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A Lie algebraic study of some integrable systems associated with root systems

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0. Introduction

In this monograph we study finite-dimensional Hamiltonian systems with Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} {p_j}^2 + V(q) \tag{0.1}$$

where $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are momentum and coordinate vectors of $\mathbb{R}^{2n}$ and the potential $V(q)$ is of the following type:

$$V(q) = g^2 \sum_{j<k} v(q_j - q_k) \tag{0.2}$$

where $g \in \mathbb{R}$ is a coupling constant and where the function $v(q)$ may be of the following form:

$$v(q) = \begin{cases} q^{-2} & \text{(I)} \\ a^2 \sinh^{-2}(aq) & \text{(II)} \\ a^2 \sin^{-2}(aq) & \text{(III)} \\ a^2 \wp(aq) & \text{(IV)} \\ q^{-2} + \omega^2 q^2 & \text{(V)} \end{cases} \tag{0.3}$$

where $\wp(x)$ is the Weierstrass function and $a, \omega \in \mathbb{R}$ are parameters. These Hamiltonians describe one-dimensional n-particle systems with pairwise interaction. The Weierstrass function $\wp(x) = \wp(x; \omega_1, \omega_2)$ is doubly periodic in x in the complex plane with halfperiods ω_1 and ω_2. In the limit in which one of the periods goes to infinity one gets, up to a constant, a potential of type II or III. If both periods go to infinity one obtains a potential of type I. So the systems of type IV are the most general. Replacing a by ia in a potential of type III, one gets a system of type II and, if one puts $a = 0$, one gets a system of type I. The system of type I is known as the Calogero-Moser model ([2], [3]) and that of type II as the Sutherland model ([4], [5]).
The potential $V(q)$ can also have the form:

$$V(q) = \sum_{j=1}^{n-1} {g_j}^2 v(q_j - q_{j+1}) \qquad \text{(VI)} \tag{0.4}$$

or

$$V(q) = \sum_{j=1}^{n} v(q_j - q_{j+1}), \qquad (q_{n+1} = q_1) \qquad \text{(VI ')} \tag{0.5}$$

where

$$v(q) = \exp 2q \tag{0.6}$$

and where the $0 < g_j \in \mathbb{R}$ are coupling constants.
These Hamiltonians describe n-particle systems with nearest-neighbour interaction and are known as the nonperiodic Toda lattice (VI) and the periodic

Toda lattice (VI $'$). One can get the nonperiodic Toda lattice as a strong coupling limit of the Sutherland model (see [53]) and the periodic Toda lattice is obtained by adding the potential energy term $\exp 2(q_n - q_1)$.
All these systems are related to the root system A_{n-1} and they are completely integrable in the sense of Liouville (see chapter 2 for a precise statement).
Now there turn out to be all kind of generalizations of these systems. One can associate integrable systems of these type to each root system. Also one can quantize these systems ([1],[15]). Furthermore there are relativistic generalizations of the classical and the quantum systems (see [16],[17],[18],[19] and [53] for a recent review). There have also been found integrable relativistic models with an external potential ([21]). For the classical type I model, mastersymmetries have been found ([20]).
All these systems are integrable in the classical or quantum sense. In [52] Opdam has proved that the quantum systems of type I, II and III are completely integrable for all root systems and for all values of the coupling constants g_α, using socalled shift operators. By taking a classical limit, using symbols of differential operators, the integrability of the classical systems follows as well, for all root systems and all values of the coupling constants. In [51] and [50] Heckman has simplified this proof, using properties of the socalled Dunkl operators.
On the other hand, Olshanetsky and Perelomov have proved in [38] and [43] the integrability of the classical systems of type I–V, using a Lax matrix and a certain functional equation. This proof however up to now only seems to work for classical root systems, and only for special values of the coupling constants (in the BC_n and B_n case).
For the models of type VI and VI$'$, the socalled Toda systems, and for numerous other integrable sytems, there exists a group-theoretical proof of the integrability, using the socalled Kostant-Adler-Symes-Reyman-Semenov-Tian-Shansky (K.A.S.R.S.) construction (see in particular [11] for a review). For these systems the solution of the Hamilton equations can be reduced to a factorization problem in a group G.
It is still not clear what kind of relation there exists between the K.A.S.R.S. theorem on the one hand, and the integrable systems that can be solved using this theorem, and the integrable systems of Calogero-Moser type on the other hand, i.e. the models of type I–V and their generalizations.
In this monograph we describe a way of obtaining the classical systems of type I–V, which is reminiscent of the K.A.S.R.S. construction and which seems to generalize that setting. In particular there seems to be no Yang-Baxter operator or double Lie algebra to describe the corresponding Poisson structure.

The setup of this book is as follows:
In chapter 1 we describe the models of type I–V associated with root system A_{n-1}. This is merely a reformulation of [1].
In chapter 2 we collect some properties of Poisson manifolds.

In chapter 3 we review some facts about real semisimple Lie algebras and observe some properties of quasi-split Lie algebras.
In chapter 4 we describe in some detail the theory of Poisson structures on Lie algebras, double Lie algebras, Yang-Baxter operators and Lie bialgebras. In section 4.5 we describe the K.A.S.R.S. theorem and illustrate it by the examples of the Toda lattice and the harmonic oscillator.
In chapter 5 we use the theory of the chapters 2–4 to construct the systems of type I–V as Hamiltonian systems on real semisimple Lie algebras. This is done in section 5.3 by constructing a (nontrivial) Poisson imbedding of the phase space M in a larger space $P = \Lambda \times \mathfrak{g}$, where Λ is the configuration space and $\mathfrak{g}$ is a real semisimple Lie algebra. In section 5.4 it is shown how the Hamilton equations can be derived.
In section 5.5 a condition is formulated for a certain constant element $\mu \in \mathfrak{g}$ (the moment), which is sufficient for integrability, and this condition is translated in a condition on the dimension of the centralizer of μ in the compact subalgebra $\mathfrak{k}$.
In section 5.6 it is shown how the Lax equation can be derived, using the condition of section 5.5 and a functional equation. In section 5.7 this leads to a (new) Lie algebraic proof of the integrability of the systems of type I–III. The only ingredients in this proof are the functional equation, the condition on μ and the properties of $\mathfrak{g}$ and Ad-invariant functions.
In chapter 6 we analyse in more detail the examples which are related to classical root systems. In section 6.1 we classify all possible μ's in the case of quasi-split Lie algebras and in the case of root systems of exceptional type. This results in a few unknown choices, corresponding to the classical root systems and to the F_4 case. All this suggests that the construction of a Lax pair, as described in chapter 5, does not work in the case of root systems of type E_8, E_7, E_6 and G_2.
In section 6.2 it is shown how the examples of μ which are known to satisfy the condition of section 5.5 can be constructed in a canonical way. They are all elements of quasi-split Lie algebras and for these cases there exists an involutive automorphism σ, commuting with the Cartan automorphism θ, which has some extra properties, which makes it possible to characterize μ in a canonical way.
In section 6.3 a Lie algebraic construction is given of the Lax pair in the case of the classical root systems, explaining why the conditions on the coupling constants are necessary.
In section 6.4 a certain property of the root system A_{n-1} is formulated that could explain why the construction of μ does not work in the case of the exceptional root systems.
In chapter 7 we try to analyse the relation between the construction of chapter 5 and the K.A.S.R.S. theorem. The upshot is that there does not seem to exist a double Lie algebra structure which "explains" the Poisson structure of chapter 5 and the corresponding Lax pair.

1. The A_{n-1} models

1.1. The models of type I – IV.

In this section we describe in more detail the systems of type I–IV (see [1] and [49] for more details). Consider first its configuration space. Because $V(q)$ becomes infinite if $q_j = q_k$ for $j \neq k$, the ordening of the particles during the evolution of the system cannot change and so one may assume that $q_j > q_k$ if $j < k$. The configuration space for the systems I and II is therefore of the form

$$\Lambda = \{q \in \mathbb{R}^n \mid q_1 > q_2 > \ldots > q_n\} \tag{1.1}$$

and for the systems III and IV it has, up to periodicity, the form

$$\Lambda = \{q \in \mathbb{R}^n \mid q_1 > q_2 > \ldots > q_n, q_1 - q_n < d\} \tag{1.2}$$

where $d = \pi/a$ for type III and $d = 2\omega/a$ for type IV, with ω a halfperiod of $V(q)$. The phase space M is given by

$$M = \Lambda \times \mathbb{R}^n \tag{1.3}$$

Now let $f_1, f_2 \in \mathcal{C}^\infty(M)$, then the canonical Poisson bracket on $M \subset \mathbb{R}^{2n}$ is defined by

$$\{f_1, f_2\} = \sum_{j=1}^{n} \left(\frac{\partial f_1}{\partial p_j}\frac{\partial f_2}{\partial q_j} - \frac{\partial f_1}{\partial q_j}\frac{\partial f_2}{\partial p_j} \right) \tag{1.4}$$

so for the coordinate functions one has

$$\{p_j, q_k\} = \delta_{jk} \tag{1.5}$$

Hamilton's equations are given by

$$\dot{q}_j = \{H, q_j\} = \frac{\partial H}{\partial p_j} = p_j$$

$$\dot{p}_j = \{H, p_j\} = -\frac{\partial H}{\partial q_j} = g^2 \sum_{k \neq j} v'(q_k - q_j) \tag{1.6}$$

where the dot denotes the time derivative. For type I these become:

$$\dot{q}_j = p_j, \quad \dot{p}_j = -2g^2 \sum_{k \neq j} (q_k - q_j)^{-3} \tag{1.7}$$

Define

$$q_{tot} = \frac{1}{n}\sum_{j=1}^{n} q_j, \qquad p_{tot} = \frac{1}{n}\sum_{j=1}^{n} p_j \tag{1.8}$$

then, because $v(q) = v(-q)$, (1.6) implies

$$\dot{q}_{tot} = p_{tot}, \qquad \dot{p}_{tot} = 0 \tag{1.9}$$

so the system is translation invariant and one has the Poisson bracket relations

$$\{H, q_{tot}\} = p_{tot}, \qquad \{H, p_{tot}\} = 0, \qquad \{p_{tot}, q_{tot}\} = n \tag{1.10}$$

Now define the matrices L and M by

$$L = \sum_j p_j e_{jj} + g \sum_{j<k} (q_j - q_k)^{-1} i(e_{jk} - e_{kj}) \tag{1.11}$$

$$M = g \sum_{j<k} (q_j - q_k)^{-2} i(e_{jk} + e_{kj} - e_{jj} - e_{kk} + \frac{2}{n} I_n) \tag{1.12}$$

where $\{e_{jk}, j, k = 1, \ldots, n\}$ is the standard basis of $gl(n, \mathbb{C})$, $i = \sqrt{-1}$, $g \in \mathbb{R}$ a coupling constant and I_n the $n \times n$ identity matrix. L is hermitian, M is traceless and skew-hermitian and a straightforward calculation, using

$$e_{jk} e_{lm} = \delta_{kl} e_{jm} \tag{1.13}$$

shows that (1.7) is equivalent with the matrix equation

$$\dot{L} = [M, L] \tag{1.14}$$

if one requires that $\dot{q}_{tot} = p_{tot}$. The Hamiltonian can be written as

$$H = \frac{1}{2} \mathrm{tr} L^2 \tag{1.15}$$

Equation (1.14) is known as a Lax equation for the Calogero-Moser model and (L, M) is called a Lax pair. Now define

$$f_k(L) = \frac{1}{k} \mathrm{tr} L^k, \qquad 1 \leq k \leq n \tag{1.16}$$

These are real-valued functions and

$$f_1 = n p_{tot}, \qquad f_2 = H \tag{1.17}$$

From the Lax equation (1.14) one easily derives with induction that

$$(\dot{L^k}) = [M, L^k] \tag{1.18}$$

and so, using the properties of the trace, one concludes that

$$\dot{f}_k(L) = 0 \tag{1.19}$$

and because $f_k(L) = \frac{1}{k} \sum_{j=1}^n {\lambda_j}^k$ are polynomials in the eigenvalues λ_j of L, it follows that the eigenvalues are conserved, so one immediately gets n constants of motion. Going to the center-of-mass coordinates $\tilde{q_j} = q_j - q_{tot}$, $\tilde{p_j} = p_j - p_{tot}$, L becomes traceless and $\{\tilde{p_j}, \tilde{q_k}\} = \delta_{jk} - \frac{1}{n}$, $\dot{\tilde{q_j}} = \tilde{p_j}$, $\dot{\tilde{p_j}} = \dot{p_j}$.

Because L is hermitian and M is skew-hermitian the Lax equation implies that there exists a one-parameter family $U(t)$ of unitary matrices such that $U(0) = I_n$ and

$$L(t) = U(t)L(0)U(t)^{-1} \tag{1.20}$$

so $L(t)$ undergoes an isospectral deformation under the action of the group $SU(n)$. Differentiating (1.20) one gets back the Lax equation (1.14) with $M = \dot{U}U^{-1}$. So the solution of the Lax equation is reduced to the construction of the one-parameter group $U(t)$. This can be solved for the Calogero-Moser model, as we shall see presently.

Now define the matrices P, Q and μ by

$$P = \operatorname{diag}(p_1, \ldots, p_n) \tag{1.21}$$

$$Q = \operatorname{diag}(q_1, \ldots, q_n), \quad q_1 > q_2 > \ldots > q_n \tag{1.22}$$

$$\mu = g \sum_{j<k} i(e_{jk} + e_{kj}) \tag{1.23}$$

From (1.9),(1.11) and (1.12) one derives the following relations

$$[Q, L] = \mu \tag{1.24}$$

$$[Q, M] = L_{\text{off}} \tag{1.25}$$

$$[M, \mu] = 0 \tag{1.26}$$

$$\dot{Q} = P = L_{\text{diag}} \tag{1.27}$$

where L_{diag} and L_{off} denote the diagonal component resp. the off-diagonal component of L. Note that (1.24)–(1.27) uniquely determine L, M in terms of the q_j, p_j.

From this one also derives:

$$\begin{aligned} &[\dot{Q}, L] + [Q, \dot{L}] \\ =&[P, L] + [Q, [M, L]] \\ =&[P, L] + [[Q, M], L] + [M, [Q, L]] \\ =&[P, L] + [L_{\text{off}}, L] + [M, \mu] = 0 \end{aligned} \tag{1.28}$$

so (1.14) and (1.27) are consistent with (1.24). Now define the matrix

$$X(t) = L(0)t + Q(0) \tag{1.29}$$

then $X(t)$ is hermitian and

$$[X(t), L(0)] = \mu \tag{1.30}$$

Because $X(t)$ is hermitian there exists a unitary matrix $U(t)$ such that

$$U(t)X(t)U(t)^{-1} = Q(t) = \mathrm{diag}(q_1(t), \ldots, q_n(t)) \tag{1.31}$$

At this point it is not yet clear that $Q(t)$ indeed satisfies (1.24)–(1.28). This has still to be shown. Now it turns out (see [46]) that it is possible to choose $U(t)$ in such a way that $U(0) = I_n$ and

$$U(t)\mu U(t)^{-1} = \mu \tag{1.32}$$

and if one now defines

$$L(t) = U(t)L(0)U(t)^{-1} \tag{1.33}$$

then $Q(t)$ and $L(t)$ satisfy

$$\begin{aligned} &[Q(t), L(t)] \\ &= [U(t)X(t)U(t)^{-1}, U(t)L(0)U(t)^{-1}] \\ &= U(t)[X(t), L(0)]U(t)^{-1} \\ &= U(t)\mu U(t)^{-1} = \mu \end{aligned} \tag{1.34}$$

which implies that $q_j(t) \neq q_k(t)$ if $j \neq k$ and $U(t)$ becomes unique by requiring that $q_1(t) > q_2(t) \ldots > q_n(t)$. Differentiating (1.31) gives

$$\dot{Q} = [\dot{U}U^{-1}, Q] + L \tag{1.35}$$

so equating diagonal and off-diagonal components yields

$$\dot{Q} = L_{\mathrm{diag}} = P \tag{1.36}$$

$$[Q, \dot{U}U^{-1}] = L_{\mathrm{off}} \tag{1.37}$$

differentiating (1.32) gives

$$[\dot{U}U^{-1}, \mu] = 0 \tag{1.38}$$

and differentiating (1.33) gives the Lax equation (1.14). From (1.37), (1.38) and (1.14) it follows that $\dot{U}U^{-1} = M$, so we have constructed a one-parameter family $U(t)$ such that $(Q(t), L(t))$ is the solution of the equations (1.14) and (1.27) which also satisfy (1.24)–(1.26) and $\{q_j(t), 1 \leq j \leq n\}$ are the eigenvalues of $X(t)$. So the solution of the Calogero-Moser model is reduced to the calculation of the eigenvalues of $X(t)$.

Example 1.1. $(n = 2)$ Let (p, q) denote the center-of-mass coordinates, with $q > 0$ then one has

$$\dot{q} = p, \quad \dot{p} = 1/4g^2q^{-3}, \quad H = p^2 + 1/4g^2q^{-2} \tag{1.39}$$

and $Q = \operatorname{diag}(q, -q)$, $P = \operatorname{diag}(p, -p)$ and

$$X(t) = (q(0) + tp(0))(e_{11} - e_{22}) + 1/2igtq(0)^{-1}(e_{12} - e_{21}) \tag{1.40}$$

and the eigenvalues of $X(t)$ are $(\lambda(t), -\lambda(t))$ with

$$\lambda(t) = \sqrt{(q(0) + tp(0))^2 + 1/4g^2t^2q(0)^{-2}} > 0 \tag{1.41}$$

and so $q(t) = \lambda(t)$, $p(t) = \dot{\lambda}(t)$ and because H is constant it follows that

$$p(\pm\infty) = \sqrt{p(0)^2 + 1/4g^2q(0)^{-2}} \tag{1.42}$$

The matrix $U(t)$ is given by

$$U(t) = \cos s I_2 + i \sin s(e_{12} + e_{21}) \tag{1.43}$$

where

$$\sin^2 s = \tfrac{1}{2}\left(\frac{1 - |q(0) + tp(0)|}{\lambda(t)}\right)$$

and so

$$s = 1/2 \arcsin\left(\frac{gt}{2q(0)\lambda(t)}\right) \tag{1.44}$$

Observe that this is well-defined because

$$\frac{g^2t^2}{4q(0)^2\lambda(t)^2} \leq 1$$

Now we return to the models of type II–IV and make the ansatz

$$L = P + g\sum_{j<k} x(q_j - q_k)i(e_{jk} - e_{kj}) \tag{1.45}$$

$$\begin{aligned} M = {} & g\sum_{j<k} y(q_j - q_k)i(e_{jk} + e_{kj}) \\ & - g\sum_{j<k} z(q_j - q_k)i(e_{jj} + e_{kk} - \frac{2}{n}I_n) \end{aligned} \tag{1.46}$$

so L is hermitian, M skew-hermitian, x, y and z are real-valued functions, with

$$x(-q) = -x(q), \quad y(-q) = y(q), \quad z(-q) = z(q) \tag{1.47}$$

and

$$v(q) = x^2(q) + \text{constant} \tag{1.48}$$

If one requires that Hamilton's equations (1.6) are equivalent with the Lax equation (1.14), under the assumption that $\dot{q}_{tot} = p_{tot}$, where L and M as in (1.45) and (1.46) then it turns out that

$$y(q) = -x'(q) \tag{1.49}$$

and $x(q)$ and $z(q)$ have to satisfy the functional equation

$$x(\xi)x'(\eta) - x(\eta)x'(\xi) = x(\xi + \eta)[z(\xi) - z(\eta)] \tag{1.50}$$

This functional equation has been solved by several people and one has

Theorem 1.2. [38] Let $x(\eta)$ be an odd meromorphic function which satisfies (1.50) and

$$\lim_{\eta\to 0} \eta x(\eta) = 1 \tag{1.51}$$

then

$$z(\eta) = x''(\eta)/2x(\eta) \tag{1.52}$$

$$(x')^2 = x^4 - 2bx^2 + c \quad b, c \in \mathbb{R} \tag{1.53}$$

and $x(\eta)$ can have the following form:

$$x(\eta) = \begin{cases} \eta^{-1} & \text{(I)} \\ a\coth(a\eta) & \text{(II a)} \\ a\sinh^{-1}(a\eta) & \text{(II b)} \\ a\cot(a\eta) & \text{(III a)} \\ a\sin^{-1}(a\eta) & \text{(III b)} \\ a\,\mathrm{cn}(a\eta)/\mathrm{sn}(a\eta), \quad a\,\mathrm{dn}(a\eta)/\mathrm{sn}(a\eta), \quad a/\mathrm{sn}(a\eta) & \text{(IV a,b,c)} \end{cases} \tag{1.54}$$

where sn, cn and dn are the Jacobi elliptic functions and $a \in \mathbb{R}$ is a parameter. □

This gives a Lax pair for the systems I–IV. From (1.52) and (1.53) one derives

$$z(\eta) = x(\eta)^2 - b \tag{1.55}$$

so the functional equation can be simplified to

$$x(\xi)x'(\eta) - x(\eta)x'(\xi) = x(\xi+\eta)(x^2(\xi) - x^2(\eta)) \tag{1.56}$$

and M becomes:

$$\begin{aligned} M = {} & g\sum_{j<k} y(q_j - q_k)i(e_{jk} + e_{kj}) \\ & - g\sum_{j<k} x^2(q_j - q_k)i(e_{jj} + e_{kk} - \frac{2}{n}I_n) \end{aligned} \tag{1.57}$$

Again $H = \frac{1}{2}\mathrm{tr}L^2$ and from the Lax equation it follows that the f_k are conserved quantities and L undergoes an isospectral deformation. One can also prove with the help of the functional equation that the f_k are in involution and are functionally independent ([49]), so the systems of type I–IV are completely integrable. In some cases one can explicitly solve these equations by constructing the one- parameter family $U(t)$. In chapter 5 we shall give another (Lie algebraic) proof of the integrability of the models of type I, II, and III.

1.2. The model of type V.

Now consider the model of type V as defined in (0.1)–(0.3) (see [1] and [49] for more details). The configuration space and phase space are the same as for the models of type I and II and Hamilton's equations are given by:

$$\dot{q}_j = p_j$$

$$\dot{p}_j = -2g^2 \sum_{k \neq j} (q_k - q_j)^{-3} + 2\omega^2 g^2 \sum_{k \neq j} (q_k - q_j) \tag{1.58}$$

Define the matrices P,Q,L,M and μ as for type I, then one can easily verify that the equations (1.58) are equivalent with the matrix equations:

$$\dot{Q} = P, \quad \dot{L} = [M, L] - \tilde{\omega}^2 Q, \quad \tilde{\omega}^2 = 2n\omega^2 g^2 \tag{1.59}$$

This is not yet a Lax equation. So define

$$L^{\pm} = L \pm i\tilde{\omega} Q \tag{1.60}$$

then

$$[Q, L^{\pm}] = \mu, \quad [L^+, L^-] = 2i\tilde{\omega}\mu \tag{1.61}$$

and

$$(L^+)^{\dagger} = L^- \tag{1.62}$$

where † denotes the hermitian adjoint. From (1.59) it follows that

$$\dot{L}^{\pm} = [M, L^{\pm}] \pm i\tilde{\omega} L^{\pm} \tag{1.63}$$

Now define

$$N_1 = L^+ L^-, \quad N_2 = L^- L^+ \tag{1.64}$$

then N_1 and N_2 are hermitian and satisfy the Lax equations

$$\dot{N}_1 = [M, N_1]$$

$$\dot{N}_2 = [M, N_2] \tag{1.65}$$

Finally define

$$N = 1/2(N_1 + N_2) = L^2 + \tilde{\omega}^2 Q^2 \tag{1.66}$$

then N is again hermitian, $H = 1/2 \mathrm{tr} N$ and N satisfies the Lax equation

$$\dot{N} = [M, N] \tag{1.67}$$

So the eigenvalues of N are conserved and N undergoes an isospectral deformation. Using N one can explicitly solve this model. To see this define

$$X(t) = Q(0)\cos\tilde{\omega}t + \tilde{\omega}^{-1} L(0) \sin\tilde{\omega}t \tag{1.68}$$

$X(t)$ is hermitian, so there exists a one-parameter family $U(t)$ of unitary matrices, such that

$$U(t)X(t)U(t)^{-1} = Q(t) \tag{1.69}$$

Of course at this point one has to show that this $Q(t)$ indeed satisfies (1.59)–(1.61). Now define

$$L(t) = U(t)(L(0)\cos\tilde{\omega}t - \tilde{\omega}Q(0)\sin\tilde{\omega}t)U(t)^{-1} \tag{1.70}$$

Again one can choose $U(t)$ in such a way that $U(t)\mu U(t)^{-1} = \mu$ and $U(0) = I_n$. Now

$$[Q(0), X(t)] = \tilde{\omega}^{-1}\sin(\tilde{\omega}t)\mu \tag{1.71}$$

and so

$$[Q(t), L(t)] = \mu \tag{1.72}$$

So again $q_j(t) \neq q_k(t)$ if $j \neq k$ and $U(t)$ becomes unique by requiring $q_1(t) > q_2(t)\ldots > q_n(t)$. Differentiating (1.69) with respect to t gives:

$$\dot{Q} = [\dot{U}U^{-1}, Q] + L \tag{1.73}$$

and equating left- and righthand sides yields:

$$\dot{Q} = P, \quad [Q, \dot{U}U^{-1}] = L_{\text{off}} \tag{1.74}$$

Differentiating (1.70) gives:

$$\dot{L} = [\dot{U}U^{-1}, L] - \tilde{\omega}^2 Q \tag{1.75}$$

and combining (1.74) and (1.75) it follows that $M = \dot{U}U^{-1}$ so we have constructed an integral curve $(Q(t), L(t))$, satisfying (1.59). Using (1.69) and (1.70) one can check that

$$N(t) = U(t)N(0)U(t)^{-1} \tag{1.76}$$

2. Poisson manifolds and completely integrable systems

In this chapter we collect some well-known facts about Poisson manifolds. We refer to the literature ([22],[23],[24],[25],[26],[39],[40]) for the proofs and further details.
Let M be an n-dimensional smooth manifold.

Definition 2.1. A Poisson structure or Poisson bracket is a skew-symmetric bilinear map $\{\ ,\ \} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ which satisfies the following properties ($f, g, h \in \mathcal{C}^\infty(M)$):

$$\{fg, h\} = f\{g,h\} + \{f,h\}g \quad \text{(Leibniz property)} \tag{2.1}$$

$$\{\{f,g\},h\} + \text{cycl.} = 0 \quad \text{(Jacobi identity)} \tag{2.2}$$

So a Poisson structure turns the commutative algebra $\mathcal{C}^\infty(M)$ into a Lie algebra such that $\{h, .\} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a derivation for all $h \in \mathcal{C}^\infty(M)$.

Definition 2.2. A Poisson manifold is a manifold with a Poisson structure.

Example 2.3. Let $M = \mathbb{R}^{2m}$ with coordinates $\{p_1, \ldots, p_m, q_1, \ldots, q_m\}$ and define

$$\{f,g\} = \sum_{j=1}^{m} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{p_j} \right) \tag{2.3}$$

This is the socalled canonical Poisson bracket and $\{p_j, q_k\} = \delta_{jk}$.

Definition 2.4. A Poisson mapping $\phi : M \to N$, with M and N Poisson manifolds, is a smooth map which satisfies:

$$\phi^*\{f,g\}_N = \{\phi^*(f), \phi^*(g)\}_M \tag{2.4}$$

for all $f, g \in \mathcal{C}^\infty(M)$, where the pullback $\phi^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ is defined by:

$$\phi^*(f)(m) = f(\phi(m)), \quad m \in M \tag{2.5}$$

A diffeomorphic Poisson mapping $\phi : M \to M$ is called an automorphism of the Poisson manifold M.

Definition 2.5. An infinitesimal endomorphism A of a Poisson manifold M is a linear map $A : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ which satisfies:

$$A\{f,g\} = \{A(f), g\} + \{f, A(g)\} \tag{2.6}$$

so it is a derivation of the Lie algebra $\mathcal{C}^\infty(M)$; it is called an inner endomorphism if there exists a function $h \in \mathcal{C}^\infty(M)$ such that

$$A(f) = \{h, f\} \tag{2.7}$$

Definition 2.6. A submanifold $N \subset M$, where M and N are Poisson manifolds, is a Poisson submanifold of M if the inclusion $i : N \to M$ is a Poisson mapping.

Definition 2.7. A function $f \in \mathcal{C}^\infty(M)$ is called a Casimir function (or distinguished function, cyclic function or invariant) if $f \in Z(\mathcal{C}^\infty(M))$, i.e. $\{f,h\} = 0$ for all $h \in \mathcal{C}^\infty(M)$.

Remark 2.8. The constant functions are Casimir functions.

Definition 2.9. With each $h \in \mathcal{C}^\infty(M)$ one can associate a vector field $v_h : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ by defining

$$v_h(f) = \{h, f\}, \quad f \in \mathcal{C}^\infty(M) \tag{2.8}$$

This is called the Hamiltonian vector field associated with the Hamiltonian h.

Lemma 2.10. The map $h \mapsto v_h$ has the following properties:

$$v_{\{f,g\}} = [v_f, v_g] \tag{2.9}$$

$$v_h\{f,g\} = \{v_h(f), g\} + \{f, v_h(g)\} \tag{2.10}$$

$$v_{fg} = f v_g + g v_f \tag{2.11}$$

Proof. This follows directly from the definition of the Poisson bracket. □

So the map $v : \mathcal{C}^\infty(M) \to \mathcal{V}(M)$ defined by $v(h) = v_h$ is a Lie algebra homomorphism from $\mathcal{C}^\infty(M)$ into the Lie algebra of inner infinitesimal endomorphisms of M.

Definition 2.11. Let v_h be a Hamiltonian vector field with Hamiltonian h. Let $\phi : \mathbb{R} \to M$ be a smooth curve in M such that

$$\dot{\phi}(t) = (v_h)(\phi(t)) \tag{2.12}$$

where $(v_h)(\phi(t))$ is the value of v_h in $\phi(t)$. Equations (2.12) are called Hamilton's equations with Hamiltonian h. The unique maximal integral curve passing through $x \in M$ is denoted by $\psi(t, x)$ and is called the flow of v_h and is often written as

$$\psi(t, x) = \exp(t v_h)(x) \tag{2.13}$$

Example 2.12. Let $M = \mathbb{R}^{2m}$ with Poisson bracket (2.3). Then the Hamiltonian vector field v_f corresponding to f is given by

$$v_f = \sum_{j=1}^{m} \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right) \tag{2.14}$$

Let $\phi(t) = (p(t), (q(t))$ be a curve in M then Hamilton's equations become:

$$\dot{\phi}(t) = \left(-\frac{\partial f}{\partial q_1}, \ldots, -\frac{\partial f}{\partial q_m}, \frac{\partial f}{\partial p_1}, \ldots, \frac{\partial f}{\partial p_m} \right) \tag{2.15}$$

written out in terms of the coordinates $(p_1, \ldots, p_m, q_1, \ldots, q_m)$ these become:

$$\dot{q_j} = \frac{\partial f}{\partial p_j}, \quad \dot{p_j} = -\frac{\partial f}{\partial q_j} \tag{2.16}$$

Take for example $f = \sum_{j=1}^m a_j p_j$, where $a_j \in \mathbb{R}$, then $v_f = \sum_{j=1}^m a_j \frac{\partial}{\partial q_j}$, which is the generator of a translation, and the corresponding flow is $\psi(t, p, q) = (p, q + ta)$, where $a = (a_1, \ldots, a_m)$.

Lemma 2.13. For each t, the flow $\exp(tv_h) : M \to M$ determines a (local) Poisson automorphism of M.

Definition 2.14. Two functions $f, g \in \mathcal{C}^\infty(M)$ are said to be in involution if $\{f, g\} = 0$.

Lemma 2.15. Let v_h denote a Hamiltonian vector field, let $f \in \mathcal{C}^\infty(M)$ and f and h in involution, then f is constant along integral curves of v_h.

Proof.

$$\begin{aligned} \frac{d}{dt} f(\phi(t)) &= df(\phi(t))(\dot{\phi}(t)) = df(\phi(t))(v_h(\phi(t))) \\ &= v_h(f)(\phi(t)) = \{h, f\}(\phi(t)) = 0 \end{aligned}$$

□

Example 2.16. Consider the Calogero-Moser model as defined in (1.6) and (1.7). In this case $\{H, p_{tot}\} = 0$, which followed from the translation-invariance. So the total momentum is conserved along integral curves of Hamilton's equations.

From the properties of the Poisson bracket it follows that in each point $m \in M$, v_h depends only on dh and $\{f, g\}$ depends only on df and dg, so there is a bundle map $B : T^*M \to TM$ such that $v_h = B(dh)$ and $v = Bd : \mathcal{C}^\infty(M) \to \mathcal{V}(M)$. B is sometimes called the Hamiltonian operator. One may also think of B as defining a contravariant skew-symmetric 2-tensor W on M, for which

$$\{f, g\} = W(df, dg) = dg(B(df)) \tag{2.17}$$

The tensor W is sometimes called a cosymplectic structure. Now let $\{x^j, j = 1, \ldots, n\}$ be local coordinates on M. Then it follows from the properties of the Poisson bracket that one can write:

$$\{f, g\} = \sum_{j,k} W^{jk} \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^k} \tag{2.18}$$

where

$$W^{jk} = \{x^j, x^k\} \tag{2.19}$$

So a Poisson structure is determined by the system of functions W^{jk}. These are sometimes called the structure functions of M relative to the local coordinates x^j, and the matrix W is called the structure matrix of M. One can also view the W^{jk} as the components of the cosymplectic structure. The Jacobi identity is equivalent to the Jacobi identity for the coordinate functions. In local coordinates one has in each point $x \in M$

$$B(dx^j) = \sum_{k=1}^{n} W^{jk}(x) \frac{\partial}{\partial x^k} \tag{2.20}$$

Example 2.17. Choose W constant and skew-symmetric.

Definition 2.18. The rank of a Poisson structure at a point $x \in M$ is defined as the rank of the linear map $B(x) : T_x^* M \to T_x M$. In local coordinates it is also the rank of the matrix $W^{jk}(x)$.

Lemma 2.19. The rank of a Poisson manifold at any point is always an even integer.

Lemma 2.20. The rank of a Poisson manifold is constant along flows of Hamiltonian vector fields.

Definition 2.21. A Poisson manifold M is called symplectic if the rank is everywhere equal to the dimension n of M.

Corollary 2.22. A symplectic manifold is even-dimensional.

If M is a symplectic manifold, then one can define a symplectic structure on M by

$$\omega(Bdf, Bdg) = dg(B(df)) \tag{2.21}$$

From the properties of B it follows that ω is a closed non-degenerate 2-form on M.

Theorem 2.23. Each Poisson manifold M naturally splits into a family of even-dimensional symplectic manifolds, the leaves of the socalled symplectic foliation. The dimension of any such leaf N equals the rank of the Poisson structure at any point $y \in N$.

Theorem 2.24. (Darboux) Let M be an n-dimensional Poisson manifold of constant rank $2m \leq n$. At each $x \in M$ there exist local coordinates

$$(p, q, z) = (p_1, \ldots, p_m, q_1, \ldots, q_m, z_1, \ldots, z_l)$$

where $2m + l = n$, in terms of which the Poisson bracket takes the form:

$$\{f, g\} = \sum_{j=1}^{m} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right) \tag{2.22}$$

so $\{p_j, q_k\} = \delta_{jk}$ and the z_j are Casimir functions. The leaves of the symplectic foliation intersect the coordinate chart in the slices $\{z_1 = c_1, \ldots, z_l = c_l\}$ determined by the distinguished coordinates z. So locally a Poisson structure always has the canonical form (2.22), but this is often not the most convenient way of viewing a Poisson structure. This is for example the case for Poisson structures on Lie algebras, which will be studied in chapter 4.

Finally consider completely integrable Hamiltonian systems (in the commutative case). These are systems which satisfy the conditions of the following well-known Liouville-theorem.

Theorem 2.25. (Liouville) Let M be a symplectic manifold of dimension $n = 2m$ and consider Hamilton's equations on M with Hamiltonian h. If there exist m functions $\{f_1, \ldots, f_m, f_1 = h\}$, which are functionally independent, such that $\{f_j, f_k\} = 0$ for all j, k then the Hamiltonian system is completely integrable, which means that there exist global action-angle coordinates $\{I_j, \phi_j, j = 1, \ldots, m\}$ on M in which the equations become:

$$\dot{I}_j = 0, \quad \dot{\phi}_j = f_j(I), \quad I = (I_1, \ldots, I_m) \tag{2.23}$$

These are $2m$ ordinary differential equations which can be integrated immediately.

Remark 2.26. The Liouville theorem does not give an explicit construction of the angle variables, but in concrete examples, for example the Calogero-Moser model, there often exists a natural construction of these action-angle variables. For example, in the case of the Calogero-Moser model ([18], [44]) the action variables are the eigenvalues λ_j of L and the angle variables are given by:

$$\phi_j = \frac{1}{j}\mathrm{tr}(QL^{j-1}), \quad 1 \leq j \leq n$$

and one can easily verify, using (1.16), (1.25), (1.27) and (1.14) and the properties of the trace, that:

$$\dot{\phi}_j = \frac{1}{j}\mathrm{tr}(L^j) = f_j(L)$$

Finally, a useful criterium to decide whether a submanifold is a Poisson submanifold is the following:

Lemma 2.27. A submanifold $N \subset M$ of a Poisson manifold M is a Poisson submanifold iff all Hamiltonian vector fields are tangent to N.

Corollary 2.28. If M is a vector space and N a subspace this simplifies to the condition:

$$v_f(x) \in N, \text{ for all } x \in N, f \in C^\infty(M) \tag{2.24}$$

3. Real semisimple Lie algebras

In this chapter $\mathfrak{g}$ is a finite-dimensional real semisimple Lie algebra (see [30],[32] and [33] for the proofs and more details). Let $<\ ,\ >$ be the Killing form and θ a Cartan involution of $\mathfrak{g}$, with corresponding eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which is orthogonal with respect to the Killing form, and commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \tag{3.1}$$

The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$ and one can define an inner product on $\mathfrak{g}$ by

$$< x, y >_\theta = - < x, \theta(y) > \tag{3.2}$$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, such that G has finite center, and let K be the subgroup corresponding to $\mathfrak{k}$. Then K is a maximal compact subgroup of G and the center of G is contained in K. Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$, which is again semisimple. Now suppose $\mathfrak{g}$ is noncompact, so $\dim \mathfrak{p} \neq 0$. Then $\mathfrak{p}$ contains a maximal abelian subspace $\mathfrak{a}$. Let $\mathfrak{h}_R$ be a maximal abelian subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}$ and which is θ-stable. Then the complexification $\mathfrak{h}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$. Also one has the orthogonal decomposition

$$\mathfrak{h}_R = (\mathfrak{h}_R \cap \mathfrak{k}) \oplus (\mathfrak{h}_R \cap \mathfrak{p}) = (\mathfrak{h}_R \cap \mathfrak{k}) \oplus \mathfrak{a} \tag{3.3}$$

and we write $\mathfrak{h}_k = \mathfrak{h}_R \cap \mathfrak{k}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, then

$$\mathfrak{h}_k \subset \mathfrak{m} \tag{3.4}$$

Now let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha \tag{3.5}$$

be the root space decomposition of $\mathfrak{g}$ with respect to ad $\mathfrak{a}$, with corresponding root system R, then

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a} \tag{3.6}$$

and we write

$$\mathfrak{g}_0{}^\perp = \sum_{\alpha \in R} \mathfrak{g}_\alpha \tag{3.7}$$

R is called the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$. Let Δ denote the simple roots with respect to some ordening, R_+ the positive roots, R_- the negative roots and define

$$\mathfrak{g}_+ = \sum_{\alpha \in R_+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_- = \sum_{\alpha \in R_-} \mathfrak{g}_\alpha \tag{3.8}$$

One also has $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and the commutation relations

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \text{ if } \alpha + \beta \in R$$

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$$

$$[\mathfrak{g}_0, \mathfrak{g}_+] \subset \mathfrak{g}_+, \quad [\mathfrak{g}_0, \mathfrak{g}_-] \subset \mathfrak{g}_-$$

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha \tag{3.9}$$

Furthermore one has

$$< \mathfrak{g}_\alpha, \mathfrak{g}_\beta >= 0 \text{ if } \alpha + \beta \neq 0 \tag{3.10}$$

Example 3.1. Let $\mathfrak{g} = sl(3,\mathbb{C})$, viewed as a real Lie algebra. It consists of all traceless 3×3-matrices over $\mathbb{C}$, and the Killing form is given by

$$< x, y >= \operatorname{Re}\operatorname{tr}(xy)$$

for $x, y \in \mathfrak{g}$. The Cartan involution θ is given by $\theta(x) = -x^\dagger$, where $\dagger$ denotes the hermitian conjugate. Then $\mathfrak{k} = \{$traceless skew-hermitian matrices$\}$, $\mathfrak{p} = \{$traceless hermitian matrices$\}$, $G = SL(3,\mathbb{C})$ and $K = SU(3)$. Choose $\mathfrak{a} = \{$real diagonal matrices with zero trace$\}$, then $\mathfrak{m} = i\mathfrak{a}$. In this case $\mathfrak{h}_k = \mathfrak{m}$, so $\mathfrak{h}_R = \mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}_0$. Let $q = \operatorname{diag}(q_1, q_2, q_3) \in \mathfrak{a}$ and define $\varepsilon_j(q) = q_j, j = 1, 2, 3$. Then the restricted root system R is given by $R = \{\alpha_{jk} := \varepsilon_j - \varepsilon_k, 1 \leq j \neq k \leq 3\}$ and is of type A_2.
Choose

$$\Delta = \{\alpha_j := \varepsilon_j - \varepsilon_{j+1}, \quad j = 1, 2\}$$

then the positive roots are $R_+ = \{\alpha_{jk} \in R, 1 \leq j < k \leq 3\}$ and $R_- = \{\alpha_{jk} \in R, 3 \geq j > k \geq 1\}$. The root spaces $\mathfrak{g}_\alpha$ are given by

$$\mathfrak{g}_{\alpha_{jk}} = \mathbb{R} \prec e_{jk}, ie_{jk} \succ$$

.

Definition 3.2. The (real) rank of $\mathfrak{g}$ is defined as $l = \dim \mathfrak{a}$ and $m_\alpha := \dim \mathfrak{g}_\alpha$ is called the multiplicity of the root α.

Remark 3.3. In contrast to the complex case R can be a non-reduced root system, m_α can be > 1 and $\mathfrak{m}$ is in general a proper non-zero subspace of $\mathfrak{g}_0$ and is not necessarily abelian.

Example 3.4. If $\mathfrak{g} = sl(3,\mathbb{C})$ then $l = 2$ and $m_\alpha = 2$ for all $\alpha \in R$. Also $\mathfrak{m}$ is abelian.

Because the Killing form restricted to $\mathfrak{a}$ is positive definite and nondegenerate, $\mathfrak{a}$ is an l-dimensional Euclidian vector space with inner product $(\ ,\ )$ and one can identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with the help of $(\ ,\ )$. If $\alpha \in \mathfrak{a}^*$ one defines $t_\alpha \in \mathfrak{a}$ by

$$(a, t_\alpha) = \alpha(a), \text{ for all } a \in \mathfrak{a} \tag{3.11}$$

and if $\alpha, \beta \in \mathfrak{a}^*$ one defines

$$(\alpha, \beta) = (t_\alpha, t_\beta) \tag{3.12}$$

Furthermore one defines

$$h_\alpha = \frac{2t_\alpha}{(\alpha, \alpha)} \tag{3.13}$$

Now choose $0 \neq e_\alpha \in \mathfrak{g}_\alpha, \alpha \in R_+$, and the following two different normalizations:

a) Define (see [33])

$$e_{-\alpha} := -\theta(e_\alpha) \in \mathfrak{g}_{-\alpha} \tag{3.14}$$

and normalize e_α in such a way that

$$< e_\alpha, e_\alpha >_\theta = \frac{2}{(\alpha, \alpha)} \tag{3.15}$$

then also

$$< e_{-\alpha}, e_{-\alpha} >_\theta = \frac{2}{(\alpha, \alpha)} \tag{3.16}$$

and one has the commutation relations

$$\begin{aligned} [e_\alpha, e_{-\alpha}] &= h_\alpha \\ [h_\alpha, e_\alpha] &= 2e_\alpha \\ [h_\alpha, e_{-\alpha}] &= -2e_{-\alpha} \end{aligned} \tag{3.17}$$

so $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ forms an $sl(2, \mathbb{R})$ subalgebra. Also

$$< e_\alpha, e_{-\alpha} > = \frac{2}{(\alpha, \alpha)} \tag{3.18}$$

From (3.14) it follows that

$$e_\alpha - e_{-\alpha} \in \mathfrak{k}, \quad e_\alpha + e_{-\alpha} \in \mathfrak{p} \tag{3.19}$$

and so

$$\begin{aligned} < e_\alpha - e_{-\alpha}, e_\alpha - e_{-\alpha} > &= -\frac{4}{(\alpha, \alpha)} \\ < e_\alpha + e_{-\alpha}, e_\alpha + e_{-\alpha} > &= \frac{4}{(\alpha, \alpha)} \end{aligned} \tag{3.20}$$

b) Now define (see [38]):

$$e_{-\alpha} := \theta(e_\alpha) \in \mathfrak{g}_{-\alpha} \tag{3.21}$$

and normalize e_α in such a way that

$$< e_\alpha, e_\alpha >_\theta = 1 \tag{3.22}$$

then also

$$< e_{-\alpha}, e_{-\alpha} >_\theta = 1 \tag{3.23}$$

and

$$< e_\alpha, e_{-\alpha} > = -1 \tag{3.24}$$

Now one has the commutation relations

$$\begin{aligned} [e_\alpha, e_{-\alpha}] &= -t_\alpha = -\frac{1}{2}(\alpha, \alpha) h_\alpha \\ [h_\alpha, e_\alpha] &= 2e_\alpha \\ [h_\alpha, e_{-\alpha}] &= -2e_{-\alpha} \end{aligned} \tag{3.25}$$

and also

$$e_\alpha + e_{-\alpha} \in \mathfrak{k}, \quad e_\alpha - e_{-\alpha} \in \mathfrak{p} \tag{3.26}$$

and

$$\begin{aligned} < e_\alpha + e_{-\alpha}, e_\alpha + e_{-\alpha} > &= -2 \\ < e_\alpha - e_{-\alpha}, e_\alpha - e_{-\alpha} > &= 2 \end{aligned} \tag{3.27}$$

Example 3.5. Let $\mathfrak{g} = sl(3, \mathbb{C})$. In this case one has:

$$t_{\alpha_{jk}} = h_{\alpha_{jk}} = d_j - d_k$$

where $d_j = e_{jj}$ and so $(\alpha, \alpha) = 2$. Now the two normalizations almost coincide, so choose the second one. If one chooses $e_{\alpha_{jk}} = ie_{jk}$, then $e_{-\alpha_{jk}} = ie_{kj}$ and if one chooses $e_{\alpha_{jk}} = e_{jk}$ then $e_{-\alpha_{jk}} = -e_{kj}$.

Definition 3.6. An element $x \in \mathfrak{p}$ is called regular if $\dim \mathrm{Cent}_{\mathfrak{g}}(\mathrm{x}) = \dim \mathfrak{g}_0$.

Lemma 3.7. An element $q \in \mathfrak{a}$ is regular iff $\alpha(q) \neq 0$ for all $\alpha \in R$. In that case $\mathrm{Cent}_{\mathfrak{g}}(q) = \mathfrak{g}_0$.

Definition 3.8. The positive Weyl chamber $\mathfrak{a}_+$ of $\mathfrak{a}$ is defined as

$$\mathfrak{a}_+ = \{a \in \mathfrak{a} \mid \alpha(a) > 0, \text{ for all } \alpha \in R_+\} \tag{3.28}$$

The Weyl alcove $\mathfrak{a}_+^d$, $d \in \mathbb{R}$, is defined as

$$\mathfrak{a}_+^d = \{a \in \mathfrak{a} \mid 0 < \alpha(a) < d, \text{ for all } \alpha \in R_+\} \tag{3.29}$$

So $\mathfrak{a}_+$ and $\mathfrak{a}_+^d$ consist of regular elements.

Definition 3.9. The character δ of $\mathfrak{g}$ is defined as:

$$\delta = \dim \mathfrak{p} - \dim \mathfrak{k} \tag{3.30}$$

But because $\dim \mathfrak{g}_0^{\perp} \cap \mathfrak{p} = \dim \mathfrak{g}_0^{\perp} \cap \mathfrak{k}$ one also has

$$\delta = l - \dim \mathfrak{m} \tag{3.31}$$

Definition 3.10. A real noncompact semisimple Lie algebra is called split if $\dim \mathfrak{m} = 0$, quasisplit if $\mathfrak{h}_k = \mathfrak{m}$ (see [42, p 172]) and a normal real form of $\mathfrak{g}_C$ if $\dim \mathfrak{h}_k = 0$.

Example 3.11. Let $\mathfrak{g} = sl(3, \mathbb{C})$, then $q = \mathrm{diag}(q_1, q_2, q_3) \in \mathfrak{a}$ is regular if $q_1 \neq q_2 \neq q_3 \neq q_1$. The positive Weyl chamber and Weyl alcove are given by:

$$\mathfrak{a}_+ = \{q \in \mathfrak{a} \mid q_1 > q_2 > q_3\}$$

$$\mathfrak{a}_+^d = \{q \in \mathfrak{a} \mid q_1 > q_2 > q_3, q_1 - q_3 < d, d \in \mathbb{R}\}$$

Because in this case $\mathfrak{m} = i\mathfrak{a}$, $\delta = 0$ and because $\mathfrak{h}_k = \mathfrak{m}$, $\mathfrak{g}$ is quasisplit.

Lemma 3.12. A real semisimple Lie algebra $\mathfrak{g}$ is split iff it is a normal real form.

Lemma 3.13. If $\mathfrak{g}$ is a normal real form and semisimple then $m_\alpha = 1$ for all $\alpha \in R$.

Now specialize to the simple noncompact Lie algebras. Then one has:

Theorem 3.14. If $\mathfrak{g}$ is a real simple noncompact Lie algebra then it belongs to one of the following two types:

Type III: the pair $(\mathfrak{g}, \theta)$ is such that $\mathfrak{g}$ is a simple noncompact real Lie algebra, the complexification $\mathfrak{g}_C$ is a simple Lie algebra over $\mathbb{C}$ and θ is a Cartan involution such that $\mathfrak{k}$ is a compactly embedded subalgebra.

Type IV: the pair $(\mathfrak{g}, \theta)$ is such that $\mathfrak{g}$ is a complex simple Lie algebra, viewed as a real Lie algebra and θ is the conjugation of $\mathfrak{g}$ with respect to a maximal compactly imbedded subalgebra.

The Lie algebras of type IV are listed in Table IV and those of type III in Table V of [30].
Now suppose $\mathfrak{g}$ is quasi-split then $\dim \mathfrak{h}_k = \dim \mathfrak{m}$ but also $\dim \mathfrak{h}_k = \mathrm{rank}\, \mathfrak{g}_C - \dim \mathfrak{a}$ and combining this with (3.31) one gets the following condition:

Lemma 3.15. A Lie algebra $\mathfrak{g}$ is quasi-split iff

$$\operatorname{rank} \mathfrak{g}_C = 2l - \delta \tag{3.32}$$

Now combining the information of Tables IV, V and VI of [30] and Tables 9.3 and 9.6 of [32] one finds the following:

Proposition 3.16. If $\mathfrak{g}$ is a real noncompact simple Lie algebra of quasi-split type, then $\mathfrak{g}$ belongs to one of the following classes:

1) $\mathfrak{g}$ is of type IV.
2) $\mathfrak{g}$ is of type III and is a normal real form of $\mathfrak{g}_C$.
3) $\mathfrak{g}$ is of type III and one of the following series:

$$\text{AIII } (p = q, p = q+1), \text{ BDI } (p = q+2), \text{ E II} \tag{3.33}$$

where we have used the notation of [30].

Proof. If $\mathfrak{g}$ is of type IV then $\mathfrak{m} = i\mathfrak{a}$, so $\mathfrak{m}$ is abelian, $\delta = 0$ and $\operatorname{rank} \mathfrak{g}_C = 2l$, so the condition (3.32) is satisfied. If $\mathfrak{g}$ is of type III and a normal real form then $\dim \mathfrak{m} = \dim \mathfrak{h}_k = 0$, so $\delta = l$ and $\operatorname{rank} \mathfrak{g}_C = l$. Now suppose $\mathfrak{g}$ is of type III and not a real form then one can check the condition (3.32) and one finds that the only cases are the ones in (3.33), with:

AIII $(p = q)$
$\mathfrak{g} = su(q,q), \delta = 1, l = q, \mathfrak{g}_C = sl(2q, \mathbb{C}), \operatorname{rank} \mathfrak{g}_C = 2q - 1 = 2l - \delta, \mathfrak{k} = s(u(q) \oplus u(q)), R = C_q$

AIII $(p = q+1)$
$\mathfrak{g} = su(q+1, q), \mathfrak{k} = s(u(q+1) \oplus u(q)), \delta = 0, l = q, \mathfrak{g}_C = sl(2q+1, \mathbb{C}), \operatorname{rank} \mathfrak{g}_C = 2q, R = BC_q$

BDI $(p = q+2)$
$\mathfrak{g} = so(q+2, q), \mathfrak{k} = so(q+2) \oplus so(q), R = B_q, \delta = q - 1, l = q, \mathfrak{g}_C = so(2q+2, \mathbb{C}), \operatorname{rank} \mathfrak{g}_C = q + 1$

EII
$\mathfrak{g} = E_6, \mathfrak{k} = su(6) \oplus su(2), R = F_4, \delta = 2, l = 4, \mathfrak{g}_C = E_6, \operatorname{rank} \mathfrak{g}_C = 6$ □

Using this classification and the tables already mentioned one also finds:

Proposition 3.17. If $\mathfrak{g}$ is a real simple noncompact Lie algebra, the following properties are equivalent:

(i) $\mathfrak{g}$ is quasi-split.
(ii) $\mathfrak{m}$ is abelian.
(iii) $m_\alpha \leq 2$ for all $\alpha \in R$.
(iv) the Satake diagram has no black nodes.

Proof. Observe that $m_\alpha = 2$ for type IV and $m_\alpha = 1$ for normal real forms. Inspection of Table VI from [30] shows that the Proposition is true. □

4. Poisson structures on Lie algebras

4.1. Introduction.

In this chapter we consider Poisson structures on Lie algebras and their relation with integrable systems, via the Kostant-Adler-Symes theorem. For most of the proofs and for more details we refer to [11],[12],[13],[14],[36],[37],[47] and [48].

Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra. Let $\phi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ be a 2-form on $\mathfrak{g}$ with values in $\mathfrak{g}$ and define $\{\ ,\ \} : \mathcal{C}^\infty(\mathfrak{g}^*) \times \mathcal{C}^\infty(\mathfrak{g}^*) \to \mathcal{C}^\infty(\mathfrak{g}^*)$ by:

$$\{f, g\}(\lambda) = \lambda(\phi(F, G)) \tag{4.1}$$

where $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*), F = df(\lambda), G = dg(\lambda), \lambda \in \mathfrak{g}^*$, and where we identified $T^*\mathfrak{g}^*$ with $\mathfrak{g}$, so we view F and G as elements of $\mathfrak{g}$.

Proposition 4.1. If ϕ satisfies the Jacobi identity

$$\phi(\phi(x, y), z) + \text{cycl.} = 0 \text{ for all } x, y, z \in \mathfrak{g} \tag{4.2}$$

then (4.1) defines a Poisson bracket on $\mathcal{C}^\infty(\mathfrak{g}^*)$ and the linear functions on $\mathfrak{g}^*$ form a Lie subalgebra of $\mathcal{C}^\infty(\mathfrak{g}^*)$

Proof. It is clear that $\{\ ,\ \}$ is skew-symmetric, bilinear and satisfies the Leibniz rule. From chapter 2 we know that the Jacobi identity is equivalent with the Jacobi identity for coordinate functions. But coordinate functions on $\mathfrak{g}^*$ belong to $(\mathfrak{g}^*)^* \cong \mathfrak{g}$, so if one takes $x, y \in \mathfrak{g}$ then $dx(\lambda) = x, dy(\lambda) = y$ and $\{x, y\}(\lambda) = \lambda(\phi(x, y))$, so $\{x, y\} \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$. From this one gets $d\{x, y\}(\lambda) = \phi(x, y)$, so

$$\{\{x, y\}, z\}(\lambda) + \text{cycl.} = \lambda(\phi(\phi(x, y), z)) + \text{cycl.} = 0 \text{ for all } \lambda \in \mathfrak{g}^*, x, y, z \in \mathfrak{g}$$

$\square$

We shall use the symbol $(\mathfrak{g}, \phi)$ to denote the vector space $\mathfrak{g}$ equipped with the bracket ϕ.

Now assume that $\mathfrak{g}$ has a nondegenerate invariant symmetric bilinear form $<\ ,\ >$. Identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using the isomorphism $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ defined by $\kappa(x)(y) =< x, y >$ for all $x, y \in \mathfrak{g}$. Now define

$$\{f, g\}(x) =< x, \phi(F, G) > \tag{4.3}$$

where $f, g \in \mathcal{C}^\infty(\mathfrak{g}), x \in \mathfrak{g}, F = \nabla f(x), G = \nabla g(x)$ and $< \nabla f(x), y >= df(x)(y)$ for all $x, y \in \mathfrak{g}$. One easily checks that this defines a Poisson bracket on $\mathcal{C}^\infty(\mathfrak{g})$ which is isomorphic with the bracket (4.1),where the isomorphism is given by the pull-back of κ, and $\mathfrak{g}^*$ forms a Lie subalgebra of $\mathcal{C}^\infty(\mathfrak{g})$. So if ϕ satisfies the Jacobi identity, then there are two Lie algebra structures on $\mathfrak{g}$ and a Lie algebra structure on $\mathfrak{g}^*$, which are in general independent of each other.

Example 4.2. Let $\phi(x,y) = [x,y]$ then the corresponding Poisson structure on $\mathcal{C}^\infty(\mathfrak{g}^*)$ is known as the Lie-Poisson bracket or Kirillov-Kostant bracket. In this case the two Lie algebra structures on $\mathfrak{g}$ coincide.

4.2. Double Lie algebras and Yang-Baxter operators.

Now make the ansatz

$$\phi(x,y) = \tfrac{1}{2}[R(x),y] + \tfrac{1}{2}[x,R(y)] := [x,y]_R \tag{4.4}$$

where $R \in \operatorname{End}\mathfrak{g}$.

Definition 4.3. A pair $(\mathfrak{g}, R)$ is called a double Lie algebra if (4.4) satisfies the Jacobi identity. The symbol $\mathfrak{g}_R$ is used to denote the vector space $\mathfrak{g}$ equipped with the bracket (4.4).

Now define

$$\omega(x,y) = R[R(x),y] + R[x,R(y)] - [R(x),R(y)], \quad x,y \in \mathfrak{g} \tag{4.5}$$

then, using the Jacobi identity for $[\ ,\ ]$, it is easy to show that the Jacobi identity for (4.4) is equivalent with

$$[\omega(x,y),z] + \text{cycl.} = 0 \tag{4.6}$$

Definition 4.4. An operator $R \in \operatorname{End}\mathfrak{g}$ is called a Yang-Baxter operator if $\omega(x,y) = \rho[x,y]$ for all $x,y \in \mathfrak{g}$, where $\rho \in \mathbb{R}$ is a constant. If $\rho = 0$ this reduces to the Yang-Baxter equation

$$\omega(x,y) = 0 \tag{4.7}$$

If $\rho \neq 0$ one can rescale R in such a way that it satisfies the modified Yang-Baxter equation

$$\omega(x,y) = [x,y] \tag{4.8}$$

The operator R is called unitary if $R^* = -R$ where R^* denotes the adjoint of R.

Proposition 4.5. If R is a Yang-Baxter then (4.4) satisfies the Jacobi identity.

Proof. Substitute (4.7) or (4.8) in (4.6) and use the Jacobi identity for $[\ ,\ ]$. □

Example 4.6. A trivial solution of (4.8) is $R = \text{id}$. In this case $\phi(x,y) = [x,y]$, so of course one refinds the situation of Example 4.2. So the Lie algebra $\mathfrak{g}$ itself is a (trivial) double Lie algebra.

Example 4.7. Take $\mathfrak{g} = sl(2,\mathbb{C})$, then all unitary R are of the form $R = \operatorname{ad} z$ for some $z = \alpha h + \beta e + \gamma f, \alpha, \beta, \gamma \in \mathbb{C}$. All these operators R are Yang-Baxter operators, because $\omega(x,y) = 4(\alpha^2 + \beta\gamma)[x,y]$, for all $x,y \in \mathfrak{g}$.

Definition 4.8. A function $f \in \mathcal{C}^\infty(\mathfrak{g})$ is called Ad-invariant if

$$f(\mathrm{Ad}h(x)) = f(x), \text{ for all } x \in \mathfrak{g}, h \in G \tag{4.9}$$

Here G is a connected Lie group with Lie algebra $\mathfrak{g}$ and Ad is the adjoint representation of G on $\mathfrak{g}$.

Lemma 4.9. Let $f, g \in \mathcal{C}^\infty(\mathfrak{g})$ be Ad-invariant functions then they have the following properties:
(i) $[\nabla f(x), x] = 0$
(ii) $[\nabla f(x), \nabla g(x)] = 0$
(iii) $\nabla f(\mathrm{Ad}h(x)) = \mathrm{Ad}h(\nabla f(x))$
(iv) $\frac{d}{dt}\big|_{t=0} \nabla f(x + t[y, x]) = [y, \nabla f(x)]$
for all $x, y \in \mathfrak{g}, h \in G$.

Proof. We first prove (iii). For all $x, y \in \mathfrak{g}$ one has:

$$\begin{aligned}
&< \nabla f(\mathrm{Ad}h(x)), y > \\
&= \tfrac{d}{dt}\big|_{t=0} f(\mathrm{Ad}h(x) + ty) \\
&= \tfrac{d}{dt}\big|_{t=0} f(x + t\mathrm{Ad}h^{-1}(y)) \\
&=< \nabla f(x), \mathrm{Ad}h^{-1}(y) > \\
&=< \mathrm{Ad}h(\nabla f(x)), y >
\end{aligned}$$

and so the result follows. Now (iv) is nothing but the infinitesimal form of (iii), (i) follows from (iv) by taking $y = x$ and (ii) follows from (iv) and (i) by taking $y = \nabla g(x)$ in (iv). □

Example 4.10. Let $\varphi : \mathfrak{g} \to \mathrm{End}\, V$ be a finite-dimensional representation of $\mathfrak{g}$ and define

$$f_k(x) = \frac{1}{k}\mathrm{tr}[\varphi(x)^k], \quad k \geq 1 \tag{4.10}$$

then the f_k are Ad-invariant functions and are homogeneous polynomials of degree k.

Now consider the Lie-Poisson bracket on $\mathcal{C}^\infty(\mathfrak{g})$

$$\{f, g\}(x) =< x, [F, G] > \tag{4.11}$$

where $f, g \in \mathcal{C}^\infty(\mathfrak{g}), x \in \mathfrak{g}$ and $F = \nabla f(x), G = \nabla g(x)$. The structure functions are given by

$$W^{jk} = \{x^j, x^k\} = c_l^{jk} x^l \tag{4.12}$$

where c_l^{jk} are the structure constants of $\mathfrak{g}$ with respect to a basis $\{T_j\}$, $\{\hat{T}_j\}$ is the dual basis and $x = x^j \hat{T}_j$ and so the Poisson bracket becomes in coordinate form:

$$\{f, g\} = c_l^{jk} x^l \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^k} \tag{4.13}$$

where we have used the sommation convention.
The map $B(x) : \mathfrak{g} \to \mathfrak{g}$ is given by

$$B(x)(y) = \operatorname{ad} x(y), \quad y \in \mathfrak{g} \tag{4.14}$$

and so the Hamiltonian vector field is given by

$$v_f(x) = [x, \nabla f(x)] \tag{4.15}$$

or in coordinate form:

$$v_f(x) = c_l^{jk} x^l \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^k} \tag{4.16}$$

and Hamilton's equations become:

$$\dot{x}^k = c_l^{jk} x^l \frac{\partial f}{\partial x^j} \tag{4.17}$$

From (4.14) one also sees that rank $B(x) = \dim O_x$, where O_x is the AdG-orbit through x. If $\mathfrak{g}$ is semisimple and x is regular, then the corresponding orbit is called a generic orbit, otherwise it is called a singular orbit (see [28], [36] and [37] for the non-semisimple case).
The following theorem has much to do with Theorem 2.23 and precizes it in this particular case.

Theorem 4.11. [28] The Lie-Poisson bracket has the following properties:
(i) the Ad-invariant functions are Casimir functions.
(ii) the symplectic leaves are the orbits of Ad G on $\mathfrak{g}$ and are the common level sets of the Ad-invariant functions.
(iii) for all $h \in G$, Ad h is a linear Poisson automorphism, which preserves the leaves of the symplectic foliation.
(iv) if $\lambda = \kappa(y) \in \mathfrak{g}^*$ then $v_\lambda = -\operatorname{ad} y$ and

$$\exp(t v_\lambda(x)) = \operatorname{Ad}(\exp(-ty))(x) \tag{4.18}$$

so the flow of v_λ is the orbit of $\exp(-ty)$ through x.

Now consider the more general situation where $\mathfrak{g}$ is a double Lie algebra. Then there is the following theorem:

Theorem 4.12. [12] Let $\mathfrak{g}_R$ be a double Lie algebra with Poisson bracket

$$\{f, g\}(x) = < x, [F, G]_R > \tag{4.19}$$

where $[F, G]_R$ as in (4.4), then:

(i) the Ad-invariant functions on $\mathfrak{g}$ are in involution with respect to (4.19)
(ii)

$$v_f(x) = \tfrac{1}{2}[x, R(F)] + \tfrac{1}{2}R^*[x, F] \tag{4.20}$$

(iii) if f is Ad-invariant then:

$$v_f(x) = \tfrac{1}{2}[x, R(F)] \tag{4.21}$$

(iv)

$$\begin{aligned} B(x) &= \operatorname{ad} x \circ R + R^* \circ \operatorname{ad} x \\ &= \tfrac{1}{2}[\operatorname{ad} x, R] \text{ if } R \text{ is unitary} \end{aligned} \tag{4.22}$$

Proof. (i) follows directly from Lemma 4.9, (ii) and (iv) follow directly from (4.19) and (4.4) and (iii) follows from Lemma 4.9 and (ii). □

So in the case of a double Lie algebra the Ad-invariant functions are in involution and if x is regular they are also functionally independent. So to construct integrable systems one has to restrict oneselves to a symplectic submanifold of dimension $2l$.

Remark 4.13. Theorem 4.11 and 4.12 can be formulated more generally in the case of (4.1) without identifying $\mathfrak{g}$ and $\mathfrak{g}^*$. Instead of the adjoint representation one has to use the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. If $\mathfrak{g}$ has a nondegenerate invariant symmetric bilinear form, then the adjoint and coadjoint representations are equivalent (see [12] and [13] for more details).

4.3. Poisson automorphisms of double Lie algebras.

Definition 4.14. A Yang-Baxter operator R is called invariant with respect to $x \in \mathfrak{g}$ if (in the ring of endomorphisms of the vector space $\mathfrak{g}$)

$$R \circ \operatorname{ad} x - \operatorname{ad} x \circ R = [R, \operatorname{ad} x] = 0 \tag{4.23}$$

If R is unitary this implies $B(x) = 0$ so $v_f(x) = 0$, which means that x is a fixed point.

Lemma 4.15. If R is a Yang-Baxter operator and $\theta \in \operatorname{Aut} \mathfrak{g}$ a Lie algebra automorphism of $\mathfrak{g}$ then $\tilde{R} := \theta R \theta^{-1}$ is again a Yang-Baxter operator and R and $\tilde{R}$ are called equivalent.

Proof. Straightforward. □

Definition 4.16. R is called invariant with respect to $\theta \in \operatorname{Aut} \mathfrak{g}$ if $R = \theta R \theta^{-1}$.

Proposition 4.17. If R is invariant with respect to $\theta \in \operatorname{Aut}\mathfrak{g}$ then $\theta \in \operatorname{Aut}\mathfrak{g}_R$, θ is a linear Poisson automorphism of (4.19) and

$$B(\theta(x)) = \theta B(x)\theta^{-1} \tag{4.24}$$

Proof. Straightforward. □

Example 4.18. If R is invariant with respect to x then R is invariant with respect to $\theta = \operatorname{Ad}(\exp(tx))$ and this is an inner Poisson automorphism.

Example 4.19. If $R = \mathrm{id}$ then it is invariant with respect to all $\theta \in \operatorname{Aut}\mathfrak{g}$.

One can generalize this to the situation in (4.1) and then one has:

Proposition 4.20. If $\theta \in \operatorname{Aut}\mathfrak{g}$ satisfies

$$\theta\phi(x,y) = \phi(\theta(x), \theta(y)) \tag{4.25}$$

where ϕ as defined in section 4.1, then $\theta \in \operatorname{Aut}(\mathfrak{g}, \phi)$ and θ is a linear Poisson automorphism of (4.1) and if $\theta = \operatorname{Ad}\exp(tz)$ is an inner automorphism of $\mathfrak{g}$ then (4.25) implies the infinitesimal identity

$$[z, \phi(x,y)] = \phi([z,x], y) + \phi(x, [z,y]) \tag{4.26}$$

If one views $\phi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ as a 2-form with values in $\mathfrak{g}$ then (4.26) is nothing else but

$$L_z\phi = 0 \tag{4.27}$$

which means that $\operatorname{ad} z$ is a derivation of the Lie algebra $(\mathfrak{g}, \phi)$.
Now let $\theta = \operatorname{Ad}\exp(tz)$ be a Poisson automorphism, θ^* the pull-back and define

$$A(f)(x) = \tfrac{d}{dt}\big|_{t=0}\, \theta^*(f)(x) = df(x)([z,x]) \tag{4.28}$$

Then $A : \mathcal{C}^\infty(\mathfrak{g}) \to \mathcal{C}^\infty(\mathfrak{g})$ is an infinitesimal endomorphism of $\mathfrak{g}$, as one checks easily. So if f is Ad-invariant, then $\theta^*(f) = f$ and $A(f) = 0$.

4.4. Properties of Yang-Baxter operators.

Now we are going to study Yang-Baxter operators in more detail.

Proposition 4.21. [12] Let $R \in \operatorname{End}\mathfrak{g}$ be a solution of (4.7) then
(i) $\frac{1}{2}R : \mathfrak{g}_R \to \mathfrak{g}$ is a Lie algebra homomorphism.
(ii) $\operatorname{Im} R$ is a subalgebra of $\mathfrak{g}$ and ker R is an abelian ideal of $\mathfrak{g}_R$.
(iii)

$$
\begin{aligned}
[\operatorname{Im} R, \operatorname{Ker} R] &\subset \operatorname{Ker} R \\
[\operatorname{Im} R, (\operatorname{Im} R)^{\perp}] &\subset (\operatorname{Im} R)^{\perp} \\
[\operatorname{Im} R, (\operatorname{Ker} R)^{\perp}] &\subset (\operatorname{Ker} R)^{\perp} \\
[\operatorname{Ker} R, (\operatorname{Ker} R)^{\perp}] &\subset (\operatorname{Im} R)^{\perp}
\end{aligned}
\tag{4.29}
$$

(iv) $(\operatorname{Ker} R)^{\perp}$ is a Poisson submanifold of $\mathfrak{g}$.

Proof. (i) and (ii) follow immediately from (4.4) and (4.7); (iii) follows from (4.7), (ii) and the properties of the scalar product; (iv) follows from (iii), (4.20) and Corollary 2.28 , because $\operatorname{Im} R^* = (\operatorname{Ker} R)^{\perp}$.

Proposition 4.22. Let $R \in \operatorname{End}\mathfrak{g}$ be a solution of (4.8) and define

$$
R_{\pm} = \tfrac{1}{2}(R \pm \mathrm{id}), \quad \mathfrak{g}_{\pm} = \operatorname{Im} R_{\pm}, \quad \mathfrak{k}_{\pm} = \operatorname{Ker} R_{\mp} \tag{4.30}
$$

then:
(i) $R_{\pm} : \mathfrak{g}_R \to \mathfrak{g}$ are Lie algebra homomorphisms.
(ii) $\mathfrak{g}_{\pm}$ are subalgebras of $\mathfrak{g}$
(iii) $\mathfrak{k}_{\pm}$ are ideals of $\mathfrak{g}_R$
(iv) Ker R is abelian in $\mathfrak{g}$
(v) $[\mathfrak{g}_{\pm}, \mathfrak{k}_{\pm}] \subset \mathfrak{k}_{\pm}$
(vi) $[\mathfrak{k}_{+}, \mathfrak{k}_{-}]_R = 0$
(vii) $\mathfrak{k}_{-}^{\perp}$ and $\mathfrak{k}_{+}^{\perp}$ are Poisson submanifolds of $\mathfrak{g}$
(viii) $\mathfrak{k}_{\pm}$ are subalgebras of $\mathfrak{g}$
(ix) $[\mathfrak{g}, \operatorname{Ker} R] \subset \operatorname{Im} R$

Proof. (i), (ii) and (iii) are direct consequences of the fact that one can rewrite (4.8) as:

$$
R_{\pm}[x,y]_R = [R_{\pm}(x), R_{\pm}(y)] \tag{4.31}
$$

(iv) follows directly from (4.8) and (v) and (vi) follow from the fact that:

$$
[x,y]_R = [R_{+}(x), y] + [x, R_{-}(y)] \tag{4.32}
$$

Now one has

$$
v_f(x) = [x, R_{+}(F)] + R_{-}^{*}[x,F] = [x, R_{-}(F)] + R_{+}^{*}[x,F] \tag{4.33}
$$

and so one can write

$$B(x) = \operatorname{ad} x \circ R_- + R_+^* \circ \operatorname{ad} x \tag{4.34}$$

Because

$$\operatorname{Im} R_\pm^* = (\operatorname{Ker} R_\pm)^\perp = \mathfrak{k}_\mp^\perp \tag{4.35}$$

and (v) implies

$$[\mathfrak{k}_+^\perp, \mathfrak{g}_+] \subset \mathfrak{k}_+^\perp, \quad [\mathfrak{k}_-^\perp, \mathfrak{g}_-] \subset \mathfrak{k}_-^\perp \tag{4.36}$$

(vii) follows from (4.33) and Corollary 2.28. Now suppose $x, y \in \operatorname{Ker} R_-$, so $R(x) = x, R(y) = y$, then using (4.8) one gets

$$[x, y] = 2R[x, y] - [x, y]$$

and so $R_-[x, y] = 0$. Finally (ix) follows directly from (4.8). □

An important property of Yang-Baxter operators is the following:

Proposition 4.23. Suppose that $\mathfrak{g}$ has a nondegenerate invariant symmetric bilinear form and $R \in \operatorname{End} \mathfrak{g}$ is a Yang-Baxter operator. Define

$$R^+ = \tfrac{1}{2}(R + R^*), \quad R^- = \tfrac{1}{2}(R - R^*) \tag{4.37}$$

where R^* is the adjoint of R with respect to the bilinear form. Suppose that R^+ satisfies (4.7) and

$$R[R^+(x), y] = [R^+(x), R(y)] \tag{4.38}$$

for all $x, y \in \mathfrak{g}$, which says that R is Im R^+-invariant, then R^- is a unitary Yang-Baxter operator and Im R^+ is abelian.

Proof. From (4.38) one easily derives, using the properties of the bilinear form:

$$R^*[R^+(x), y] = [R^+(x), R^*(y)] \tag{4.39}$$

and so also

$$R^+[R^+(x), y] = [R^+(x), R^+(y)] \tag{4.40}$$

$$R^-[R^+(x), y] = [R^+(x), R^-(y)] \tag{4.41}$$

From (4.41) one obtains the dual form, using the bilinear form:

$$R^+[R^-(x), y] + R^+[x, R^-(y)] = 0 \tag{4.42}$$

Now combine (4.40) and (4.7) to obtain

$$[R^+(x), R^+(y)] = 0 \tag{4.43}$$

so Im R^+ is abelian. Now a straightforward calculation, using (4.7), (4.40), (4.41) and (4.42), shows that R^- is a Yang-Baxter operator. □

Remark 4.24. If (4.38) is true then:

$$\operatorname{Im} R^+ \text{ is abelian} \iff R^+ \text{ satisfies (4.7)}$$

because of (4.40).

An important and interesting solution of (4.8) is given by the following

Proposition 4.25. Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ (vector space direct sum), where $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras, let π_a and π_b denote the corresponding projections and define

$$R = \pi_a - \pi_b \tag{4.44}$$

then R satisfies (4.8), and in this case one has:

$$[x, y]_R = [\pi_a(x), \pi_a(y)] - [\pi_b(x), \pi_b(y)] \tag{4.45}$$

and

$$R_+ = \pi_a, \quad R_- = -\pi_b, \quad \mathfrak{g}_+ = \mathfrak{k}_+ = \mathfrak{a}, \quad \mathfrak{g}_- = \mathfrak{k}_- = \mathfrak{b} \tag{4.46}$$

Proof. Straightforward calculation. □

Corollary 4.26. Let $\mathfrak{n}_a$ denote the normalizer of $\mathfrak{a}$ in $\mathfrak{b}$, then R is invariant with respect to $\mathfrak{n}_a$, where $\mathfrak{a}$, $\mathfrak{b}$ and R are as in Prop. 4.25.

Proof. Suppose $x \in \mathfrak{n}_a \subset \mathfrak{b}$ then $[x, a] \in \mathfrak{a}$ for all $a \in \mathfrak{a}$. So for all $y \in \mathfrak{g}$:

$$\begin{aligned}
&R[x, y] - [x, R(y)] \\
=&R[x, \pi_a y + \pi_b y] - [x, R(\pi_a y + \pi_b y)] \\
=&R[x, \pi_a y] + R[x, \pi_b y] - [x, \pi_a y - \pi_b y] \\
=&[x, \pi_a y] - [x, \pi_b y] - [x, \pi_a y] + [x, \pi_b y] = 0
\end{aligned}$$

□

Corollary 4.27. R is x-invariant $\iff$ $\pi_a(x) \in \mathfrak{n}_b$ and $\pi_b(x) \in \mathfrak{n}_a$.

Proof. Straightforward. □

Example 4.28. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ be the triangular root space decomposition as defined in Chapter 3. Then $\mathfrak{a} = \mathfrak{g}_+ \oplus \mathfrak{g}_0$ and $\mathfrak{b} = \mathfrak{g}_-$ are subalgebras and so

$$R = \pi_+ + \pi_0 - \pi_- \tag{4.47}$$

is a Yang-Baxter operator, where π_0, π_+ and π_- denote the projections on the corresponding subspaces. Now $\pi_0^* = \pi_0, \pi_+^* = \pi_-, \mathfrak{a}^\perp = \mathfrak{g}_+, \mathfrak{b}^\perp = \mathfrak{g}_- \oplus \mathfrak{g}_0$ and

$$R^* = \pi_- + \pi_0 - \pi_+ \tag{4.48}$$

and so $R^- = \pi_+ - \pi_-$ and $R^+ = \pi_0$. If $\mathfrak{g}$ is quasi-split then $\mathfrak{g}_0$ is abelian, R^+ satisfies (4.7), and the commutation relations (3.9) imply that R^* and R satisfy (4.38), so $R^- = \pi_+ - \pi_-$ is a unitary Yang-Baxter operator. Observe that $R^2 = \mathrm{id}$, but $(R^-)^2 \neq \mathrm{id}$!

4.5. The K.A.S.R.S. theorem.

An important connection between solutions of (4.8) of the form (4.44) and integrable systems is given by the following theorem.

Theorem 4.29. [12] (Factorization theorem of Kostant-Adler-Symes-Reyman-Semenov-Tian-Shansky) Let $\mathfrak{g}$ be a real Lie algebra with a nondegenerate invariant symmetric bilinear form and identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using this form. Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ with $\mathfrak{a}$ and $\mathfrak{b}$ subalgebras. Let $R = \pi_a - \pi_b$ be the corresponding Yang-Baxter operator and $\{\ ,\ \}$ the corresponding Lie-Poisson bracket (4.19). Then one has (4.45) and (4.46). Let $h \in C^\infty(\mathfrak{g})$ be an Ad-invariant function, with $H = \nabla h(x)$ and consider Hamilton's equations with Hamiltonian h:

$$\dot{x} = v_h(x) = [x, \pi_a(H)] = -[x, \pi_b(H)] \tag{4.49}$$

Let A and B be the connected subgroups of G corresponding to the subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ and let $(g_a(t), g_b(t))$ be the solution (for small t) of the factorization problem

$$\exp[-tH(0)] = g_b(t)^{-1} g_a(t) \tag{4.50}$$

where $H(0) = \nabla h(x(0))$, then the solution of the Lax equation (4.49) is given by:

$$x(t) = g_b(t)x(0)g_b(t)^{-1} = g_a(t)x(0)g_a(t)^{-1} \tag{4.51}$$

Proof. Differentiating (4.51) gives

$$\dot{x} = [\dot{g_b} {g_b}^{-1}, x] = [\dot{g_a} {g_a}^{-1}, x] \tag{4.52}$$

and

$$H(t) = \nabla h(g_b(t)x(0)g_b(t)^{-1}) = g_b(t)H(0)g_b(t)^{-1} \tag{4.53}$$

where we have used Lemma 4.9. Now differentiate (4.50):

$$\begin{aligned}
\dot{g}_b(t)g_b(t)^{-1} &= \frac{d}{dt}(g_a(t) \exp tH(0))g_b(t)^{-1} \\
&= \dot{g}_a(t)\exp(tH(0))g_b(t)^{-1} + g_a(t)\exp(tH(0))H(0)g_b(t)^{-1} \\
&= \dot{g}_a(t)g_a(t)^{-1}g_b(t)g_b(t)^{-1} + g_b(t)H(0)g_b(t)^{-1} \\
&= \dot{g}_a(t)g_a(t)^{-1} + H(t)
\end{aligned} \tag{4.54}$$

and so $H = \dot{g}_b {g_b}^{-1} - \dot{g}_a {g_a}^{-1}$, which implies

$$\pi_a(H) = -\dot{g}_a {g_a}^{-1}, \quad \pi_b(H) = \dot{g}_b {g_b}^{-1} \tag{4.55}$$

because $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras, and substituting this in (4.52) gives (4.49).

□

So the solution of the Lax equation (4.49) is reduced to a factorization problem in the Lie group G. This is the finite-dimensional group-theoretical analogue of the Riemann-Hilbert problem in the case of partial differential equations.

From Prop. 4.22 it follows that one can restrict (4.49) to the Poisson submanifold $\mathfrak{a}^\perp$ or $\mathfrak{b}^\perp$.

Remark 4.30. One can formulate Theorem 4.29 for any real Lie algebra, without identifying $\mathfrak{g}$ and $\mathfrak{g}^*$, using the coadjoint representation (see [12]).

We shall now consider two applications of Theorem 4.29.

Example 4.31. (The finite nonperiodic Toda lattice) Let $\mathfrak{g}$ be a normal real form of a simple complex Lie algebra, then $\dim \mathfrak{m} = 0$ and $m_\alpha = 1$ for all $\alpha \in R$. Choose $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ with normalization (3.15). Now consider the following decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l} \text{ (Iwasawa decomposition)} \tag{4.56}$$

where $\mathfrak{l} = \mathfrak{g}_0 \oplus \mathfrak{g}_-$ (Borel subalgebra) and $\mathfrak{k}$ is the maximal compact subalgebra as defined in chapter 3. Then one has in terms of the projections π_0, π_+, π_- and the Cartan involution θ:

$$\begin{aligned} \pi_k &= \pi_+ + \theta\pi_+ \\ \pi_l &= \pi_0 + \pi_- - \theta\pi_+ \end{aligned} \tag{4.57}$$

Also

$$\mathfrak{g} = \mathfrak{k}^\perp \oplus \mathfrak{l}^\perp \tag{4.58}$$

where $\mathfrak{k}^\perp = \mathfrak{p}$ and $\mathfrak{l}^\perp = \mathfrak{g}_-$ and one has

$$\pi_0^* = \pi_0, \quad \pi_+^* = \pi_-, \quad \theta^* = \theta \tag{4.59}$$

so

$$\begin{aligned} \pi_k^* &= \pi_{l^\perp} = \pi_- + \pi_-\theta \\ \pi_l^* &= \pi_p = \pi_0 + \pi_+ - \pi_-\theta \end{aligned} \tag{4.60}$$

Now the Yang-Baxter operator $R = \pi_k - \pi_l$ becomes:

$$R = \pi_+ + 2\theta\pi_+ - \pi_0 - \pi_- \tag{4.61}$$

and

$$R^* = \pi_- + 2\pi_-\theta - \pi_0 - \pi_+ \tag{4.62}$$

but $\theta\pi_+ = \pi_-\theta$ and so one gets

$$R^- = \pi_+ - \pi_-, \quad R^+ = 2\theta\pi_+ - \pi_0 \tag{4.63}$$

so one sees that the skew-symmetric part of R is the unitary Yang-Baxter operator of Example 4.28. Now take as phase space the Poisson submanifold $\mathfrak{p}$ and as Hamiltonian $h(x) = \frac{1}{2} < x, x >$ then $\nabla h(x) = x$ and Hamilton's equations become

$$\dot{x} = [x, \pi_k(x)] \tag{4.64}$$

with respect to the Yang-Baxter operator R. But because $x \in \mathfrak{p}$ one has $\theta(x_+) = -x_-$ and so

$$\pi_k(x) = x_+ + \theta(x_+) = x_+ - x_- = R^-(x) \tag{4.65}$$

Now one gets the Toda lattice by restricting the equations (4.64) to the Poisson submanifold O_μ, consisting of the Ad$^*L$-orbit through the element μ, defined by

$$\mu = \sum_{\alpha\in\Delta} g_\alpha(e_\alpha + e_{-\alpha}), \quad 0 < g_\alpha \in \mathbb{R} \tag{4.66}$$

Here we have identified $\mathfrak{l}^*$ and $\mathfrak{k}^\perp = \mathfrak{p}$ via the Killing form, and L is the connected subgroup of G with Lie algebra $\mathfrak{l}$. A general element of O_μ can be written as:

$$L = P + \sum_{\alpha\in\Delta} g_\alpha \exp\alpha(Q)(e_\alpha + e_{-\alpha}) \tag{4.67}$$

where $P, Q \in \mathfrak{a}$. The Hamiltonian H is given by

$$H = \tfrac{1}{2} < P, P > + \sum_{\alpha\in\Delta} \frac{2g_\alpha^2}{(\alpha,\alpha)} \exp 2\alpha(Q) \tag{4.68}$$

and the Lax equation (4.64) becomes

$$\dot{L} = [L, M] \tag{4.69}$$

where

$$M = L_+ - L_- = \sum_{\alpha\in\Delta} g_\alpha \exp\alpha(Q)(e_\alpha - e_{-\alpha}) \tag{4.70}$$

If one specializes this to the case $\mathfrak{g} = sl(n, \mathbb{R})$, one gets:

$$Q = \mathrm{diag}(q_1, \ldots, q_n), \quad P = \mathrm{diag}(p_1, \ldots, p_n)$$

$$\mu = \sum_{j=1}^{n-1} g_j(e_{j,j+1} + e_{j+1,j}), \quad g_j > 0$$

$$H = \tfrac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} g_j^2 \exp 2(q_j - q_{j+1})$$

$$L = P + \sum_{j=1}^{n-1} g_j \exp(q_j - q_{j+1})(e_{j,j+1} + e_{j+1,j})$$

$$M = \sum_{j=1}^{n-1} g_j \exp(q_j - q_{j+1})(e_{j,j+1} - e_{j+1,j}) \tag{4.71}$$

and Hamilton's equations become

$$\dot{q}_j = p_j, \quad \dot{p}_j = 2g_{j-1}^2 \exp 2(q_{j-1} - q_j) - 2g_j^2 \exp 2(q_j - q_{j+1}) \tag{4.72}$$

with the convention that $q_0 = -\infty$, $q_{n+1} = \infty$. In this case the orbit O_μ consists of the socalled Jacobi matrices. These have simple spectrum and the factorization in $SL(n, \mathbb{R})$ in this case is known as the ortho-triangular decomposition or QL factorization (see [8],[9],[10] and [25] for more details).
One can also take the Yang-Baxter operator R^-, because the Jacobi matrices are again a Poisson submanifold (but not the subspace $\mathfrak{p}$!). One gets the same Lax equation, but the factorization is somewhat more difficult. Observe that the Lie-brackets $[\ ,\ ]_R$ and $[\ ,\ ]_{R^-}$ are non-isomorphic and that

$$\theta R^- \theta = -R^- \tag{4.73}$$

and so

$$\theta[x, y]_{R^-} = -[\theta(x), \theta(y)]_{R^-} \tag{4.74}$$

so θ is an anti-automorphism of $\mathfrak{g}_{R^-}$.

Example 4.32. (Harmonic oscillator) Consider the decomposition ($\mathfrak{g}$ is a real semisimple Lie algebra)

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{b} \tag{4.75}$$

where $\mathfrak{n} = \mathfrak{g}_+$ and $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_-$ (Borel subalgebra) and the corresponding Yang-Baxter operator (cf. Example 4.28)

$$R = \pi_n - \pi_b = \pi_+ - \pi_0 - \pi_- \tag{4.76}$$

and $\mathfrak{g} = \mathfrak{n}^\perp \oplus \mathfrak{b}^\perp$, where $\mathfrak{n}^\perp = \mathfrak{g}_+ \oplus \mathfrak{g}_0$, $\mathfrak{b}^\perp = \mathfrak{g}_-$ and also $R_+ = \pi_n$, $R_- = -\pi_b$, so the Hamiltonian vector field on $\mathfrak{n}^\perp$ becomes

$$\begin{aligned} v_f(x) &= [x, \pi_n(F)] - \pi_b^*[x, F] \\ &= [x, \pi_+(F)] - (\pi_0 + \pi_+)[x, F] \end{aligned} \tag{4.77}$$

If one now takes as Hamiltonian the function $f(x) = \frac{1}{2} < x, x >$, then Hamilton's equations become

$$\dot{x}_0 = 0, \quad \dot{x}_+ = [x_0, x_+] \tag{4.78}$$

Now take $\mathfrak{g} = sl(n, \mathbb{C})$ and take as the phase space the elements $x \in \mathfrak{n}^{\perp}$ of the form:

$$x = i \sum_{j=1}^{n} \omega_j e_{jj} + \sum_{j<k} a_{jk} e_{jk} \tag{4.79}$$

where $\omega_j \in \mathbb{R}$, $a_{jk} = p_{jk} + i(\omega_j - \omega_k)q_{jk}$ and $p_{jk}, q_{jk} \in \mathbb{R}$, then

$$f(x) = \tfrac{1}{2}\mathrm{Re}[\mathrm{tr}(x^2)] = -\tfrac{1}{2} \sum_{j=1}^{n} \omega_j^2 \tag{4.80}$$

with $\sum_{j=1}^{n} \omega_j = 0$ and Hamilton's equations become:

$$\dot{\omega}_j = 0, \quad \dot{q}_{jk} = p_{jk}, \quad \dot{p}_{jk} = -(\omega_j - \omega_k)^2 q_{jk}, \quad (j < k) \tag{4.81}$$

These equations describe $\frac{1}{2}n(n-1)$ harmonic oscillators, which for $\mathfrak{g} = sl(2, \mathbb{C})$ reduce to the well-known one-dimensional harmonic oscillator , by taking $\omega_1 = -\omega_2 = \frac{1}{2}\omega$, $q_{12} = q$ and $p_{12} = p$. Factorizing $\exp[-tx(0)]$ one gets:

$$g_b(t) = g_m(t) = \mathrm{diag}(\exp it\omega_1, \ldots, \exp it\omega_n) \tag{4.82}$$

so

$$x(t) = \mathrm{Ad}(g_m(t))x(0) \tag{4.83}$$

is the solution of (4.81).
Of course in this special case the Lax equation (4.78) can be integrated immediately, because $\dot{x}_0 = 0$, so one does not need the factorization theorem. But one should observe that with respect to the given Poisson bracket p_{jk} and q_{jk} are not canonically conjugated variables. Instead, for example in the case of $n = 2$, one has

$$\{p, q\} = 2q^2, \quad \{\omega, p\} = 2\omega q, \quad \{\omega, q\} = -2\omega^{-1}p \tag{4.84}$$

and if one defines $P = p$ and $Q = -\frac{1}{2}q^{-1}$ then

$$\{P, Q\} = 1 \tag{4.85}$$

$Z = p^2 + \omega^2 q^2$ belongs to the center and the Hamiltonian is given by $H = -\frac{1}{4}\omega^2$.

Remark 4.33. One can also take $R = \pi_+ - \pi_-$, because then $\mathfrak{n}^{\perp}$ is again a Poisson submanifold, since $\mathrm{Ker}(R_-) = \mathfrak{n}$ and one gets the same Lax equation.

4.6. Lie bialgebras.

Consider the Poisson bracket on $\mathcal{C}^\infty(\mathfrak{g})$ as defined in (4.3). Because $\mathfrak{g}^*$ is a Lie subalgebra of $\mathcal{C}^\infty(\mathfrak{g})$, this induces a Lie algebra structure on $\mathfrak{g}^*$, which is isomorphic with the Lie algebra $(\mathfrak{g}, \phi)$, so we shall denote this bracket also by ϕ. Define the dual map $\phi^* : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ by

$$< \phi^*(x), y \otimes z > = < x, \phi(y, z) >, \quad x, y, z \in \mathfrak{g} \tag{4.86}$$

Definition 4.34. [47] The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called a Lie bialgebra, if ϕ^* is a one-cocycle with values in $\mathfrak{g} \otimes \mathfrak{g}$, i.e.:

$$d\phi^*(x, y) := \phi^*([x, y]) - x.\phi^*(y) + y.\phi^*(x) = 0 \tag{4.87}$$

where $\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by:

$$x.(y \otimes z) = [x, y] \otimes z + y \otimes [x, z], \quad x, y, z \in \mathfrak{g} \tag{4.88}$$

The connection between Lie bialgebras and integrable systems is given by the following Proposition:

Proposition 4.35. Let $h(t) = \exp ty \in G$ denote a one-parameter subgroup of G and let $\theta(t) = \mathrm{Ad}h(t)$. Let $f, g \in \mathcal{C}^\infty(\mathfrak{g})$ be Ad-invariant, then:

$$\begin{aligned} A\{f, g\}(x) &= d\{f, g\}(x)([y, x]) \\ &= < x, (L_y\phi)(F, G) > \\ &= < d\phi^*(y, x), F \otimes G > \end{aligned} \tag{4.89}$$

for all $x, y \in \mathfrak{g}$.

Proof.

$$\begin{aligned} \theta^*\{f, g\}(x) &= \{f, g\}(\mathrm{Ad}h(x)) \\ &= < \mathrm{Ad}h(x), \phi(\nabla f(\mathrm{Ad}h(x)), \nabla g(\mathrm{Ad}h(x))) > \\ &= < \mathrm{Ad}h(x), \phi(\mathrm{Ad}h(F), \mathrm{Ad}h(G)) > \end{aligned}$$

where we have used Lemma 4.9, and so

$$\begin{aligned} A\{f, g\}(x) &= \tfrac{d}{dt}\Big|_{t=0} \theta^*\{f, g\}(x) \\ &= < [y, x], \phi(F, G) > + < x, \phi([y, F], G) + \phi(F, [y, G]) > \end{aligned}$$

and this can be rewritten in the form (4.89). □

Proposition 4.36. Suppose $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra and G is a connected, simply connected Lie group, then the Ad-invariant functions form a Lie subalgebra of $\mathcal{C}^\infty(\mathfrak{g})$.

Proof. Let f, g be Ad-invariant functions, then, because $d\phi^* = 0$, Prop. 4.35 implies $A\{f, g\} = 0$, and this implies $\theta^*\{f, g\} = \{f, g\}$, so $\{f, g\}$ is Ad-invariant. □

The connection between Lie bialgebras and double Lie algebras is given by the following

Proposition 4.37. [12] A double Lie algebra $(\mathfrak{g}, R)$ is a Lie bialgebra iff:
(i) there is a nondegenerate invariant bilinear form on $\mathfrak{g}$
(ii) R is unitary.

Proposition 4.38. Suppose $\mathfrak{g}$ is a complex or real semisimple Lie algebra and $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then $\mathfrak{g}$ is also a double Lie algebra.

Proof. Because $\mathfrak{g}$ is real or complex semisimple each 1-cocycle is a coboundary, so ϕ^* is exact, which means that there exists a skew-symmetric tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$, such that $\phi^* = dr$. Here $dr(x) = [x \otimes \mathrm{id} + \mathrm{id} \otimes x, r]$. Now identify r with a unitary $R \in \operatorname{End} \mathfrak{g}$ via the Killing form, i.e.:

$$< r, x \otimes y >=< x, R(y) >, \quad x, y \in \mathfrak{g} \tag{4.90}$$

then one finds, using the invariance of the Killing form that

$$\phi(x, y) = -[R(x), y] - [x, R(y)]$$

so $(\mathfrak{g}, R)$ is a double Lie algebra. □

Remark 4.39. In this case the Ad-invariant functions form an abelian Lie subalgebra of $\mathcal{C}^\infty(\mathfrak{g})$, but in general this is not true.

Recall that $(\mathfrak{g}, R)$ is a double Lie algebra if R is a Yang-Baxter operator. In general this is not a necessary condition, but one has:

Proposition 4.40. [47] Suppose $\mathfrak{g}$ is a complex simple Lie algebra, $(\mathfrak{g}, R)$ is a double Lie algebra and R is unitary, then R is a Yang-Baxter operator.

Proof. Define the 3-form $< R, R >$ with values in $\mathbb{C}$ by:

$$< R, R > (x, y, z) =< x, [R(y), R(z)] > +\text{cycl.}, \quad \text{x, y, z} \in \mathfrak{g} \tag{4.91}$$

Because R is skew-symmetric $< R, R >$ is also skew-symmetric, and for all $x, y, z, u \in \mathfrak{g}$:

$$\begin{aligned} &L_u < R, R > (x, y, z) \\ &=< [u, x], [R(y), R(z)] > + < x, [R[u, y], R(z)] > \\ &+ < x, [R(y), R[u, z]] > +\text{cycl.} \\ &=< u, [z, [R(x), R(y)] - R[R(x), y] - R[x, R(y)]] + \text{cycl.} >= 0 \end{aligned}$$

because of (4.6) and (4.5). But if $\mathfrak{g}$ is a complex simple Lie algebra, the only ad-invariant 3-form is $\alpha < x, [y, z] >$, with $\alpha \in \mathbb{C}$, so

$$< R, R > (x, y, z) = \alpha < x, [y, z] > \tag{4.92}$$

and rewriting this yields

$$R[R(x), y] + R[x, R(y)] - [R(x), R(y)] = \alpha[x, y]$$

so R is a Yang-Baxter operator. □

Remark 4.41. The 3-form $< R, R >$ is known as the Schouten bracket of R with itself (see [48]).

Proposition 4.42. Suppose ϕ^* is $\mathrm{ad}\, z$-invariant with $z \in \mathfrak{g}$, which means $\phi^*([z,x]) = z.\phi^*(x)$ for all $x \in \mathfrak{g}$, then $\theta(t) = \mathrm{Ad}\exp(tz)$ is a Poisson automorphism of (4.3).

Proof. This is nothing but the dual version of Prop. 4.20. $\square$

5. Hamiltonian systems of type I–V on Lie algebras

5.1. Introduction.

In this chapter we are going to construct Hamiltonian systems of type I–V for all root systems. This will be done by defining a phase space and a Poisson bracket on it. Also we shall derive a Lax equation in the case of the classical root systems and give a Lie algebraic proof of the integrability for the systems of type I, II, and III.

Let R be a root system in an n-dimensional Euclidian vector space E and let Δ, R_+ and $(\ , \ )$ be defined as in chapter 3. Let $\alpha \mapsto g_\alpha$ be a Weyl group-invariant mapping of R into $[0, \infty)$. This implies that $g_\alpha = g_\beta$ if $(\alpha, \alpha) = (\beta, \beta)$. Let $p, q \in E$ and $q_\alpha = (\alpha, q)$ and define:

$$H = \tfrac{1}{2}(p,p) + \sum_{\alpha \in R_+} {g_\alpha}^2 v(q_\alpha) \tag{5.1}$$

with $v(x)$ as in (0.3). Then H is determined by the root system R.
Let $\{d_j, j = 1, \ldots, n\}$ be an orthonormal basis of E; denote by p_j, q_j the components of p resp. q w.r.t. this basis and by α_j the components of α. Define the configuration space Λ by:

$$\Lambda = \{q \in E \mid q_\alpha > 0, \alpha \in R_+\} \text{ (positive Weyl chamber)} \tag{5.2}$$

for type I, II and V, and

$$\Lambda = \{q \in E \mid q_\alpha > 0, \alpha \in R_+, \Omega^*(q) < d\} \text{ (Weyl alcove)} \tag{5.3}$$

for type III and IV. Here is Ω the maximal positive root, $d = \pi/a$ for type III and $d = 2\omega/a$ for type IV. So Λ consists of regular elements. The phase space is now defined by $M = \Lambda \times E$. Take the canonical Poisson bracket on $M \subset E \times E = \mathbb{R}^{2n}$; i.e.

$$\{f, g\} = \sum_j \left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} \right) \tag{5.4}$$

for $f, g \in \mathcal{C}^\infty(M)$. Then Hamilton's equations become:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = p_j$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = -\sum_{\alpha \in R_+} {g_\alpha}^2 v'(q_\alpha)\alpha_j \tag{5.5}$$

These equations are the generalizations of (1.6) for abstract root systems.
If $v(x)$ is of type I–IV then, using (1.48) and (1.49), (5.5) becomes:

$$\dot{q}_j = p_j, \quad \dot{p}_j = 2 \sum_{\alpha \in R_+} {g_\alpha}^2 x(q_\alpha) y(q_\alpha) \alpha_j \tag{5.6}$$

where the functions x and y are defined in (1.49) and (1.54).

5.2. The Weierstrass $\wp$-function.

Now consider the Weierstrass function $\wp(z;\omega_1,\omega_2)$ with half-periods ω_1 and ω_2. It satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \tag{5.7}$$

The constants $g_2, g_3 \in \mathbb{C}$ are called the invariants of $\wp$ and should not be confused with the coupling constants g_α. The discriminant Δ is defined by

$$\Delta = {g_2}^3 - 27{g_3}^2 \tag{5.8}$$

Let e_1, e_2, e_3 denote the roots (in $\mathbb{C}$) of the cubic equation

$$4z^3 - g_2 z - g_3 = 0 \tag{5.9}$$

then Δ can be expressed in terms of e_1, e_2, e_3 by

$$\Delta = 16(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2 \tag{5.10}$$

Thus all roots are different if $\Delta \neq 0$. Also

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_1+\omega_2), \quad e_3 = \wp(\omega_2) \tag{5.11}$$

and of course the e_1, e_2, e_3 satisfy the symmetric function relations:

$$e_1+e_2+e_3 = 0, \quad e_1e_2e_3 = 1/4g_3, \quad e_1e_2+e_2e_3+e_1e_3 = -1/4g_2 \tag{5.12}$$

The type IV case corresponds with $\Delta \geq 0$. Then e_1, e_2, e_3 are real and so are g_2, g_3. If $\Delta > 0$ the roots are numbered in such a way that $e_1 > e_2 > e_3$ and there is the following correspondence between tuples $(\wp, \omega_1, \omega_2, g_2, g_3, e_1, e_2, e_3)$ on the one side and tuples $(x(\eta), k, k', K, K', b, c)$ on the other side (with $x(\eta)$ of type IVc):

$$v(\eta) = a^2\wp(a\eta;\omega_1,\omega_2) = x(\eta)^2 - 1/3a^2(k^2+1) \tag{5.13}$$

$$\begin{gathered} e_1 = 1/3(2-k^2), \quad e_2 = 1/3(2k^2-1), \quad e_3 = -1/3(k^2+1) \\ \Delta = 16k^4(1-k^2)^2, \quad b = \tfrac{1}{2}a^2(k^2+1), \quad c = a^4k^2 \\ g_2 = 4/3\left((1+k^2)^2 - 3k^2\right) > 0 \\ g_3 = 4/27(1+k^2)(2-k^2)(1-2k^2) \end{gathered} \tag{5.14}$$

and $\omega_1 = K$, $\omega_2 = iK'$.
Let z_1, z_2 be the roots of the equation

$$z^2 - 2bz + c = 0 \tag{5.15}$$

and $D = 4(b^2 - c)$ its discriminant, then

$$z_1 = a^2, \quad z_2 = a^2k^2, \quad D = a^4(1-k^2)^2 \tag{5.16}$$

and so $z_1 > z_2 > 0$ because $0 < k^2 < 1$ and also

$$e_1 = 1/3a^{-2}(2z_1 - z_2), \quad e_2 = 1/3a^{-2}(2z_2 - z_1), \quad e_3 = -1/3a^{-2}(z_1 + z_2)$$

$$\Delta a^{12} = 16Dc^2 \tag{5.17}$$

From (5.14) one sees that indeed $e_1 > e_2 > e_3$ and moreover $e_1 > 0,\ e_3 < 0$ and

$$\begin{aligned} e_2 &= 0 \text{ if } k^2 = \tfrac{1}{2} \\ &< 0 \text{ if } 0 < k^2 < \tfrac{1}{2} \\ &> 0 \text{ if } \tfrac{1}{2} < k^2 < 1 \end{aligned} \tag{5.18}$$

Also $g_2 > 0$ because $0 < k^2 < 1$ and

$$\begin{aligned} g_3 &= 0 \text{ if } e_2 = 0 \\ &> 0 \text{ if } e_2 < 0 \\ &< 0 \text{ if } e_2 > 0 \end{aligned} \tag{5.19}$$

The type II and III cases correspond with a limit situation in which $\Delta = 0$, so $k^2 = 0$ or $k^2 = 1$.

Type II:

$k = 1, k' = 0, e_1 = e_2 = 1/3, e_3 = -2/3, g_2 = 4/3, g_3 = -8/27, q = 1, K = \infty, K' = (1/2)\pi, b = a^2, c = a^4, D = 0$

$$a^2\wp(a\eta) = a^2\coth^2 a\eta - 2/3a^2 \tag{5.20}$$

so one gets the type IIa case and the IIIa case can be obtained by the substitution $a \rightarrow ia$.

Type III:

$$k = 0, k' = 1, e_1 = 2/3, e_2 = e_3 = -1/3, K = 1/2\pi, K' = \infty, q = 0$$

$$g_2 = 4/3, g_3 = 8/27, b = 1/2a^2, c = 0, D = a^4$$

$$a^2\wp(a\eta) = a^2\sin^{-2}(a\eta) - 1/3a^2 \tag{5.21}$$

so one gets the type IIIb case and the IIb case by the substitution $a \rightarrow ia$. The type I potential can be obtained by taking the limit $a \rightarrow 0$ and in this case

$$g_2 = g_3 = 0, e_1 = e_2 = e_3 = 0, K = K' = \infty, b = c = D = \Delta = 0 \tag{5.22}$$

This can be derived by scaling the constants g_2, g_3, e_1, e_2, e_3 and using

$$\wp(\eta; g_2, g_3) = a^2\wp(a\eta; g_2a^{-4}, g_3a^{-6}) \tag{5.23}$$

Then

$$v'(\eta)^2 = 4v(\eta)^3 - g_2'v(\eta) - g_3' \tag{5.24}$$

with $g_2' = a^4g_2,\ g_3' = a^6g_3$ and from this one derives $e_1' = a^2e_1, e_2' = a^2e_2, e_3' = a^2e_3, \Delta' = a^{12}\Delta, \omega_1' = \omega_1/a, \omega_2' = \omega_2/a$.

5.3. The Poisson structure.

Let $\mathfrak{g}$ be a real noncompact semisimple Lie algebra with restricted root system R (cf. chapter 3) and define $P = \Lambda \times \mathfrak{g}$. Let $(q, z) \in P$, then $T_{(q,z)}P \cong \mathfrak{a} \oplus \mathfrak{g}$, so the Killing form induces a non-degenerate scalar product on $T_{(q,z)}P$ by

$$< (a_1, z_1), (a_2, z_2) >=< a_1, a_2 > + < z_1, z_2 > \tag{5.25}$$

We will identify $\mathfrak{a}$ and $\mathfrak{a}^* = E$ using the Killing form. One can view P as a trivial vector bundle over Λ with fiber $\mathfrak{g}$ and projection $\pi : P \to \Lambda$, given by $\pi(q, z) = q$, where $(q, z) \in P$.
Now define the linear map $X(q) : \mathfrak{g} \to \mathfrak{g}$ by:

$$X(q)(z) = 0 \text{ if } z \in \mathfrak{g}_0, \quad X(q)(e_\alpha) = x(\alpha(q))e_\alpha \tag{5.26}$$

and choose the function x in such a way that

$$x(q_\alpha) > 0 \text{ if } q \in \Lambda \tag{5.27}$$

This can be realized by choosing it of type I, II b, III b, and IV c in (1.54). Then $X(q)$ is semisimple, and because of (5.27) $\operatorname{Ker} X(q) = \mathfrak{g}_0$ and $X(q) : \mathfrak{g}_0^\perp \to \mathfrak{g}_0^\perp$ is an isomorphism. Now define $X^{-1}(q)$ by

$$X^{-1}(q)(z) = 0 \text{ if } z \in \mathfrak{g}_0, \quad X^{-1}(q)(e_\alpha) = x^{-1}(\alpha(q))e_\alpha \tag{5.28}$$

In the same way define the map $Y(q) : \mathfrak{g} \to \mathfrak{g}$ by

$$Y(q)(z) = 0 \text{ if } z \in \mathfrak{g}_0, \quad Y(q)(e_\alpha) = y(\alpha(q))e_\alpha \tag{5.29}$$

and $R := -YX^{-1}$ (From now on we suppress the q-dependence). Because $x(\eta)$ is uneven and $y(\eta)$ is even, it follows that

$$X : \mathfrak{k} \to \mathfrak{p}, \mathfrak{p} \to \mathfrak{k}, \quad Y : \mathfrak{k} \to \mathfrak{k}, \mathfrak{p} \to \mathfrak{p} \tag{5.30}$$

Lemma 5.1. The maps X, Y satisfy the following properties:
(i) X is skew-symmetric and Y is symmetric with respect to the Killing form.
(ii) X and Y commute.
(iii) X and Y commute with $\operatorname{ad} z$ for all $z \in \mathfrak{g}_0$.
(iv)

$$\begin{gathered} \pi_0[X(x), y] = -\pi_0[x, X(y)] \\ \pi_0[Y(x), y] = \pi_0[x, Y(y)] \text{ for all } x, y \in \mathfrak{g} \end{gathered} \tag{5.31}$$

where π_0 denotes the projection on $\mathfrak{g}_0$.

Proof. (i) is clear because $x(\eta)$ is uneven and $y(\eta)$ is an even function. (ii) is trivial, (iii) follows from the definition and from the fact that $[\mathfrak{g}_0, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$ and (iv) follows from (i) and (iii). □

Now define

$$[x,y]_R = [\pi_{\mathfrak{a}}(x), R(y)] + [R(x), \pi_{\mathfrak{a}}(y)], \quad x, y \in \mathfrak{g} \tag{5.32}$$

where $\pi_{\mathfrak{a}}$ denotes the projection on $\mathfrak{a}$.

Proposition 5.2. Formula (5.32) defines a new Lie bracket on $\mathfrak{g}$

Proof.

$$\begin{aligned}
&[[x,y]_R, z]_R + \text{cycl.} \\
&= [[\pi_{\mathfrak{a}}(x), R(y)] + [R(x), \pi_{\mathfrak{a}}(y)], z]_R + \text{cycl.} \\
&= [R[\pi_{\mathfrak{a}}(x), R(y)] + R[R(x), \pi_{\mathfrak{a}}(y)], \pi_{\mathfrak{a}}(z)] + \text{cycl.} \\
&= [[\pi_{\mathfrak{a}}(x), R^2(y)] + [R^2(x), \pi_{\mathfrak{a}}(y)], \pi_{\mathfrak{a}}(z)] + \text{cycl.} \\
&= [[\pi_{\mathfrak{a}}(x), R^2(y)], \pi_{\mathfrak{a}}(z)] + [[R^2(y), \pi_{\mathfrak{a}}(z)], \pi_{\mathfrak{a}}(x)] + \text{cycl.} \\
&= [[\pi_{\mathfrak{a}}(x), \pi_{\mathfrak{a}}(z)], R^2(y)] + \text{cycl.} = 0
\end{aligned}$$

where we have used Lemma 5.1(iii),(iv) and the Jacobi identity for $[\ ,\ ]$. □

Now consider the corresponding Lie-Poisson bracket on $\mathcal{C}^\infty(\mathfrak{g})$. This can be extended to a Poisson bracket on P in the following way. Let $f, g \in \mathcal{C}^\infty(P)$ and F_1 and F_2 the components of $F = \nabla f(q,x)$ (with respect to the decomposition $P = \Lambda \times \mathfrak{g}$), then define

$$\{f,g\}_1(q,x) = < x, [F_2, G_2]_R > \tag{5.33}$$

where $(q,x) \in P$ and $R = R(q)$.

Proposition 5.3. Formula (5.33) defines a Poisson bracket on P, which is linear on each fiber $\mathfrak{g}$.

Proof. The skew-symmetry and Leibniz rule are clear, so from chapter 2 we know that it is suffient to prove the Jacobi identity for coordinate functions. For coordinate functions on $\mathfrak{g}$ it reduces to the Jacobi identity for the Lie bracket $[\ ,\ ]_R$ on $\mathfrak{g}$. Now observe that

$$\mathcal{C}^\infty(\Lambda) \subset \mathcal{Z}(\mathcal{C}^\infty(P)) \tag{5.34}$$

□

Now define a second bracket on $\mathcal{C}^\infty(P)$ by setting for $(q,x) \in P$ and $f, g \in \mathcal{C}^\infty(P)$

$$\{f,g\}_0(q,x) = < \pi_{\mathfrak{a}}(F_2), G_1 > - < F_1, \pi_{\mathfrak{a}}(G_2) > \tag{5.35}$$

Proposition 5.4. Formula (5.35) defines a Poisson structure on P.

Proof. Trivial, because the corresponding structure matrix is constant and skew-symmetric. □

Proposition 5.5. The brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_1$ form a Hamiltonian pair, which means that $\{\ ,\ \}_0 + \lambda\{\ ,\ \}_1$ satisfies the Jacobi identity for all $\lambda \in \mathbb{R}$.

Proof. Because $\{\ ,\ \}_0$ and $\{\ ,\ \}_1$ are Poisson brackets, one only has to prove that

$$\{\{f,g\}_0, h\}_1 + \{\{f,g\}_1, h\}_0 + \text{cycl.} = 0 \tag{5.36}$$

To prove this it is sufficient to prove it for coordinate functions and then it is clear that the first term of (5.36) is zero because the structure matrix of $\{\ ,\ \}_0$ is constant, so one only has to prove:

$$\{\{f,g\}_1, h\}_0 + \text{cycl.} = 0 \tag{5.37}$$

First we prove the following Lemma:

Lemma 5.6. Let f, g be coordinate functions on P and $(q,z) \in P$, then

$$\pi_1 \nabla\{f,g\}_1(q,z) = \pi_{\mathfrak{a}}[R'(G_2), [z, \pi_{\mathfrak{a}}(F_2)]] + \pi_{\mathfrak{a}}[R'(F_2), [\pi_{\mathfrak{a}}(G_2), z]] \tag{5.38}$$

$$\pi_2 \nabla\{f,g\}_1(q,z) = [F_2, G_2]_R \tag{5.39}$$

where

$$R'(q)(e_\alpha) = r'(\alpha(q))e_\alpha, \quad r(\eta) = -x^{-1}(\eta)y(\eta) = x^{-1}(\eta)x'(\eta)$$

Proof. Let $(q(t), z+ty)$ be a curve in P through (q,z), then

$$\begin{aligned}
&< \nabla\{f,g\}_1(q,z), (\dot q, y) > \\
&= \tfrac{d}{dt}\big|_{t=0} \{f,g\}_1(q(t), z+ty) \\
&= \tfrac{d}{dt}\big|_{t=0} < z+ty, [\pi_2\nabla f(q(t), z+ty), \pi_2\nabla g(q(t), z+ty)]_{R(q(t))} > \\
&=< y, [\pi_2\nabla f(q,z), \pi_2\nabla g(q,z)]_{R(q)} > \\
&+ < z, \tfrac{d}{dt}\big|_{t=0} [\pi_2\nabla f(q(t), z+ty), \pi_2\nabla g(q(t), z+ty)]_{R(q(t))} > \\
&=< y, [F_2, G_2]_{R(q)} > + < z, [\pi_{\mathfrak{a}}F_2, [\dot q, R'(q)(G_2)]] > \\
&+ < z, [[\dot q, R'(q)(F_2)], \pi_{\mathfrak{a}}G_2] > \\
&=< y, [F_2, G_2]_{R(q)} > \\
&+ < \dot q, [R'(q)(G_2), [z, \pi_{\mathfrak{a}}F_2]] + [R'(q)(F_2), [\pi_{\mathfrak{a}}G_2, z]] >
\end{aligned}$$

where we have used Lemma 5.1. This proves Lemma 5.6. □

Now substitute (5.38) and (5.39) in (5.37) to obtain:

$$< \pi_2\nabla\{f,g\}_1(q,z), H_1 > - < \pi_1\nabla\{f,g\}_1(q,z), H_2 > +\text{cycl.}$$

The first term is zero because $H_1 \in \mathfrak{a}$ and $[F_2, G_2]_R \in \mathfrak{g}_0^\perp$ and the second term becomes:

$$\begin{aligned}
&- < [R'(G_2), [z, \pi_{\mathfrak{a}}F_2]] + [R'(F_2), [\pi_{\mathfrak{a}}G_2, z]], H_2 > +\text{cycl.} \\
&=< z, [\pi_{\mathfrak{a}}F_2, [R'(G_2), \pi_{\mathfrak{a}}H_2]] + [[R'(F_2), \pi_{\mathfrak{a}}H_2], \pi_{\mathfrak{a}}G_2] > +\text{cycl.} = 0
\end{aligned}$$

for the same reason as in the proof of Prop. 5.2. This ends the proof of Prop. 5.5. □

Now define

$$\{f,g\} = \{f,g\}_1 + \{f,g\}_0 \tag{5.40}$$

This is a Poisson structure on P, and the Hamiltonian vector field corresponding to a function f is given by:

$$v_f(q,x) = (\pi_{\mathfrak{a}} F_2, \pi_{\mathfrak{a}}[x, R(F_2)] - [R(x), \pi_{\mathfrak{a}} F_2] - F_1) = B(q,x)(F_1, F_2) \tag{5.41}$$

where $(q,x) \in P$.

Now choose $\mu \in \mathfrak{g}_0^{\perp}$ and define

$$P_\mu = \{(q, X(q)\mu + x_0), q \in \Lambda, x_0 \in \mathfrak{g}_0\} \tag{5.42}$$

Remark 5.7. P_μ can be viewed as a trivial fibre subbundle of $P = \Lambda \times \mathfrak{g}$, because $\psi : \Lambda \times \mathfrak{g}_0 \to P_\mu \subset P$, defined by $\psi(q, x_0) = (q, X(q)\mu + x_0)$ is an isomorphism, which is linear in the fibres.

Lemma 5.8. We have

$$\mathcal{C}^\infty(\mathfrak{m}) \subset \mathcal{Z}(\mathcal{C}^\infty(P))$$

Proof. Let $f \in \mathcal{C}^\infty(\mathfrak{m})$ then $F_1 = 0$ and $F_2 \in \mathfrak{m}$, so from (5.41) it follows that $v_f(q,x) = 0$ for all $(q,x) \in P$, and so $f \in \mathcal{Z}(\mathcal{C}^\infty(P))$. □

Proposition 5.9. P_μ is a Poisson submanifold of P.

Proof. Let $(q(t), x_0(t) + X(q(t))\mu)$ denote a curve in P_μ with $\dot{q} \in \mathfrak{a}, \dot{x}_0 \in \mathfrak{g}_0$. Differentiating this with respect to t gives

$$(\dot{q}(t), [Y(q(t))\mu, \dot{q}(t)] + \dot{x}_0(t)) \in T_{(q,x)}P_\mu \tag{5.43}$$

and comparing this with (5.41) one sees that $v_f(q,x) \in T_{(q,x)}P_\mu$ for all $(q,x) \in P$ and $f \in \mathcal{C}^\infty(P)$, so using Lemma 2.27 we conclude that P_μ is a Poisson submanifold of P. □

One can rewrite Hamilton's equations on P_μ, using Lemma 5.1 and the definition of R and one gets:

$$\dot{q} = \pi_{\mathfrak{a}} F_2, \quad \dot{x} = [Y\mu, \pi_{\mathfrak{a}} F_2] + \pi_{\mathfrak{a}}[Y\mu, F_2] - F_1 \tag{5.44}$$

Proposition 5.10. If $\mu \in \mathfrak{k}$ then P_μ is a Poisson manifold of constant rank $2l$, where $l = \dim \mathfrak{a}$.

Proof. Consider the map $B(q,x) : T_{(q,x)}P_\mu \to T_{(q,x)}P_\mu$ with $(q,x) \in P_\mu$. From Lemma 5.8 and Prop. 5.9 it follows that

$$\mathfrak{m} \subset \operatorname{Ker} B(q,x), \quad \operatorname{Im} B(q,x) \subset T_{(q,x)}P_\mu \tag{5.45}$$

and also $\dim T_{(q,x)}P_\mu = 3l - \delta$ (where δ as defined in chapter 3). Because $\dim \operatorname{Ker} B(q,x) + \dim \operatorname{Im} B(q,x) = \dim T_{(q,x)}P_\mu$ it is sufficient to prove that $\operatorname{Ker} B(q,x) = \mathfrak{m}$. So let $v(q,x) = (\dot{q}, [Y\mu, \dot{q}] + \dot{x}_0) \in T_{(q,x)}P_\mu$ and assume that $v(q,x) \in \operatorname{Ker} B(q,x)$, then one has:

$$\pi_{\mathfrak{a}}\dot{x}_0 = 0, \quad \dot{q} = \pi_{\mathfrak{a}}[Y\mu, [Y\mu, \dot{q}]], \quad \dot{q} \in \mathfrak{a}, \dot{x}_0 \in \mathfrak{g}_0 \tag{5.46}$$

but this implies

$$\begin{aligned} 0 \leq <\dot{q}, \dot{q}> &= <\dot{q}, \pi_{\mathfrak{a}}[Y\mu, [Y\mu, \dot{q}]]> \\ &= <[\dot{q}, Y\mu], [Y\mu, \dot{q}]> \\ &= -<[\dot{q}, Y\mu], [\dot{q}, Y\mu]> \leq 0 \end{aligned} \tag{5.47}$$

because $Y\mu \in \mathfrak{k}$ and the Killing form is positive-definite om $\mathfrak{p}$. Thus $<\dot{q}, \dot{q}>= 0$ and so $\dot{q} = 0$, which implies that $v(q,x) \in \mathfrak{m}$. □

Corollary 5.11. The symplectic leaves of P_μ have dimension $2l$ and they intersect P_μ in the coordinate slices $m_j =$ const., where $\{m_j, j = 1, \ldots, \dim \mathfrak{m}\}$ are coordinates on $\mathfrak{m}$.

Remark 5.12. We will denote by P_μ^0 the symplectic leaf, for which $m_j = 0$. In that case one has $x \in \mathfrak{p}$ and a general element of P_μ^0 will often be denoted by (Q, L), where $Q \in \mathfrak{a}$, $L \in \mathfrak{p}$ and $P \in \mathfrak{a}$.

5.4. Derivation of Hamilton's equations.

Now we are ready to give a (generalized), K.A.S.R.S type construction of the Hamiltonian systems as defined in Section 5.1.

Choose $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in R_+$ with normalization (3.21) and (3.22) and define

$$\mu = \sum_{\alpha \in R_+} g_\alpha(e_\alpha + e_{-\alpha}) \in \mathfrak{k} \tag{5.48}$$

where g_α is a Weyl group invariant function of α. Because in general $\dim \mathfrak{g}_\alpha = m_\alpha > 1$, this choice is not unique. To describe the models of type I–IV define on P_μ the following function

$$f(q,x) = \tfrac{1}{2} <x, x> \tag{5.49}$$

where $(q,x) \in P_\mu$. Using (3.27) one finds

$$\begin{aligned}
f(q,x) &= \tfrac{1}{2} < x_0 + X\mu, x_0 + X\mu > \\
&= \tfrac{1}{2} < x_0, x_0 > + \tfrac{1}{2} < \sum_{\alpha \in R_+} X(e_\alpha + e_{-\alpha}), \sum_{\beta \in R_+} X(e_\beta + e_{-\beta}) > \\
&= \tfrac{1}{2} < x_0, x_0 > + \tfrac{1}{2} \sum_{\alpha,\beta \in R_+} g_\alpha g_\beta x(\alpha(q)) x(\beta(q)) < e_\alpha - e_{-\alpha}, e_\beta - e_{-\beta} > \\
&= \tfrac{1}{2} < x_0, x_0 > + \sum_{\alpha \in R_+} {g_\alpha}^2 x(\alpha(q))^2 \\
&= \tfrac{1}{2} < p, p > + \sum_{\alpha \in R_+} {g_\alpha}^2 v(q_\alpha) + \tfrac{1}{2} < x_m, x_m >
\end{aligned} \tag{5.50}$$

where $x_0 = p + x_m, p \in \mathfrak{a}, x_m \in \mathfrak{m}$. Restricting this to a symplectic leaf, where $x_m =$ const., gives the Hamiltonian (5.1) of type I–IV.
From (5.49) one derives

$$F_1 = 0, \quad F_2 = x \tag{5.51}$$

and substituting this in (5.44) one finds:

$$\dot{q} = p, \quad \dot{x} = [Y\mu, p] + \pi_\mathfrak{a}[Y\mu, X\mu] \tag{5.52}$$

and so

$$\begin{aligned}
\dot{p} &= \pi_\mathfrak{a}\dot{x} = \pi_\mathfrak{a}[Y\mu, X\mu] \\
&= \pi_\mathfrak{a}[\sum_{\alpha \in R_+} g_\alpha Y(e_\alpha + e_{-\alpha}), \sum_{\beta \in R_+} g_\beta X(e_\beta + e_{-\beta})] \\
&= \sum_{\alpha,\beta \in R_+} g_\alpha g_\beta y(\alpha(q)) x(\beta(q)) \pi_\mathfrak{a}[e_\alpha + e_{-\alpha}, e_\beta - e_{-\beta}] \\
&= \sum_{\alpha \in R_+} {g_\alpha}^2 y(q_\alpha) x(q_\alpha) t_\alpha
\end{aligned} \tag{5.53}$$

Now take the components with respect to the orthonormal basis $\{d_j, j = 1, \ldots, n\}$ of E and one gets back the Hamilton equations (5.5) and (5.6). Here we make constantly use of the identification of $\mathfrak{a}$ and $\mathfrak{a}^* = E$. Under this identification one has the correspondence

$$\alpha \leftrightarrow t_\alpha, \quad \alpha^\nu := \frac{\alpha}{2(\alpha,\alpha)} \leftrightarrow h_\alpha \tag{5.54}$$

To describe the system of type V one has to take as Hamiltonian the function:

$$H = f(q,x) = \tfrac{1}{2}\tilde{\omega}^2 < q, q > + \tfrac{1}{2} < x, x > \tag{5.55}$$

where $\tilde{\omega} \in \mathbb{R}$ is a constant, and then

$$F_1 = \tilde{\omega}^2 q, \quad F_2 = x \tag{5.56}$$

Substituting this in (5.39) one finds

$$\dot{q} = p, \quad \dot{x} = [Y\mu, p] + \pi_{\mathfrak{a}}[Y\mu, X\mu] - \tilde{\omega}^2 q \tag{5.57}$$

and one easily verifies that these are the same equations as (1.58) in the case of a root system of type A_{n-1} (if one identifies q with Q, p with P and x with L). So we shall view (5.55) as the generalization of the type V Hamiltonian for abstract root systems.

5.5. A condition on μ.

Hamilton's equations (5.52) and (5.57) are not yet in Lax form. In order to derive the Lax equation one has to impose conditions on the element μ, where $\mu \in \mathfrak{k} \cap \mathfrak{m}^{\perp}$. It turns out that a sufficient condition is the following:

$$(\forall k \in \mathfrak{k})(\pi_{\mathfrak{m}}[k, \mu] = 0 \longrightarrow \exists m \in \mathfrak{m}, [k + m, \mu] = 0) \tag{5.58}$$

In chapter 6 it is shown that such elements do exist. In the split case, when $\mathfrak{m} = \{0\}$, the condition is trivial and implies that $\mu \in \mathcal{Z}(\mathfrak{k})$. In section 6.1 the nontrivial cases, where $\mathfrak{k}$ is not semisimple, are listed. In this section we will assume that $\mathfrak{g}$ is not of split type, so $\mathfrak{m} \neq \{0\}$.
Now consider the following chain of maps:

$$\mathrm{Cent}_{\mathfrak{k}}(\mu) \xrightarrow{\pi_0^{\perp}} \mathfrak{k} \cap \mathfrak{g}_0^{\perp} \xrightarrow{A} \mathfrak{m} \tag{5.59}$$

where $A = \pi_{\mathfrak{m}} \circ \operatorname{ad}\mu$, $\pi_{\mathfrak{m}}$ denotes the projection on $\mathfrak{m}$ and $\pi_0^{\perp}$ denotes the projection on $\mathfrak{g}_0^{\perp}$. Then (5.58) is equivalent with :

$$\operatorname{Ker} A \subset \operatorname{Im} \pi_0^{\perp} \tag{5.60}$$

but one also has:

$$\operatorname{Im} \pi_0^{\perp} \subset \operatorname{Ker} A \tag{5.61}$$

because $\mu \in \mathfrak{k} \cap \mathfrak{g}_0^{\perp}$.

Lemma 5.13. $\mathfrak{m} = \mathrm{Cent}_{\mathfrak{m}}(\mu) \oplus \mathrm{Im\,A}$

Proof.

$$\begin{aligned} \operatorname{Im} A &= (\operatorname{Ker} A^*)^{\perp} = \operatorname{Ker}(\operatorname{ad}\mu \circ \pi_{\mathfrak{m}})^{\perp} \\ &= (\mathfrak{m}^{\perp} \oplus \mathrm{Cent}_{\mathfrak{m}}(\mu))^{\perp} \\ &= \mathrm{Cent}_{\mathfrak{m}}(\mu)^{\perp} \cap \mathfrak{m} \end{aligned}$$

and $\mathfrak{m} = \mathrm{Cent}_{\mathfrak{m}}(\mu) \oplus \mathrm{Cent}_{\mathfrak{m}}(\mu)^{\perp} \cap \mathfrak{m}$ □

Using Lemma 5.13 and (5.61) one obtains the following relations:

$$\dim \mathrm{Cent}_{\mathfrak{k}}(\mu) = \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) + \dim \mathrm{Im}\, \pi_o^{\perp}$$

$$\dim \mathfrak{k} \cap \mathfrak{g}_0^{\perp} = \dim \mathrm{Ker}\, A + \dim \mathrm{Im}\, A = \dim \mathrm{Ker}\, A + \dim \mathfrak{m} - \dim \mathrm{Cent}_{\mathfrak{m}}(\mu)$$

$$\dim \mathrm{Im}\, \pi_0^{\perp} \leq \dim \mathrm{Ker}\, A \tag{5.62}$$

Using these relations one can prove:

Proposition 5.14. Suppose $\mu \in \mathfrak{k} \cap \mathfrak{m}^{\perp}$, then the following properties are equivalent:
(i) $\dim \mathrm{Ker}\, A = \dim \mathrm{Im}\, \pi_0^{\perp}$ (which implies (5.58))
(ii) $\dim \mathrm{Cent}_{\mathfrak{k}}(\mu) = \dim \mathfrak{k} - 2 \dim \mathfrak{m} + 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu)$

Proof.
(i)⇒(ii)

$$\begin{aligned} \dim \mathrm{Cent}_{\mathfrak{k}}(\mu) &= \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) + \dim \mathrm{Im}\, \pi_0^{\perp} \\ &= \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) + \dim \mathrm{Ker\, A} \\ &= \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) + \dim \mathfrak{k} \cap \mathfrak{g}_0^{\perp} - \dim \mathfrak{m} + \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \\ &= \dim \mathfrak{k} - 2 \dim \mathfrak{m} + 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \end{aligned}$$

(ii)⇒(i)

$$\begin{aligned} \dim \mathrm{Ker}\, A &= \dim \mathfrak{k} - 2 \dim \mathfrak{m} + \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \\ &= \dim \mathrm{Cent}_{\mathfrak{k}}(\mu) - \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \\ &= \dim \mathrm{Im}\, \pi_0^{\perp} \end{aligned}$$

□

Remark 5.15. So we are looking for elements $\mu \in \mathfrak{k} \cap \mathfrak{m}^{\perp}$ for which the dimension of the $\mathrm{Ad}K$-orbit is equal to $2 \dim \mathfrak{m} - 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu)$. In [45] Adler also derives some sufficient conditions on μ. He remarks that his properties A and B imply that one should look for μ's whose orbits have dimension $2 \dim \mathfrak{t}$. However, his construction only seems to work for Lie algebras of type IV, and in this case his condition coincides with ours if one requires that $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$. So in this case our condition is more general and we will show in section 6.1 that the only μ which satisfies this condition is the one already known for the A_{n-1} case. Moreover, our condition can be formulated for all real semisimple Lie algebras and our proof of the integrability assumes no other special properties of $\mathfrak{g}$. Adler also requires that $g_{\alpha} \neq 0$, but in the BC_n case this is not necessary.

Because $\mathfrak{m} \subset \operatorname{Ker} A$ and $\operatorname{Im} A \subset \mathfrak{m}$ it follows that $\operatorname{Im} A \subset \operatorname{Ker} A$, so $A^2 = 0$. Now

$$\begin{aligned}\mathfrak{k} &= \operatorname{Ker} A \oplus (\operatorname{Ker} A)^{\perp} \\ &= \operatorname{Ker} A \oplus \operatorname{Im}(\operatorname{ad}\mu \circ \pi_{\mathfrak{m}})\end{aligned}$$

and if one of the conditions of Prop. 5.14 is satisfied one can define a linear map

$$\mathcal{M} : \operatorname{Ker} A \to \operatorname{Im} A \tag{5.63}$$

as follows: if $k \in \operatorname{Ker} A$ then according to (5.58) there exists an element $m \in \mathfrak{m}$ such that $[k + m, \mu] = 0$. But because of Lemma 5.13 one may take the component of m in $\operatorname{Im} A$ and still $[k+m, \mu] = 0$. This uniquely defines a linear map $\mathcal{M} : \operatorname{Ker} A \to \operatorname{Im} A$ with the property that $[\mathcal{M}(k) + k, \mu] = 0$, in other words:

$$\mathcal{M} + \mathrm{id} : \operatorname{Ker} A \to \operatorname{Cent}_{\mathfrak{k}}(\mu) \tag{5.64}$$

We also observe the following

Proposition 5.16. If μ satisfies one of the properties in Prop. 5.14, then one has the following exact sequence of maps:

$$0 \to \operatorname{Cent}_{\mathfrak{m}}(\mu) \to \operatorname{Cent}_{\mathfrak{k}}(\mu) \xrightarrow{\pi_0^{\perp}} \mathfrak{k} \cap \mathfrak{g}_0^{\perp} \xrightarrow{A} \mathfrak{m} \xrightarrow{\pi} \operatorname{Cent}_{\mathfrak{m}}(\mu) \to 0 \tag{5.65}$$

Proof. Trivial. □

5.6. Derivation of the Lax equation.

Now we are ready to derive a Lax equation for the models of type I–V. Recall the functional equation

$$x(\eta)y(\xi) - x(\xi)y(\eta) = x(\xi + \eta)(x^2(\xi) - x^2(\eta)) \tag{5.66}$$

Let $q \in \Lambda, \alpha, \beta \in R$ and $\alpha + \beta \in R$ and take $\eta = \alpha(q)$, $\xi = \beta(q)$, then one gets the following equation for the maps X and Y:

$$[Xe_\alpha, Ye_\beta] - [Ye_\alpha, Xe_\beta] = X[e_\alpha, X^2 e_\beta] - X[X^2 e_\alpha, e_\beta] \tag{5.67}$$

Now let $z_1, z_2 \in \mathfrak{g}_0^{\perp}$, with $z_1 = \sum_{\alpha\in R} \lambda^\alpha e_\alpha, z_2 = \sum_{\beta \in R} \mu^\beta e_\beta$, then, because of linearity, one easily derives that

$$\pi_0^{\perp}[Xz_1, Yz_2] - \pi_0^{\perp}[Yz_1, Xz_2] = X[z_1, X^2 z_2] - X[X^2 z_1, z_2] \tag{5.68}$$

for all $z_1, z_2 \in \mathfrak{g}_0^{\perp}$.
Now suppose that $\mu \in \mathfrak{k} \cap \mathfrak{g}_0^{\perp}$ satisfies (5.58). Choose $z_1 = z_2 = \mu$ in (5.68), then

$$\pi_0^{\perp}[X\mu, Y\mu] = X[\mu, X^2\mu] \tag{5.69}$$

Now

$$\pi_{\mathfrak{m}}[\mu, X^2\mu] = -\pi_{\mathfrak{m}}[X\mu, X\mu] = 0$$

so $X^2\mu \in \operatorname{Ker} A$ and thus

$$[X^2\mu + \mathcal{M}X^2\mu, \mu] = 0 \tag{5.70}$$

where $\mathcal{M} : \operatorname{Ker} A \to \operatorname{Im} A$ is the map which was defined in section 5.5. And so

$$\begin{aligned} X[\mu, X^2\mu] &= X[\mathcal{M}X^2\mu, \mu] \\ &= [\mathcal{M}X^2\mu, X\mu] \end{aligned} \tag{5.71}$$

where we used Lemma 5.1.

Proposition 5.17. Suppose μ satisfies (5.58) and X and Y satisfy the functional equation (5.67), then Hamilton's equations (5.52) on P_μ can be written in the form:

$$\dot{q} = p, \quad \dot{x} = [M, x - x_m] \tag{5.72}$$

where $(q, x) \in P_\mu$, x_m denotes the component of x in $\mathfrak{m}$, and $M = \mathcal{M}X^2\mu + Y\mu$. So on the symplectic leaf P_μ^0, $\dot{x}$ is in Lax form, i.e. $\dot{x} = [M, x]$. In general one has on P_μ $\dot{x}_p = [M, x_p]$, because $\dot{x}_m = 0$. Here x_p denotes the projection of x on $\mathfrak{p}$.

Proof. Substitute (5.69) and (5.71) in (5.52) to find

$$\begin{aligned} \dot{x} &= [Y\mu, p] + \pi_{\mathfrak{a}}[Y\mu, X\mu] \\ &= [Y\mu, p] + [Y\mu, X\mu] - \pi_0^\perp[Y\mu, X\mu] \\ &= [Y\mu, x - x_m] + X[\mu, X^2\mu] \\ &= [Y\mu, x - x_m] + [\mathcal{M}X^2\mu, X\mu] \\ &= [\mathcal{M}X^2\mu + Y\mu, x - x_m] = [M, x - x_m] \end{aligned}$$

where $M = \mathcal{M}X^2\mu + Y\mu$. □

Using this result one can also easily derive a Lax equation for the type V model. Indeed, on P_μ^0 Hamilton's equations (5.57) become

$$\dot{q} = p, \quad \dot{x} = [M, x] - \tilde{\omega}^2 q \tag{5.73}$$

Now let $\mathfrak{g}$ be the Lie algebra of an associative algebra and define $N = x^2 + \tilde{\omega}^2 q^2$, then

$$\begin{aligned} \dot{N} &= \dot{x}x + x\dot{x} + \tilde{\omega}^2(\dot{q}q + q\dot{q}) \\ &= [M, x]x - \tilde{\omega}^2 qx + x[M, x] - \tilde{\omega}^2 xq + \tilde{\omega}^2(pq + qp) \\ &= [M, x^2] - \tilde{\omega}^2\{(X\mu)q + q(X\mu)\} \end{aligned} \tag{5.74}$$

On the other hand one has, because in this case $Y = X^2$

$$\begin{aligned}[M, N] &= [M, x^2 + \tilde{\omega}^2 q^2] \\ &= [M, x^2] + \tilde{\omega}^2[X^2\mu, q^2] \\ &= [M, x^2] + \tilde{\omega}^2 q[X^2\mu, q] + \tilde{\omega}^2[X^2\mu, q]q \\ &= [M, x^2] - \tilde{\omega}^2\{q(X\mu) + (X\mu)q\}\end{aligned} \tag{5.75}$$

so $\dot{N}$ is in Lax form.

5.7. A proof of the integrability for type I, II, and III.

To prove that the Ad-invariant functions are in involution on P_μ we first derive another form of the functional equation (5.66). For notational convenience we write f for the function x.

Proposition 5.18. Suppose the function f satisfies the functional equation

$$f(x)f'(y) - f(y)f'(x) = f(x+y)(f(x)^2 - f(y)^2) \tag{5.76}$$

and

$$f'(x)^2 = f(x)^4 - 2bf(x)^2 + c, \quad f(x) \neq 0 \tag{5.77}$$

then f also satisfies the functional equation

$$f(x+y)\left(\frac{f'(x)}{f(x)} + \frac{f'(y)}{f(y)}\right) = \frac{c}{f(x)f(y)} - f(x)f(y) \tag{5.78}$$

Proof.

$$\begin{aligned}&f(x+y)\left(\frac{f'(x)}{f(x)} + \frac{f'(y)}{f(y)}\right)(f(x)^2 - f(y)^2) \\ &= f(x+y)(f(x)^2 - f(y)^2)(f(x)f'(y) + f(y)f'(x))[f(x)f(y)]^{-1} \\ &= (f(x)f'(y) - f(y)f'(x))(f(x)f'(y) + f(y)f'(x))[f(x)f(y)]^{-1} \\ &= (f(x)^2 f'(y)^2 - f(y)^2 f'(x)^2)[f(x)f(y)]^{-1} \\ &= \{f(x)^2(f(y)^4 - 2bf(y)^2 + c) - f(y)^2(f(x)^4 - 2bf(x)^2 + c)\}[f(x)f(y)]^{-1} \\ &= (c - f(x)^2 f(y)^2)(f(x)^2 - f(y)^2)[f(x)f(y)]^{-1}\end{aligned}$$

which implies (5.78). □

In fact, given (5.77), (5.78) and (5.76) are equivalent, because

Proposition 5.19. Suppose f satisfies (5.77) and (5.78), then it also satisfies (5.76).

Proof. Differentiating (5.78) to x resp. y and subtracting, one gets

$$\begin{aligned}
&2f(x+y)\left(\frac{f''(x)}{2f(x)}-\frac{f''(y)}{2f(y)}\right)\\
&=f(x+y)\left(\frac{f'(x)^2}{f(x)^2}-\frac{f'(y)^2}{f(y)^2}\right)-c[f(x)f(y)]^{-1}\left(\frac{f'(x)}{f(x)}-\frac{f'(y)}{f(y)}\right)\\
&+f(x)f'(y)-f'(x)f(y)\\
&=\left(\frac{c}{f(x)f(y)}-f(x)f(y)\right)\left(\frac{f'(x)}{f(x)}-\frac{f'(y)}{f(y)}\right)\\
&-\frac{c}{f(x)f(y)}\left(\frac{f'(x)}{f(x)}-\frac{f'(y)}{f(y)}\right)+f(x)f'(y)-f'(x)f(y)\\
&=2(f(x)f'(y)-f(y)f'(x))
\end{aligned}$$

which implies (5.76), because (5.77) implies

$$\frac{f''(x)}{2f(x)}=f(x)^2-b \tag{5.79}$$

□

Corollary 5.20. If f satisfies (5.78) and (5.77), then $R=-YX^{-1}$ and X satisfy the equation

$$X[Rz_1,z_2]+X[z_1,Rz_2]=c\pi_0^\perp[X^{-1}z_1,X^{-1}z_2]-\pi_0^\perp[Xz_1,Xz_2] \tag{5.80}$$

for all $z_1,z_2\in\mathfrak{g}_0^\perp$.

Now let $f\in\mathcal{C}^\infty(\mathfrak{g})$ be an Ad-invariant function, viewed as a function on P_μ, by defining $f(q,x)=f(x)$. Then $F_1=0$ and $F_2=F=\nabla f(x)$.

Lemma 5.21. Let f be an Ad-invariant function, viewed as a function on P_μ, then $\pi_\mathfrak{m}[\mu,X(\pi_\mathfrak{p}F)]=-[\pi_\mathfrak{m}F,\pi_\mathfrak{m}x]$.

Proof. Because f is Ad-invariant one has $[F,x]=0$. Taking the $\mathfrak{k}$-component of this gives $[\pi_\mathfrak{p}F,\pi_\mathfrak{p}x]=[\pi_\mathfrak{m}x,\pi_\mathfrak{k}F]$. Using this one gets:

$$\begin{aligned}
\pi_\mathfrak{m}[\mu,X(\pi_\mathfrak{p}F)]&=-\pi_\mathfrak{m}[X\mu,\pi_\mathfrak{p}F]=-\pi_\mathfrak{m}[\pi_\mathfrak{p}x,\pi_\mathfrak{p}F]\\
&=-\pi_\mathfrak{m}[\pi_\mathfrak{k}F,\pi_\mathfrak{m}x]=-[\pi_\mathfrak{m}F,\pi_\mathfrak{m}x]
\end{aligned}$$

where we have used extensively the commutation relations (3.9). □

Corollary 5.22. $X(\pi_\mathfrak{p}F)\in\operatorname{Ker}A$ if one of the following conditions is satisfied:

(i) $\mathfrak{g}$ is quasi-split.

(ii) f is restricted to the symplectic leaf P_μ^0.

So if μ satisfies (5.58), f is Ad-invariant, and one of the conditions in Corollary 5.22 is satisfied, one can write

$$[\mu, X(\pi_{\mathfrak{p}}F)] = [\mathcal{M}X(\pi_{\mathfrak{p}}F), \mu] \tag{5.81}$$

and so

$$\begin{aligned} &X[\mu, X(\pi_{\mathfrak{p}}F)] \\ &= X[\mathcal{M}X(\pi_{\mathfrak{p}}F), \mu] \\ &= [\mathcal{M}X(\pi_{\mathfrak{p}}F), X\mu] \\ &= [\mathcal{M}X(\pi_{\mathfrak{p}}F), x] \end{aligned} \tag{5.82}$$

By taking $F = x$ one gets back (5.71).
Using (5.80) one can rewrite the bracket $\{\ ,\ \}_1$ on P_μ as follows, where we write $F = F_2$ and $G = G_2$:

$$\begin{aligned} \{f,g\}_1(q,x) =& < x, [\pi_{\mathfrak{a}}F, R(G)] + [R(F), \pi_{\mathfrak{a}}G] > \\ =& < x, [\pi_{\mathfrak{a}}F, R(\pi_{\mathfrak{p}}G)] + [R(\pi_{\mathfrak{p}}F), \pi_{\mathfrak{a}}G] > \\ =& < x, [\pi_{\mathfrak{p}}F, R(\pi_{\mathfrak{p}}G)] + [R(\pi_{\mathfrak{p}}F), \pi_{\mathfrak{p}}G] > \\ &- < x, [\pi_0^\perp \pi_{\mathfrak{p}}F, R(\pi_{\mathfrak{p}}G)] + [R(\pi_{\mathfrak{p}}F), \pi_0^\perp \pi_{\mathfrak{p}}G] > \end{aligned} \tag{5.83}$$

The second term in (5.83) becomes, using Lemma 5.1:

$$\begin{aligned} &- < X\mu, [\pi_0^\perp \pi_{\mathfrak{p}}F, R(\pi_{\mathfrak{p}}G)] + [R(\pi_{\mathfrak{p}}F), \pi_0^\perp \pi_{\mathfrak{p}}G] > \\ &=< \mu, X[\pi_0^\perp F_p, R(G_p)] + X[R(F_p), \pi_0^\perp G_p] > \\ &= c < \mu, \pi_0^\perp [X^{-1}(F_p), X^{-1}(G_p)] > - < \mu, \pi_0^\perp [X(F_p), X(G_p)] > \\ &= c < \mu, [X^{-1}(F_p), X^{-1}(G_p)] > - < \mu, [X(F_p), X(G_p)] > \end{aligned} \tag{5.84}$$

where F_p resp. G_p denote the components of F resp. G in $\mathfrak{p}$. So the Poisson bracket on P_μ can be written as:

$$\begin{aligned} \{f,g\}_1(q,x) =& < x, [R(F_p), G_p] + [F_p, R(G_p)] > \\ &+ c < \mu, [X^{-1}(F_p), X^{-1}(G_p)] > \\ &- < \mu, [X(F_p), X(G_p)] > \end{aligned} \tag{5.85}$$

so the Hamilton vectorfield corresponding to a function f becomes

$$\begin{aligned} v_f(q,x) = (&\pi_{\mathfrak{a}}F, \pi_{\mathfrak{p}}[x, R(F_p)] - \pi_{\mathfrak{p}}R[x, F_p] \\ &- cX^{-1}[\mu, X^{-1}(F_p)] + X[\mu, X(F_p)] - F_1) \end{aligned} \tag{5.86}$$

Theorem 5.23. Let P_μ be the phase space of the models of type I, II or III, with Poisson bracket (5.40). Let $f, g \in C^\infty(\mathfrak{g})$ be Ad-invariant functions, viewed as functions on P_μ, where μ satisfies (5.58). Then they are in involution on P_μ^0 and the Hamiltonian vectorfield on P_μ^0 corresponding to such an Ad-invariant f can be written as:

$$v_f(q,x) = (\pi_{\mathfrak{a}} F, [\mathcal{M}X(F_p) - R(F_p), x]) \tag{5.87}$$

Proof. For the models of type I, II and III, one has $c = 0$, if the function $x(\eta)$ is chosen of type I, IIb and IIIb. If $(q,x) \in P_\mu^0$ then $x \in \mathfrak{p}$, so if f is Ad-invariant then $[F_p, x] = 0$. Using this and (5.82), one can rewrite (5.86) as:

$$\begin{aligned} \dot{x} &= [x, R(F_p)] + X[\mu, X(F_p)] \\ &= [x, R(F_p)] + [\mathcal{M}X(F_p), x] \end{aligned}$$

Now the involution of f and g follows as a corollary, because $[G, x] = 0$. □

Remark 5.24. For the general model of type IV this proof does not work, because it is not clear how the c-term in (5.86) can be recast in Lax form.

6. Construction of examples

6.1. Classification of μ.

So we have shown in the previous chapter that (5.58) is a sufficient condition to derive the Lax equation and to prove that the Ad-invariant functions are in involution. In section 5.5 it was shown that (5.58) is equivalent with a restriction on the dimension of the AdK-orbit through μ. From this condition we derive a necessary inequality and classify all elements μ which satisfy that inequality. In the case of the classical root systems we thereby restrict ourselves to the Lie algebras of quasi-split type. One reason for this restriction is that all the examples that are found up to now, are connected with quasi-split Lie algebras. Moreover, in these cases μ can be defined in a canonical way, as will be shown in section 6.2 and 6.3. For the root systems of exceptional type we also list the possible elements in the non-quasi-split case.
So let $\mathfrak{g}$ be a real simple noncompact Lie algebra with Cartan involution θ. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ which contains $\mathfrak{h}_k$. From the classification in [30] it follows that $\mathfrak{k}$ will be of the form:

$$\mathfrak{k} = \mathfrak{c} \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 = \mathfrak{c} \oplus \mathfrak{k}_s \tag{6.1}$$

where $\mathfrak{c}$ denotes the center of $\mathfrak{k}$, with $\dim \mathfrak{c} \leq 1$ and $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are simple. Evidently one has $\mathfrak{c} \subset \mathfrak{t}$ and so

$$\mathfrak{t} = \mathfrak{c} \oplus \mathfrak{t}_1 \oplus \mathfrak{t}_2 \tag{6.2}$$

where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are Cartan subalgebras of $\mathfrak{k}_1$ and $\mathfrak{k}_2$. From Prop. 5.14 it follows that it is sufficient to look for elements $\mu \in \mathfrak{k} \cap \mathfrak{m}^\perp$ which satisfy

$$\dim \mathrm{Cent}_{\mathfrak{k}}(\mu) = \dim \mathfrak{k} - 2 \dim \mathfrak{m} + 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \tag{6.3}$$

but because $\mathrm{Cent}_{\mathfrak{k}}(\mu) = \mathfrak{c} \oplus \mathrm{Cent}_{\mathfrak{k}_s}(\mu)$ this becomes:

$$\dim \mathrm{Cent}_{\mathfrak{k}_s}(\mu) = \dim \mathfrak{k}_s - 2 \dim \mathfrak{m} + 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \tag{6.4}$$

Furthermore we assume that $\mathfrak{g}$ is not split, so $\mathfrak{m} \neq \{0\}$, otherwise $\mu \in \mathfrak{c}$. Because $\mathfrak{k} = \mathrm{Ad}K(\mathfrak{t})$ and because dimension relations are the same for the whole orbit, we are looking for elements $t \in \mathfrak{t}$, of the form:

$$t = t_c + t_1 + t_2 = t_c + t_s \tag{6.5}$$

which satisfy:

$$\dim \mathrm{Cent}_{\mathfrak{k}_s}(\mathrm{t}) = \dim \mathfrak{k}_s - 2 \dim \mathfrak{m} + 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \tag{6.6}$$

where t_c is the component of t in $\mathfrak{c}$ and t_s the component of t in $\mathfrak{t}_1 \oplus \mathfrak{t}_2$.
If $\mu \in \mathfrak{c}$ then μ satisfies (6.3), but because we also require that $\mu \in \mathfrak{m}^\perp$ this is only a nontrivial solution if $\mathfrak{c} \subset \mathfrak{m}^\perp$, so:

Lemma 6.1. Suppose $\dim \mathfrak{c} = 1$, then $\mu \in \mathfrak{c}$ satisfies (6.3) and $0 \neq \mu \in \mathfrak{m}^\perp$ iff $\mathfrak{c} \subset \mathfrak{m}^\perp$.

In general $\mathfrak{c} \not\subset \mathfrak{m}^\perp$ but:

Lemma 6.2. $\mathfrak{c} \cap \mathfrak{m} = \{0\}$.

Proof. Suppose $c \in \mathfrak{c} \cap \mathfrak{m}$, choose $q \in \mathfrak{a}_+$. Normalize the root vectors e_α such that $e_\alpha - e_{-\alpha} \in \mathfrak{p}$, then

$$[q, [c, e_\alpha - e_{-\alpha}]] = [c, [q, e_\alpha - e_{-\alpha}]] = 0$$

because $[q, e_\alpha - e_{-\alpha}] \in \mathfrak{k}$ and $c \in \mathfrak{c}$. So $[c, e_\alpha - e_{-\alpha}] \in \mathfrak{a}$ because q is regular. But because $c \in \mathfrak{m}$ one also has $[c, e_\alpha - e_{-\alpha}] \in \mathfrak{a}^\perp$, which implies $[c, e_\alpha] = 0$ for all $\alpha \in R_+$, because $[\mathfrak{m}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$. But this means that $c \in \mathcal{Z}(\mathfrak{g}) = \{0\}$. $\square$

Now consider the complexification $\mathfrak{k}_C = \mathfrak{k}_s \oplus i\mathfrak{k}_s$, which is again semisimple, with Cartan subalgebra $\mathfrak{h} = \mathfrak{t}_s \oplus i\mathfrak{t}_s$. Let $\mathfrak{h}_1 = \mathfrak{t}_1 \oplus i\mathfrak{t}_1$ and $\mathfrak{h}_2 = \mathfrak{t}_2 \oplus i\mathfrak{t}_2$. Let Φ denote the root system of the pair $(\mathfrak{k}_C, \mathfrak{h})$ with root space decomposition

$$\mathfrak{k}_C = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha \tag{6.7}$$

Then $\dim \mathfrak{g}^\alpha = 1$. For each $\alpha \in \mathfrak{h}^*$ define $t_\alpha \in \mathfrak{h}$ such that

$$\kappa(t_\alpha, h) = \alpha(h) \text{ for all } h \in \mathfrak{h} \tag{6.8}$$

where $\kappa(,)$ is the Killing form of $\mathfrak{k}_C$. Then $i\mathfrak{t}_s = \mathbb{R} \prec t_\alpha, \alpha \in \Phi \succ$. So it is clear that our problem is equivalent to finding an element $h \in i\mathfrak{t}_s$ which satisfies

$$\dim \mathrm{Cent}_{\mathfrak{k}_C}(\mathrm{h}) = \dim \mathfrak{k}_s - 2 \dim \mathfrak{m} + 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \tag{6.9}$$

Now it is easy to see that

$$\dim \mathrm{Cent}_{\mathfrak{k}_C}(\mathrm{h}) = \dim_C \mathfrak{h} + |\{\alpha \in \Phi \mid \alpha(\mathrm{h}) = 0\}| \tag{6.10}$$

but $\dim_C \mathfrak{h} = \dim \mathfrak{t}_s$, so (6.9) becomes

$$\dim \mathfrak{k}_s - \dim \mathfrak{t}_s - 2 \dim \mathfrak{m} + 2 \dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = |\{\alpha \in \Phi \mid \alpha(\mathrm{h}) = 0\}| \tag{6.11}$$

Now let $\tilde{\Delta}$ denote the simple roots of Φ, Φ_+ the positive roots. Identify $\mathfrak{h}$ with $\mathfrak{h}^*$, using the Killing form of $\mathfrak{k}_C$. Define the fundamental dominant weights $\lambda_j \in \mathfrak{h}^*$ by $(\lambda_j, \alpha_k) = \frac{1}{2}(\alpha_k, \alpha_k)\delta_{jk}$, where $\alpha_k \in \tilde{\Delta}$.
Because of the Weyl group invariance of our problem and because each weight $\lambda \in \mathfrak{h}^*$ is conjugate under the Weyl group to a dominant weight, it is sufficient to find a a dominant weight

$$\lambda = \sum_{j=1}^{\dim \mathfrak{h}} c_j \lambda_j, \quad c_j \geq 0 \tag{6.12}$$

which satisfies

$$2b(\lambda) = \dim \mathfrak{k}_s - \dim \mathfrak{t}_s - 2\dim \mathfrak{m} + 2\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) \tag{6.13}$$

where

$$b(\lambda) = |\{\alpha \in \Phi_+ \mid (\alpha, \lambda) = 0\}| \tag{6.14}$$

One can calculate $b(\lambda)$ as follows: colour the vertices of the Dynkin diagram of Φ black which correspond to the λ_j for which $c_j \neq 0$, then $2b(\lambda)$ equals the total dimension of the subdiagrams which are formed by the white vertices.

Example 6.3. Consider $\Phi = A_3$ and take $\lambda = c_1\lambda_1$, then $2b(\lambda) = |A_2| = 6$. Here $|A_2|$ denotes the total number of roots in A_2.

From (6.13) one can derive the following necessary condition:

$$2b(\lambda) \geq \dim \mathfrak{k}_s - \dim \mathfrak{t}_s - 2\dim \mathfrak{m} \tag{6.15}$$

First consider the Lie algebras of type IV. Then $\mathfrak{k}$ is semisimple and $\dim \mathfrak{m} = \dim \mathfrak{t} = l$, so (6.15) becomes:

$$2b(\lambda) \geq \dim \mathfrak{k} - 3l \tag{6.16}$$

1. Consider $\mathfrak{g} = \mathbf{sl(n+1,\mathbb{C})}$, viewed as a real Lie algebra, then $\mathfrak{k} = su(n+1), R = \Phi = A_n, l = n, \dim \mathfrak{k} = n(n+2)$. It is easy to see that $2b(\lambda) \leq \dim \mathfrak{k} - 3l$ and one has equality if $\lambda = c_1\lambda_1$ or $\lambda = c_n\lambda_n$, with $c_1, c_n \neq 0$, because $|A_{n-1}| = n(n-1) = n(n+2) - 3n$. So, combining this with (6.13), this means that one must require that $\mathrm{Cent}_{\mathfrak{m}}(\mu) = \{0\}$.

It turns out that this is the only solution in the case of the type IV algebras, because, using the information in [29],[30] and [35] and the inequalities:

$$|B_n| = |C_n|$$

$$|D_n| \leq |A_n| \leq |B_n| \text{ for } n \leq 3$$

$$|A_n| < |D_n| < |B_n| \text{ for } n > 3$$

it is easy to verify that for all the other root systems

$$2b(\lambda) < \dim \mathfrak{k} - 3l$$

Example 6.4. Take $\mathfrak{g} = G_2$, then $\dim \mathfrak{k} = 14$, but $2b(\lambda) \leq |A_1| < \dim \mathfrak{k} - 3l = 8$.

Next consider the normal real forms. Then, cf. Lemma 3.12, $\mathfrak{g}$ is split, i.e. $\dim \mathfrak{m} = 0$, so (6.3) implies that $\mu \in \mathfrak{c}$. So all the possible solutions correspond with the cases where $\mathfrak{k}$ is not semisimple. This gives the following examples:

2.**AI** ($n = 2$): $\mathfrak{g} = sl(2,\mathbb{R}), \mathfrak{k} = so(2), \dim \mathfrak{k} = 1, R = A_1$ (isomorphic with AIII ($p = q = 1$)).

3. **DI** ($p = q+2$): $\mathfrak{g} = so(2,2), \mathfrak{k} = so(2) \oplus so(2), R = D_2 \cong A_1 \times A_1$ (isomorphic with AI ($n = 2$)$\times$AI ($n = 2$)).

4. **CI** ($n \geq 1$): $\mathfrak{g} = sp(n,\mathbb{R}), \mathfrak{k} = u(n), R = C_n$ (the case $n = 1$ is isomorphic with AI ($n = 2$)).

5. **BI** ($q = 1, p = 2$): $\mathfrak{g} = so(2,1), \mathfrak{k} = so(2), R = B_1 = A_1$ (isomorphic with AI ($n = 2$)).

6. **BI** ($q = 2, p = 3$): $\mathfrak{g} = so(3,2), \mathfrak{k} = so(3) \oplus so(2), R = B_2$ (isomorphic with CI ($n = 2$)). So the only non-isomorphic case is CI ($n \geq 2$), with $\mu \in \mathfrak{c}$.

Next consider the quasi-split Lie algebras of type III which are not normal real forms.

7. **AIII** ($p = q = n \geq 1$): $\mathfrak{g} = su(n,n), \mathfrak{k} = \mathfrak{c} \oplus su(n) \oplus su(n), R = C_n, \Phi = A_{n-1} \oplus A_{n-1}, \dim \mathfrak{k} = 2n^2 - 1, \dim \mathfrak{m} = n - 1, \dim \mathfrak{t} = 2n - 1$, so $\dim \mathfrak{k}_s - \dim \mathfrak{t}_s - 2 \dim \mathfrak{m} = 2(n-1)^2$ and the only solutions are:

$$\lambda = (c_1\lambda_1, 0), \quad \lambda = (c_{n-1}\lambda_{n-1}, 0)$$

Here λ_1 and λ_{n-1} denote fundamental dominant weights and the notation $(c_1\lambda_1, 0)$ means that the component of λ in $\mathfrak{h}_1^*$ is $c_1\lambda_1$ and the component of λ in $\mathfrak{h}_2^*$ is zero. These two cases are conjugate under the Weyl group so consider the first one. If $c_1 = 0$ then $\mu \in \mathfrak{c}$ and because in this case $\mathfrak{c} \subset \mathfrak{m}^\perp$, this is a solution. If $c_1 \neq 0$ then

$$2b(\lambda) = |A_{n-2}| + |A_{n-1}| = (n-1)(n-2) + (n-1)n = 2(n-1)^2$$

so one has equality in (6.15) and so one must require that $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$. Of course one also has the solutions $\lambda = (0, c_1\lambda_1)$ and $\lambda = (0, c_{n-1}\lambda_{n-1})$ with $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$.
From this it is clear that for all other choices of λ one has $2b(\lambda) < 2(n-1)^2$.

8. **AIII** ($p = q + 1 = n + 1 \geq 2$): $\mathfrak{g} = su(n+1, n)$ and $\mathfrak{k} = \mathfrak{c} \oplus su(n+1) \oplus su(n), \dim \mathfrak{k} = 2n(n+1), R = BC_n, \Phi = A_n \oplus A_{n-1}, \dim \mathfrak{m} = n, \dim \mathfrak{t} = 2n$, so $\dim \mathfrak{k}_s - \dim \mathfrak{t}_s - 2 \dim \mathfrak{m} = 2n(n-1)$ and one has the solution $\lambda = (c_1\lambda_1, 0)$ or the conjugate solution $\lambda = (c_n\lambda_n, 0)$. In this case $\mathfrak{c} \not\subset \mathfrak{m}^\perp$, so one must take $c_1 \neq 0$ and then $2b(\lambda) = 2|A_{n-1}| = 2n(n-1)$. So one has again equality in (6.15) and one must require that $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$. Another solution is $\lambda = (0, c_1\lambda_1)$ or $\lambda = (0, c_{n-1}\lambda_{n-1})$, with $c_1, c_{n-1} \neq 0$ and $2b(\lambda) = 2(n^2 - n + 1) \geq 2n(n-1)$. In this case one must require that $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 1$. One can easily verify that for $n \geq 5$ these are the only solutions and for all other choices of λ we have $2b(\lambda) < 2n(n-1)$. For $n = 3$ one has furthermore the solution $\lambda = (0, c_1\lambda_1 + c_2\lambda_2)$, with $c_1 \neq 0 \wedge c_2 \neq 0$ and $2b(\lambda) = 12$ so one must have $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$. For $n = 4$ there is the solution $\lambda = (0, c_2\lambda_2)$ with $2b(\lambda) = 24$ so again $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$.

9. **BDI** ($p = q + 2, q = n \geq 1$): $\mathfrak{g} = so(n+2, n), \mathfrak{k} = so(n+2) \oplus so(n), \dim \mathfrak{k} = n(n+1) + 1, \dim \mathfrak{m} = 1, R = B_n$ and

$$\begin{aligned} \dim \mathfrak{t} &= n + 1 \text{ if } n \text{ is even} \\ &= n \text{ if } n \text{ is uneven} \end{aligned}$$

The $n = 1$ case is isomorphic with the case $\mathfrak{g} = sl(2, \mathbb{C})$ and the $n = 2$ case is isomorphic with AIII ($p = q = 2$), so let $n \geq 3$. For $n = 3$ one has $\dim \mathfrak{k} - \dim \mathfrak{t} - 2 \dim \mathfrak{m} = 8$ and $\mathfrak{k} \cong B_2 \oplus A_1$ and the only solution is $\lambda = (0, c_1\lambda_1)$ for which $2b(\lambda) = 8$. Because one has equality in (6.15) one must require $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$. But in this case $\mathfrak{m} \subset so(5)$, so $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 1$, which means that there is no λ satisfying (6.13).
For $n = 4$ one has $\dim \mathfrak{k} - \dim \mathfrak{t} - 2 \dim \mathfrak{m} = 14$ and $\mathfrak{k} = so(6) \oplus so(4) \cong A_3 \oplus D_2$ and $D_2 \cong A_1 \times A_1$. The only λ which satisfies (6.15) is $\lambda = (0, c_1\lambda_1, 0)$ and in this case $2b(\lambda) = |A_3| + |A_1| = 14$. Again one has equality in (6.15) so one must require $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$, but because $\mathfrak{m} \subset so(6)$ one has $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 1$. So again there is no λ which satisfies (6.13). One can easily check that for $n \geq 5$ one has $2b(\lambda) < \dim \mathfrak{k} - \dim \mathfrak{t} - 2 \dim \mathfrak{m}$.

10. **EII**: $\mathfrak{g} = E_6, R = F_4, \mathfrak{k} = su(6) \oplus su(2), \dim \mathfrak{k} = 38, \dim \mathfrak{m} = 2, \dim \mathfrak{t} = 6$, so $\dim \mathfrak{k} - \dim \mathfrak{t} - 2 \dim \mathfrak{m} = 28$ and because $\Phi = A_5 \oplus A_1$, the only possibility is $\lambda = (0, c_1\lambda_1)$ with $c_1 \neq 0$ and then $2b(\lambda) = |A_5| = 30$, so for (6.13) to be satisfied one must have $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 1$. But $\mathfrak{m} \subset su(6)$ so this cannot be the case.

Let us finally consider the non-quasi-split cases where R is of exceptional type.

11. **EVI**: $\mathfrak{g} = E_7, R = F_4, \mathfrak{k} = so(12) \oplus su(2), \Phi = D_6 \oplus A_1, \dim \mathfrak{k} = 69, \dim \mathfrak{m} = 9, \dim \mathfrak{t} = 7$, so $\dim \mathfrak{k} - \dim \mathfrak{t} - 2 \dim \mathfrak{m} = 44$, and the only solution is $\lambda = (0, c_1\lambda_1)$ with $c_1 \neq 0$ and $2b(\lambda) = |D_6| = 60$, so one must require that $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 8$. But $\mathfrak{m} \subset so(12)$ so this cannot be the case.

12. **EIX**: $\mathfrak{g} = E_8, R = F_4, \mathfrak{k} = E_7 \oplus su(2), \Phi = E_7 \oplus A_1, \dim \mathfrak{k} = 136, \dim \mathfrak{m} = 28, \dim \mathfrak{t} = 8$, so one must require $2b(\lambda) \geq 72$. There are two possibilities:
$\lambda = (0, c_1\lambda_1)$ with $c_1 \neq 0$ and $2b(\lambda) = |E_7| = 126$, so $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 27$
and
$\lambda = (c_1\lambda_1, c_1'\lambda_1)$ with $c_1 \neq 0 \wedge c_1' \neq 0$ and $2b(\lambda) = |E_6| = 72$, so one must require $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = 0$.

So let $\mathfrak{g}$ be a real semisimple noncompact Lie algebra with restricted root system of exceptional type, then the only possible choices for μ which cannot be ruled out correspond with the root system F_4 and are given in 12.

Remark 6.5. This implies that the construction of the Lax pair in chapter 5 does not work for the cases E_6, E_7, E_8 and G_2.

Remark 6.6. Another reason for restricting to the quasi-split case is the following: the analysis in this section determines dominant weights $\lambda \in \mathfrak{h}^*$ which satisfy the inequality (6.15) and so corresponding elements $t \in \mathfrak{t}$ which satisfy (6.15). To obtain from this an element $\mu \in \mathfrak{m}^\perp$ which satisfies (6.3) one has to be sure that the AdK-orbit through t intersects $\mathfrak{m}^\perp$, and that the constants c_j can be chosen in such a way that $\mathrm{Cent}_{\mathfrak{m}}(\mu)$ has the right dimension. If $\mathfrak{g}$ is quasi-split one has $\mathfrak{m} \subset \mathfrak{t}$ so $\mathfrak{t}^\perp \subset \mathfrak{m}^\perp$. So one only has to know that the AdK-orbit through t intersects $\mathfrak{t}^\perp$, and this follows from a convexity theorem of Kostant (see [31, p. 473]).

6.2. A construction of μ for quasi-split Lie algebras.

Now let σ be a (nontrivial) involutive automorphism of $\mathfrak{g}$, i.e. $\sigma^2 = \mathrm{id}$, which commutes with the Cartan involution θ. The corresponding eigenspace decomposition is denoted by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, with commutation relations:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}, \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h} \tag{6.17}$$

Because θ and σ commute one also has:

$$\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}), \quad \mathfrak{p} = (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \tag{6.18}$$

and all these decompositions are orthogonal with respect to the Killing form. Now choose $\mathfrak{a}$ and $\mathfrak{h}_R$ σ-stable. Then $\mathfrak{m}$ is σ-stable too and one has the orthogonal decomposition:

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{q}), \quad \mathfrak{m} = (\mathfrak{m} \cap \mathfrak{h}) \oplus (\mathfrak{m} \cap \mathfrak{q}) \tag{6.19}$$

Furthermore consider the involutive automorphism $\sigma^a = \sigma\theta$ with eigenspace decomposition $\mathfrak{g} = \mathfrak{h}^a \oplus \mathfrak{q}^a$ and

$$\mathfrak{h}^a = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}, \quad \mathfrak{q}^a = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h} \tag{6.20}$$

Now suppose that the triple $(\mathfrak{g}, \sigma, \theta)$ satisfies one of the following conditions:

$$\text{(i) } \dim(\mathfrak{g}_\alpha \cap \mathfrak{q}^a) = 1 \text{ for all } \alpha \in R \tag{6.21}$$

$$\text{(ii) } \dim(\mathfrak{g}_\alpha \cap \mathfrak{h}^a) = 1 \text{ for all } \alpha \in R \tag{6.22}$$

If this is the case, define root vectors e_α and $e_{-\alpha}$ as follows:
(i) Choose for all $\alpha \in R_+$ the root vector $e_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{q}^a$ which satisfies (3.22) and define $e_{-\alpha} = \theta(e_\alpha) \in \mathfrak{g}_{-\alpha}$. Then one has $\sigma(e_\alpha) = -\theta(e_\alpha) = -e_{-\alpha}$, so

$$e_\alpha + e_{-\alpha} \in \mathfrak{k} \cap \mathfrak{q}, \quad e_\alpha - e_{-\alpha} \in \mathfrak{p} \cap \mathfrak{h} \tag{6.23}$$

Now define:

$$\mu = \sum_{\alpha \in R_+} g_\alpha (e_\alpha + e_{-\alpha}) \in \mathfrak{k} \cap \mathfrak{q} \tag{6.24}$$

(ii) Choose for all $\alpha \in R_+$ the root vector $e_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{h}^a$ which satisfies (3.22) and define $e_{-\alpha} = \theta(e_\alpha) \in \mathfrak{g}_{-\alpha}$. Then one has $\sigma(e_\alpha) = \theta(e_\alpha) = e_{-\alpha}$, so

$$e_\alpha + e_{-\alpha} \in \mathfrak{k} \cap \mathfrak{h}, \quad e_\alpha - e_{-\alpha} \in \mathfrak{p} \cap \mathfrak{q} \tag{6.25}$$

Now define:

$$\mu = \sum_{\alpha \in R_+} g_\alpha (e_\alpha + e_{-\alpha}) \in \mathfrak{k} \cap \mathfrak{h} \tag{6.26}$$

There is a strong relation between triples $(\mathfrak{g}, \sigma, \theta)$ which satisfy (6.21) or (6.22) and quasi-split Lie algebras. Indeed, comparing Prop. 3.16 with Table IV in [41] one can verify that the following is true:

Proposition 6.7. Let $\mathfrak{g}$ be a real noncompact simple Lie algebra, then the following properties are equivalent:
(i) $\mathfrak{g}$ is quasi-split.
(ii) there exists an involutive automorphism σ and a Cartan involution θ which commutes with σ such that (6.21) or (6.22) is true.

In the next section we will see that all the examples of μ that are known to satisfy (5.58) belong to quasi-split Lie algebras and can be constructed as in (6.24) and (6.26). If (6.21) or (6.22) is true then one also has:

Lemma 6.8. Suppose $(\mathfrak{g}, \sigma, \theta)$ satisfies (6.21) or (6.22) then:
(i) $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$
(ii) $\mathfrak{m} \subset \mathfrak{k} \cap \mathfrak{q}$

Proof. We prove it for the case (6.21). The other case is similar.
(i) Observe that $\mathfrak{a}$ is spanned by the t_α and

$$\sigma(t_\alpha) = [\sigma(e_{-\alpha}), \sigma(e_\alpha)] = -[e_{-\alpha}, e_\alpha] = -t_\alpha$$

so $t_\alpha \in \mathfrak{q}$.
(ii) Choose for $\alpha \in R_+$ a root vector $f_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{h}^a$ which satisfies (3.22) and define $f_{-\alpha} = \theta(f_\alpha) \in \mathfrak{g}_{-\alpha}$. Then $\sigma\theta(f_\alpha) = f_\alpha$, so $\sigma(f_{-\alpha}) = f_\alpha$. Now define

$$\tilde{h}_\alpha = \tfrac{1}{2}[e_\alpha, f_{-\alpha}] + \tfrac{1}{2}[e_{-\alpha}, f_\alpha] \tag{6.27}$$

then $\theta(\tilde{h}_\alpha) = \tilde{h}_\alpha$, so $\tilde{h}_\alpha \in \mathfrak{k}$ and for all $a \in \mathfrak{a}$ one has $[a, \tilde{h}_\alpha] = 0$, so $\tilde{h}_\alpha \in \mathfrak{g}_0 \cap \mathfrak{k} = \mathfrak{m}$ and also $\sigma(\tilde{h}_\alpha) = -\tilde{h}_\alpha$, so $\tilde{h}_\alpha \in \mathfrak{m} \cap \mathfrak{q}$ and, beause $\dim \mathfrak{g}_\alpha \leq 2$, $\mathfrak{m}$ is spanned by the $\tilde{h}_\alpha$'s and so we conclude that $\mathfrak{m} \subset \mathfrak{k} \cap \mathfrak{q}$. □

Now suppose one has the situation of (6.21) and $\mu \in \mathfrak{k} \cap \mathfrak{q}$ is defined as in (6.24). Then $\mathfrak{k} \cap \mathfrak{q} \subset \operatorname{Ker} A$ because $\mathfrak{m} \subset \mathfrak{k} \cap \mathfrak{q}$ and because of the commutation relations. To prove that μ satisfies (5.58) it is sufficient to prove it for all $z \in \mathfrak{k} \cap \mathfrak{q}$, because:

Lemma 6.9. Suppose one has the situation of (6.21), and suppose that $\mu \in \mathfrak{k} \cap \mathfrak{q}$ satisfies (5.58) for all $z \in \mathfrak{k} \cap \mathfrak{q}$. Then also the following is true:

$$w \in \mathfrak{k} \cap \mathfrak{h} \cap \operatorname{Ker} A \Rightarrow [w, \mu] = 0$$

and so μ satisfies (5.58) for all $z \in \mathfrak{k}$.

Proof. For all $z \in \mathfrak{k} \cap \mathfrak{q}$ one has

$$\begin{aligned} &< [w, \mu], z >=< w, [\mu, z] >=< w, [\mu, m(z)] > \\ &=< [w, \mu], m(z) >=< \pi_{\mathfrak{m}}[w, \mu], m(z) >= 0 \end{aligned}$$

so we conclude that $[w, \mu] = 0$. □

Now suppose one has the situation of (6.22) and $\mu \in \mathfrak{k} \cap \mathfrak{h}$ is defined as in (6.26). Then $\mathfrak{k} \cap \mathfrak{h} \in \operatorname{Ker} A$, because of the commutation relations and so μ can only satisfy (5.58) if $\mathfrak{k} \cap \mathfrak{h} \in \operatorname{Cent}_{\mathfrak{k}}(\mu)$.

Example 6.10. Let $\mathfrak{g}$ be a normal real form and $\sigma = \theta$ then $\mathfrak{k} = \mathfrak{h}$ and $\mathfrak{p} = \mathfrak{q}$, so $\mathfrak{h}^a = \mathfrak{g}$ and $\mathfrak{q}^a = \{0\}$ and clearly condition (6.22) is satisfied. If now μ satisfies (5.58) then $\mathfrak{k} = \operatorname{Cent}_{\mathfrak{k}}(\mu)$, so μ must be a central element of $\mathfrak{k}$ and this is only possible if $\mathfrak{k}$ is not semisimple. Observe that this also follows from condition (ii) of Prop. 5.14 because $\dim \mathfrak{m} = 0$.

6.3. Explicit construction of examples.

In this section we will give an explicit construction of μ and the corresponding Lax pair for the classical root systems. In all these cases $\mathfrak{g}$ is quasi-split and μ can be defined as in (6.24) and (6.26).

6.3.1 The A_{n-1} case

Let $\mathfrak{g} = sl(n, \mathbb{C})$, viewed as a real Lie algebra, with the Killing form given by

$$< x, y >= \mathrm{Re}\,\mathrm{tr}(xy) \tag{6.28}$$

The Cartan involution θ is given by $\theta(x) = -x^\dagger$, where $\dagger$ denotes the hermitian conjugate, and we choose $\sigma(x) = -x^t$. Then $\mathfrak{k} = \{\text{skew-hermitian matrices}\}$, $\mathfrak{p} = \{\text{hermitian matrices}\}$, $\mathfrak{h} = \{\text{complex skew-symmetric matrices}\}$, $\mathfrak{q} =$- $\{\text{ complex symmetric matrices}\}$. Choose $\mathfrak{a} = \{\text{real diagonal matrices}\}$, then $\mathfrak{a}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{m} = i\mathfrak{a}$. Moreover $\mathfrak{a}, \mathfrak{m} \subset \mathfrak{q}$. Let $q = \mathrm{diag}(q_1, \ldots, q_n) \in \mathfrak{a}$ and define $\varepsilon_j(q) = q_j$. Then the restricted root system R is given by $R = \{\alpha_{jk} := \varepsilon_j - \varepsilon_k, j \neq k\}$ and is of type A_{n-1}.
Choose

$$\Delta = \{\alpha_j := \varepsilon_j - \varepsilon_{j+1}, 1 \leq j \leq n-1\} \tag{6.29}$$

then the positive roots are $R_+ = \{\alpha_{jk} \in R, j < k\}$ and $R_- = \{\alpha_{jk} \in R, j > k\}$. The root spaces $\mathfrak{g}_\alpha$ are given by

$$\mathfrak{g}_{\alpha_{jk}} = \mathrm{Span} \prec e_{jk}, ie_{jk} \succ \tag{6.30}$$

and so $m_\alpha = 2$ for all $\alpha \in R$. Also

$$\mathfrak{g}_{\alpha_{jk}} \cap \mathfrak{q}^a = \mathrm{Span} \prec ie_{jk} \succ$$

and

$$\mathfrak{g}_{\alpha_{jk}} \cap \mathfrak{h}^a = \mathrm{Span} \prec e_{jk} \succ \tag{6.31}$$

so $\dim \mathfrak{g}_\alpha \cap \mathfrak{q}^a = \dim \mathfrak{g}_\alpha \cap \mathfrak{h}^a = 1$ for all $\alpha \in R$.
Following the construction in Section 6.2 choose $e_{\alpha_{jk}} = ie_{jk}$ and $f_{\alpha_{jk}} = e_{jk}$ which satisfy the normalization (3.22). So $e_{-\alpha_{jk}} = ie_{kj}$, $f_{-\alpha_{jk}} = -e_{kj}$, and define, as in (6.24):

$$\mu = g \sum_{j<k} i(e_{jk} + e_{kj}), \quad g \in \mathbb{R} \tag{6.32}$$

which is precisely the element μ as defined in (1.23). Because A_{n-1} has only one root length, there is only one coupling constant, which is denoted by g.
Moreover:

$$\begin{aligned} \mathfrak{k} \cap \mathfrak{q} \cap \mathfrak{m}^\perp &= \mathrm{Span} \prec e_\alpha + e_{-\alpha}, \alpha \in R_+ \succ \\ \mathfrak{k} \cap \mathfrak{h} &= \mathrm{Span} \prec f_\alpha + f_{-\alpha}, \alpha \in R_+ \succ \\ \mathfrak{p} \cap \mathfrak{h} &= \mathrm{Span} \prec e_\alpha - e_{-\alpha}, \alpha \in R_+ \succ \\ \mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{a}^\perp &= \mathrm{Span} \prec f_\alpha - f_{-\alpha}, \alpha \in R_+ \succ \end{aligned} \tag{6.33}$$

Furthermore one has:

$$t_{\alpha_{jk}} = h_{\alpha_{jk}} = d_j - d_k \tag{6.34}$$

where $d_j = e_{jj}$ and so $(\alpha, \alpha) = 2$. Here t_α and h_α are as defined in (3.9) and (3.13).
The positive Weyl chamber and Weyl alcove are given by:

$$\mathfrak{a}_+ = \{q \in \mathfrak{a} \mid q_j - q_k > 0 \text{ if } j < k\}$$

$$\mathfrak{a}_+^d = \{q \in \mathfrak{a} \mid 0 < q_j - q_k < d \text{ if } j < k, d \in \mathbb{R}\} \tag{6.35}$$

Lemma 6.11. $\mathrm{Cent}_{\mathfrak{g}_0}(\mu) = \{0\}$ if $g \neq 0$.
Proof. Let $q \in \mathfrak{a}$ and $[q, \mu] = 0$ then $g \sum_{\alpha \in R_+} \alpha(q)(e_\alpha - e_{-\alpha}) = 0$ which implies $\alpha(q) = 0$ for all $\alpha \in R$ and so $q = 0$. Because $\mathfrak{m} = i\mathfrak{a}$ the Lemma follows. □

Proposition 6.12. The element μ, as defined in (6.32), satisfies (5.58) and the map $\mathcal{M} : \mathfrak{k} \cap \mathfrak{q} \cap \mathfrak{m}^\perp \to \mathfrak{m}$ is given by:

$$\mathcal{M}(e_{\alpha_{jk}} + e_{-\alpha_{jk}}) = i(\frac{2}{n} I_n - d_j - d_k) \tag{6.36}$$

Proof. From Lemma 6.9 it follows that it is sufficient to construct the map $\mathcal{M}$ for $\mathfrak{k} \cap \mathfrak{q}$ and one easily checks that $[\mu, (\mathrm{id} + \mathcal{M})(e_\alpha + e_{-\alpha})] = 0$. □

So μ satisfies (5.58) which was sufficient to prove integrability. Now use (6.36) in (5.72) and one gets back the Lax pair (L, M) as defined in (1.45) and (1.57).

Remark 6.13. One can write μ in the dyadic (see [34] for properties of dyads) form:

$$\mu = ig(e \otimes e - I_n) \tag{6.37}$$

where $e = (1, \ldots, 1) \in \mathbb{R}^n$ and one easily checks that μ is conjugate under $\mathrm{Ad}K$ to $ngi\lambda_1$ if $g \geq 0$ and to $-ngi\lambda_{n-1}$ if $g \leq 0$. Here λ_1 and λ_{n-1} are fundamental dominant weights of $sl(n, \mathbb{C})$, with

$$\lambda_1 = \frac{1}{n}\mathrm{diag}(n-1, -1, \ldots, -1), \quad \lambda_{n-1} = \frac{1}{n}\mathrm{diag}(1, 1, \ldots, 1, 1-n) \tag{6.38}$$

This example corresponds to case 1 of section 6.1.

6.3.2 The BC_n and B_n case

Let $\mathfrak{g} = su(n+1,n)$ which consists of matrices of the form:

$$X = \begin{pmatrix} A & w & B \\ -v^\dagger & ia & -w^\dagger \\ C & v & -A^\dagger \end{pmatrix} \tag{6.39}$$

where $A \in gl(n,\mathbb{C}), B, C \in u(n), a \in \mathbb{R}, v, w \in \mathbb{C}^n$ and $\mathrm{tr}(A - A^\dagger) + ia = 0$. An element $X \in \mathfrak{g}$ will be denoted as $X = (A, B, C, w, v, ia)$.

Remark 6.14. The standard representation of $su(n+1,n)$ is given by matrices of the form

$$\tilde{X} = \begin{pmatrix} \tilde{A} & \tilde{w} & \tilde{C} \\ -\tilde{w}^\dagger & ia & \tilde{v}^\dagger \\ \tilde{C}^\dagger & \tilde{v} & \tilde{B} \end{pmatrix} \tag{6.40}$$

where $\tilde{C} \in gl(n,\mathbb{C}), \tilde{A}, \tilde{B} \in u(n), \tilde{v}, \tilde{w} \in \mathbb{C}^n, a \in \mathbb{R}$. The relation between these two representations is given by $\tilde{X} = gXg^{-1}$, where $g \in SO(2n+1)$ is the element

$$g = \begin{pmatrix} \frac{1}{\sqrt{2}}I_n & 0 & \frac{1}{\sqrt{2}}I_n \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}}I_n & 0 & \frac{1}{\sqrt{2}}I_n \end{pmatrix} \tag{6.41}$$

so that the correspondence is given by:

$$\begin{aligned} \tilde{w} &= \frac{1}{\sqrt{2}}(v+w) \\ \tilde{v} &= \frac{1}{\sqrt{2}}(v-w) \\ \tilde{A} &= \tfrac{1}{2}(B+C) + \tfrac{1}{2}(A - A^\dagger) \\ \tilde{B} &= -\tfrac{1}{2}(B+C) + \tfrac{1}{2}(A - A^\dagger) \\ \tilde{C} &= \tfrac{1}{2}(B-C) - \tfrac{1}{2}(A + A^\dagger) \end{aligned} \tag{6.42}$$

In the representation (6.39) one can take $\theta(X) = -X^\dagger, \sigma(X) = -X^t$ and

$$\begin{aligned} \mathfrak{k} &= \{(A, B, B, w, w, ia), A \in u(n)\} \\ \mathfrak{p} &= \{(A, B, -B, w, -w, 0), A^\dagger = A\} \\ \mathfrak{c} &= \mathbb{R} \prec i(-I_n, (2n+1)I_n, (2n+1)I_n, 0, 0, 2n) \succ \\ \mathfrak{h} &= \{(A, B, -B^t, w, \bar{w}, 0), A^t = -A\} \\ \mathfrak{q} &= \{(A, B, B^t, w, -\bar{w}, ia), A^t = A\} \end{aligned} \tag{6.43}$$

For $\mathfrak{a}$ one can take:

$$\mathfrak{a} = \{(D, 0, 0, 0, 0, 0), D = \mathrm{diag}(q_1, \ldots, q_n)\} \tag{6.44}$$

so $\mathfrak{a} \subset \mathfrak{q}$ and

$$\mathfrak{m} = \{(iD, 0, 0, 0, 0, ia), D \text{ real diagonal}, 2\text{tr}D + a = 0\} \tag{6.45}$$

so $\mathfrak{m} \subset \mathfrak{q}$ and $\mathfrak{m}$ is abelian. From (6.44) and (6.45) it follows that $l = n$ and $\dim \mathfrak{m} = n$, so $\delta = 0$, as expected. Also $\mathfrak{m} \cap \mathfrak{k}_1 = \mathfrak{m} \cap \mathfrak{k}_2 = \mathfrak{c} \cap \mathfrak{m} = \{0\}$, $\dim \mathfrak{c} = 1$ and $\mathfrak{c} \not\subset \mathfrak{m}^\perp$.
For the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ one can choose:

$$\mathfrak{t} = \{(iD_1, iD_2, iD_2, 0, 0, ia), D_1, D_2 \text{ real diagonal}, \text{tr}D_1 + a = 0\} \tag{6.46}$$

so $\mathfrak{m} \subset \mathfrak{t}, \mathfrak{c} \subset \mathfrak{t}$ and $\dim \mathfrak{t} = 2n$. The Killing form is given by:

$$< x, y >= \tfrac{1}{2}\text{tr}(xy) \tag{6.47}$$

Let $q = (D, 0, 0, 0, 0, 0) \in \mathfrak{a}$ and define $\varepsilon_j(q) = q_j$, then the restricted root system is given by

$$R = \{\pm\varepsilon_j, \pm 2\varepsilon_j, \pm(\varepsilon_j + \varepsilon_k), \pm(\varepsilon_j - \varepsilon_k), j < k\} \tag{6.48}$$

and is of type BC_n. For the simple roots Δ one can take $\Delta = \{\varepsilon_j - \varepsilon_{j+1}, 1 \leq j \leq n-1, \varepsilon_n\}$ and then

$$\begin{aligned} R_+ &= \{\varepsilon_j, 2\varepsilon_j, \varepsilon_j - \varepsilon_k, \varepsilon_j + \varepsilon_k, j < k\} \\ R_- &= \{-\varepsilon_j, -2\varepsilon_j, \varepsilon_k - \varepsilon_j, -(\varepsilon_j + \varepsilon_k), j < k\} \end{aligned} \tag{6.49}$$

Now write $\alpha_{jk} = \varepsilon_j - \varepsilon_k, \beta_{jk} = \varepsilon_j + \varepsilon_k$, then the corresponding root spaces are given by:

$$\begin{aligned} \mathfrak{g}_{\varepsilon_j} &= \text{Span} \prec (0,0,0,e_j,0,0), (0,0,0ie_j,0,0) \succ \\ \mathfrak{g}_{-\varepsilon_j} &= \text{Span} \prec (0,0,0,0,e_j,0), (0,0,0,0,ie_j,0) \succ \\ \mathfrak{g}_{2\varepsilon_j} &= \text{Span} \prec (0, ie_{jj}, 0,0,0,0) \succ \\ \mathfrak{g}_{-2\varepsilon_j} &= \text{Span} \prec (0,0, ie_{jj}, 0,0,0) \succ \\ \mathfrak{g}_{\alpha_{jk}} &= \text{Span} \prec (e_{jk},0,0,0,0,0), (ie_{jk},0,0,0,0,0), j < k \succ \\ \mathfrak{g}_{-\alpha_{jk}} &= \text{Span} \prec (e_{kj},0,0,0,0,0), (ie_{kj},0,0,0,0,0), j < k \succ \\ \mathfrak{g}_{\beta_{jk}} &= \text{Span} \prec (0, i(e_{jk}+e_{kj}),0,0,0,0), (0, e_{jk}-e_{kj},0,0,0,0), j < k \succ \\ \mathfrak{g}_{-\beta_{jk}} &= \text{Span} \prec (0,0, i(e_{jk}+e_{kj}),0,0,0), (0,0,e_{jk}-e_{kj},0,0,0), j < k \succ \end{aligned} \tag{6.50}$$

so $m_{\varepsilon_j} = m_{\alpha_{jk}} = m_{\beta_{jk}} = 2$ and $m_{2\varepsilon_j} = 1$. Also one has $t_{\varepsilon_j} = d_j := (e_{jj}, 0,0,0,0,0)$ and $< d_j, d_k >= \delta_{jk}$ and so:

$$t_{\varepsilon_j} = d_j, \quad (\varepsilon_j, \varepsilon_j) = 1, \quad h_{\varepsilon_j} = 2d_j$$

$$t_{2\varepsilon_j} = 2d_j, \quad (2\varepsilon_j, 2\varepsilon_j) = 4, \quad h_{2\varepsilon_j} = \tfrac{1}{2}d_j$$

$$t_{\alpha_{jk}} = d_j - d_k, \quad (\alpha_{jk}, \alpha_{jk}) = 2, \quad h_{\alpha_{jk}} = d_j - d_k$$
$$t_{\beta_{jk}} = d_j + d_k, \quad (\beta_{jk}, \beta_{jk}) = 2, \quad h_{\beta_{jk}} = d_j + d_k \tag{6.51}$$

In this case the involution σ^a is given by $\sigma^a(x) = \bar{x}$ and from (6.43) and (6.50) it follows that $\dim(\mathfrak{g}_\alpha \cap \mathfrak{q}^a) = 1$ for all $\alpha \in R$, so following the construction in 6.2 define the root vectors by:

$$\begin{aligned}
e_{\varepsilon_j} &= (0,0,0,ie_j,0,0), \quad e_{-\varepsilon_j} = (0,0,0,0,ie_j,0)\\
f_{\varepsilon_j} &= (0,0,0,e_j,0,0), \quad f_{-\varepsilon_j} = (0,0,0,0,e_j,0)\\
e_{2\varepsilon_j} &= (0,\sqrt{2}ie_{jj},0,0,0,0), \quad e_{-2\varepsilon_j} = (0,0,\sqrt{2}ie_{jj},0,0,0)\\
f_{2\varepsilon_j} &= f_{-2\varepsilon_j} = 0\\
e_{\alpha_{jk}} &= (ie_{jk},0,0,0,0,0), \quad e_{-\alpha_{jk}} = (ie_{kj},0,0,0,0,0)\\
f_{\alpha_{jk}} &= (e_{jk},0,0,0,0,0), \quad f_{-\alpha_{jk}} = (-e_{kj},0,0,0,0,0)\\
e_{\beta_{jk}} &= (0,i(e_{jk}+e_{kj}),0,0,0,0), \quad e_{-\beta_{jk}} = (0,0,i(e_{jk}+e_{kj}),0,0,0)\\
f_{\beta_{jk}} &= (0,e_{jk}-e_{kj},0,0,0,0), \quad f_{-\beta_{jk}} = (0,0,e_{jk}-e_{kj},0,0,0)
\end{aligned} \tag{6.52}$$

where $j < k$.
Now define as in (6.24)

$$\mu = \sum_{\alpha \in R_+} g_\alpha (e_\alpha + e_{-\alpha}) \tag{6.53}$$

where g_α is a Weyl group invariant function from R to $\mathbb{R}$. Because BC_n has three different root lengths, there are three different coupling constants. Using (6.51) and (6.52) this becomes:

$$\begin{aligned}
\mu &= g\sum_{j<k}(e_{\alpha_{jk}} + e_{-\alpha_{jk}}) + g\sum_{j<k}(e_{\beta_{jk}} + e_{-\beta_{jk}})\\
&+ g_1\sum_j (e_{\varepsilon_j} + e_{-\varepsilon_j}) + g_2\sum_j (e_{2\varepsilon_j} + e_{-2\varepsilon_j})\\
&= (\hat{\mu}, \hat{\mu} + ig_2\sqrt{2}I_n, \hat{\mu} + ig_2\sqrt{2}I_n, ig_1 e, ig_1 e, 0)
\end{aligned} \tag{6.54}$$

where $\hat{\mu}$ is the element defined in (6.32), $e = (1, \ldots, 1) \in \mathbb{R}^n$ and $g, g_1, g_2 \in \mathbb{R}$ are constants, with $g_1 \neq 0$.

Lemma 6.15. If $\mu \neq 0$ then $\mathrm{Cent}_{\mathfrak{a}}(\mu) = \{0\}$.
Proof. Let $q \in \mathfrak{a}$ and $[q, \mu] = 0$ then

$$g(q_j - q_k) = g(q_j + q_k) = g_1 q_j = 2g_2 q_j = 0$$

for all $j < k$ and this implies $q = 0$. □

Proposition 6.16. The element μ, as defined in (6.54), is conjugate under AdK to

$$(n+1)(2g-\frac{1}{n+1}\sqrt{2}g_2)i(\lambda_1,0)+\frac{1}{n+1}g_2\sqrt{2}c \text{ if } 2g-\frac{1}{n+1}\sqrt{2}g_2\geq 0$$

and to

$$-(n+1)(2g-\frac{1}{n+1}\sqrt{2}g_2)i(\lambda_n,0)+\frac{1}{n+1}g_2\sqrt{2}c \text{ if } 2g-\frac{1}{n+1}\sqrt{2}g_2\leq 0 \quad (6.55)$$

iff

$$g_1{}^2 = 2g^2-\sqrt{2}g_2g \quad (6.56)$$

where $\lambda_1=\frac{1}{n+1}\text{diag}(n,-1,\ldots,-1)$ and $\lambda_n=\frac{1}{n+1}\text{diag}(1,1,\ldots,1,-n)$ are fundamental dominant weights of $sl(n+1,\mathbb{C})$ and $c=i\text{diag}(nI_{n+1},-(n+1)I_n)\in\mathfrak{c}$, where we have used the representation (6.40).

Proof. First transform μ to the standard representation (6.40). After splitting off the $\mathfrak{c}$-component one gets a matrix $i\tilde{\mu}$ in $su(n+1)$, where $\tilde{\mu}$ is of the form:

$$\tilde{\mu}=\begin{pmatrix} b & \ldots & b & c\\ \vdots & \ddots & \vdots & \vdots\\ b & \ldots & b & c\\ c & \ldots & c & -na\end{pmatrix}+\text{diag}(a-b,\ldots,a-b,0)$$

where $a=\frac{1}{n+1}g_2\sqrt{2}$, $b=2g$ and $c=g_1\sqrt{2}$, so $c\neq 0$. With induction one can easily prove that this matrix has eigenvalues $a-b$ with multiplicity $n-1$ and eigenvalues ρ_1 and ρ_2 with multiplicity 1, where ρ_1 and ρ_2 are the roots of the equation:

$$\lambda^2+(n-1)(a-b)\lambda-n(a^2+(n-1)ab+c^2)=0$$

so

$$\rho_1=\tfrac{1}{2}(n-1)(b-a)+\tfrac{1}{2}\sqrt{D},\quad \rho_2=\tfrac{1}{2}(n-1)(b-a)-\tfrac{1}{2}\sqrt{D}$$

where D is the disriminant. Now D can be rewritten as:

$$\begin{aligned}D&=(n-1)^2(a-b)^2+4n(a^2+(n-1)ab+c^2)\\&=[(n+1)a+(n-1)b]^2+4nc^2\end{aligned}$$

and because $c\neq 0$ this implies $D>0$, so $\rho_1>\rho_2$.
Now consider the following cases:
(i) Suppose $\sqrt{D}\geq(n+1)\,|b-a|$ then $c^2\geq b^2-(n+1)ab$ and $\rho_1\geq a-b\geq\rho_2$. So $\tilde{\mu}$ is conjugate to

$$\text{diag}(\rho_1,a-b,\ldots,a-b,\rho_2)=\tfrac{1}{2}[(n+1)(b-a)+\sqrt{D}]\lambda_1+\tfrac{1}{2}[(n+1)(a-b)+\sqrt{D}]\lambda_n$$

where λ_1 and λ_n are fundamental dominant weights of $sl(n+1,\mathbb{C})$.

(ii) Now suppose $0 < \sqrt{D} \leq (n+1)\,|b-a|$ then $c^2 \leq b^2 - (n+1)ab$. Now consider two subcases:
a) Suppose $b-a \geq 0$ then $\sqrt{D} \leq (n+1)(b-a)$ and this implies $\rho_1 > \rho_2 \geq a-b$, so $\tilde{\mu}$ is conjugate to:

$$\mathrm{diag}(\rho_1, \rho_2, a-b, \ldots, a-b) = \sqrt{D}\lambda_1 + \tfrac{1}{2}[(n+1)(b-a) - \sqrt{D}]\lambda_2$$

b) Now suppose $b - a \leq 0$ then $\sqrt{D} \leq (n+1)(a-b)$ and this implies $a - b \geq \rho_1 > \rho_2$, so $\tilde{\mu}$ is conjugate to:

$$\mathrm{diag}(a-b, \ldots, a-b, \rho_1, \rho_2) = \tfrac{1}{2}[(n+1)(a-b) - \sqrt{D}]\lambda_{n-1} + \sqrt{D}\lambda_n$$

From this it is clear that $\tilde{\mu}$ is of the form $c_1\lambda_1$ or $c_n\lambda_n$ if and only if $\sqrt{D} = (n+1)\,|b-a|$, but this is equivalent with $c^2 = b^2 - (n+1)ab$, which is nothing else but (6.56). If $\sqrt{D} = (n+1)\,|b-a|$ then $\tilde{\mu}$ is conjugate to:

$$\begin{gathered}(n+1)(b-a)\lambda_1 + ac \text{ if } b-a \geq 0\\ (n+1)(a-b)\lambda_n + ac \text{ if } b-a \leq 0\end{gathered}$$

Because $\sqrt{D} > 0$ this also implies $b - a \neq 0$. From this the results about μ follow. □

So combining this with the discussion in 6.1 it is clear that one must require:

$$2g - \frac{1}{n+1}\sqrt{2}g_2 \neq 0 \tag{6.57}$$

and

$$\mathrm{Cent}_{\mathfrak{m}}(\mu) = \{0\} \tag{6.58}$$

but (6.57) follows from (6.56) and because $g_1 \neq 0$ (6.56) also implies $g \neq 0$ and one can check that this implies $\mathrm{Cent}_{\mathfrak{m}}(\mu) = \{0\}$.

Remark 6.17. This element μ corresponds to the first example of case 8 in Section 6.1.

Proposition 6.18. Suppose $g_1 \neq 0$ and (6.56) holds, then the map $\mathcal{M} : \mathfrak{k} \cap \mathfrak{q} \cap \mathfrak{m}^{\perp} \to \mathfrak{m}$ is given by:

$$\begin{aligned}
\mathcal{M}(e_{\alpha_{jk}} + e_{-\alpha_{jk}}) &= i(\frac{4}{2n+1}I_n - e_{jj} - e_{kk}, 0, 0, 0, 0, \frac{4}{2n+1})\\
\mathcal{M}(e_{\beta_{jk}} + e_{-\beta_{jk}}) &= i(\frac{4}{2n+1}I_n - e_{jj} - e_{kk}, 0, 0, 0, 0, \frac{4}{2n+1})\\
\mathcal{M}(e_{2\varepsilon_j} + e_{-2\varepsilon_j}) &= i\sqrt{2}(\frac{2}{2n+1}I_n - e_{jj}, 0, 0, 0, 0, \frac{2}{2n+1})\\
\mathcal{M}(e_{\varepsilon_j} + e_{-\varepsilon_j}) &= i(bI_n - g_1 g^{-1} e_{jj}, 0, 0, 0, 0, a)
\end{aligned}$$

where

$$a = \frac{2}{2n+1}\left(\frac{g_1}{g} - 2n\frac{g}{g_1}\right) \tag{6.59}$$

and

$$b = \frac{2}{2n+1}\left(\frac{g}{g_1} + \frac{g_1}{g}\right) \tag{6.60}$$

Proof. Let $X_1, X_2 \in \mathfrak{k} \cap \mathfrak{q}$ with

$$X_1 = i(A_1, B_1, B_1, w_1, w_1, a_1), \quad X_2 = i(A_2, B_2, B_2, w_2, w_2, a_2)$$

and A_1, A_2, B_1, B_2 real symmetric, $w_1, w_2 \in \mathbb{R}^n, a_1, a_2 \in \mathbb{R}$ and $a_1 + 2\mathrm{tr}A_1 = a_2 + 2\mathrm{tr}A_2 = 0$ then

$$[X_1, X_2] = (A_3, B_3, B_3, w_3, w_3, 0) \tag{6.61}$$

with

$$\begin{aligned} A_3 &= [A_2, A_1] + [B_2, B_1] + w_2 \otimes w_1 - w_1 \otimes w_2 \\ B_3 &= [A_2, B_1] + [B_2, A_1] + w_2 \otimes w_1 - w_1 \otimes w_2 \\ w_3 &= (A_2 + B_2 - a_2\mathrm{id})w_1 - (A_1 + B_1 - a_1\mathrm{id})w_2 \end{aligned} \tag{6.62}$$

Taking $X_2 = \mu$, the condition $[X_1, X_2] = 0$ reduces to the equations:

$$g[e \otimes e, A_1 + B_1] + g_1 e \otimes w_1 - g_1 w_1 \otimes e = 0$$

$$g_1(A_1 + B_1)e - (g_1 a_1 + 2g < e, w_1 >)e = (\sqrt{2}g_2 - 2g)w_1 \tag{6.63}$$

where we have used (6.37) and where $<,>$ denotes the standard inner product in $\mathbb{R}^n$. Using (6.63) one can verify that

$$[(\mathrm{id} + \mathcal{M})(e_\alpha + e_{-\alpha}), \mu] = 0 \text{ for all } \alpha \in R_+ \tag{6.64}$$

□

Now one can apply the construction of chapter 5 and conclude that using (6.54) and (6.64) one can construct a Lax pair for Hamilton's equations for the BC_n-Hamiltonian:

$$\begin{aligned} H =& \tfrac{1}{2}\sum_{j=1}^{n} {p_j}^2 + g^2 \sum_{j<k} [x(q_j - q_k)^2 + x(q_j + q_k)^2] \\ &+ {g_1}^2 \sum_{j=1}^{n} x(q_j)^2 + {g_2}^2 \sum_{j=1}^{n} x(2q_j)^2 \end{aligned} \tag{6.65}$$

for $g_1 \neq 0, {g_1}^2 = 2g^2 - \sqrt{2}gg_2$.
This gives a more transparant Lie algebraic interpretation of the construction of Olshanetsky and Perelomov in [38] and [43].

One can now consider three special cases:

(i) $g_2 \neq 0 \Rightarrow BC_n$-model
(ii) $g_2 = 0 \Rightarrow {g_1}^2 = 2g^2$. This corresponds to the B_n-model for special values of the coupling constants.
(iii) $g_2 \neq 0, g_1 = g = \sqrt{2}g_2$. This can be viewed as a reduced A_{2n}-model by imposing the conditions:

$$q_{n+1} = 0, \quad q_{2n+2+j} = -q_j, \quad 1 \leq j \leq n \tag{6.66}$$

So one does not get a Lax representation for the B_n-model for all values of the coupling constants.

Proposition 6.19. One can easily verify that in the general case

$$[e_{\varepsilon_j} + e_{-\varepsilon_j} + \mathcal{M}(e_{\varepsilon_j} + e_{-\varepsilon_j}), \mu] = \nu := g^{-1}({g_1}^2 - 2g^2 + g_2 g\sqrt{2})(f_{\varepsilon_j} + f_{-\varepsilon_j})$$

and

$$[e_\alpha + e_{-\alpha} + \mathcal{M}(e_\alpha + e_{-\alpha}), \mu] = 0$$

for all other roots $\alpha \in R$. This implies that Hamilton's equations for the general BC_n model can be written in the following way:

$$\dot{x} = [Y\mu + \mathcal{M}X^2\mu, x] - X^3\nu$$

Now define:

$$\begin{aligned}\mu &= g\sum_{j<k}(e_{\alpha_{jk}} + e_{-\alpha_{jk}}) - g\sum_{j<k}(e_{\beta_{jk}} + e_{-\beta_{jk}}) \\ &= (\hat{\mu}, -\hat{\mu}, -\hat{\mu}, 0, 0, 0) \\ &= \begin{pmatrix} \hat{\mu} & 0 & -\hat{\mu} \\ 0 & 0 & 0 \\ -\hat{\mu} & 0 & \hat{\mu} \end{pmatrix}\end{aligned}$$

where $\hat{\mu} = ig\sum_{j<k}(e_{jk} + e_{kj}) \in su(n)$. In the standard representation this takes the form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\hat{\mu} \end{pmatrix}$$

so using the results in the A_n-case, it is clear that μ is conjugate under $\mathrm{Ad}K$ to $-2ngi(0, \lambda_{n-1})$ if $g \leq 0$ and to $2ngi(0, \lambda_1)$ if $g \geq 0$, where λ_1 and λ_{n-1} are fundamental dominant weights of $sl(n, \mathbb{C})$.
One can also easily verify that:

$$\mathrm{Cent_m}(\mu) = \mathbb{R} \prec (\mathrm{iI_n}, 0, 0, 0, 0, -2\mathrm{in}) \succ$$

so $\dim \mathrm{Cent_m}(\mu) = 1$. So this μ satisfies (5.58) and corresponds to the second example of case 8 of section 6.1. Using this μ one gets another Lax representation of the D_n-model. Of course one can also view μ as an element of $su(n, n)$. Then $\dim \mathrm{Cent_m}(\mu) = 0$ and this corresponds to the example of case 7 in section 6.1.

6.3.3 The C_n and D_n case

To obtain a Lax representation of the C_n and D_n-models we have to take $\mathfrak{g} = su(n,n)$, which corresponds to removing the $n+1^{th}$ column and row in (6.39) and taking

$$v = w = a = 0, \quad g_1 = 0 \tag{6.67}$$

Then the restricted root system R is of type C_n. The element μ takes the form:

$$\mu = (\hat{\mu}, \hat{\mu} + ig_2\sqrt{2}I_n, \hat{\mu} + ig_2\sqrt{2}I_n) \tag{6.68}$$

In the same way as in the $su(n+1,n)$ case one can check that

$$\mathrm{Cent}_{\mathfrak{a}}(\mu) = \{0\} \text{ iff } g \neq 0 \vee g_2 \neq 0, \quad \mathrm{Cent}_{\mathfrak{m}}(\mu) = \{0\} \text{ iff } g \neq 0 \tag{6.69}$$

If $g = 0$ then $\mu \in \mathfrak{c}$, $\dim \mathrm{Cent}_{\mathfrak{k}}(\mu) = \dim \mathfrak{k}$ and $\dim \mathrm{Cent}_{\mathfrak{m}}(\mu) = \dim \mathfrak{m}$, so μ still satisfies (5.58). If $g \neq 0$ the map $\mathcal{M}$ is given by:

$$\mathcal{M}(e_{\alpha_{jk}} + e_{-\alpha_{jk}}) = i(\frac{2}{n}I_n - e_{jj} - e_{kk}, 0, 0)$$

$$\mathcal{M}(e_{\beta_{jk}} + e_{-\beta_{jk}}) = i(\frac{2}{n}I_n - e_{jj} - e_{kk}, 0, 0)$$

$$\mathcal{M}(e_{2\varepsilon_j} + e_{-2\varepsilon_j} = i\sqrt{2}(\frac{1}{n}I_n - e_{jj}, 0, 0) \tag{6.70}$$

The corresponding Hamiltonian is given by (6.65) with $g_1 = 0$ and $g \neq 0 \vee g_2 \neq 0$, so using (6.70) and the construction of chapter 5 one gets a Lax representation of the following models:

(i) $g \neq 0 \wedge g_2 \neq 0 \Rightarrow C_n$-model
(ii) $g \neq 0, g_2 = 0 \Rightarrow D_n$-model
(iii) $\sqrt{2}g_2 = g \neq 0 \Rightarrow$ this can be viewed as a reduced A_{2n-1}-model by imposing the conditions:

$$q_j + q_{2n+1-j} = 0, \quad 1 \leq j \leq n \tag{6.71}$$

(iv) $g = 0, g_2 \neq 0 \Rightarrow$ this can be viewed as n uncoupled A_1-models.

Proposition 6.20. In these cases μ is conjugate under AdK to

$$2ngi(\lambda_1, 0) + g_2\sqrt{2}c \text{ if } g \geq 0$$

and to

$$-2ngi(\lambda_{n-1}, 0) + g_2\sqrt{2}c \text{ if } g \leq 0$$

where $c = i\mathrm{diag}(I_n, -I_n) \in \mathfrak{c}$ and λ_1 and λ_{n-1} are fundamental dominant weights of $sl(n, \mathbb{C})$.

Proof. First transform μ to the standard representation. After splitting off the $\mathfrak{c}$-component one is left with the matrix:

$$\begin{pmatrix} 2\hat{\mu} & 0 \\ 0 & 0 \end{pmatrix}$$

where $\hat{\mu} = ig\sum_{j<k}(e_{jk} + e_{kj})$. And so the result follows. □

Now consider the case **CI** ($n \geq 2$) with $\mathfrak{g} = sp(n, \mathbb{R})$, which consists of the matrices of the form:

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \tag{6.72}$$

where $A, B, C \in gl(n, \mathbb{R})$ and B, C are real symmetric. Denote a general element X by $X = (A, B, C)$, then

$$\mathfrak{k} = \{(A, B, -B), A \in so(n), B \text{ real symmetric}\}$$

$$\mathfrak{p} = \{(A, B, B), A, B \text{ real symmetric}\} \tag{6.73}$$

For $\mathfrak{a}$ one can choose:

$$\mathfrak{a} = \{(D, 0, 0), D = \operatorname{diag}(q_1, \ldots, q_n)\} \tag{6.74}$$

and then

$$\mathfrak{m} = \{0\}, \quad \mathfrak{c} = \operatorname{Span}(0, I_n, -I_n) \tag{6.75}$$

For the Killing form one can take (6.47) and the restricted root system is of type C_n. Now define

$$\mu = g_2\sqrt{2}(0, I_n, -I_n) \tag{6.76}$$

The Cartan involution is given by

$$\theta(X) = \theta(A, B, C) = (-A^t, -C, -B) \tag{6.77}$$

In this case one can take $\sigma = \theta$ and then (6.22) is true. Now define the root vectors as in 6.2 and then

$$\mu = g_2 \sum_{j=1}^{n} (e_{2\varepsilon_j} + e_{-2\varepsilon_j}) \tag{6.78}$$

so one gets another Lax representation of case (iv) above, where the map $\mathcal{M}$ is trivial, because $\mu \in \mathfrak{c}$.
This example corresponds with case 4 of section 6.1.

6.4. Analysis of the map $\mathcal{M}$.

So in all the examples of section 6.3 μ can be constructed in the same way:

$$\mu = \sum_{\alpha \in R_+} g_\alpha (e_\alpha + e_{-\alpha}) \tag{6.79}$$

where g_α is Weyl group invariant and where the root vectors are defined as in section 6.2. The map $\mathcal{M}$ is given by:

$$m(\alpha) := \mathcal{M}(e_\alpha + e_{-\alpha}) \in \mathfrak{m} \tag{6.80}$$

so that

$$[e_\alpha + e_{-\alpha} + m(\alpha), \mu] = 0 \tag{6.81}$$

This arises the question whether such a construction is possible for all quasi-split Lie algebras, especially for the algebras of type IV.
In this section we analyse in more detail the map $\mathcal{M}$ in the case of $sl(n, \mathbb{C})$, as defined in (6.36), but now considered as a complex Lie algebra. Define

$$\mu = g \sum_{j<k} (e_{jk} + e_{kj}) \tag{6.82}$$

with $g \in \mathbb{C}$. Recall from (6.36) that

$$m(\alpha_{jk}) = \mathcal{M}(e_{jk} + e_{kj}) = \frac{2}{n} I_n - e_{jj} - e_{kk}, \quad j < k \tag{6.83}$$

then one still has

$$[e_{jk} + e_{kj} + m(\alpha_{jk}), \mu] = 0$$

Now $\mathfrak{h}_C = \mathfrak{m} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and the Killing form is given by $< x, y >= \mathrm{tr}(xy)$. Take a Chevalley basis $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ which satisfies the commutation relations:

$$\begin{aligned} [e_\alpha, e_{-\alpha}] &= h_\alpha \\ [h_\alpha, e_\alpha] &= 2e_\alpha \\ [h_\alpha, e_{-\alpha}] &= -2e_{-\alpha} \end{aligned}$$

and define the fundamental dominant weights $\lambda_j \in \mathfrak{h}_C^*$ by:

$$(\lambda_j, \alpha_k) = \tfrac{1}{2}(\alpha_k, \alpha_k)\delta_{jk} \tag{6.84}$$

Using these weights and the structure constants $c_{\alpha,\beta}$ one can give a more abstract characterization of $m(\alpha)$. First observe:

Proposition 6.21. The map $m(\alpha)$ as defined in (6.83) has the following properties:
(i) $m(\alpha) = m(-\alpha)$
(ii) $\alpha(m(\alpha)) = 0$
(iii) $\beta(m(\alpha)) = 0$ if $(\alpha, \beta) = 0$
(iv)

$$\begin{aligned} \beta(m(\alpha)) &= c_{\alpha,-\beta} \text{ if } (\alpha, \beta) = 1 \\ &= -c_{\alpha,\beta} \text{ if } (\alpha, \beta) = -1 \end{aligned}$$

Proof. (i), (ii) and (iii) are clear. Now suppose $\alpha = \varepsilon_j - \varepsilon_k$ and $\beta = \varepsilon_m - \varepsilon_n$, with $m < n, j < k$, and suppose $(\alpha, \beta) = -1$. Then $\alpha + \beta \in R$ and there are two possibilities:
a) $j = n \Rightarrow m < n < k$ and

$$[e_\alpha, e_\beta] = [e_{jk}, e_{mj}] = -e_{mk} = -e_{\alpha+\beta}$$

so $c_{\alpha,\beta} = -1$ and $\beta(m(\alpha)) = 1$ and this agrees with (6.83).
b) $m = k \Rightarrow j < k < n$ and

$$[e_\alpha, e_\beta] = [e_{jk}, e_{kn}] = e_{jn} = e_{\alpha+\beta}$$

so $c_{\alpha,\beta} = 1$ and $\beta(m(\alpha)) = -1$ and again this agrees with (6.83).
If $(\alpha, \beta) = 1$ then $\alpha - \beta \in R$ and one can easily check that the proposition is true. □

Corollary 6.22. Using the fundamental dominant weights $m(\alpha)$ can be written as:

$$m(\alpha) = \sum_{j=1}^{l} (\alpha, \alpha_j)(c_{\alpha,\alpha_j} + c_{\alpha,-\alpha_j})\lambda_j \tag{6.85}$$

where $\alpha_j \in \Delta$.
In this form the definition of $m(\alpha)$ is valid for all root systems, but the question arises which of the properties survive. Still $m(\alpha) = m(-\alpha)$ because the structure constants satisfy $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$.
So one would like to know why (6.81) is not true for the other root systems. So let R be of type A, D or E and choose a Chevalley basis. Let $\alpha, \beta \in R$ and let $\beta - r\alpha, \ldots, \beta + q\alpha$ be the α-string through β, then $r - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ and $[e_\alpha, e_\beta] = 0$ if $q = 0$ and $[e_\alpha, e_\beta] = \pm(r+1)e_{\alpha+\beta}$ if $q \geq 1$. Now for type A, D or E one has $q + r \leq 1$ so if $q = 1$ then $r = 0$. So $c_{\alpha,\beta}{}^2 = 1$ if $\alpha + \beta \in R$. Also one has $(\alpha, \alpha) = 2$, so $r - q = (\alpha, \beta)$. Combining these bits of information there are the following possibilities:

$$\begin{aligned}
q = 0, r = 0 &\Rightarrow (\alpha, \beta) = 0, \alpha + \beta \notin R, \alpha - \beta \notin R \\
q = 0, r = 1 &\Rightarrow (\alpha, \beta) = 1, \beta + \alpha \notin R, \beta - \alpha \in R \\
q = 1, r = 0 &\Rightarrow (\alpha, \beta) = -1, \beta + \alpha \in R, \beta - \alpha \notin R
\end{aligned} \tag{6.86}$$

Using this one can compute for $\alpha \in R_+$:

$$\begin{aligned}
&[e_\alpha + e_{-\alpha} + m(\alpha), \mu] \\
&= \sum_{\beta \in R_+} [e_\alpha + e_{-\alpha} + m(\alpha), e_\beta + e_{-\beta}] \\
&= \sum_{\beta \in R_+} c_{\alpha,\beta}(e_{\alpha+\beta} - e_{-\alpha-\beta}) + \sum_{\beta \in R_+} c_{\alpha,-\beta}(e_{\alpha-\beta} - e_{\beta-\alpha}) \\
&+ \sum_{\beta \in R_+} \beta(m(\alpha))(e_\beta - e_{-\beta}) \\
&= \sum_{\gamma \in R_+, (\gamma,\alpha)=1} c_{-\gamma,\alpha}(e_\gamma - e_{-\gamma}) \\
&- \sum_{\gamma \in R_+, (\gamma,\alpha)=-1} c_{\gamma,\alpha}(e_\gamma - e_{-\gamma}) \\
&+ \sum_{\gamma \in R_+} \gamma(m(\alpha))(e_\gamma - e_{-\gamma})
\end{aligned} \tag{6.87}$$

where we have used

$$c_{\alpha,\beta} = c_{\beta,-\alpha-\beta} = c_{-\alpha-\beta,\alpha} \tag{6.88}$$

and it is clear that (6.87) is zero if $m(\alpha)$ satisfies the properties of Prop. 6.21.

Proposition 6.23. Define $m(\alpha)$ as in (6.85), then it satisfies the conditions of 6.21, if $\alpha, \beta \in \Delta$.

Proof. Choose $\alpha = \alpha_k, \beta = \alpha_m$, then $\alpha - \beta \notin R$ and so

$$m(\alpha_k) = \sum_{j=1}^{l} (\alpha_k, \alpha_j) c_{\alpha_k,\alpha_j} \lambda_j \tag{6.89}$$

Now properties (ii), (iii) and (iv) follow directly from (6.89). □

Proposition 6.24. A necessary condition for Prop. 6.21 to hold is:

$$c_{\alpha_k,\alpha_m+\alpha_n} = c_{\alpha_k,\alpha_n} \tag{6.90}$$

for $\alpha_k, \alpha_m, \alpha_n \in \Delta, (\alpha_m, \alpha_n) = -1, (\alpha_m, \alpha_k) = 0, (\alpha_n, \alpha_k) = -1$.

Proof. Let $\beta = \alpha_m + \alpha_n$, then $(\beta, \alpha_k) = -1$ and so

$$\beta(m(\alpha_k)) = (\alpha_k, \alpha_m) c_{\alpha_k,\alpha_m} + (\alpha_k, \alpha_n) c_{\alpha_k,\alpha_n} = -c_{\alpha_k,\alpha_n}$$

and because of Prop. 6.21 (iv) this must be equal to $-c_{\alpha_k,\beta}$. □

Apparently this property of the structure constants is only satisfied in the case of the root system A_n and this could 'explain' why the construction only works for $sl(n, \mathbb{C})$.

7. Lie bialgebras and Yang-Baxter operators

In this chapter we want to compare the construction in chapter 5 and 6 with the construction of integrable systems in chapter 4 and see whether the Poissonstructure can be understood in terms of an underlying double Lie algebra structure and a corresponding Yang-Baxter operator.

7.1. The co-Lie-algebra structure.

Let $\mathfrak{g}$ be quasi-split, satisfying (6.21) and define the root vectors as in section 6.2. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the skew-symmetric tensor corresponding to the map R, so

$$< r, x \otimes y > = < x, R(y) > \tag{7.1}$$

for all $x, y \in \mathfrak{g}$, where we extended the Killing form to an $\mathfrak{g}$-invariant scalar product on $\mathfrak{g} \otimes \mathfrak{g}$. Now consider the Poisson structure on $\mathcal{C}^\infty(P)$ as defined in (5.33) and define the map ϕ by

$$\phi(y, z) = [\pi_\mathfrak{a} y, R(z)] + [R(y), \pi_\mathfrak{a} z] \tag{7.2}$$

The dual map ϕ^* is defined by

$$< \phi^*(x), y \otimes z > = < x, \phi(y, z) > \tag{7.3}$$

Using (7.1) one finds

$$\phi^*(x) = (\pi_\mathfrak{a} \otimes \mathrm{id})[r, x \otimes I] + (\mathrm{id} \otimes \pi_\mathfrak{a})[r, I \otimes x] \tag{7.4}$$

which can also be written as:

$$\phi^*(x) = -(\pi_\mathfrak{a} \otimes \mathrm{id} + \mathrm{id} \otimes \pi_\mathfrak{a}) dr(x) \tag{7.5}$$

where $dr(x) = [x \otimes \mathrm{id} + \mathrm{id} \otimes x, r]$. Let $\{T_j\}$ be a basis of $\mathfrak{g}$, $\{\hat{T}_j\}$ the dual basis, then one can write (7.4) as:

$$\phi^*(x) = \sum_j \pi_\mathfrak{a}[R(T_j), x] \otimes \hat{T}_j - \sum_j \hat{T}_j \otimes \pi_\mathfrak{a}[R(T_j), x] \tag{7.6}$$

Lemma 7.1. Let $\mathfrak{g}$ be quasi-split, satisfying (6.21), and define the root vectors as in section 6.2, then:
(i) $< e_\alpha, f_{-\alpha} > = 0$ for all $\alpha \in R$
(ii) $[e_\alpha, f_{-\alpha}] = [e_{-\alpha}, f_\alpha]$

Proof.
(i) Because $e_\alpha \in \mathfrak{q}^a$ and $f_{-\alpha} \in \mathfrak{h}^a$ one has $\sigma^a(e_\alpha) = -e_\alpha$ and $\sigma^a(f_{-\alpha}) = f_\alpha$ and so $< e_\alpha, f_{-\alpha} > = 0$ because σ^a is an automorphism.
(ii) $[e_\alpha, f_{-\alpha}] - [e_{-\alpha}, f_\alpha] \in \mathfrak{a}$, but for all $a \in \mathfrak{a}$

$$< a, [e_\alpha, f_{-\alpha}] - [e_{-\alpha}, f_\alpha] > = < [a, e_\alpha], f_{-\alpha} > - < [a, e_{-\alpha}], f_\alpha > = 0$$

because of (i) and so $[e_\alpha, f_{-\alpha}] - [e_{-\alpha}, f_\alpha] = 0$ □

Corollary 7.2. $\tilde{h}_\alpha = [e_\alpha, f_{-\alpha}]$, where $\tilde{h}_\alpha$ as defined in (6.27).

Now we can compute ϕ^*, using $R = -YX^{-1}$.

Proposition 7.3. The dual map ϕ^* is given by:
(i) $\phi^*(x) = 0$ for all $x \in \mathfrak{g}_0$.
(ii) $\phi^*(e_\alpha) = r(\alpha(q))(t_\alpha \otimes e_\alpha - e_\alpha \otimes t_\alpha)$ for all $\alpha \in R$.
(iii) $\phi^*(f_\alpha) = r(\alpha(q))(t_\alpha \otimes f_\alpha - f_\alpha \otimes t_\alpha)$ for all $\alpha \in R$,
where $r(\eta) = x^{-1}(\eta)x'(\eta)$.

Proof. Straightforward calculation, using the commutation relations (3.9) and (3.25). □

Now consider the boundary of the chain $\phi^* : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, defining the co-Lie-algebra structure, defined by:

$$d\phi^*(x,y) = \phi^*([x,y]) - x.\phi^*(y) + y.\phi^*(x) \tag{7.7}$$

where $\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ as in (4.88).

Proposition 7.4.
(i) ϕ^* is $\operatorname{ad}\mathfrak{g}_0$-invariant, so $d\phi^*(x,y) = 0$ if $x \in \mathfrak{g}_0$ or $y \in \mathfrak{g}_0$.
(ii) $d\phi^*(e_\alpha, e_{-\alpha}) = d\phi^*(f_\alpha, f_{-\alpha}) = 0$
(iii) $d\phi^*(e_\alpha, f_{-\alpha}) = d\phi^*(e_{-\alpha}, f_\alpha) = 2r(\alpha(q))([e_\alpha, f_{-\alpha}] \otimes t_\alpha - t_\alpha \otimes [e_\alpha, f_{-\alpha}])$
(iv) $d\phi^*(e_\alpha, e_\beta) = 0$ if $0 \neq \alpha + \beta \notin R$ and $(\alpha, \beta) = 0$.

Proof.
(i) If $x, y \in \mathfrak{g}_0$ it is clear, because $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$. So let $x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}_\alpha$, then (i) follows from the commutation relation $[\mathfrak{g}_0, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$, and because ϕ^* is linear $\phi^*([x,y]) = x.\phi^*(y)$ for all $x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}$.
(ii) $d\phi^*(e_\alpha, e_{-\alpha}) = e_{-\alpha}.\phi^*(e_\alpha) - e_\alpha.\phi^*(e_{-\alpha})$ and

$$\begin{aligned} e_{-\alpha}.\phi^*(e_\alpha) &= r(\alpha(q))e_{-\alpha}.(t_\alpha \otimes e_\alpha - e_\alpha \otimes t_\alpha) \\ &= (\alpha,\alpha)r(\alpha(q))(e_{-\alpha} \otimes e_\alpha - e_\alpha \otimes e_{-\alpha}) \end{aligned}$$

and so $e_{-\alpha}.\phi^*(e_\alpha) - e_\alpha.\phi^*(e_{-\alpha}) = 0$ because r is uneven. In the same way one can prove that $d\phi^*(f_\alpha, f_{-\alpha}) = 0$.
(iii) $d\phi^*(e_\alpha, f_{-\alpha}) = f_{-\alpha}.\phi^*(e_\alpha) - e_\alpha.\phi^*(f_{-\alpha})$ and

$$\begin{aligned} f_{-\alpha}.\phi^*(e_\alpha) &= r(\alpha(q))f_{-\alpha}.(t_\alpha \otimes e_\alpha - e_\alpha \otimes t_\alpha) \\ &= \tfrac{1}{2}(\alpha,\alpha)r(\alpha(q))\{2f_{-\alpha} \otimes e_\alpha - 2e_\alpha \otimes f_{-\alpha} \\ &\quad + h_\alpha \otimes [f_{-\alpha}, e_\alpha] - [f_{-\alpha}, e_\alpha] \otimes h_\alpha\} \end{aligned}$$

which implies, using Lemma 7.1:

$$d\phi^*(e_\alpha, f_{-\alpha}) = (\alpha,\alpha)r(\alpha(q))([e_\alpha, f_{-\alpha}] \otimes h_\alpha - h_\alpha \otimes [e_\alpha, f_{-\alpha}])$$

and so $d\phi^*(e_\alpha, f_{-\alpha}) = d\phi^*(e_{-\alpha}, f_\alpha)$.
(iv) $d\phi^*(e_\alpha, e_\beta) = e_\beta.\phi^*(e_\alpha) - e_\alpha.\phi^*(e_\beta)$ where

$$\begin{aligned} e_\beta.\phi^*(e_\alpha) &= r(\alpha(q))e_\beta.(t_\alpha \otimes e_\alpha - e_\alpha \otimes t_\alpha) \\ &= \beta(t_\alpha)r(\alpha(q))(e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha) \end{aligned} \tag{7.8}$$

and because $(\alpha, \beta) = 0$, $d\phi^*(e_\alpha, e_\beta) = 0$. □

Corollary 7.5. Let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$, then $\mathrm{Ad}g, g \in G_0$ is a Poisson automorphism of $\mathcal{C}^\infty(P)$.

Proof. This follows from Prop. 4.42, Prop. 7.3 and Prop. 7.4. □

Corollary 7.6. Let $f, g \in \mathcal{C}^\infty(\mathfrak{g})$ be $\mathrm{Ad}G$-invariant, then $\{f,g\}_1$ is $\mathrm{Ad}G_0$-invariant.

Proof. This follows from Prop. 4.35 and Prop. 7.4. □

Proposition 7.7. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is not a Lie bialgebra.

Proof. For $(\mathfrak{g}, \mathfrak{g}^*)$ to be a Lie bialgebra, the chain ϕ^* must be a cocycle, i.e. $d\phi^* = 0$. Now take $0 \neq \alpha + \beta \notin R$ and $(\alpha, \beta) \neq 0$ then from (7.8) one derives

$$d\phi^*(e_\alpha, e_\beta) = (\alpha, \beta)(r(\alpha(q)) + r(\beta(q)))(e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha) \tag{7.9}$$

and this is in general not equal to zero. □

7.2. The Poisson bracket on P_μ.

Now consider the $c = 0$ case, then R and X satisfy the functional equation:

$$X[R(z_1), z_2] + X[z_1, R(z_2)] = -\pi_0^\perp[X(z_1), X(z_2)] \tag{7.10}$$

for all $z_1, z_2 \in \mathfrak{g}_0^\perp$.

Proposition 7.8. Let $\mathfrak{g}$ be quasi-split, satisfying (6.21). Suppose that μ as defined in (6.24) satisfies (5.58), then the Poisson bracket on P_μ^0 can be rewritten as

$$\{f,g\}(q,x) =< x, [S(F), G] + [F, S(G)] > + < F, G_1 > - < F_1, G > \tag{7.11}$$

where the map $S \in \mathrm{End}\,\mathfrak{g}$ is defined by

$$S = R\pi_\mathfrak{p} - \mathcal{M}X\pi_{\mathfrak{p}\cap\mathfrak{h}} \tag{7.12}$$

Proof. Using (5.85) the bracket $\{,\}_1$ on P_μ^0 can be rewritten in the following way:

$$\{f,g\}_1(q,x) =< x, [R(F_p), G_p] + [F_p, R(G_p)] > - < \mu, [X(F_p), X(G_p)] > \tag{7.13}$$

Because μ satisfies (5.58) and $\mu \in \mathfrak{k} \cap \mathfrak{q}$, $\mathfrak{k} \cap \mathfrak{q} \in \mathrm{Ker}\,A$ and so the map $\mathcal{M}$ is defined on the whole space $\mathfrak{k} \cap \mathfrak{q}$. Using this the second term of (7.13) can be rewritten as:

$$\begin{aligned}
&- < \mu, [X(F_{\mathfrak{p}\cap\mathfrak{h}}), X(G_{\mathfrak{p}\cap\mathfrak{q}})] + [X(F_{\mathfrak{p}\cap\mathfrak{q}}), X(G_{\mathfrak{p}\cap\mathfrak{h}})] > \\
&=- < [\mu, X(F_{\mathfrak{p}\cap\mathfrak{h}})], X(G_{\mathfrak{p}\cap\mathfrak{q}}) > - < X(F_{\mathfrak{p}\cap\mathfrak{q}}), [X(G_{\mathfrak{p}\cap\mathfrak{h}}), \mu] > \\
&= < [\mu, \mathcal{M}X(F_{\mathfrak{p}\cap\mathfrak{h}})], X(G_{\mathfrak{p}\cap\mathfrak{q}}) > - < X(F_{\mathfrak{p}\cap\mathfrak{q}}), [\mu, \mathcal{M}X(G_{\mathfrak{p}\cap\mathfrak{h}})] > \\
&=- < G_{\mathfrak{p}\cap\mathfrak{q}}), X[\mu, \mathcal{M}X(F_{\mathfrak{p}\cap\mathfrak{h}})] > + < F_{\mathfrak{p}\cap\mathfrak{q}}), X[\mu, \mathcal{M}X(G_{\mathfrak{p}\cap\mathfrak{h}})] > \\
&= < G_{\mathfrak{p}\cap\mathfrak{q}}, [\mathcal{M}X(F_{\mathfrak{p}\cap\mathfrak{h}}), x] > + < F_{\mathfrak{p}\cap\mathfrak{q}}, [x, \mathcal{M}X(G_{\mathfrak{p}\cap\mathfrak{h}})] > \\
&= < x, [G_{\mathfrak{p}\cap\mathfrak{q}}, \mathcal{M}X(F_{\mathfrak{p}\cap\mathfrak{h}})] + [\mathcal{M}X(G_{\mathfrak{p}\cap\mathfrak{h}}), F_{\mathfrak{p}\cap\mathfrak{q}}] > \\
&= < x, [G, \mathcal{M}X(F_{\mathfrak{p}\cap\mathfrak{h}})] + [\mathcal{M}X(G_{\mathfrak{p}\cap\mathfrak{h}}), F] >
\end{aligned} \tag{7.14}$$

and combining this with (7.13) gives (7.11). □

This suggests that S could be a possible candidate for a Yang-Baxter operator. So define

$$\psi(y,z) = [S(y),z] + [y,S(z)] \tag{7.15}$$

Because S is zero on $\mathfrak{k}$, it is clear that

$$\psi(y,z) = 0 \text{ if } y,z \in \mathfrak{k} \tag{7.16}$$

and $\psi : \mathfrak{k} \otimes \mathfrak{p} \to \mathfrak{k}, \mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{p}$. Now define:

$$\omega(y,z) = S\psi(y,z) - [S(y),S(z)] \tag{7.17}$$

then

$$\omega(y,z) = 0 \text{ if } y,z \in \mathfrak{k} \text{ or } y \in \mathfrak{k}, z \in \mathfrak{p} \tag{7.18}$$

and $\omega : \mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{k}$.

Proposition 7.9. The map S is not a Yang-Baxter operator.

Proof. If it is a Yang-Baxter operator then S must satisfy:

$$\omega(y,z) = \lambda(q)[y,z] \tag{7.19}$$

where $\lambda(q)$ is a constant which may depend on q, but not on α. Now take $y = h_\alpha \in \mathfrak{a}$ and $z = f_\alpha - f_{-\alpha} \in \mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{a}^\perp$ then (7.19) reduces to

$$r(\alpha(q))^2 = \lambda(q) \text{ for all } \alpha \in R \tag{7.20}$$

which implies r =constant, and this is not true for root systems other than A_1. □

Proposition 7.10. The map ψ defines a Lie bracket on $\mathfrak{g}$ iff S is a Yang-Baxter operator.

Proof. We already know from chapter 4 that if S is a Yang-Baxter operator then ψ defines a Lie bracket. Now suppose that ψ defines a Lie bracket. This is equivalent with the condition:

$$[\omega(y,z),w] + \text{cycl.} = 0 \tag{7.21}$$

for all $y,z,w \in \mathfrak{g}$. For $y,z \in \mathfrak{p}, w \in \mathfrak{k}$ this reduces to

$$[\omega(y,z),w] = 0 \tag{7.22}$$

for all $w \in \mathfrak{k}$, so $\omega(y,z) \in \mathcal{Z}(\mathfrak{k})$. If $\mathfrak{k}$ is semisimple then $\omega(y,z) = 0$ for all $y,z \in \mathfrak{p}$. Combining this with (7.18) it follows that S is a Yang-Baxter operator. If $\mathfrak{k}$ is reductive (7.22) implies that $\omega(y,z) = c \in \mathfrak{c}$ for all $y,z \in \mathfrak{p}$. Now choose $y,z,w \in \mathfrak{p}$ in (7.21). Then one gets:

$$[c, y+z+w] = 0 \text{ for all } y,z,w \in \mathfrak{p}$$

and so $c \in \mathcal{Z}(\mathfrak{g}) = \{0\}$ so $\omega(y,z) = 0$ for all $y,z \in \mathfrak{g}$ and so S is a Yang-Baxter operator. □

Corollary 7.11. The pair $(\mathfrak{g}, S)$ is not a double Lie algebra.

Because S is zero on $\mathfrak{k}$ and $S : \mathfrak{p} \to \mathfrak{k}$, the dual S^* is zero on $\mathfrak{p}$ and $S^* : \mathfrak{k} \to \mathfrak{p}$. Using this one can rewrite the Poisson bracket on P_μ as:

$$\begin{aligned} & < x, [S(F), G] + [F, S(G)] > \\ = & < x, [S(F), G_p] + [F_p, S(G)] > \\ = & < x, [(S - S^*)(F), G_p] + [F_p, (S - S^*)(G)] > \end{aligned} \tag{7.23}$$

so the restricion of ϕ^* to P_μ can be written as:

$$\phi^*(x) = -(\pi_{\mathfrak{p}} \otimes \mathrm{id} + \mathrm{id} \otimes \pi_{\mathfrak{p}}) ds(x) \tag{7.24}$$

where $s \in \mathfrak{g} \otimes \mathfrak{g}$ is the skew-symmetric tensor corresponding to the map $S - S^*$. So the restriction of ϕ^* to fibres F_q is the projection of a coboundary.

Proposition 7.12. The map ϕ^* has the following properties:

(i) ϕ^* is constant on fibres.
(ii) $\phi^*(X e_\alpha) = (X \otimes \mathrm{id} + \mathrm{id} \otimes X)\phi^*(e_\alpha)$

Proof. This follows directly from Lemma 5.1 and Prop. 7.3. □

Proposition 7.13. The map $S - S^*$ is not a Yang-Baxter operator.

Proof. The same as for Prop. 7.9. □

All this suggests that there does not exist a double Lie algebra structure on $\mathfrak{g}$ such that P_μ is a Poisson submanifold and Hamilton's equations with respect to an Ad-invariant function are of type I, II, III or V.

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