zero map from T to \mathbb{R}^m . Recall from (9) Section III, the definition of maps $f_C(u, w, \cdot)$ and $y_0^{f,\Phi}(u, w, \cdot)$.

Lemma 1: If Φ has a hybrid kernel representation, then the functions $K_w^{f,\Phi}$, $G_{w,j}^{f,\Phi}$, $f \in \Phi$, $w \in \Gamma^*$, $j = 1, \dots, |w| + 1$, $f \in \Phi$ are uniquely defined and their high-order derivatives at 0 are of the form

$$\begin{aligned} D^\alpha K_w^{f,\Phi} &= D^\alpha f_C(0, w, \cdot), \\ D^\xi G_{w,l}^{f,\Phi} e_j &= D^\beta y_0^{f,\Phi}(e_j, w, \cdot) \end{aligned} \quad (15)$$

where $\beta = (0, 0, \dots, 0, \xi_1 + 1, \xi_2, \dots, \xi_l)$ and, $\alpha \in \mathbb{N}^{|w|+1}$, $\xi \in \mathbb{N}^l$, and $l = 1, \dots, |w| + 1$, and $j = 1, \dots, m$.

Proof: Formula (15) follows from the formula $\frac{d}{dt} \int_0^t f(t, \tau) d\tau = f(t, t) + \int_0^t \frac{d}{dt} f(t, \tau) d\tau$, (see [15]), and (7). Assume that both $K_w^{f,\Phi}$, $G_{w,j}^{f,\Phi}$ and $\tilde{K}_w^{f,\Phi}$, $\tilde{G}_{w,j}^{f,\Phi}$ are analytic functions which satisfy (7). Then by (15) for each $\alpha \in \mathbb{N}^{|w|+1}$, the high-order derivatives $D^\alpha K_w^{f,\Phi}$ and $D^\alpha \tilde{K}_w^{f,\Phi}$ coincide with $D^\alpha f_C(0, w, \cdot)$ and hence with each other. Similarly, for each $l = 1, \dots, |w| + 1$, and $\xi \in \mathbb{N}^l$, and $j = 1, \dots, m$, the j th column of the high-order derivatives $D^\xi G_{w,l}^{f,\Phi}$ and $D^\xi \tilde{G}_{w,l}^{f,\Phi}$ are equal to $D^\beta y_0^{f,\Phi}(e_j, w, \cdot)$ and hence to each other. Since the functions $K_w^{f,\Phi}$, $G_{w,j}^{f,\Phi}$, $\tilde{K}_w^{f,\Phi}$ and $\tilde{G}_{w,j}^{f,\Phi}$ are analytic and their high-order derivatives coincide, we get that $K_w^{f,\Phi} = \tilde{K}_w^{f,\Phi}$ and $G_{w,j}^{f,\Phi} = \tilde{G}_{w,j}^{f,\Phi}$. \blacksquare

Proposition 1: Let H be a linear hybrid system of the form (1) and let μ be a map of the form $\mu : \Phi \rightarrow \mathcal{H}_H$. The pair (H, μ) is a realization of Φ if and only if Φ has a hybrid kernel representation and for each $w \in \Gamma^*$, $f \in \Phi$, $j = 1, 2, \dots, m$ and $\alpha \in \mathbb{N}^{|w|+1}$ the following holds

$$\begin{aligned} D^\alpha y_0^{f,\Phi}(e_j, w, \cdot) &= D^\beta G_{w,k+2-l}^{f,\Phi} e_j = \\ C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l-1} B_{q_{l-1}} e_j \\ D^\alpha f_C(0, w, \cdot) &= D^\alpha K_w^{f,\Phi} = \\ C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} x_0 \\ f_D(w) &= \lambda(q_0, w) \end{aligned} \quad (16)$$

where l is the smallest index such that $\alpha_l > 0$, i.e. $\alpha_1 = \dots = \alpha_{l-1} = 0$ and $\alpha_l > 0$, e_j is the j th unit vector of \mathbb{R}^m , $\beta = (\alpha_l - 1, \dots, \alpha_{|w|+1})$ and $w = \gamma_1 \cdots \gamma_k$, $\gamma_1, \dots, \gamma_k \in \Gamma$, and $\mu(f) = (q_0, x_0)$.

The proposition above says that Φ is realized by a hybrid system if and only if for each $f \in \Phi$, the discrete valued component f_D is realized by the automaton of the hybrid systems, and the high-order derivatives of the continuous-valued component f_C can be expressed as products of the system matrices.

Proof: [Proposition 1] Recall the statement of Remark 2 and (8). If (H, μ) is a realization of Φ , then $y_0^{f,\Phi} = \Pi_{\mathbb{R}^p} \circ v_H((\mu_D(f), 0), \cdot)$. By (15) we get that $D^\alpha y_0^{f,\Phi}(e_j, w, \cdot) = D^\beta G_{w,l}^{f,\Phi} e_j$ and $D^\alpha f(0, w, \cdot) = D^\alpha K_w^{f,\Phi}$. If we compute the high-order derivatives, then we see that (8) implies (16). Assume that (16) holds. Then the high-order derivatives $D^\alpha K_w^{f,\Phi}$ and $D^\beta G_{w,k+2-l}^{f,\Phi}$ equal the corresponding high-order derivatives of the right-hand sides of the expressions in (8). Notice that due to their analyticity the high-order derivatives, $D^\alpha K_w^{f,\Phi}$ and $D^\beta G_{w,k+2-l}^{f,\Phi} e_j$ determine $K_w^{f,\Phi}$ and $G_{w,k+2-l}^{f,\Phi}$ uniquely. Hence, (16) implies (8), which implies that (H, μ) is a realization of Φ . \blacksquare

V. FORMAL POWER SERIES

The material of this section is based on the classical theory of formal power series, see [13], [8]. Unlike in the classical case where rationality of a single formal power series is studied, we will be interested in rationality of a *set of formal power series*. Therefore, the original framework [13], [8] has to be extended. This extension is relatively straightforward. Therefore, we will only formulate the most important results and we will omit the proofs, which can be found in [16], [3] and are quite similar to the classical ones, see [13], [8].

Let J be an arbitrary set and let Σ be a finite set, which will be referred to as the *alphabet*. Recall the notation introduced in Section II. A *rational representation with the index set J* , or simply a *representation*, is a tuple $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in \Sigma}, B, C)$ such that **(1)** \mathcal{X} is a finite-dimensional vector space, i.e. $\dim \mathcal{X} < +\infty$, **(2)** $C : \mathcal{X} \rightarrow \mathbb{R}^p$ is a linear map, **(3)** for each letter $\sigma \in \Sigma$, $A_\sigma : \mathcal{X} \rightarrow \mathcal{X}$ is a linear map, and **(4)** $B = \{B_j \in \mathcal{X} \mid j \in J\}$ is a set of elements of \mathcal{X} indexed by J . The number $\dim \mathcal{X}$ is called the *dimension* of R and it is denoted by $\dim R$. In the sequel the following short-hand notation will be used: for each word $w = \sigma_1 \sigma_2 \cdots \sigma_k \in \Sigma^*$, $\sigma_1, \dots, \sigma_k \in \Sigma$, $k > 0$, denote by A_w the linear map obtained by composition of the linear maps

$A_{\sigma_1}, A_{\sigma_2}, \dots, A_{\sigma_k}$, i.e. $A_w = A_{\sigma_k} A_{\sigma_{k-1}} \cdots A_{\sigma_1}$; and A_ϵ will be identified with the identity map. Define the following subspaces of \mathcal{X}

$$\begin{aligned} W_R &= \text{Span}\{A_w B_j \in \mathcal{X} \mid w \in \Sigma^*, j \in J\} \text{ and} \\ O_R &= \bigcap_{w \in \Sigma^*} \ker C A_w \end{aligned} \quad (17)$$

The representation R is called *reachable* if $\dim W_R = \dim R$ and R is called *observable* if $O_R = \{0\}$. The space O_R is analogous to the kernel of the observability matrix of a linear system, and W_R is analogous to the image of the reachability matrix of a linear system. For a subspace $W \subseteq \mathcal{X}$, the representation R is said to be *W-observable* if $W \cap O_R = \{0\}$. It is clear that if R is observable, then R is *W-observable* for any subspace W . Let $R_i = (\mathcal{X}_i, \{A_{i,\sigma}\}_{\sigma \in \Sigma}, B_i, C_i)$, $i = 1, 2$ be two representations with the *same* index set J , and assume that $B_i = \{B_{i,j} \in \mathcal{X}_i \mid j \in J\}$, $i = 1, 2$. A *representation morphism* $T : R_1 \rightarrow R_2$ is a linear map $T : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ such that **(1)** for any letter $\sigma \in \Sigma$, $T A_{1,\sigma} = A_{2,\sigma} T$, **(2)** for any index $j \in J$, $T B_{1,j} = B_{2,j}$, and **(3)** $C_1 = C_2 T$. The morphism T is called *surjective*, *injective*, *isomorphism* if T is a surjective, injective or isomorphism respectively, if considered as a linear map.

A *formal power series* S with coefficients in \mathbb{R}^p is a map $S : \Sigma^* \rightarrow \mathbb{R}^p$, i.e. it is simply a function which maps finite words over Σ to vectors in \mathbb{R}^p . We denote by $\mathbb{R}^p \ll \Sigma^* \gg$ the set of all formal power series with coefficients in \mathbb{R}^p . Notice that the set $\mathbb{R}^p \ll \Sigma^* \gg$ can be regarded a vector space with point-wise addition and multiplication [8]. More precisely, if $S, T \in \mathbb{R}^p \ll \Sigma^* \gg$ and $\alpha, \beta \in \mathbb{R}$, then define the linear combination $\alpha T + \beta S \in \mathbb{R}^p \ll \Sigma^* \gg$ as $(\alpha T + \beta S)(w) = \alpha T(w) + \beta S(w)$. Consider the indexed set of formal power series $\Psi = \{S_j \in \mathbb{R}^p \ll \Sigma^* \gg \mid j \in J\}$. We will often refer to indexed sets of formal power series as *families* of formal power series. The family Ψ is called *rational* if there exists a representation $R = (\mathcal{X}, \{A_\sigma\}_{\sigma \in \Sigma}, B, C)$ with the index set J , such that for any sequence $\sigma_1, \dots, \sigma_k \in \Sigma, k \geq 0$,

$$S_j(\sigma_1 \sigma_2 \cdots \sigma_k) = C A_{\sigma_k} A_{\sigma_{k-1}} \cdots A_{\sigma_1} B_j \quad (18)$$

for each index $j \in J$. If (18) holds, then we will say that R is (*rational*) *representation* of Ψ . A representation R_{\min} of Ψ is called *minimal* if for each representation R of Ψ it holds that $\dim R_{\min} \leq \dim R$. Define the Hankel matrix of H_Ψ of Ψ as follows. The columns of H_Ψ are indexed by pairs (w, j) where $j \in J$ is an arbitrary index, and $w \in \Sigma^*$ is an arbitrary word. The rows of H_Ψ are indexed by pairs (v, i) where $i = 1, \dots, p$ and v is an arbitrary word $v \in \Sigma^*$.

The element of H_Ψ indexed by the column index (w, j) and the row index (v, i) is the i th row of the column vector $S_j(wv)$. That is, if $(S_j(wv))_i$ denotes the i th row of $S_j(wv) \in \mathbb{R}^p$, then

$$(H_\Psi)_{(v,i),(w,j)} = (S_j(wv))_i \quad (19)$$

for all $v, w \in \Sigma^*$, $i = 1, \dots, p$ and $j \in J$. Let $w \in \Sigma^*$ be a word over Σ and for any formal power series $S \in \mathbb{R}^p \ll \Sigma^* \gg$ define the *left shift* $w \circ S \in \mathbb{R}^p \ll \Sigma^* \gg$ of S by w as $(w \circ S)(v) = S(wv)$ for all $v \in \Sigma^*$. Define the subspace W_Ψ of $\mathbb{R}^p \ll \Sigma^* \gg$ as the space spanned by all the formal power series of the form $w \circ S_j$ with $j \in J$ and $w \in \Sigma^*$, i.e. $W_\Psi = \text{Span}\{w \circ S_j \in \mathbb{R}^p \ll \Sigma^* \gg \mid j \in J, w \in \Sigma^*\}$. Notice that there is a one-to-one correspondence between the columns of H_Ψ indexed by (w, j) and the formal power series $w \circ S_j$, hence the vector space spanned by the columns of H_Ψ and the space W_Ψ are isomorphic. The dimension of the vector space spanned by the columns of H_Ψ will be called *the rank of H_Ψ* .

Theorem 2 ([16], [3]): With the notation above the following holds.

- **Rationality.** Ψ is rational if and only if the Hankel-matrix of Ψ is finite, i.e. $\dim W_\Psi = \text{rank } H_\Psi < +\infty$.
- **Minimality.** If Ψ is rational, then there exists a minimal rational representation of Ψ . A representation R_{\min} of Ψ is minimal if and only if one of the following equivalent conditions holds; **(i)** R_{\min} is reachable and observable, or **(ii)** if R is a reachable representation of Ψ , then there exists a surjective representation morphism $T : R \rightarrow R_{\min}$. In addition all minimal representations of R are isomorphic.

Remark 3: If Ψ is rational, i.e. $\text{rank } H_\Psi < +\infty$, then we can construct a minimal representation R_f of Ψ on W_Ψ , or, which is the same, on the column space of the Hankel matrix H_Ψ , as follows; $R_f = (W_\Psi, \{A_\sigma\}_{\sigma \in \Sigma}, B, C)$, where for each $\sigma \in \Sigma$, A_σ is the linear map corresponding to the left shift by σ , i.e. for any formal power series $T \in W_\Psi$, $A_\sigma(T) = \sigma \circ T$; the indexed set B is indexed by J and it is equal to Ψ , that is, $B = \{S_j \in W_\Psi \mid j \in J\}$; and the map C simply evaluates each formal power series at the empty sequence, that is, for any T , $C(T) = T(\epsilon)$. It can be shown that R_f is minimal.

Notice that the construction of R_f can be carried out on the column space of H_Ψ , i.e. on a space spanned by column vectors of *infinite length*. It is possible to replace the column vectors of infinite length by column vectors of finite length, and thus to obtain a *partial*

realization theory. The details can be found in [3], here we confine ourselves to sketching the main ideas. Let $N, L > 0$ and consider the *finite* sub-matrix $H_{\Psi, L, N}$ of the Hankel-matrix H_{Ψ} which is formed by the intersection of all the columns indexed by (w, j) and all the rows indexed by (v, i) such that w, v are words of length at most N and L respectively. That is, $(H_{\Psi, L, N})_{(v, i), (w, j)} = (H_{\Psi})_{(v, i), (w, j)} = (S_j(wv))_i$. If $\text{rank } H_{\Psi, N, N} = \text{rank } H_{\Psi}$, then we can compute a minimal representation of Ψ by factorizing the finite sub-matrix $H_{\Psi, N+1, N}$ of H_{Ψ} . In particular, if $\text{rank } H_{\Psi} \leq N$, then $\text{rank } H_{\Psi, N, N} = \text{rank } H_{\Psi}$ and hence we can compute a minimal representation of Ψ from the *finite* matrix $H_{\Psi, N, N}$. In fact, we have a complete partial realization theory for rational formal power series, similar to that of for linear systems. For more on computational issues for representations and partial realization theory, see [3], [4].

VI. FINITE MOORE-AUTOMATON

Realization theory of Moore-automata is among the oldest results in automata theory, see [11], [10]. However, in this paper we are interested in realization of *families of input-output maps* rather than single input output maps and this is not covered by the classical theory. In this section we will only state the results on realization of families of input-output maps by Moore-automata. The proofs of these results are completely analogous to the proofs of the corresponding classical results [11], [10] and they can also be found in [3].

Let J be an arbitrary set and let $\mathcal{D} = \{\phi_j : \Gamma^* \rightarrow O \mid j \in J\}$ be an indexed set of input-output maps. For an arbitrary map $\phi : \Gamma^* \rightarrow O$ and for an arbitrary sequence $w \in \Gamma^*$ define *the left shift of ϕ by w* as the map $w \circ \phi : \Gamma^* \ni v \mapsto \phi(wv) \in O$. The definition above can be found in the classical literature [11], [10]. Denote by $W_{\mathcal{D}}$ the set formed by all the maps of the form $w \circ \phi_j$ with $j \in J$, and $\phi_j \in \mathcal{D}$, i.e. $W_{\mathcal{D}} = \{w \circ \phi_j : \Gamma^* \rightarrow O \mid w \in \Gamma^*, j \in J\}$. Notice that if we apply the definition of $W_{\mathcal{D}}$ to the set $\mathcal{D} = \Phi_D = \{f_D : \Gamma^* \rightarrow O \mid f \in \Phi\}$, then we obtain the set W_{Φ_D} already defined in (14).

Theorem 3 (Realization Theory [3], [11], [10]): With the notation above,

- **Existence of a Realization.** \mathcal{D} has a realization by a finite Moore-automaton if and only if $W_{\mathcal{D}}$ is finite, i.e. $\text{card}(W_{\mathcal{D}}) < +\infty$.
- **Minimality.** If \mathcal{D} has a Moore-automaton realization, then it also has a minimal Moore-automaton realization. A realization (\mathcal{A}, ζ) of \mathcal{D} is minimal if and only if one of the following equivalent conditions holds; **(i)** (\mathcal{A}, ζ) is reachable and observable, or **(ii)** for

each reachable realization (\mathcal{A}', ζ') of \mathcal{D} there exists a surjective automaton morphism $T : (\mathcal{A}', \zeta') \rightarrow (\mathcal{A}, \zeta)$.

In addition, any two minimal Moore-automaton realizations of \mathcal{D} are isomorphic.

Remark 4: If $W_{\mathcal{D}}$ is finite, then we can define a minimal realization $(\mathcal{A}_{can}, \zeta_{can})$ of \mathcal{D} with the state-space $W_{\mathcal{D}}$ as follows; $\mathcal{A}_{can} = (W_{\mathcal{D}}, \Gamma, O, L, T)$, $\zeta_{can}(j) = \phi_j$ for all $j \in J$, and $L(\phi, \gamma) = \gamma \circ \phi$ for all $\gamma \in \Gamma$, $\phi \in W_{\mathcal{D}}$, and $T(\phi) = \phi(\epsilon)$ for all $\phi \in W_{\mathcal{D}}$. It can be shown that $(\mathcal{A}_{can}, \zeta_{can})$ is minimal.

From the construction of $(\mathcal{A}_{can}, \zeta_{can})$ it follows that one can construct a Moore-automaton realization of \mathcal{D} from the finite set $W_{\mathcal{D}}$ of input-output maps. However, each such input-output map contains infinite data points. It turns out that one can construct a Moore-automaton realization of \mathcal{D} from finite data, i.e. one has *partial realization theory*. We will just sketch the construction below, the reader is referred to [3] for details. Let $M, L > 0$ and denote by $W_{\mathcal{D}, L, M}$ the table whose rows are indexed by words v over Γ of length at most L and whose columns are indexed by pairs (w, j) , where w is a word over Γ of length at most M , and $j \in J$. The entry of $W_{\mathcal{D}, L, M}$ indexed by $(v, (w, j))$ equals to $\psi_j(wv)$. If $\text{card}(W_{\mathcal{D}, M, M}) = \text{card}(W_{\mathcal{D}})$ then it is possible to compute a minimal Moore-automaton realization from $W_{\mathcal{D}, M+1, M}$. In particular, if $\text{card}(W_{\mathcal{D}}) \leq M$, then $\text{card}(W_{\mathcal{D}, M, M}) = \text{card}(W_{\mathcal{D}})$ and hence it is possible to construct a realization from $W_{\mathcal{D}, M+1, M}$. For more details on algorithms for Moore-automata see [3], [4] and the references therein.

VII. EXISTENCE OF A LINEAR HYBRID SYSTEM REALIZATION: PROOF OF THEOREM 1

Let Φ be a set of input-output maps. It follows from Proposition 1 that if Φ has a linear hybrid realization, then Φ has a hybrid kernel representation. Therefore, we can assume that Φ has a hybrid kernel representation. The rest of the proof relies on the following steps.

Step 1. We will construct a certain family of formal power series Ψ_{Φ} and an indexed set of discrete input-output maps \mathcal{D}_{Φ} from Φ . The family Ψ_{Φ} will have the property that the Hankel matrix of Φ equals the Hankel-matrix of Ψ_{Φ} , i.e. $H_{\Phi} = H_{\Psi_{\Phi}}$. In addition, in Lemma 2 we will show that the indexed set \mathcal{D}_{Φ} can be realized by a Moore-automaton if and only if the sets $H_{\Phi, O}$ and W_{Φ_D} are finite.

Step 2. Let (H, μ) be a realization, where H is of the form (1) and $\mu : \Phi \rightarrow \mathcal{H}_H$ is a map assigning initial states. In Theorem 4 we will show that if (H, μ) is a realization of Φ , then we

can construct from (H, μ) a representation $R_{H, \mu}$, and a Moore-automaton $\bar{\mathcal{A}}_H$, such that $R_{H, \mu}$ is a representation of Ψ_Φ and $(\bar{\mathcal{A}}_H, \mu_D)$ is a realization of \mathcal{D}_Φ . Here, μ_D denotes the discrete Q -valued part of μ , that is, for all $f \in \Phi$, $\mu(f) = (\mu_D(f), \mu_C(f))$ and $\mu_D(f) \in Q$. That is, if (H, μ) is a realization of Φ , then Ψ_Φ is rational and \mathcal{D}_Φ has a realization by a Moore-automaton. We will call $R_{H, \mu}$ the *representation associated with the linear hybrid system realization* (H, μ) . We will call $(\bar{\mathcal{A}}_H, \mu_D)$ the *Moore-automaton realization associated with the realization* (H, μ) .

Step 3. In Theorem 5 we will show that if R is an observable representation of Ψ_Φ , and $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of \mathcal{D}_Φ , then we can construct from R and $(\bar{\mathcal{A}}, \zeta)$ a linear hybrid system realization $(H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ of Φ , and we will call the linear hybrid system realization $(H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ the *linear hybrid system realization associated with R and $(\bar{\mathcal{A}}, \zeta)$* . By Theorem 2, if Ψ_Φ is rational, then it has a minimal representation R and this representation is observable. Similarly, from Theorem 3 it follows that if \mathcal{D}_Φ has a Moore-automaton realization, then there exists a minimal, and hence reachable, Moore-automaton realization $(\bar{\mathcal{A}}, \zeta)$ of \mathcal{D}_Φ . Then $(H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ is a well-defined linear hybrid system realization of Φ . Hence, if Ψ_Φ is rational and \mathcal{D}_Φ has a Moore-automaton realization, then we can construct a linear hybrid system realization of Φ .

Step 4. From **Step 2** and **Step 3** it follows that Φ can be realized by a linear hybrid system if and only if Ψ_Φ is rational and \mathcal{D}_Φ has a realization by a Moore-automaton. From this, the statement of the theorem follows easily, by noticing that by Theorem 2, Ψ_Φ is rational if and only if $\text{rank } H_{\Psi_\Phi} = \text{rank } H_\Phi < +\infty$; and by Lemma 2 \mathcal{D}_Φ has a Moore-automaton realization if and only if $\text{card}(W_{\mathcal{D}_\Phi}) < +\infty$ and $\text{card}(H_{\Phi, O}) < +\infty$.

Below we will carry out the steps outlined above more formally. Recall from Section III the definition of set $\tilde{\Gamma} = \Gamma \cup \{e\}$, $e \notin \Gamma$ and recall the maps Z_f and $Z_{f,j}$, where $f \in \Phi$ and $j = 1, \dots, m$. It is easy to see that Z_f and $Z_{f,j}$ are formal power series over the alphabet $\Sigma = \tilde{\Gamma}$ with the coefficients in \mathbb{R}^p , i.e. $Z_f, Z_{f,j} \in \mathbb{R}^p \ll \tilde{\Gamma}^* \gg$. Consider the index set $I_\Phi = \Phi \cup (\Phi \times \{1, 2, \dots, m\})$ formed by elements of Φ and pairs (f, j) where $f \in \Phi$ and $j = 1, \dots, m$.

Definition 2: Define the set of formal power series Ψ_Φ associated with Φ as

$$\Psi_\Phi = \{Z_j \in \mathbb{R}^p \ll \tilde{\Gamma}^* \gg \mid j \in I_\Phi\}$$

where we identify $Z_{(f,j)}$ with $Z_{f,j}$ for $f \in \Phi$ and $j = 1, \dots, m$.

That is, Ψ_Φ is the indexed set of formal power series formed by the formal power series $Z_f, Z_{f,j}$

and indexed by the elements of I_Φ . It is easy to see that the Hankel-matrix H_Φ of Φ defined in Section III and the Hankel-matrix of Ψ_Φ are equal, i.e. $H_\Phi = H_{\Psi_\Phi}$. Consider the linear hybrid system H of the form (1) and let $\mu : \Phi \rightarrow \mathcal{H}_H$ be a map assigning initial states.

Construction 1: Define the representation $R_{H,\mu}$ associated with (H, μ) from **Step 2.** as

$$R_{H,\mu} = (\mathcal{X}, \{M_\sigma\}_{\sigma \in \tilde{\Gamma}}, \tilde{B}, \tilde{C}), \text{ where} \quad (20)$$

State-space \mathcal{X} . Assume that Q has N elements, i.e. $\text{card}(Q) = N$, and fix a basis $\{e_{q,j} \mid q \in Q, j = 1, \dots, m\}$ in \mathbb{R}^{Nm} . Define \mathcal{X} as the direct sum of the vector spaces \mathcal{X}_q , $q \in Q$ and \mathbb{R}^{Nm} , i.e. $\mathcal{X} = (\bigoplus_{q \in Q} \mathcal{X}_q) \oplus \mathbb{R}^{Nm}$.

Notice that the vector spaces \mathcal{X}_q , $q \in Q$ and \mathbb{R}^{Nm} can be viewed as subspaces of \mathcal{X} .

Linear maps. The linear maps $\tilde{C} : \mathcal{X} \rightarrow \mathbb{R}^p$, $M_e : \mathcal{X} \rightarrow \mathcal{X}$ and $M_\gamma : \mathcal{X} \rightarrow \mathcal{X}$, $\gamma \in \Gamma$, are defined as follows.

For all $q \in Q$ and $x \in \mathcal{X}_q$: $\tilde{C}x = C_q x$, $M_e x = A_q x \in \mathcal{X}_q$, and $M_\gamma x = M_{\delta(q,\gamma),\gamma,q} x \in \mathcal{X}_{\delta(q,\gamma)}$.

For all $q \in Q$, $j = 1, \dots, m$: $\tilde{C}e_{q,j} = 0$, $M_e e_{q,j} = B_q e_j \in \mathcal{X}_q$, and $M_\gamma e_{q,j} = e_{\delta(q,\gamma),j} \in \mathbb{R}^{Nm}$.

That is, the restriction of M_e , M_γ and \tilde{C} to any of the subspaces \mathcal{X}_q of \mathcal{X} equals A_q , $M_{\delta(q,\gamma),\gamma,q}$ and C_q respectively. The application of M_e to each $e_{q,j} \in \mathbb{R}^{Nm}$ yields the j th column of B_q , the restriction of \tilde{C} to \mathbb{R}^{Nm} is the constant zero map, and the restriction of M_γ to \mathbb{R}^{Nm} simulates the discrete-state transition map.

Initial states. The indexed set $\tilde{B} = \{\tilde{B}_f \in \mathcal{X} \mid f \in I_\Phi\}$ is defined by $\tilde{B}_f = \mu_C(f) \in \mathcal{X}_{\mu_D(f)}$ and $\tilde{B}_{f,l} = e_{\mu_D(f),l}$, for each $f \in \Phi$, $l = 1, 2, \dots, m$. That is, \tilde{B}_f is the continuous component of the initial state $\mu(f)$ and $\tilde{B}_{f,l}$ is the vector $e_{\mu_D(f),l}$, where $\mu_D(f)$ is the discrete component of the initial state $\mu(f)$. Notice that \tilde{B}_f is always an element of $\mathcal{X}_{\mu_D(f)}$.

The idea behind the choice of $R_{H,\mu}$ is the following. By "stacking up" the matrices A_q , M_{q_1,γ,q_2} and taking the "state-space" $\bigoplus_{q \in Q} \mathcal{X}_q$, we encoded most of the information on the discrete-state dynamics which has an effect on the continuous input-output behaviour. But we still need to keep track of the matrices B_q , and for that we need to simulate the discrete-state transitions. This is done by introducing the vectors $e_{q,j}$ and defining the action of M_γ on these vectors accordingly.

Let $\bar{O} = \underbrace{\mathbb{R}^p \ll \tilde{\Gamma}^* \gg \times \dots \times \mathbb{R}^p \ll \tilde{\Gamma}^* \gg}_{m\text{-times}}$ be the set of m tuples of formal power series from $\mathbb{R}^p \ll \tilde{\Gamma}^* \gg$. Define for each $f \in \Phi$ the map ψ_f as a pair, whose first component is simply the discrete-values part f_D of f and the second component maps each sequence of discrete inputs

w to the m tuple of shifted formal power series $w \circ Z_{f,j}$, $j = 1, \dots, m$, i.e.

$$\psi_f(w) = (f_D(w), (w \circ Z_{f,1}, w \circ Z_{f,2}, \dots, w \circ Z_{f,m})) \in O \times \bar{O}$$

for all $w \in \Gamma^*$.

Definition 3: Define the set of discrete valued maps \mathcal{D}_Φ associated with Φ as the indexed set formed by all the ψ_f , $f \in \Phi$ and indexed by elements of Φ . More formally,

$$\mathcal{D}_\Phi = \{\psi_f : \Gamma^* \rightarrow O \times \bar{O} \mid f \in \Phi\}$$

Let H be a hybrid system of the form (1) and let $\mu : \Phi \rightarrow \mathcal{H}_H$. Define the automaton realization $(\bar{\mathcal{A}}_H, \mu_D)$ associated with the realization (H, μ) described in **Step 2.** as follows

Construction 2: The automaton $\bar{\mathcal{A}}_H$ is of the form $\bar{\mathcal{A}}_H = (Q, \Gamma, O \times \bar{O}, \delta, \bar{\lambda})$, i.e. the state space and state-transition map of $\bar{\mathcal{A}}_H$ is the same as that of \mathcal{A}_H and the readout map $\bar{\lambda} : Q \rightarrow O \times \bar{O}$ is defined by $\bar{\lambda}(q) = (\lambda(q), (Z_{q,1}, \dots, Z_{q,m}))$ for all $q \in Q$, where the formal power series $Z_{q,j} \in \mathbb{R}^p \ll \tilde{\Gamma}^* \gg$ are defined as

$$\begin{aligned} Z_{q,j}(e^{\alpha_1} \gamma_1 \dots e^{\alpha_k} \gamma_k e^{\alpha_{k+1}}) = \\ C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \dots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l-1} B_{q_{l-1}} e_j \end{aligned} \quad (21)$$

for all $j = 1, \dots, m$, for all $\alpha_1, \dots, \alpha_k \in \mathbb{N}$, $k \geq 0$, $\gamma_1, \dots, \gamma_k \in \Gamma$, where l is such that $\alpha_1 = \alpha_2 = \dots = \alpha_{l-1} = 0$ and $\alpha_l > 0$. The map $\mu_D : \Phi \rightarrow Q$ is simply the Q -valued part of μ , for all $f \in \Phi$, i.e. $\mu(f) = (\mu_D(f), \mu_C(f))$.

The intuition behind the definition of $\bar{\mathcal{A}}_H$ is the following. For each discrete state $q \in Q$, the continuous valued part $\Pi_{\mathbb{R}^p} \circ v_H((q, 0), \cdot)$ of the input-output map induced by the hybrid state $(q, 0)$ contains information which cannot be encoded by continuous states only. That is why we have to consider it as an additional discrete output associated with the discrete state q . Since there is a one-to-one correspondence between $\Pi_{\mathbb{R}^p} \circ v_H((q, 0), \cdot)$ and the formal power series $Z_{q,j}$, $j = 1, \dots, m$, we can replace $\Pi_{\mathbb{R}^p} \circ v_H((q, 0), \cdot)$ with the m -tuple formed by $Z_{q,j}$, $j = 1, \dots, m$. With the above definitions we can formulate the following theorem.

Theorem 4: If (H, μ) is a realization of Φ , then $R_{(H, \mu)}$ is a representation of Ψ_Φ and $(\bar{\mathcal{A}}_H, \mu_D)$ is a realization of \mathcal{D}_Φ .

Proof:

Assume that (H, μ) is a realization of Φ . By Proposition 1 the first two equations of (16) hold. Notice that by construction of $R_{H, \mu}$, for each $x \in \mathcal{X}_{q_0}$, for each discrete state $q_0 \in Q$, and

for each word $s = e^{\alpha_1}\gamma_1 e^{\alpha_2}\gamma_2 \cdots e^{\alpha_k}\gamma_k e^{\alpha_{k+1}} \in \tilde{\Gamma}^*$,

$$\begin{aligned} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} x &= M_s x \in \mathcal{X}_{q_k}, \\ \text{and } C_{q_k} z &= \tilde{C}z \text{ for all } z \in \mathcal{X}_{q_k} \end{aligned} \quad (22)$$

and $B_{q_{l-1}} e_j = M_e M_{\gamma_{l-1}} M_{\gamma_{l-2}} \cdots M_{\gamma_1} e_{q_0, j}$ for all $j = 1, \dots, m$ and $q_{l-1} = \delta(q_0, \gamma_1 \gamma_2 \cdots \gamma_{l-1})$.

Hence, for $q_0 = \mu_D(f)$ we get

$$\begin{aligned} Z_{f,j}(\gamma_1 \gamma_2 \cdots \gamma_{l-1} e^{\alpha_l} \gamma_l \cdots \gamma_k e^{\alpha_{k+1}}) &= C M_e^{\alpha_{k+1}} M_{\gamma_k} \cdots \\ &\cdots M_{\gamma_l} M_e^{\alpha_l-1} M_e M_{\gamma_{l-1}} \cdots M_{\gamma_1} \tilde{B}_{f,j} \\ Z_f(e^{\alpha_1} \gamma_1 e^{\alpha_2} \cdots \gamma_k e^{\alpha_{k+1}}) &= \\ &= C M_e^{\alpha_{k+1}} M_{\gamma_k} \cdots M_{\gamma_1} M_e^{\alpha_1} \tilde{B}_f \end{aligned} \quad (23)$$

for all $j = 1, \dots, m$, $l = 1, \dots, k$, $\gamma_1, \dots, \gamma_k \in \Gamma$, $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{N}$, $k \geq 0$. Notice that $Z_{f,j}(\gamma_1 \cdots \gamma_{l-1}) = 0 = \tilde{C} e_{q_{l-1}, j} = \tilde{C} M_{\gamma_{l-1}} \cdots M_{\gamma_1} \tilde{B}_{f,j}$. Hence, $R_{H, \mu}$ is indeed a representation of Ψ_Φ . We will show that $(\bar{\mathcal{A}}_H, \mu_D)$ is a realization of \mathcal{D}_Φ ; (21) implies that for all $q \in Q$, $j = 1, \dots, m$, $w \circ Z_{q,j}(s) = Z_{q,j}(ws) = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_l, \gamma_l, q_{l-1}} A^{\alpha_l} B_{q_{l-1}} e_j = Z_{\delta(q, w), j}(s)$ for all $w \in \Gamma^*$ and $s \in \tilde{\Gamma}^*$, such that s is of the form $s = \gamma_1 \cdots \gamma_{l-1} e^{\alpha_l+1} \gamma_l e^{\alpha_{l+1}} \cdots \gamma_k e^{\alpha_{k+1}}$ and $q_{l-1} = \delta(q, w \gamma_1 \cdots \gamma_{l-1})$. That is, $w \circ Z_{q,j} = Z_{\delta(q, w), j}$. Assume that $q = \mu_D(f)$. Then (21) and Proposition 1 imply that $Z_{f,j} = Z_{q,j}$ for all $j = 1, \dots, m$, and hence $w \circ Z_{f,j} = w \circ Z_{q,j} = Z_{\delta(q, w), j}$. Therefore, $\bar{\lambda}(\mu_D(f), w) = (\lambda(\mu_D(f), w), (Z_{\delta(\mu_D(f), w), 1}, \dots, Z_{\delta(\mu_D(f), w), m})) = (f_D(w), (w \circ Z_{f,1}, \dots, w \circ Z_{f,m})) = \psi_f(w)$. \blacksquare

Let $R = (\mathcal{X}, \{M_\sigma\}_{\sigma \in \tilde{\Gamma}}, \tilde{B}, \tilde{C})$ be an observable representation of Ψ_Φ and let $(\bar{\mathcal{A}}, \zeta)$ be a reachable Moore-automaton realization of \mathcal{D}_Φ . Then define the linear hybrid realization $(H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ associated with R and $(\bar{\mathcal{A}}, \zeta)$ as follows.

Construction 3: Require $H_{R, \bar{\mathcal{A}}, \zeta}$ to be a linear hybrid system of the form (1), and require that the system parameters of $H_{R, \bar{\mathcal{A}}, \zeta}$ and the map $\mu_{R, \bar{\mathcal{A}}, \zeta}$ are defined as follows.

Moore-automaton. Assuming that $\bar{\mathcal{A}}$ is of the form $\bar{\mathcal{A}} = (Q, \Gamma, O \times \bar{O}, \delta, \bar{\lambda})$, define the automaton \mathcal{A} of $(H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ as $\mathcal{A} = (Q, \Gamma, O, \delta, \lambda)$, where the discrete state space and the state-transition map of \mathcal{A} are the same as those of $\bar{\mathcal{A}}$, and the readout map λ of \mathcal{A} is defined as $\lambda = \Pi_O \circ \bar{\lambda}$. That is, the value of $\lambda(q)$ is the first (O -valued) component of the value of $\bar{\lambda}(q)$ for each discrete state $q \in Q$.

Continuous state space. For each $q \in Q$, the continuous state-space component \mathcal{X}_q belonging

to q is defined as follows. Denote by $RS(q, f)$ the set of all strings s over $\tilde{\Gamma}$ such that if s is of the form $s = e^{\alpha_1}\gamma_1 e^{\alpha_2}\gamma_2 \cdots e^{\alpha_k}\gamma_k e^{\alpha_{k+1}}$ with $k \geq 0$ and $\gamma_1, \dots, \gamma_k \in \Gamma$, then $q = \delta(\mu_D(f), \gamma_1\gamma_2 \cdots \gamma_k)$. That is, q can be reached from $\mu_D(f)$ by the string $\gamma_1 \cdots \gamma_k$ in the automaton \mathcal{A} . Let \mathcal{X}_q be the subset of \mathcal{X} spanned by all elements of \mathcal{X} of the form $M_s M_e M_v \tilde{B}_{f,j}$ and $M_h \tilde{B}_f$ with $f \in \Phi$, $j = 1, \dots, m$, $s, h \in \tilde{\Gamma}^*$, $v \in \Gamma^*$ such that $ves, h \in RS(q, f)$, i.e.

$$\begin{aligned} \mathcal{X}_q = \text{Span}\{ & M_s M_e M_v \tilde{B}_{f,j}, M_h \tilde{B}_f \mid \\ & s, h \in \tilde{\Gamma}^*, v \in \Gamma^*, f \in \Phi, \text{ and } ves, h \in RS(q, f)\} \end{aligned} \quad (24)$$

It is clear that \mathcal{X}_q is a finite-dimensional subspace of \mathcal{X} . Assume that $n_q = \dim \mathcal{X}_q$ and fix a basis in \mathcal{X}_q . By identifying the elements of \mathcal{X}_q with the vector of their coordinates in this basis, we can identify \mathcal{X}_q with \mathbb{R}^{n_q} , and we can identify linear maps from \mathcal{X}_{q_1} to \mathcal{X}_{q_2} or to \mathbb{R}^p with $n_{q_2} \times n_{q_1}$, or $p \times n_{q_1}$ matrices respectively.

System Matrices. For each $q \in Q$, define the matrices $A_q \in \mathbb{R}^{n_q \times n_q}$, $C_q \in \mathbb{R}^{p \times n_q}$ and $M_{\delta(q,\gamma),\gamma,q} \in \mathbb{R}^{n_{\delta(q,\gamma)} \times n_q}$, $\gamma \in \Gamma$ as follows. We will view A_q , C_q , $M_{\delta(q,\gamma),\gamma,q}$ as linear maps $A_q : \mathcal{X}_q \rightarrow \mathcal{X}_q$, $C_q : \mathcal{X}_q \rightarrow \mathbb{R}^p$ and $M_{\delta(q,\gamma),\gamma,q} : \mathcal{X}_q \rightarrow \mathcal{X}_{\delta(q,\gamma)}$, $\gamma \in \Gamma$ which are defined as restrictions of M_e , \tilde{C} and respectively M_γ to \mathcal{X}_q . That is, for all $x \in \mathcal{X}_q$,

$$A_q x = M_e x \in \mathcal{X}_q, C_q x = \tilde{C}x \in \mathbb{R}^p, \text{ and}$$

$$M_{\delta(q,\gamma),\gamma,q} = M_\gamma x \in \mathcal{X}_{\delta(q,\gamma)} \text{ for all } \gamma \in \Gamma$$

Notice that the subspace \mathcal{X}_q is M_e invariant by construction, i.e. $M_e(\mathcal{X}_q) \subseteq \mathcal{X}_q$, and M_γ maps elements \mathcal{X}_q to elements of $\mathcal{X}_{\delta(q,\gamma)}$, i.e. $M_\gamma(\mathcal{X}_q) \subseteq \mathcal{X}_{\delta(q,\gamma)}$, for all $\gamma \in \Gamma$. Define the matrix $B_q \in \mathbb{R}^{n_q \times m}$ as the matrix such that for all $j = 1, \dots, m$, the j th column of B_q , viewed as an element of $\mathbb{R}^{n_q} \cong \mathcal{X}_q$, equals $M_e M_w \tilde{B}_{f,j}$ for some $f \in \Phi$ and $w \in \Gamma^*$ such that $\delta(\zeta(f), w) = q$, i.e. $B_q e_j = M_e M_w \tilde{B}_{f,j} \in \mathcal{X}_q$. Notice that B_q is indeed well-defined for each $q \in Q$. Indeed, since $(\bar{\mathcal{A}}, \zeta)$ is reachable, it follows that for each $q \in Q$ there exists a map $f \in \Phi$ and a word $w \in \Gamma^*$ such that $q = \delta(\zeta(f), w)$. Hence, it is left to show that the definition of B_q is independent of the choice of w and f . If $q = \delta(\zeta(f), w) = \delta(\zeta(g), v)$, then $\psi_g(v) = \psi_f(w)$, since $\bar{\mathcal{A}}$ is a realization of \mathcal{D}_Φ . But then $\Pi_{\bar{O}}(\psi_g(v)) = \Pi_{\bar{O}}(\psi_f(w))$, i.e. for all $j = 1, \dots, m$, $v \circ Z_{g,j} = w \circ Z_{f,j}$. Since R is a representation of Ψ_Φ we get that $v \circ Z_{g,j}(es) = Z_{g,j}(ves) = Z_{f,j}(wes) = \tilde{C} M_s M_e M_w \tilde{B}_{f,j} = \tilde{C} M_s M_e M_v \tilde{B}_{g,j}$ for each $s \in \tilde{\Gamma}^*$. Hence, observability of R implies that $M_e M_w \tilde{B}_{f,j} = M_e M_v \tilde{B}_{g,j}$.

The map $\mu_{R,\bar{\mathcal{A}},\zeta}$. Define the map $\mu_{R,\bar{\mathcal{A}},\zeta}(f)$ as follows. For each $f \in \Phi$, let $\mu_{R,\bar{\mathcal{A}},\zeta}(f) = (\zeta(f), \tilde{B}_f) \in \{\zeta(f)\} \times \mathcal{X}_{\zeta(f)}$, where \tilde{B}_f is viewed as an element of $\mathcal{X}_{\zeta(f)}$.

It should be clear now why we needed observability of R and reachability of $(\bar{\mathcal{A}}, \zeta)$. If R was not observable, we could have multiple choices for the matrices B_q . If $(\bar{\mathcal{A}}, \zeta)$ was not reachable, we could have discrete states $q \in Q$ for which we would have trouble defining a continuous state space. It is also clear that if $(\bar{\mathcal{A}}, \zeta)$ was not realization of \mathcal{D}_Φ , but only a realization of $\{f_D \mid f \in \Phi\}$, then the following scenario could take place: $f_D = g_D$, $\zeta(f) = \zeta(g)$, but $y_0^{f,\Phi} \neq y_0^{g,\Phi}$ for some $f, g \in \Phi$; hence, we would have trouble choosing the correct B_q .

Theorem 5: If R is an observable representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a reachable realization of \mathcal{D}_Φ , then $(H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$ is a realization of Φ .

Proof: Let $(H, \mu) = (H_{R,\bar{\mathcal{A}},\zeta}, \mu_{R,\bar{\mathcal{A}},\zeta})$. From the definition of (H, μ) it follows that for all $q_0 \in Q$ and $x \in \mathcal{X}_q$,

$$\begin{aligned} M_s x &= A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} x \in \mathcal{X}_{q_k} \\ \tilde{C} M_s x &= C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} x \end{aligned} \quad (25)$$

for all $s \in \tilde{\Gamma}^*$ of the form $s = e^{\alpha_1} \gamma_1 e^{\alpha_2} \gamma_2 \cdots e^{\alpha_k} \gamma_k e^{\alpha_{k+1}}$ for some $k \geq 0$, $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{N}$, $\gamma_1, \dots, \gamma_k \in \Gamma$. Moreover, $\tilde{B}_f \in \mathcal{X}_{\zeta(f)}$ and for each $w \in \Gamma^*$, $M_e M_w \tilde{B}_{f,j} = B_{\delta(\zeta(f), w)} e_j$. First, we will show that (H, μ) is a realization of Φ . Consider the string $s = e^{\alpha_1} \gamma_1 e^{\alpha_2} \cdots \gamma_k e^{\alpha_{k+1}}$ from above. Assume that $l > 0$ is such that $\alpha_1 = \dots = \alpha_{l-1} = 0$ and $\alpha_l > 0$ and let $v = \gamma_1 \cdots \gamma_{l-1}$. Consider an arbitrary $f \in \Phi$ and let $q_0 = \mu_D(f)$. Denote by q_{l-1} the discrete state $q_{l-1} = \delta(q_0, v)$. Since R is a representation of Ψ_Φ , we get that

$$\begin{aligned} Z_{f,j}(s) &= \tilde{C} M_w M_e M_v \tilde{B}_{f,j} = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} A_{q_{k-1}}^{\alpha_k} \cdots M_{q_l, \gamma_l, q_{l-1}} A_{q_{l-1}}^{\alpha_l-1} B_{q_{l-1}} e_j \\ Z_f(s) &= \tilde{C} M_s \tilde{B}_f = C_{q_k} A_{q_k}^{\alpha_{k+1}} M_{q_k, \gamma_k, q_{k-1}} \cdots M_{q_1, \gamma_1, q_0} A_{q_0}^{\alpha_1} \mu(f) \end{aligned} \quad (26)$$

for each $f \in \Phi$ and $j = 1, \dots, m$. If $(\bar{\mathcal{A}}, \zeta)$ is a realization of \mathcal{D}_Φ , we get that for each $f \in \Phi$, $w \in \Gamma^*$, $f_D(w) = \Pi_O \circ \psi_f(w) = \Pi_O \circ \bar{\lambda}(\zeta(f), w) = \lambda(\mu_D(f), w)$. This, (26), and Proposition 1 imply that (H, μ) is a realization of Φ . ■

In the sequel we will formulate conditions for existence of a Moore-automata realization of \mathcal{D}_Φ . Recall from Section III the definition of the set $H_{\Phi,O}$ given by (13), and of the set W_{Φ_D} given by (14).

Lemma 2: Assume that Φ has a hybrid kernel representation. Then \mathcal{D}_Φ has a realization by a finite Moore-automaton if and only if $\text{card}(W_{\Phi_D}) < +\infty$ and $\text{card}(H_{\Phi,0}) < +\infty$.

Proof: By Theorem 3 \mathcal{D}_Φ has a realization by a Moore-automaton if and only if $W_{\mathcal{D}_\Phi} = \{w \circ \psi_f \mid f \in \Phi, w \in \Gamma^*\}$ is a finite set. It is easy to see that $W_{\mathcal{D}_\Phi}$ is finite if and only if the sets $W_{\Phi_D} = \{w \circ f_D \mid f \in \Phi, w \in \Gamma^*\}$ and $W_K = \{w \circ \kappa_f \mid w \in \Gamma^*, f \in \Phi\}$ are finite sets, where $\kappa_f(v) = \Pi_{\bar{O}}(\psi_f(v))$, i.e. $\psi_f(v) = (f_D(v), \kappa_f(v))$ for all $v \in \Gamma^*$. Notice that $w \circ \kappa_f(v) = (wv \circ Z_{f,1} \dots, wv \circ Z_{f,m})$, and there is one to one correspondence between $w \circ Z_{f,j}$, and the column of $H_{\Phi,0}$ indexed by $(w, (f, j))$. Therefore, W_K is finite if and only if $H_{\Phi,0}$ is finite. ■

Corollary 1: If R is a minimal representation of Ψ_Φ and $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of \mathcal{D}_Φ , then $(H, \mu) = (H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ is well-defined and it is a linear hybrid realization of Φ .

Proof: If R is minimal, then by Theorem 2 it is observable. If $(\bar{\mathcal{A}}, \zeta)$ is a minimal realization of \mathcal{D}_Φ , then by Theorem 3 it is reachable. Hence, the linear hybrid system realization (H, μ) is well-defined and by Theorem 5 it is a realization of Φ . ■

Remark 5: In fact, in Part II we will show that the realization (H, μ) from Corollary 1 is a *minimal linear hybrid system* realizing Φ .

Remark 6 (Construction of a realization from the Hankel-matrix): Recall from Remark 3, Section V that we can construct a minimal representation R_f of Φ from the column space of the Hankel-matrix H_Φ of Φ . Recall from Remark 4 of Section VI that we can construct a minimal Moore-automaton realization $(\mathcal{A}_{can}, \zeta_{can})$ of \mathcal{D}_Φ from the infinite set $W_{\mathcal{D}_\Phi}$. Notice that $W_{\mathcal{D}_\Phi}$ is completely determined by the collection of discrete-valued input-output maps f_D , $f \in \Phi$ and by those columns of the Hankel-matrix H_Φ which are indexed by elements of the form $(w, (f, j))$, $w \in \Gamma^*$, $f \in \Phi$, $j = 1, \dots, m$. That is, $(\mathcal{A}_{can}, \zeta_{can})$ can be constructed from the columns of the Hankel-matrix H_Φ and from the values of the discrete-valued input-output maps f_D , $f \in \Phi$. Since both R_f is minimal and $(\mathcal{A}_{can}, \zeta_{can})$ is minimal, it follows that $(H_f, \mu_f) = (H_{R_f, \mathcal{A}_{can}, \zeta_{can}}, \mu_{R_f, \mathcal{A}_{can}, \zeta_{can}})$ is a well-defined realization of Φ . That is, *a linear hybrid system realization of Φ can be constructed from the columns of the Hankel-matrix H_Φ and from the collection of discrete-valued input-output maps $f_D, f \in \Phi$.*

VIII. REALIZATION ALGORITHMS

In this section we will briefly sketch a realization algorithm for linear hybrid systems. The interested reader can find a more detailed account on the topic in [3], [4]. Note that the realization algorithm was implemented.

Assume that Φ is finite. Since we know how to compute a Moore-automaton realization and a minimal rational representation from finite data, we would like to use Corollary 1 to compute a linear hybrid realization of Φ . However, the values of ψ_f live in the set $O \times \bar{O}$ which is infinite, and hence a minimal realization of \mathcal{D}_Φ cannot be computed. We will solve this problem by replacing the values in \bar{O} with finite vectors of real numbers. To this end, let N be such that $\text{rank } H_\Phi \leq N$ and for each $f \in \Phi$, define the map $\psi_{f,N} : w \mapsto (f_D(w), (w \circ Z_{f,1}^N, w \circ Z_{f,2}^N, \dots, w \circ Z_{f,m}^N))$ where $w \circ Z_{f,i}^N$ denotes the *finite* vector formed by all the values $Z_{f,i}(ws)$ where $s \in \tilde{\Gamma}^*$ is any word of length at most N , i.e. $|s| \leq N$, and denote by $\mathcal{D}_{\Phi,N}$ the indexed set $\mathcal{D}_{\Phi,N} = \{\psi_{f,N} \mid f \in \Phi\}$ formed by all the maps of the form $\psi_{f,N}$.

Proposition 2 ([3], [4]): If $(\bar{\mathcal{A}}_N, \zeta)$ is a minimal realization of $\mathcal{D}_{\Phi,N}$ and R is a minimal representation of Ψ_Φ , then we can compute a minimal linear hybrid realization $(H_N, \mu_N) = (H_{R, \bar{\mathcal{A}}_N, \zeta}, \mu_{R, \bar{\mathcal{A}}_N, \zeta})$ of Φ by repeating literally the same steps as for the construction of $(H_{R, \bar{\mathcal{A}}, \zeta}, \mu_{R, \bar{\mathcal{A}}, \zeta})$ described in Construction 3, but using the automaton $\bar{\mathcal{A}}_N$ instead of the automaton $\bar{\mathcal{A}}$.

Notice that we have not defined the concept of minimality for linear hybrid systems so far, for the definition of minimality see [3], [4] or Part II of the current series of papers. If we want to construct a realization of Φ we can proceed as follows. We choose N such that $\text{rank } H_\Phi \leq N$. Recall the definition of $W_{\mathcal{D}_{\Phi,N}, L, M}$ from Section VI and recall the definition of $H_{\Psi_\Phi, L, K}$ from Section V. We choose K so that $\text{rank } H_{\Psi_\Phi, K, K} = \text{rank } H_\Phi$ and we choose D such that $\text{card}(W_{\mathcal{D}_{\Phi,N}, D, D}) = \text{card}(W_{\mathcal{D}})$. In particular, if we can assume that Φ has a (unknown) linear hybrid realization H of the form (1), such that the number of discrete states is d , and the sum of dimensions of the continuous components $\sum_{q \in Q} n_q$ is M ,² and $D = K = N \geq dm + M$, then the (in)equalities above always hold, i.e. $\text{rank } H_\Phi \leq N$, $\text{card}(W_{\mathcal{D}_{\Phi,N}, N, N}) = \text{card}(W_{\mathcal{D}})$ and $\text{rank } H_{\Phi, N, N} = \text{rank } H_\Phi$. We build the finite table $W_{\mathcal{D}_{\Phi,N}, D+1, D}$, as described in Section VI, using *finitely many* discrete values and high-order time derivatives of elements of Φ . We build the finite matrix $H_{\Phi_\Psi, K+1, K}$, as described in Section V, using *finitely many* high-order time derivatives of elements of Φ . Then we compute a minimal representation R of Φ_Ψ from $H_{\Phi_\Psi, K+1, K}$ and a minimal Moore-automaton realization $(\bar{\mathcal{A}}_N, \zeta)$ of $\mathcal{D}_{\Phi,N}$ from $W_{\mathcal{D}_{\Phi,N}, D+1, D}$.

²In Part II we will define the dimension of the hybrid system H as the pair (d, M) .

Finally, we compute the linear hybrid realization $(H_{R,\bar{A}_N,\zeta}, \mu_{R,\bar{A}_N,\zeta})$ of Φ .

To demonstrate the procedure above, consider the following numerical example.

Example 2: Consider the linear hybrid system H defined in Example 1. Recall from Example 1 the definition of the set of input-output maps Φ . Consider the upper-left block $H_{\Phi,5,4}$ of Hankel matrix H_Φ of Φ . It can be shown that $\text{rank } H_{\Phi,5,4} = \text{rank } H_\Phi$. Consider the finite table $\text{card}(W_{\mathcal{D}_{\Phi,4,1,1}})$. It can be shown that $\text{card}(W_{\mathcal{D}_{\Phi,4,1,1}}) = \text{card}(W_{\mathcal{D}_\Phi})$. The linear hybrid realization (H_f, μ_f) computed from $H_{\Phi,5,4}$ and $W_{\mathcal{D}_{\Phi,4,1,1}}$ is of the following form

$$H^f = (\mathcal{A}^f, \mathbb{R}^m, \mathbb{R}^p, (\mathcal{X}_q^f, A_q^f, B_q^f, C_q^f)_{q \in Q^f}, \{M_{\delta^f(q,\gamma),\gamma,q}^f \mid q \in Q^f, \gamma \in \Gamma\})$$

where $\mathcal{A}_f = (Q^f, \Gamma, O, \delta^f, \lambda^f)$ with $Q^f = \{q_1, q_2, q_3\}$, and state transition, $\delta^f(q_3, b) = q_2$, $\delta^f(q_2, b) = q_2$, $\delta^f(q_1, b) = q_1$, $\delta^f(q_3, a) = q_3$, $\delta^f(q_1, a) = q_1$, $\delta^f(q_2, a) = q_3$, $\lambda^f(q_1) = o$, $\lambda^f(q_2) = d$, $\lambda^f(q_3) = g$. The reset maps and the linear subsystems are of the form:

$$\begin{aligned} A_{q_1}^f &= \begin{bmatrix} -3.03 & -0.37 & 0.01 \\ 0.08 & -1.97 & 0.1 \\ 0.04 & 0 & -1 \end{bmatrix}, B_{q_1}^f = \begin{bmatrix} -0.02 \\ 0.13 \\ 1.23 \end{bmatrix}, C_{q_1}^f = \begin{bmatrix} -0.31 & 0.4 & 0.77 \end{bmatrix}, \\ M_{q_1,b,q_1}^f &= \begin{bmatrix} 0.22 & -0.28 & 0.02 \\ -0.62 & 0.78 & 0.01 \\ 0.01 & 0 & 1 \end{bmatrix}, M_{q_1,a,q_1}^f = \begin{bmatrix} 0.92 & -1.15 & 0.12 \\ -0.07 & 0.08 & 0.09 \\ 0 & 0.02 & 1 \end{bmatrix}, \\ A_{q_2}^f &= \begin{bmatrix} -1 & 0.16 \\ 0 & 0 \end{bmatrix}, B_{q_2}^f = \begin{bmatrix} -0.22 \\ -1.37 \end{bmatrix}, C_{q_2}^f = \begin{bmatrix} 0.65 \\ -0.1 \end{bmatrix}^T, \\ M_{q_2,b,q_2}^f &= \begin{bmatrix} 1 & -0 \\ 0 & 1 \end{bmatrix}, M_{q_3,a,q_2}^f = \begin{bmatrix} 0 & -1.09 \end{bmatrix}, \\ A_{q_3}^f &= \begin{bmatrix} -1 \end{bmatrix}, B_{q_3}^f = 0, C_{q_3}^f = 0, M_{q_2,b,q_3}^f = \begin{bmatrix} 1.03 \\ 0 \end{bmatrix}, M_{q_3,a,q_3}^f = 1. \end{aligned}$$

The initial states are $\mu_f(f_1) = (q_1, (-0.69, 1.9, 0.02)^T)$ and $\mu_f(f_2) = (q_2, (0, 0)^T)$. We compared the output responses of (H, μ) and (H_f, μ_f) for ten different timed sequences of discrete inputs and for generated random white noise continuous input. The responses are essentially identical, the small numerical error is caused by accumulation of numerical errors during the computation. This is in accordance with the theory, which implies that both (H, μ) and (H_f, μ_f) are realizations of the same input-output maps $\{f_1, f_2\}$, hence the output responses should be identical. As an illustration see Fig. 2.

IX. CONCLUSIONS AND FUTURE WORK

The paper is the first part of a series of papers. In this paper the solution to the realization problem for linear hybrid systems has been presented. The realization problem considered was to find a realization of a family of input-output maps. The paper combines the theory of formal power series with the classical automata theory to derive the results. In Part II of the current

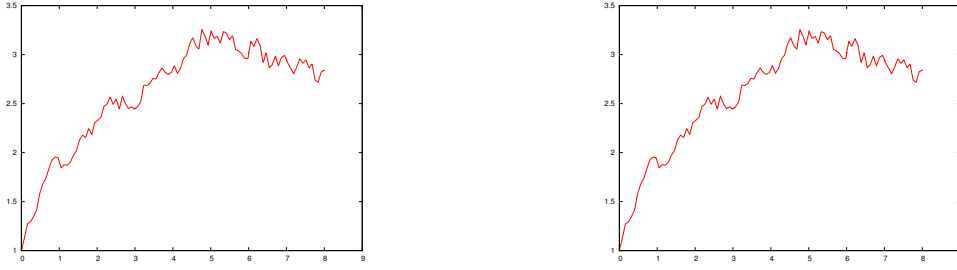


Fig. 1. The value of the input-output map f_1 for white noise continuous input and timed sequence of discrete inputs $(b, 1)(a, 2)(a, 3)(b, 1), 1$. The left-hand side figure shows the continuous response of the original system H from the initial state $\mu(f_1)$, the right-hand side figure shows the continuous response of the system H_f from the initial state $\mu_f(f_1)$

series of papers we will address the issue of minimality, observability and reachability for linear hybrid systems.

Topics of further research include realization theory for piecewise-affine systems on polytopes, and general non-linear hybrid systems without guards. We would also like to work on subspace identification and model reduction for hybrid systems, using the presented results.

Acknowledgement. The research described in this paper was carried out during Mihály Petreczky's stay as a PhD student at the Centrum voor Wiskunde en Informatica (CWI) in Amsterdam, The Netherlands. Authors thank their colleagues Pieter Collins and Luc Habets of CWI.

REFERENCES

- [1] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*. Springer-Verlag London, 2000.
- [2] J. Willems and J. Polderman, *An Introduction to Mathematical Systems Theory: A Behavioral Approach*. Springer Verlag, New York, 1998.
- [3] M. Petreczky, "Realization theory of hybrid systems," Ph.D. dissertation, Vrije Universiteit, Amsterdam, 2006, available at <http://www.cwi.nl/~mpetrec>.
- [4] —, "Realization theory for linear and bilinear hybrid systems," CWI, Tech. Rep. MAS-R0502, 2005.
- [5] T. Henzinger, R. Alur, and etc., "The algorithmic analysis of hybrid systems," *Theoretical Computer Science*, vol. 138, 1995.
- [6] G. Pola, A. J. van der Schaft, and M. D. Di Benedetto, "Bisimulation theory of switching linear systems," in *Proceedings CDC2004*, 2004.
- [7] M. Petreczky, "Hybrid formal power series and their application to realization theory of hybrid systems," in *Proc. MTNS2006*, 2006.
- [8] J. Berstel and C. Reutenauer, *Rational series and Their Languages*. Springer-Verlag, 1984.
- [9] W. Kuich and A. Salomaa, in *Semirings, Automata, Languages*, ser. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1986.

- [10] F. Gécseg and I. Peák, *Algebraic theory of automata*. Akadémiai Kiadó, Budapest, 1972.
- [11] S. Eilenberg, *Automata, Languages and Machines*. Academic Press, New York, London, 1974.
- [12] B. Jakubczyk, “Realization theory for nonlinear systems, three approaches,” in *Algeb. & Geom. Methods in Nonlin. Contr. Theory*, 1986.
- [13] E. D. Sontag, “Realization theory of discrete-time nonlinear systems: Part I – the bounded case,” *IEEE Transaction on Circuits and Systems*, vol. CAS-26, no. 4, April 1979.
- [14] A. Isidori, *Nonlinear Control Systems*. Springer Verlag, 1989.
- [15] F. M. Callier and C. A. Desoer, *Linear System Theory*. Springer-Verlag, 1991.
- [16] M. Petreczky, “Realization theory of linear and bilinear switched systems: A formal power series approach,” CWI, Tech. Rep. MAS-R0403, 2005.