Representations of Gaussian processes with stationary increments

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To Kasia
For everything

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Paweł Zaręba
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Contents

Introduction

In recent years, analysis of experimental data in several areas showed the occurrence of self-similarity and long-range dependence phenomena. Important examples are telecommunication and computer network traffic (cf. [42]), prices of financial assets (cf. [90]) or hydrology (cf. [65]), to mention just a few. Self-similarity refers to the invariance of the statistical properties of a process under a suitable change of scale. Long-range dependence indicates that even after a long time a process is still influenced by its past. Popular and attractive applications and the fact that classical models often do not incorporate these phenomena brought great attention to the class of random processes exhibiting self-similarity and/or long-range dependence.

The single most important process having both properties is the *fractional Brownian motion* (fBm). It is a zero-mean, Gaussian stochastic process with stationary increments that is self-similar in the sense that for some $H \in (0,1)$, the rescaled process $(a^{-H}X_{at})_{t\in\mathbb{R}}$ has the same distribution as the original process $(X_t)_{t\in\mathbb{R}}$. Fractional Brownian motion is long-range dependent for $H > 1/2$. The parameter $H \in (0,1)$ is the so-called *Hurst index* of the fBm. The covariance function of fBm is given by

$$\mathbb{E}X_tX_s = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad s,t \in \mathbb{R}.$$

Note that if $H = 1/2$, the fractional Brownian motion is the ordinary Brownian motion, with covariance $\mathbb{E}X_tX_s = s \wedge t$. The history of fractional Brownian motion goes back to Kolmogorov [37], who in 1940 was studying the "spirals of Wiener". In 1968 Mandelbrot and van Ness [55] proposed the present name of the process. After a relatively quiet period, the recent interest in self-similarity and long-range dependence contributed to the present renaissance of fractional Brownian motion. For instance, in queuing theory and telecommunications the fBm arises in heavy traffic limit theorems for modern models (cf. [88]), and is also used directly to model queues with self-similar input (e.g. [67]). In mathematical finance, the fBm is sometimes considered as an alternative for ordinary Brownian motion (e.g. [79]). We refer to the monograph of Doukhan et al. [12] for a more elaborate overview of the various areas of applications of the fBm.

Unless $H = 1/2$, the fractional Brownian motion is not a semimartingale, therefore we can not use the usual stochastic calculus to analyze it. It is also not a Markov process, hence the analysis of the fBm can be rather complicated. One of the ways to approach it is to represent fBm in terms of some simpler, better understood processes such as ordinary Brownian motion. The spectral representation of the covariance of fBm

$$\mathbb{E}X_tX_s = c_H \int_{\mathbb{R}} (e^{i\lambda t} - 1)(e^{-i\lambda s} - 1)|\lambda|^{-1-2H} d\lambda, \tag{0.0.1}$$

where $c_H = (\Gamma(1+2H)\sin \pi H)/(2\pi)$, allows the integral representation

$$X_t = \sqrt{c_H} \int_{\mathbb{R}} (e^{i\lambda t} - 1)|\lambda|^{-1/2-H} dW_\lambda, \qquad t \in \mathbb{R},$$

where W is an ordinary Brownian indexed by $\mathbb{R}$. Note that the integration over whole line implies that the random variable X_t depends on the whole past and whole future of the Brownian motion W. This is not the most desirable situation. Pinsker and Yaglom [72] found the representation

$$X_t = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{t} \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] dW_u,$$

where $x_+ = \max\{x, 0\}$ and W is an ordinary Brownian motion. Note that this formula was used in [55] as a definition of the fBm. Here the random variable X_t only depends on the whole past of the Brownian motion W.

Of great interest is a so-called (finite past) moving average representation of the form

$$X_t = \int_0^t f_t(u) dW_u,$$

where f_t is some deterministic function and W denotes an ordinary Brownian motion. Such a representation allows us to obtain numerous results regarding for instance prediction, maximal inequalities, stochastic calculus, equivalence of probability measures, etc. (e.g. [10], [58], [68], [69], [73], [80]). In 1969

Golosov and Molchan [28], [61] obtained the moving average representation of the fractional Brownian motion X, given by

$$X_t = \frac{C_H}{2H} \int_0^t x_t(u) u^{1/2-H} dW_u, \tag{0.0.2}$$

where

$$C_H^2 = \frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)},$$

$$x_t(u) = 2H \left[t^{H-1/2}(t-u)^{H-1/2} - \int_u^t (u-v)^{H-1/2} dv^{H-1/2} \right] 1_{(0,t)}(u), \quad u \geq 0,$$

and W is an ordinary Brownian motion. This representation is invertible in the sense that there exists some deterministic function w_t such that $W_t = \int_0^t w_t(u) dX_u$. The aforementioned representation, among other results, was rediscovered at the end of the last century by many authors, see for instance [4], [27], [10], [68], [70], [73], [18]. Various methods were used to obtain the result, including Malliavin calculus, Lamperti transforms, spectral theory, etc.

Another kind of representations that are useful for theoretical and practical purposes are series expansions of the form

$$X_t = \sum_k \phi_k(t) Z_k, \qquad t \in [0,T], \tag{0.0.3}$$

where the Z_k are independent Gaussian random variables and the $\phi_k(t)$ certain deterministic functions. According to the general theory of Gaussian random variables in Banach spaces such a representation always exists if $(X_t)_{t\in[0,T]}$ is a Borel measurable random element in a separable Banach space (see e.g. [41], [52]). The expansion (0.0.3) is not unique however, there are various ways of expanding a given process in such a series. One of the known representations is the Karhunen-Loève series expansion. It is of the form

$$X_t = \sum_k \sqrt{\lambda_k} \psi_k(t) Z_k, \qquad t \in [0,T],$$

where the $\psi_k(t)$ are the eigenfunctions corresponding to the nonnegative eigenvalues λ_k of the linear integral equation

$$\int_0^T \mathbb{E} X_t X_s \, \psi(s) ds = \lambda \psi(t), \qquad t \in [0,T], \tag{0.0.4}$$

and the Z_k are independent standard Gaussian random variables. This result is a combination of Mercer's theorem [59] and Karhunen's theorem [34], for the complete statement see for instance [94], Section 26.1.

The Karhunen-Loève expansion for fractional Brownian motion remains unknown. A series expansion of fBm of the form

$$X_t = \sum_{n \in \mathbb{Z}} \frac{e^{2i\omega_n t} - 1}{2i\omega_n} Z_n, \qquad t \in [0,1], \tag{0.0.5}$$

where ω_n are the real-valued zeros of the Bessel function of the first kind J_{1-H} and the Z_n are independent Gaussian random variables with mean zero and variances that can be expressed explicitly in terms of Bessel functions and their zeros, was obtained in [20] and can be viewed as a generalization of the classical Paley-Wiener series expansion of ordinary Brownian motion (see [71]). A different interesting series expansion was obtained in [19] by considering odd and even part of fBm separately.

Series expansions of stochastic processes are important not only from the theoretical point of view. The simulation of sample paths of processes is one of the significant applications of series representations. Expansions of this type are also relevant in connection with the study of the small ball probabilities (see e.g. [48], [50]).

The main goal of this text is to establish a theory that allows us to obtain (finite past) moving average representations and series expansions for the whole class of Gaussian processes with stationary increments. A theory that includes the representations (0.0.2) and (0.0.5) of fractional Brownian motion as a special cases, and that can be applied to any stationary increments process.

All our results originate in the spectral representation

$$\mathbb{E} X_s X_t = \int_{\mathbb{R}} \frac{(e^{i\lambda t} - 1)(e^{-i\lambda s} - 1)}{\lambda^2} \, \mu(d\lambda) \qquad s, t \geq 0 \tag{0.0.6}$$

of the continuous, centered, second-order Gaussian stochastic process $X = (X_t)_{t \geq 0}$ with stationary increments (see e.g. [11], Section XI.11). The measure μ is a unique symmetric Borel measure on the line satisfying

$$\int_{\mathbb{R}} \frac{\mu(d\lambda)}{1 + \lambda^2} < \infty. \tag{0.0.7}$$

The next step in the approach is the use of the Krein-de Branges theory, which was pioneered by M.G. Krein in 1950's (cf. [38]). From our perspective, the most significant result of Krein is the fact that there is a 1-1 relationship between symmetric Borrel measure μ on the line satisfying (0.0.7) and differential operator $\mathcal{G}f = df'/dm$ associated with a string with mass distribution m. Roughly speaking, the measure μ can be seen as to describe the kinetic energy of a string with a certain mass distribution as it vibrates at different frequencies. This association allows us to apply the theory of reproducing kernel Hilbert spaces of entire functions developed by de Branges [9] and obtain results concerning the structure of the space $L^2(\mu)$ and certain important subspaces. The general Krein-de Branges theory is extensively described in Dym and McKean [17]. This thesis includes, interprets and sometimes extends results of [17] and shows how the Krein-de Branges theory can be used to solve particular problems connected with the study of stationary and stationary increments Gaussian processes.

Given a Gaussian process X with stationary increments (si) we use Krein's result to associate with the spectral measure μ, defined by (0.0.6), a unique string with mass distribution m. Relation (0.0.6) gives rise to an isometry, mapping X_t to $(\exp(i\lambda t)-1)/i\lambda$, between the closed linear span of X_t, $t \in [0, T]$ and the closure in $L^2(\mu)$ of the linear span of the set

$$\{\lambda \mapsto (\exp(i\lambda t) - 1)/i\lambda : t \in [0, T]\}.$$

The latter space is denoted by $\mathscr{L}_T$ and the described mapping is called the spectral isometry. Based on the string corresponding to the measure μ we find a reproducing kernel and an orthonormal basis of the space $\mathscr{L}_T$, which allow us to obtain various result for the elements of this space (e.g. series expansions). The results obtained for the elements of $\mathscr{L}_T$ are transferred into results on the structure of the process X itself using the spectral isometry.

Unfortunately, Krein's result associating the measure μ with a unique string proves only the existence of the latter. To be able to use the results in concrete cases, we have to compute the explicit form of the string associated with a given spectral measure. Krein, in his original works, described some ways of dealing with this problem. In particular, he exhibited the mass distribution corresponding to the rational spectral densities. In the present text we discuss the result of [22], where the string associated with a power spectrum was computed. This allows one to find the string of the fractional Brownian motion, which turns out to be of the form $m(x) = C_H x^{(1-H)/H}$, for some explicitly given constant C_H.

The method used for the stationary increments processes can be applied as well to the class of stationary processes. This is due to the spectral representation

$$\mathbb{E}Y_sY_t = \int_{\mathbb{R}} e^{i\lambda t}e^{-i\lambda s}\mu(d\lambda), \qquad s,t \geq 0, \tag{0.0.8}$$

of the centered stationary process Y and the fact that if the measure μ is finite the closed linear span of the set $\{\lambda \mapsto \exp(i\lambda t) : t \in [0,T]\}$ coincides with the space $\mathscr{L}_T$. Therefore, we take the results on the structure of the space $\mathscr{L}_T$ from the si-case and translate them in terms of the structure of the process Y using the isometric relation $Y_t \leftrightarrow e^{i\lambda t}$, implied by (0.0.8).

The Krein measure-string association is also used to obtain representation results for random fields. If the centered Gaussian random field X_t, $t \in \mathbb{R}^N$ has homogenous increments, that is the distribution of $X_t - X_s$ depends only on $t-s$, and is isotropic in a sense that its distribution is invariant to rotations of $\mathbb{R}^N$ around the origin we have the representation

$$\mathbb{E}X_sX_t = \int_{\mathbb{R}^N} \left(e^{i\langle v,t\rangle} - 1\right)\left(e^{-i\langle v,s\rangle} - 1\right)\varrho(dv), \qquad s,t \in \mathbb{R}^N, \tag{0.0.9}$$

where ϱ is a Borel measure satisfying

$$\int_{\mathbb{R}^N} \frac{\|v\|^2}{\|v\|^2+1}\varrho(dv) < \infty, \tag{0.0.10}$$

(cf. [94]). The fact that the field is isotropic, hence the measure ϱ is spherically symmetric, allows us to rewrite this representation as an integral with respect to a measure μ on $\mathbb{R}$ satisfying (0.0.7). Therefore, we can associate with μ a unique string with mass distribution m. The Krein-de Branges theory then provides a methodology to obtain a series expansion and a moving average-type integral representation of the random field X.

Similar to the one-dimensional case, the method used for isotropic random fields with homogenous increments can be adapted to obtain results for isotropic homogenous fields. The centered random field $(Y_t)_{t\in\mathbb{R}^N}$ is said to be homogenous if the covariance function $\mathbb{E}Y_tY_s$ depends only on the vector $t-s$. We have the representation

$$\mathbb{E}Y_sY_t = \int_{\mathbb{R}^N} e^{i\langle v,t\rangle}e^{-i\langle v,s\rangle}\varrho(dv), \qquad s,t \in \mathbb{R}^N, \tag{0.0.11}$$

for some finite Borel measure ϱ on $\mathbb{R}^N$ (cf. [92]). The isotropic property implies that we can rewrite the covariance function in terms of a measure on the real line satisfying the condition (0.0.7) and therefore, associate with this measure a unique string. Reasoning analogously to homogeneous increments case we obtain a series expansion and a moving average-type integral representation of the random field Y.

Overview of the chapters

We start Chapter 1 by introducing the basic probabilistic notions used in the whole text. We provide a short introduction to spectral representations of stationary Gaussian processes and processes with stationary increments. The spectral measure and spectral isometry are defined therein. Then we move to the examples of stochastic processes that we will use to illustrate the general results throughout the whole text. They include ordinary Brownian motion, fractional Brownian motion, the Ornstein-Uhlenbeck process and processes with Matérn-type spectral densities. From Section 1.2 on we consider isotropic random fields. The definition and description of the so-called spherical harmonics precedes the representation of the covariance of an isotropic random field in which they are involved. Chapter 1 is completed by introduction of two main examples, namely Lévy's ordinary and fractional Brownian motion on $\mathbb{R}^N$ for $N \geq 2$.

Chapter 2 is entirely devoted to the Krein-de Branges theory and its application to our setup. We start with the definition of a string with mass distribution m and length l. We describe the domain on which the differential operator

$$\mathcal{G}f = \frac{df'}{dm} \tag{0.0.12}$$

acts and introduce its two eigenfunctions satisfying

$$\mathcal{G}f = -\lambda^2 f \tag{0.0.13}$$

with different sets of initial conditions. Section 2.5 explains the 1-1 relation between a string with mass distribution m and a symmetric, Borel measure μ on the line satisfying condition (0.0.7). This is illustrated by the basic example of the string associated with Lebesgue measure. The existence of the isometric relation between the space of square integrable functions with respect to the mass distribution $L^2(m)$ and a space of even functions in $L^2(\mu)$ is established in Section 2.6. It is defined by

$$f \to \int_0^l f(x)A(x,\lambda)dm(x), \qquad f \in L^2(m), \tag{0.0.14}$$

where $A(x, \lambda)$ is the eigenfunction of the operator $\mathcal{G}$. A similar relation is presented for the space of odd functions in $L^2(\mu)$. Then we prove that the subspace $L^2_T(\mu)$ of $L^2(\mu)$ defined as a closed span of the set

$$\{(e^{i\lambda t} - 1)/i\lambda : |t| \leq T\}, \tag{0.0.15}$$

is a reproducing kernel Hilbert space and its reproducing kernel is given by

$$K(\omega, \lambda) = \frac{\overline{A(l, \omega)}B(l, \lambda) - \overline{B(l, \omega)}A(l, \lambda)}{\pi(\lambda - \overline{\omega})}, \tag{0.0.16}$$

where $B(x, \lambda) = -\frac{d}{dx}A(x, \lambda)/\lambda$. In addition, the set $\{K(\omega_n, \cdot)\}_n$ where ω_n are real zeros of the function $B(l, \cdot)$ constitutes an orthogonal basis of $L^2_T(\mu)$. The aforementioned results of Chapter 2 are essentially classical results that can be found in [38], [33] or [17]. Section 2.7 gathers some new (or at least not directly presented in the literature) results concerning bases of the spaces connected with the string (e.g. $L^2(m)$). Section 2.8 contains a complete proof of some important properties of the zeros of the functions A and B that are included in [17] in a form of an exercise. This proof is a combination of several results from [16] and [17].

In chapter 3 we compute the strings associated with particular spectral measures. The first step is to describe how the string is affected by multiplying the spectral measure by a constant. This result is used to compute string corresponding with the spectral measure of ordinary Brownian motion. Next, we consider spectral measures with finite moments of the form

$$\mu(d\lambda) = \frac{d\lambda}{(1 + \lambda^2)^n}, \qquad n \in \mathbb{N}.$$

We show that they correspond to infinitely long strings whose mass function starts with a number of jumps and 'mass-free' intervals and then continues linearly. The number of jumps depends on the power n. The magnitudes and points of occurrence of the jumps are computed explicitly. Examples include the Ornstein-Uhlenbeck process and processes with Matérn-type densities. The results of Chapter 3 mentioned so far are known and can be found in [38] and [17]. The last section of this chapter presents a new result that was first published in [22] and is devoted to power spectral measures. We show that the spectral measure with density $f(\lambda) = |\lambda|^{(p-1)/(p+1)}$ corresponds to the string with power mass distribution $m(x) = x^p$ multiplied by some constant. This allows to find the spectral measure of the fractional Brownian motion, whose spectral measure is given by

$$\mu_H(d\lambda) = c_H|\lambda|^{1-2H}d\lambda, \qquad c_H = (\Gamma(1+2H)\sin\pi H)/(2\pi). \tag{0.0.17}$$

The mass distribution of the resulting string is then given by $m(x) = C_H x^{(1-H)/H}$, where the constant C_H is computed explicitly.

Chapter 4 contains the main results of this text, namely the representations of processes and fields. This chapter is largely based on the papers [22] and [23]. We start with a series representation of a Gaussian process with stationary increments for which the rates of convergence of the truncated series

$$X_t^n = \sum_{|k|\le n} \phi_k(t) Z_k$$

are provided as well. This is a general result for the whole class of Gaussian processes with stationary increments which includes the series expansion of fractional Brownian motion obtained in [20]. The rates of convergence depend on the asymptotic behavior of the reproducing kernel of the space $\mathscr{L}_T$ defined as a closed linear span of $(e^{i\lambda t}-1)/i\lambda$ for $t \in [0,T]$. In the case of the fBm the obtained rate of convergence in the sup-norm is $n^{-H}\sqrt{\log n}$, which is optimal according to [39].

In Section 4.1.2 we show that the series results for processes with stationary increments are easily adaptable to the stationary case. In particular, we obtain a series representation of the Ornstein-Uhlenbeck process that appears to be unknown and prove the optimality of the rate of convergence $n^{-1/2}$ in L^2-norm.

Then, we present a 'space-domain' moving average representation for general Gaussian stationary and si-processes. Unlike the proper 'time-domain' moving average representation, here the integration in

$$X_t = \int f_t dM$$

takes place over the location on the string associated with the process. To obtain the 'time-domain' moving average representation we have to consider a narrower class of processes, namely, processes for which the associated mass distribution is smooth. The results of this section are applied to re-obtain the well-know (finite past) moving average representation (0.0.2) of fractional Brownian motion.

Starting from Section 4.2, we follow a similar path for random fields obtaining series expansion and moving average representation of isotropic random fields with homogenous increments. Our approach to this subject was inspired by the investigations of homogenous isotropic random fields in [93], [92] or [43].

The moving average representation obtained for Lévy's Brownian motion coincides with the result of [57]. Application of our general moving average result to Lévy's fractional Brownian motion gives a new representation of the latter. It is different from the recent result of [54].

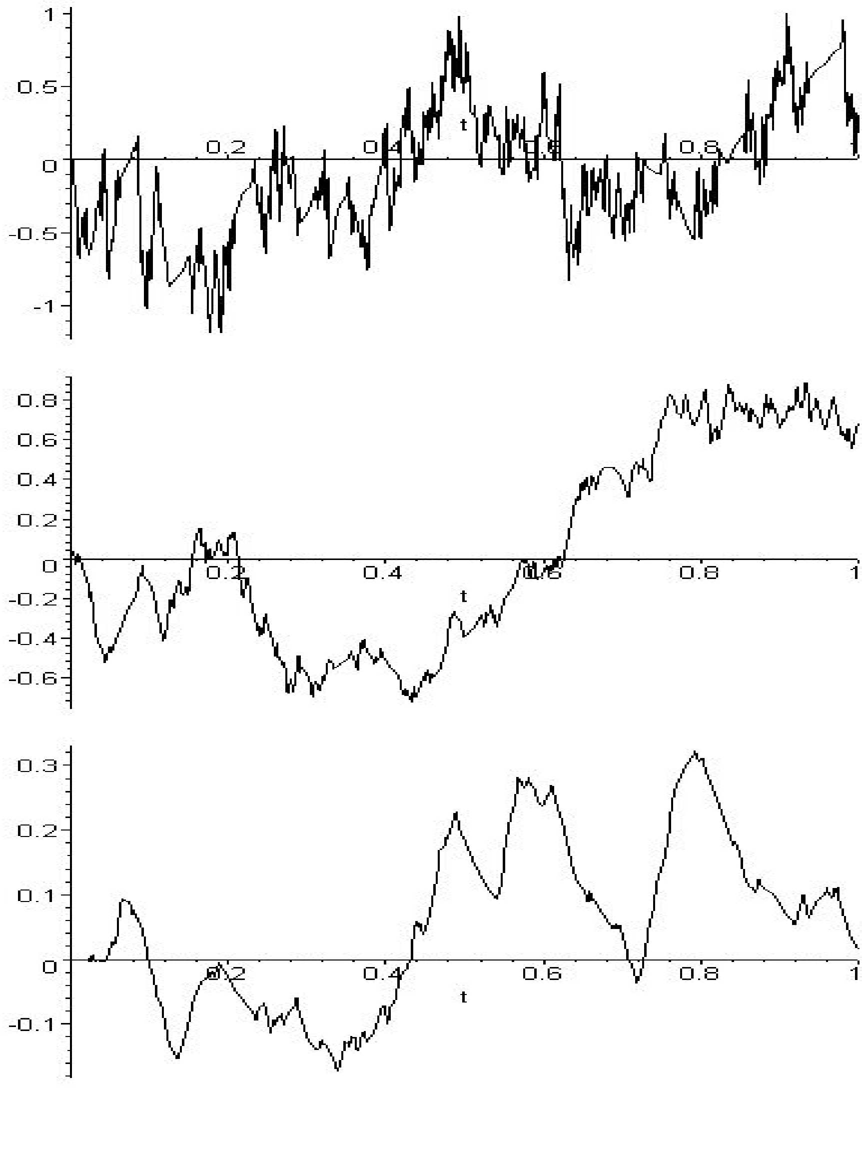
The closing Chapter 5 is devoted to the small ball probabilities. We investigate the behavior of the quantity

$$-\log \mathbb{P}\left(\|X\| < \varepsilon\right)$$

as $\varepsilon \to 0$ for various norms $\|\cdot\|$. This topic is closely related to the convergence speed of the series expansions. The convergence results obtained in Chapter 4 are used here, hence the obtained results also depend on the asymptotic behavior of the reproducing kernel of the space $\mathscr{L}_T$. After recalling some classical results from this field, we provide small ball probability bounds for the whole class of Gaussian stationary and si-processes with respect to various norms. It turns out that our bounds are sharp with respect to the L^2-norm for both fractional Brownian motion and Ornstein-Uhlenbeck processes, according to the well known results that can be found in [78] and [46]. In case of the supremum norm we obtain an additional logarithmic factor compared to the known sharp small ball estimates (cf. [49] and [47]).

Appendix A contains definitions and basic properties of several special function used throughout the text. In Appendix B we recall the basic definitions of fractional calculus.

Figure 1: Fractional Brownian motion for $H = 1/4,\, 1/2,\, 3/4$



Chapter 1

Frequency domain representations of Gaussian stochastic processes

1.1 Preliminaries

In this section we will recall the basic probabilistic notions. Let us thus start with the underlying probability space which implicitly, throughout the whole book, will be the space on which all the processes will be defined. The *underlying probability space* is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of an sample space Ω, a σ-algebra of subsets of Ω called $\mathcal{F}$ and the probability measure $\mathbb{P}$ acting on $\mathcal{F}$.

A real-valued N-dimensional *random vector* X (referred to as a random variable if $N = 1$) is a measurable function

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N)),$$

where $\mathcal{B}(\mathbb{R}^N)$ denotes a σ-algebra of Borel sets in $\mathbb{R}^N$. The *probability distribution* $\mathbf{P}$ of a random vector X is the measure on $\mathcal{B}(\mathbb{R}^N)$ defined by the formula

$$\mathbf{P}(B) = \mathbb{P}(X^{-1}(B)), \quad B \in \mathcal{B}(\mathbb{R}^N).$$

A random vector with distribution $\mathbf{P}$ is called *Gaussian* if its characteristic function

$$\varphi_X(u) = \int_{\mathbb{R}^N} e^{i\langle u,x\rangle}\mathbf{P}(dx), \quad u \in \mathbb{R}^N,$$

has the form

$$\varphi_X(u) = \exp\left\{i\langle u, a\rangle - \frac{1}{2}\langle \mathbf{B}u, u\rangle\right\} \tag{1.1.1}$$

for some vector $a \in \mathbb{R}^N$ called the *mean* and a linear self-adjoint nonnegative definite operator $\mathbf{B}$ called a *covariance operator*. The matrix $\{b_{k,j}\}$ defining $\mathbf{B}$ is said to be the *covariance matrix*. In the one-dimensional case the characteristic function takes the form

$$\varphi_X(u) = \exp\left\{iua - \frac{1}{2}\sigma^2 u^2\right\}, \quad a \in \mathbb{R}, \sigma > 0. \tag{1.1.2}$$

Let $\mathbf{T}$ be a subset of $\mathbb{R}^N$. Then X is called a *stochastic process* indexed by T when it is a function

$$X : \Omega \times \mathbf{T} \longrightarrow \mathbb{R} \tag{1.1.3}$$

such that, for every $t \in \mathbf{T}$, $X_t := X(\cdot, t)$ is a random variable. For fixed $\omega \in \Omega$ the function $X(\omega, \cdot) : \mathbf{T} \to \mathbb{R}$ is called a *sample path* or a *realization* of the stochastic process. We say that a process is *continuous* if the probability of the set of all ω's, for which the sample path is a continuous function, is one.

To denote a stochastic process we will often use $(X_t)_{t\in\mathbf{T}}$. If the set $\mathbf{T}$ is more than one-dimensional we will use the name *random field*.

A stochastic process $(X_t)_{t\in\mathbf{T}}$ is called *Gaussian* if for every $n \in \mathbb{N}$ and every $t_1, t_2, ..., t_n \in \mathbf{T}$ a random vector $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ is a n-dimensional Gaussian random vector (distributions of such a vectors are called the *finite-dimensional distributions* (fdd) of the process). This is equivalent to the fact that every linear combination $\sum_{i=1}^n \alpha_i X_{t_i}$, $\alpha_i \in \mathbb{R}$, is a Gaussian random variable.

Another important notion connected with stochastic processes is the *covariance function*. It is defined as a function of two variables

$$\mathbf{T} \times \mathbf{T} \ni (t, s) \longrightarrow \mathbb{E}\left[(X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)\right], \tag{1.1.4}$$

provided the right-hand side exists. The above function measures the level of linear dependence between the variables X_t and X_s. To recall, they are called uncorrelated if $\mathbb{E}X_tX_s = \mathbb{E}X_t\mathbb{E}X_s$. Independence always implies uncorrelation, in general the converse is not true. However, in the special case when the vector (X_t, X_s) is Gaussian, those two notions are equivalent. If the process X is zero-mean, i.e. $\mathbb{E}X_t = 0\ \forall t$, the covariance function is given by

$$R(s, t) = \mathbb{E}X_tX_s. \tag{1.1.5}$$

For the time being we restrict our considerations only to stochastic processes with one-dimensional time, i.e. $\mathbf{T} \subset \mathbb{R}$. We will go back to random fields in Section 1.2.

An important dependence feature that a stochastic process can possess are *independent increments.* We say that a stochastic process $(X_t)_{t \in T}$, $T \subset \mathbb{R}$, has independent increments if for every choice of $t_1 < t_2 < ... < t_n$ from T, $n \in \mathbb{N}$, the random variables

$$X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, ..., X_{t_n} - X_{t_{n-1}}$$

are independent.

At this point we will introduce two important classes of stochastic processes. We say that a second order stochastic process (i.e. such that $\mathbb{E}X_t^2 < \infty$) $(X_t)_{t \in T}$ is a *stationary process* if for all $t, h \in \mathbf{T}$ such that $t + h \in \mathbf{T}$ we have

$$\mathbb{E}X_{t+h} = \mathbb{E}X_t \quad \text{and} \quad \mathbb{E}X_{t+h}X_h = \mathbb{E}X_t X_0 \tag{1.1.6}$$

In other words, it means that the mean of such a process is constant and the correlation and covariance depend on the time points s and t only through the difference $|t - s|$. Similar as with independence we can copy this notion to the increments of the process. A stochastic process $(X_t)_{t \in T}$ is said to have *stationary increments* (si) if for all $h \in \mathbf{T}$ such that $t + h \in \mathbf{T}$ the processes $(X_{t+h} - X_h)_{t \in \mathbf{T}}$ and $(X_t - X_0)_{t \in \mathbf{T}}$ are equal in distribution.

1.1.1 Spectral representation

All the processes considered in this section are mean-square continuous, centered, second order and Gaussian.

Let $Y = (Y_t)_{t \in \mathbf{T}}$ be a stationary process. Stationarity implies that the covariance has the property $R(t, s) = R(t - s)$. Hence, we can consider the covariance function as a function of one real variable. According to Bochner's theorem there exists a unique positive, symmetric, finite Borel measure μ on the real line, such that

$$R(t) = \int_{\mathbb{R}} e^{i\lambda t} \mu(d\lambda). \tag{1.1.7}$$

(see for instance [32], I.6; [94], 2.9). The measure μ is called the *spectral measure* of the stationary process Y. If there exists a function f such that

$$\mu(d\lambda) = f(\lambda) d\lambda \tag{1.1.8}$$

we call it the *spectral density* of the process of interest. Given (1.1.7) and using stationarity we can deduce the following

$$\mathbb{E}Y_tY_s = \int_{\mathbb{R}} e^{i\lambda t}e^{-i\lambda s}\mu(d\lambda). \tag{1.1.9}$$

Before we step to the consequences of the above representation let us define two important spaces.

If V is a linear space and W is a subset of V, we denote the closure of the span of W by $\overline{\mathrm{sp}}\,W$. First, define the *linear space of the process* Y as

$$\mathscr{H}_{\mathbf{T}} := \overline{sp}\,\{Y_t \;:\; t \in \mathbf{T}\} \tag{1.1.10}$$

where the closure takes place in $L^2(\Omega, \mathcal{F}, \mathbb{P}) = L^2(\mathbb{P})$. The second space is defined as

$$\mathscr{L}_{\mathbf{T}} = \overline{sp}\{\lambda \mapsto e^{i\lambda t} \;:\; t \in \mathbf{T}\}, \tag{1.1.11}$$

where the closure takes place in the space $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu) = L^2(\mu)$.

Since the left-hand side of (1.1.9) can be viewed as an inner product in the space $L^2(\mathbb{P})$ and the right-hand side is nothing else than an inner product in $L^2(\mu)$, the relation (1.1.9) defines an isometry between the spaces $\mathscr{H}_{\mathbf{T}}$ and $\mathscr{L}_{\mathbf{T}}$. Under this so-called *spectral isometry* we have a one-to-one correspondence

$$\mathscr{H}_{\mathbf{T}} \ni Y_t \;\longleftrightarrow\; \left[\lambda \longmapsto e^{i\lambda t}\right] \in \mathscr{L}_{\mathbf{T}}. \tag{1.1.12}$$

This isometry is of great importance for this whole text. To indicate its role, everything that we would like to know about the linear structure of the process, i.e. its linear space $\mathscr{H}_{\mathbf{T}}$, can be easily translated, via the spectral isometry, in terms of the well-studied space $L^2(\mu)$. Then we are able to use some available tools therein. After we obtain desired results for the elements of the space $\mathscr{L}_{\mathbf{T}}$, we can go back to $\mathscr{H}_{\mathbf{T}}$.

As we have already indicated in the previous section, an integral of a stationary process is a si-process. Based on this, one can already suspect that we would have similar results also for processes with stationary increments. Indeed, they can be obtained by, in some sense, 'integrating' the stationary case.

At this point it is useful to introduce some additional notation. Note that

$$\hat{1}_{(0,t]}(\lambda) = (e^{i\lambda t} - 1)/i\lambda, \tag{1.1.13}$$

where 1_A is the indicator function of the set A and $\hat{f}$ denotes the Fourier transform of the function f, defined by

$$\hat{f}(\lambda) = \int e^{iu\lambda} f(u)\, du.$$

If $X = (X_t)_{t \in \mathbf{T}}$ is a second order, continuous, mean zero, stochastic process with stationary increments there exists a unique symmetric Borel measure μ on the real line satisfying (compared to finiteness in stationary case)

$$\int_{\mathbb{R}} \frac{\mu(d\lambda)}{1+\lambda^2} < \infty, \tag{1.1.14}$$

such that

$$\mathbb{E} X_t X_s = \int_{\mathbb{R}} \hat{1}_{(0,t]}(\lambda) \overline{\hat{1}_{(0,s]}(\lambda)} \mu(d\lambda) \tag{1.1.15}$$

for $s, t \in \mathbf{T}$ (see e.g. [11], XI.11). The measure μ is the *spectral measure* of a stochastic process with stationary increments. Similarly, we can view (1.1.15) as an equivalence of two inner products

$$\mathbb{E} X_t X_s = \langle \hat{1}_{(0,t]}, \hat{1}_{(0,s]} \rangle_\mu, \tag{1.1.16}$$

where $\langle f, g \rangle_\mu = \int_{\mathbb{R}} f(\lambda) \overline{g(\lambda)} \mu(d\lambda)$ for $f, g \in L^2(\mu)$. Therefore, it gives rise to an analogous spectral isometry. First observe that, provided μ is a finite measure, the space

$$\overline{sp}\{\lambda \mapsto \hat{1}_{(0,t]}(\lambda) \ : \ t \in \mathbf{T}\} \subset L^2(\mu) \tag{1.1.17}$$

is exactly the same space as the one defined in (1.1.11). Therefore, we use here the same notation $\mathscr{L}_\mathbf{T}$. The linear space $\mathscr{H}_\mathbf{T}$ of si-process X is defined exactly as in (1.1.10). The spectral isometry maps the elements

$$X_t \ \longleftrightarrow \ \hat{1}_{(0,t]} \tag{1.1.18}$$

of the spaces $\mathscr{H}_\mathbf{T}$ and $\mathscr{L}_\mathbf{T}$, respectively, to each other.

Observe that if Y is a stationary process with spectral measure μ, according to the relation $\hat{1}_{(0,t]}(\lambda) = \int_0^t e^{i\lambda u} du$, the integrated process

$$X_t = \int_0^t Y_s\, ds$$

is a si-process with spectral measure μ.

To study a stationary or si-process we will use the fact that its spectral measure μ is the so-called *principle spectral function* of a unique *string*. This will provide useful information about the structure of the space $\mathscr{L}_T$ and, if we use the spectral isometry described above, about the space $\mathscr{H}_\mathbf{T}$, and hence about the process of interest.

1.1.2 Examples

In this section we present examples of stochastic processes that throughout the whole text will serve us as illustration of obtained results. We begin with two processes with stationary increments: the well-studied Brownian motion and its very useful and interesting generalization - fractional Brownian motion. After that we move to the class of stationary processes of Matérn type, preceded by its simplest version - the Ornstein-Uhlenbeck process.

Example 1.1.1. *Brownian motion*

Historically, the Brownian motion process arose as an attempt to explain the phenomenon of extremely irregular motions of a small particle suspended in a fluid. It was observed in 1827 by Robert Brown and named in his honor. The values of a coordinate of such a particle recorded at definite time interval give us a realization of some stochastic process. As some time passed, it was formalized, and now we call a *(standard) Brownian motion* a continuous Gaussian stochastic process $W = (W_t)_{t\in\mathbb{R}_+}$ with stationary, independent increments, satisfying

- $W_0 = 0,$
- $\mathbb{E}W_t = 0, \qquad t \in \mathbb{R}_+,$
- $\mathbb{E}W_sW_t = s \wedge t, \qquad s, t \in \mathbb{R}_+.$

This definition implies that every increment $W_{t+s}-W_s$ has a normal distribution with mean 0 and variance t. Now, since

$$\mathbb{E}W_sW_t = s \wedge t = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(e^{i\lambda t}-1)(e^{-i\lambda s}-1)}{\lambda^2}d\lambda, \tag{1.1.19}$$

it is easy to see that the spectral measure of Brownian motion is a scaled Lebesgue measure given by

$$\mu(d\lambda) = \frac{1}{2\pi}d\lambda. \tag{1.1.20}$$

Example 1.1.2. *Fractional Brownian motion*

At a certain point, it appeared that the Brownian motion (or a general processes with Markov property, i.e. processes which future development after given time t are dependent only on the situation at t and independent of what happened before that time) are no good theoretical model for a lot of real-life situations. In many applications (such as finance, telecommunication networks, biology, etc.) real data exhibit a so-called *long-range dependence* structure: the behavior of the process after given time t not only depends on the state of the process at t but also on the whole history up to time t. To model this behavior Mandelbrot and van Ness [55] proposed a process that they called fractional Brownian motion (fBm). This process was already known few decades before - the history goes back to Kolmogorov who in several papers from early 40's considered it in the studies of turbulence. For more historical comments see [63].

A Gaussian stochastic process $X = (X_t)_{t\in\mathbb{R}_+}$ with stationary increments is called *(standard) fractional Brownian motion* with *Hurst index* $H \in (0,1)$ if it has zero mean, continuous sample paths, $X_0 = 0$ and its covariance function is of the form

$$\mathbb{E}X_sX_t = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad s,t \geq 0. \tag{1.1.21}$$

This process is *H-self similar*, meaning that the processes $(X_{at})_{t\in\mathbb{R}_+}$ and $(a^HX_t)_{t\in\mathbb{R}_+}$ have the same distribution. If $H = 1/2$ it becomes an ordinary Brownian motion. Note that for $H \neq 1/2$ we indeed have dependent increments. The correlation of the increments behaves as a negative power of the time difference:

$$\mathbb{E}(X_{t+h} - X_t)(X_{s+h} - X_s) \sim |t-s|^{2H-2}, \quad \text{as} \quad |t-s| \to \infty, \tag{1.1.22}$$

see e.g. [62], [77], Section 7.2.

The spectral measure of fractional Brownian motion is given by

$$\mu_H(d\lambda) = c_H|\lambda|^{1-2H}d\lambda, \tag{1.1.23}$$

where $c_H = (\Gamma(1+2H)\sin \pi H)/(2\pi)$ (see for instance [77], 7.2).

Now, we turn to the examples of stationary processes.

Example 1.1.3. *Ornstein-Uhlenbeck process*

One of the simplest and best-known stationary processes is the Ornstein-Uhlenbeck process. A stationary, Gaussian process $(Y_t)_{t\in\mathbb{R}_+}$ is called an *Ornstein-Uhlenbeck process* (OU) if it is continuous, has zero mean and covariance

$$\mathbb{E}Y_tY_s = \pi e^{-|t-s|}, \tag{1.1.24}$$

for $s, t \geq 0$. It can be obtained as a solution of the following stochastic differential equation

$$dY_t = -Y_t dt + \sqrt{2\pi} dW_t, \tag{1.1.25}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. Standard Fourier theory shows that the spectral measure μ of the OU process is given by

$$\mu(d\lambda) = \frac{d\lambda}{\lambda^2+1}, \tag{1.1.26}$$

cf. [29], formula 17.23.12.

Example 1.1.4. *Matérn processes*

A *Matérn process* is a stationary, mean zero, Gaussian process with a covariance function

$$\mathbb{E}Y_tY_s = \frac{1}{\Gamma(\kappa)\sqrt{\pi}}(|t-s|/2)^{\kappa-\frac{1}{2}}K_{\kappa-\frac{1}{2}}(|t-s|), \quad \kappa > \frac{1}{2} \tag{1.1.27}$$

where K_ν is a modified Bessel function of the second kind of order ν. The spectral measure of such a process is of the form

$$\mu(d\lambda) = \frac{d\lambda}{(\lambda^2+1)^\kappa} \tag{1.1.28}$$

(see for instance [91]). For $\kappa = 1$ we obtain the aforementioned ordinary OU process. If $\kappa = 2$ the covariance simplifies to

$$\mathbb{E}Y_tY_s = \frac{\pi}{2}e^{-|t-s|}(1+|t-s|) \tag{1.1.29}$$

(see for instance [94], 2.10). Then the associated spectral measure is given by

$$\mu(d\lambda) = \frac{d\lambda}{(\lambda^2+1)^2}. \tag{1.1.30}$$

We will also obtain some results for a class of processes with a spectral densities of the form (1.1.28) but with $\kappa \in \mathbb{N}$.

1.2 Isotropic random fields

In this section we will present results for the generalization of stochastic processes indexed by one-dimensional parameter, i.e. *random fields*. The role of parameter $t \in \mathbb{R}$, usually interpreted as time, is replaced here by $t = (t_1, t_2, ..., t_N) \in \mathbb{R}^N$. The most popular applications involve the cases $t = (t_1, t_2)$ and $t = (t_1, t_2, t_3)$ interpreted simply as a planar or spatial coordinates, respectively. However, there are also situations where one of the N coordinates is interpreted as time and the remaining ones as a coordinates in $\mathbb{R}^{N-1}$. Realizations of random fields are surfaces in $\mathbb{R}^{N+1}$. Very often experimental data from many diverse disciplines resembles a realization of a random field. That is the reason they are commonly used in areas like turbulence, oceanography, geology, optics, seismology and many more.

1.2.1 Basic definitions

Let $X = (X_t)_{t\in\mathbb{R}^N}$ be a zero-mean, mean-square continuous Gaussian random field starting from the origin, i.e. $X(0) = 0$. We say that the random field X is *isotropic* if for any A from the group of orthogonal matrices on $\mathbb{R}^N$ it holds that X has the same finite-dimensional distributions as the process $(X(At))_{t\in\mathbb{R}^N}$. The random field X is said to have *homogenous increments* if for every $s \in \mathbb{R}^N$, $(X_t - X_s)_{t\in\mathbb{R}^N}$ and $(X_{t-s})_{t\in\mathbb{R}^N}$ have the same finite-dimensional distributions. If X has homogenous increments and is isotropic, its covariance function admits the representation

$$\mathbb{E}X_sX_t = \int_{\mathbb{R}^N} \left(e^{i\langle v,t\rangle} - 1\right)\left(e^{-i\langle v,s\rangle} - 1\right) \varrho(dv) \tag{1.2.1}$$

(see e.g. [94]). Here $\langle \cdot\,, \cdot\rangle$ is the usual inner product on $\mathbb{R}^N$, and ϱ is a Borel measure satisfying the condition

$$\int_{\mathbb{R}^N} \frac{\|v\|^2}{1 + \|v\|^2}\varrho(dv) < \infty. \tag{1.2.2}$$

With the spectral measure ϱ on $\mathbb{R}^N$ appearing in (1.2.1) we associate the symmetric Borel measure μ on the line defined by

$$\mu(d\lambda) = \frac{\Gamma(N/2)}{2\pi^{N/2}}\lambda^2 d\Phi(\lambda), \tag{1.2.3}$$

where

$$\Phi(y) = \int_{\|v\|\le y} \varrho(dv), \qquad y \ge 0. \tag{1.2.4}$$

According to the above definition, $d\Phi(y) = \Phi(y + dy) - \Phi(y)$ may be viewed as a ϱ-measure of the spherical shell $y \leq \|v\| \leq y + dy$. Due to (1.2.2) the measure μ satisfies the integrability condition

$$\int \frac{\mu(d\lambda)}{1 + \lambda^2} < \infty. \tag{1.2.5}$$

A centered, mean-square continuous Gaussian random field $Y = (Y_t)_{t \in \mathbb{R}}$ is said to be *homogenous* if it has a constant mean and its covariance $\mathbb{E}Y_t Y_s$ depends only on the vector $t - s$. The covariance can be represented as

$$\mathbb{E}Y_t Y_s = \int_{\mathbb{R}^N} e^{i\langle v,t\rangle} e^{-i\langle v,s\rangle} \varrho(dv) \tag{1.2.6}$$

where ϱ is a finite Borel measure on $\mathbb{R}^N$. Similarly to the case of homogenous increments, if Y is isotropic, we define the measure Φ as

$$\Phi(y) = \int_{\|v\| \leq y} \varrho(dv), \qquad y \geq 0. \tag{1.2.7}$$

Clearly,

$$\int_0^\infty d\Phi(y) = \varrho(\mathbb{R}^N) < \infty. \tag{1.2.8}$$

1.2.2 Spherical harmonics and spherical Bessel functions

In order to present the representation of the covariance function of the random field X we have to step aside and devote some space to the so-called *spherical harmonics*. These are classical special functions, constituting an orthonormal basis of the space of square integrable functions on the unit sphere in $\mathbb{R}^N$.

Spherical harmonics are restrictions to the unit sphere s^{N-1} in $\mathbb{R}^N$ of homogenous polynomials that satisfy the N-dimensional Laplace equation

$$\Delta u = 0, \tag{1.2.9}$$

where $u(x) = u(x_1, ..., x_N)$ and the *Laplace operator* is defined as $\Delta = \sum_{j=1}^N (\partial^2 / \partial x_j^2)$.

A polynomial $H_l(x)$ is said to be homogenous of degree l if it is of the form

$$H_l(x) = \sum_{\alpha_1 + ... + \alpha_N = l} c_{\alpha_1,...,\alpha_N} x_1^{\alpha_1} \cdot ... \cdot x_N^{\alpha_N}$$

where $\alpha_i \in \mathbb{N}$ and $c_{\alpha_1,\dots,\alpha_N} \in \mathbb{C}$. This definition implies that

$$H_l(rx) = r^l H_l(x), \qquad r > 0.$$

The polynomial H_l is said to be *harmonic* if $\Delta H_l(x) = 0$. There are

$$h(l, N) = \frac{(2l + N - 2)(l + N - 3)!}{(N - 2)!l!} \tag{1.2.10}$$

linearly independent, homogenous harmonic polynomials of degree l in N variables (see e.g. [25], Chapter XI or [30], p. 170). In general, any homogenous harmonic polynomial can be written as

$$H_l(x) = \|x\|^l \, S_l\left(\frac{x}{\|x\|}\right),$$

for $x \in \mathbb{R}^N$, where S_l is a homogenous harmonic polynomial on the unit sphere in $\mathbb{R}^N$. The function S_l is called a *spherical harmonic*. Consider the Hilbert space $L^2(s^{N-1}, d\sigma_N)$ of real-valued functions f on the unit sphere satisfying

$$\int_{s^{N-1}} f^2(x) d\sigma_N(x) < \infty, \tag{1.2.11}$$

where $d\sigma_N(x) = \sin^{N-2}\theta_1 \cdot \ldots \cdot \sin\theta_{N-2} d\theta_1 \ldots d\theta_{N-2} d\phi$ is a surface area element of the unit sphere. The inner product of functions $f, g \in L^2(s^{N-1}, d\sigma_N)$ is defined as

$$\langle f, g \rangle_{s^{N-1}} = \int_{s^{N-1}} f(x) g(x) d\sigma_N(x). \tag{1.2.12}$$

Any two spherical harmonics of different degree are orthogonal, i.e.

$$\langle S_l, S_k \rangle_{s^{N-1}} = 0, \qquad l \neq k. \tag{1.2.13}$$

(see e.g. [3], p. 451, [30], p. 172).

Let $\{S_l\}$ be a set of $h(l, N)$ linearly independent spherical harmonics of degree l. Using Gram-Schmidt procedure we can construct an orthonormal set

$$S_l^1(\cdot), S_l^2(\cdot), \dots, S_l^{h(l,N)}(\cdot) \tag{1.2.14}$$

of linearly independent real spherical harmonics of degree l. Then

$$\langle S_l^j, S_l^k \rangle_{s^{N-1}} = \delta_j^k. \tag{1.2.15}$$

Moreover, the set $\{S_l^m\}$, $m = 1, ..., h(l, N)$, $l = 0, 1, ...$ is a complete set in the Hilbert space $L^2(s^{N-1}, d\sigma_N)$ (see [25], 11.3, Theorem 3).

For any orthonormal set $\{S_l^m\}$ of $h(l, N)$ linearly independent real spherical harmonics of degree l and unit vectors ξ and η it holds that

$$\frac{C_l^{(N-2)/2}(\langle \xi, \eta \rangle)}{C_l^{(N-2)/2}(1)} = \frac{h(l,N)}{|s^{N-1}|} \sum_{m=1}^{h(l,N)} S_l^m(\xi) S_l^m(\eta), \tag{1.2.16}$$

where C_l^α is the Gegenbauer polynomial (see Appendix A) and $|s^{N-1}| = 2\pi^{N/2}/\Gamma(N/2)$. For the proof see for instance [25], [3], [30].

To compute the explicit form of the spherical harmonics consider the Laplace operator in terms of the spherical coordinates $(r, \theta_1, \theta_2, ..., \theta_{N-2}, \phi)$, defined for $N \geq 2$ as

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ \ldots & \quad \ldots \\ x_{N-1} &= r\sin\theta_1\sin\theta_2\cdot\ldots\cdot\sin\theta_{N-2}\cos\phi \\ x_N &= r\sin\theta_1\sin\theta_2\cdot\ldots\cdot\sin\theta_{N-2}\sin\phi \end{aligned} \tag{1.2.17}$$

where $r = \|x\|$. In these coordinates the Laplace operator takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0\,, \tag{1.2.18}$$

where Δ_0 is the *Laplace-Beltrami operator* on the sphere:

$$\Delta_0 = \sum_{j=1}^{N-2} \frac{1}{q_j \sin^{N-j-1}\theta_j} \frac{\partial}{\partial\theta_j}\left(\sin^{N-j-1}\theta_j \frac{\partial}{\partial\theta_j}\right) + \frac{1}{q_{N-1}}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\phi}\right), \tag{1.2.19}$$

$$q_1 = 1, \quad q_j = (\sin\theta_1\sin\theta_2\cdots\sin\theta_{j-1})^2, \quad j \geq 2. \tag{1.2.20}$$

According to [25], 11.2, the differential equation (1.2.9) has solutions of the form $u(x) = R_l(r)S_l(x/r)$, where $r = \|x\|$, S_l is an arbitrary spherical harmonic of degree l and $R_l(r)$ is a solution of the ordinary differential equation

$$\frac{\partial^2 R}{\partial r^2} + \frac{N-1}{r}\frac{\partial R}{\partial r} - \frac{l(l+N-2)}{r^2}R = 0. \tag{1.2.21}$$

After inserting such a u into Laplace equation with Laplace operator written as in (1.2.18) and using (1.2.21) we see that the spherical harmonic of degree l is an eigenfunction corresponding to the eigenvalue $-l(l+N-2)$ of the Laplace-Beltrami operator, i.e.

$$\Delta_0 S_l = -l(l+N-2)S_l. \tag{1.2.22}$$

For a given l, the geometric multiplicity of this eigenvalue is $h(l,N)$ (cf. [24]).

The explicit expressions of the spherical harmonics for arbitrary N are rather complicated, see e.g. [85], [25], but the special cases $N=1,2,3$ of obvious physical meaning are simply described. Note that the set of linearly independent, orthonormal real spherical harmonics $\{S_l^m\}_{m=1}^{h(l,N)}$ is not uniquely defined. For that reason we present below the most commonly used choices.

If $N=1$, then the unit sphere degenerates to the set $s^0=\{-1,1\}$ and the orthonormal spherical harmonics are $S_0^1(x)=1/\sqrt{2}$ and $S_1^1(x)=x/\sqrt{2}$.

If $N=2$, the angular part of the Laplace operator reduces to $\frac{\partial^2}{\partial\phi^2}$. Hence, as a solution of

$$\frac{\partial^2 U}{\partial\phi^2} = -l^2 U \tag{1.2.23}$$

we obtain $h(l,2)=2$ real orthonormal spherical harmonics

$$S_l^1(\phi) = \frac{\cos(l\phi)}{\sqrt{2\pi}}, \qquad S_l^2(\phi) = \frac{\sin(l\phi)}{\sqrt{2\pi}}. \tag{1.2.24}$$

If $N=3$, equation (1.2.22) takes the form

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d^2\Theta}{d\theta^2}\right) + l(l+1)\sin^2\theta = \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \tag{1.2.25}$$

where $U(\theta,\phi)=\Theta(\theta)\Phi(\phi)$. We separate the variables again, with separation constant $-m^2$. The equation for Φ is then

$$\frac{d^2\Phi}{d\phi^2} - m^2\Phi = 0 \tag{1.2.26}$$

and has an orthonormal set of solutions $\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi}$. The equation for Θ is then the so-called *Legendre equation*

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d^2\Theta}{d\theta^2}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \tag{1.2.27}$$

Above equation has a solutions $\Theta(\theta) = P_l^m(\cos\theta)$ $(m = -l, ..., l)$ where P_l^m is a *Legendre polynomial* defined by

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\left(\frac{d}{dt}\right)^{l+m}(x^2-1)^l.$$

Hence, the set of the orthonormal solutions of the equation (1.2.25), consists of $2l+1 = h(l,3)$ real functions given by

$$A_0 P_l^0(\cos\theta), \qquad A_m P_l^m(\cos\theta)\cos m\phi, \qquad A_m P_l^m(\cos\theta)\sin m\phi, \quad m = 1, ..., 2l$$

with the normalizing constant $A_m = \sqrt{\frac{(l-m)!(2l+1)}{(l+m)!2\pi}}$.

The *spherical Bessel function* defined by

$$j_l(u) := 2^{(N-2)/2}\Gamma(N/2)\frac{J_{l+(N-2)/2}(u)}{u^{(N-2)/2}}, \qquad l \geq 0 \tag{1.2.28}$$

where J_ν denotes the Bessel function of the first kind of order ν and $j_l(0) = \delta_0^l$, arises as the solution of the so-called Helmholtz equation

$$\Delta\Xi(x) = \lambda^2\Xi(x), \tag{1.2.29}$$

whose solution is given by

$$\Xi(x) = j_l(\lambda\|x\|)S_l(\tfrac{x}{\|x\|}). \tag{1.2.30}$$

The two types j_l and S_l of spherical functions are related to each other via the Fourier transform

$$i^l S_l\left(\frac{t}{\|t\|}\right) j_l(\lambda\|t\|) = \frac{1}{|s^{N-1}(\lambda)|}\int_{s^{N-1}(\lambda)} e^{i\langle v,t\rangle} S_l(v)d\sigma_N(v), \tag{1.2.31}$$

where $d\sigma_N(v) = \lambda^{N-1}\sin^{N-2}\theta_1 \cdot ... \cdot \sin\theta_{N-2}d\theta_1...d\theta_{N-2}d\phi$ is the surface area element of the sphere $s^{N-1}(\lambda)$ with radius λ in $\mathbb{R}^N$ and

$$\left|s^{N-1}(\lambda)\right| = \frac{2\pi^{N/2}}{\Gamma(N/2)}\lambda^{N-1} \tag{1.2.32}$$

is its surface area (cf. [3], 9.10). We will need below only the following partial result

$$j_0(\lambda\|t\|) = \frac{1}{|s^{N-1}(\lambda)|}\int_{s^{N-1}(\lambda)} e^{i\langle v,t\rangle} d\sigma_N(v). \tag{1.2.33}$$

Using (A.0.28) and (1.2.16) we obtain that given any orthonormal set of linearly independent spherical harmonics $\{S_l^m\}_{m=1}^{h(l,N)}$ we have the following addition formula

$$j_0(\lambda\|t-s\|) = \left|s^{N-1}\right| \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) j_l(\lambda\|t\|) j_l(\lambda\|s\|) \quad (1.2.34)$$

(as is given e.g. on p. 370 of [94] or on p. 20 of [43]).

For notational convenience we set

$$G_l(r,\lambda) = \frac{j_l(0) - j_l(r\lambda)}{\lambda}. \quad (1.2.35)$$

By using the integral representation of the Bessel function, the so-called *Poisson formula*, as well as it's consequence, *Gegenbauer's formula* (see e.g. [85], chapter XI, formulas 3.2.5 and 3.3.7 respectively or [3], section 4.7), we arrive at the following representations

$$G_0(r,\lambda) = \frac{1}{B(\frac{1}{2}, \frac{N-1}{2})} \int_{-r}^{r} \left(1 - \frac{u^2}{r^2}\right)^{\frac{N-3}{2}} \frac{1-\cos(u\lambda)}{r\lambda} du \quad (1.2.36)$$

and for $l > 0$

$$-G_l(r,\lambda) = \frac{(-i)^{l-1} B(l, N-1)}{B(\frac{1}{2}, \frac{N-1}{2})} \int_{-r}^{r} \left(1 - \frac{u^2}{r^2}\right)^{\frac{N-1}{2}} C_{l-1}^{N/2}\left(\frac{u}{r}\right) e^{i\lambda u} du, \quad (1.2.37)$$

where C_l^γ are the *Gegenbauer polynomials.* These integral representations show, in particular, that G_l's are alternately odd ($l = 0, 2, \ldots$) and even ($l = 1, 3, \ldots$) function of λ. Moreover, by virtue of the Paley–Wiener theorem (cf. [9] or [17]) we see from (1.2.36) and from the real and imaginary parts of the equation (1.2.37) that all the functions $G_l(r, \cdot)$ are of exponential type at most r, according to the definition

Definition 1.2.1. (cf.[8], [17]) The entire function $f(z)$ is said to be of exponential type τ if

$$\limsup_{R\to\infty} R^{-1} \max_{|z|=R} \log|f(z)| = \tau. \quad (1.2.38)$$

Thus, we have

Lemma 1.2.2. *For each $r \in \mathbb{R}_+$, the function $G_l(r,\lambda)$ of $\lambda \in \mathbb{R}$ is an odd function for $l = 0, 2, \ldots$ and an even function for $l = 1, 3, \ldots$. Moreover, it is an analytic function of finite exponential type less or equal r.*

1.2.3 Representations of covariance

Our next task is to derive representations of covariance for both homogenous and homogenous increments random fields. Let us begin with the latter and show how we obtain the representation (1.2.44) for the random field X with homogenous increments.

Observe first that due to the homogeneity of the increments, $\mathbb{E}X_sX_t = \frac{1}{2}\left(\mathbb{E}\left|X_s\right|^2 + \mathbb{E}\left|X_t\right|^2 - \mathbb{E}\left|X(t-s)\right|^2\right)$. Since, in addition, our field is isotropic, the variance $\mathbb{E}\left|X_t\right|^2$ is a function only of the norm of t. Denoting this function (called in [94] the *structure function*) by D we thus write $D(\|t\|) = \mathbb{E}\left|X_t\right|^2$. With this notation the covariance can be rewritten as

$$\mathbb{E}X_sX_t = \tfrac{1}{2}\left(D(\|s\|) + D(\|t\|) - D(\|t-s\|)\right). \tag{1.2.39}$$

By putting $t = s$ in (1.2.1), we get the following spectral representation for the structure function

$$D(\|t\|) = 2\int_{\mathbb{R}^N}\left(1 - e^{i\langle v,t\rangle}\right)\varrho(dv) = 2\int_{\mathbb{R}^N}\left(1-\cos\langle v,t\rangle\right)\varrho(dv) \tag{1.2.40}$$

(the imaginary part vanishes, since our field X is real, cf. [94], p. 435).

It is useful to associate with the spectral measure ϱ the bounded non-decreasing function Φ defined by (1.2.4). Note that condition (1.2.2) implies

$$\int_0^\infty \frac{\lambda^2}{1+\lambda^2}\,d\Phi(\lambda) < \infty. \tag{1.2.41}$$

By rewriting the variable $v = (v_1, \dots, v_N)$ in polar coordinates with radius $\lambda = \|v\|$, we get $\left|s^{N-1}(\lambda)\right|\varrho(dv) = d\sigma_N(v)d\Phi(\lambda)$, cf. (1.2.33) and (1.2.32). Due to formula (1.2.33), the representation (1.2.40) can be rewritten in polar coordinates as

$$D(r) = 2\int_0^\infty \left(1 - j_0(r\lambda)\right)d\Phi(\lambda). \tag{1.2.42}$$

Formula (1.2.39) for the covariance function then becomes

$$\mathbb{E}X_sX_t = \int_0^\infty \left[1 - j_0(\lambda\|t\|) - j_0(\lambda\|s\|) + j_0(\lambda\|t-s\|)\right]d\Phi(\lambda). \tag{1.2.43}$$

The following representation of the covariance function is implicit in [54]. Since it serves as starting point in our considerations, we provide an explicit proof.

Theorem 1.2.3. *The covariance function of the isotropic Gaussian random field X with homogeneous increments can be represented as follows:*

$$\begin{aligned}\mathbb{E}X_sX_t &= |s^{N-1}|^2 \sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\Big(\frac{t}{\|t\|}\Big) S_l^m\Big(\frac{s}{\|s\|}\Big) \\ &\times \int_0^\infty G_l(\|t\|,\lambda)G_l(\|s\|,\lambda)\mu(d\lambda). \end{aligned} \tag{1.2.44}$$

Proof. Note that $h(0,N)=1$, $S_0^1(\cdot)$ is a constant function for every N and since the spherical harmonics are orthonormal this constant is given by $S_0^1(\cdot)\equiv 1/\sqrt{|s^{N-1}|}$. Hence, (1.2.44) is equivalent to

$$\begin{aligned}&\mathbb{E}X_sX_t - \int_0^\infty (1-j_0(\lambda\|t\|))(1-j_0(\lambda\|s\|))d\Phi(\lambda) \\ &= |s^{N-1}| \sum_{l=1}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\Big(\frac{t}{\|t\|}\Big) S_l^m\Big(\frac{s}{\|s\|}\Big)\int_0^\infty j_l(\lambda\|t\|)j_l(\lambda\|s\|)d\Phi(\lambda),\end{aligned} \tag{1.2.45}$$

which we are now going to prove. The addition formula (1.2.34) implies

$$\begin{aligned}&j_0(\lambda\|t-s\|) - j_0(\lambda\|t\|)j_0(\lambda\|s\|) \\ &= |s^{N-1}| \sum_{l=1}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\Big(\frac{t}{\|t\|}\Big) S_l^m\Big(\frac{s}{\|s\|}\Big) j_l(\lambda\|t\|)j_l(\lambda\|s\|).\end{aligned}$$

Taking the integral with respect to $d\Phi(\lambda)$ on both sides we see that the expression on the right in (1.2.45) equals to the integral

$$\int_0^\infty \Big(j_0(\lambda\|t-s\|) - j_0(\lambda\|t\|)j_0(\lambda\|s\|)\Big)d\Phi(\lambda).$$

But in view of (1.2.43) we see that also the left-hand side of (1.2.45) equals to the latter integral. Thus (1.2.45) holds true. □

Let us now consider an isotropic homogenous random field Y. Similar to the above case, we can rewrite the representation (1.2.6) as

$$\mathbb{E}Y_sY_t = \int_0^\infty \frac{1}{|s^{N-1}(\lambda)|}\int_{s^{N-1}(\lambda)} e^{i\langle v,t-s\rangle}\sigma_N(dv)d\Phi(\lambda) \tag{1.2.46}$$

where $\lambda = \|v\|$. By virtue of the formula (1.2.33) we obtain

$$\mathbb{E}Y_sY_t = \int_0^\infty j_0(\lambda\|t-s\|)d\Phi(\lambda). \tag{1.2.47}$$

A simple application of the addition formula (1.2.34) allows us to prove the following well-known representation (cf. e.g. [92], p. 6, [94] p. 370, [43] p. 21).

Theorem 1.2.4. *The covariance function of the homogenous isotropic Gaussian random field Y can be represented as*

$$\begin{aligned} \mathbb{E}Y_sY_t &= |s^{N-1}|\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right)S_l^m\left(\frac{s}{\|s\|}\right) \\ &\times \int_0^\infty j_l(\lambda\|t\|)j_l(\lambda\|s\|)d\Phi(\lambda), \end{aligned} \tag{1.2.48}$$

where the measure Φ is as defined by (1.2.7).

1.2.4 Examples

Example 1.2.5. *Lévy's Brownian motion*

Paul Lévy [44] defined the Brownian motion on $\mathbb{R}^N$ as a centered, mean-square continuous Gaussian random field with homogenous increments, with covariance

$$\mathbb{E}X_tX_s = \frac{1}{2}\left(\|t\| + \|s\| - \|t-s\|\right),$$

for $s, t \in \mathbb{R}^N$ and $X_0 = 0$. Properties of this field were investigated by several authors, see for instance [6], [57] and [60]. Since the structure function in this case is simply $D(r) = r$ we can easily verify via formula (1.2.42) that the corresponding spectral measure is given by $\lambda^2\Phi'(\lambda) = \left|s^{N-1}\right| / \left|s^N\right|$. To see this rewrite (1.2.42) in the form

$$r = -2\int_0^r du \int_0^\infty j_0'(u\lambda)d\Phi(\lambda) = \frac{|s^{N-1}|}{|s^N|}2^{N/2}\Gamma(N/2)\int_0^r du\int_0^\infty \frac{J_{N/2}(z)}{z^{N/2}}dz \tag{1.2.49}$$

and apply formula 6.561.14 of [29] to evaluate the last integral.

Thus by (1.2.3) we have $\mu(d\lambda) = d\lambda/|s^N|$.

Example 1.2.6. *Lévy's fractional Brownian motion*

Lévy's fractional Brownian motion is defined on $\mathbb{R}^N$ as a centered, mean-square continuous Gaussian random field with homogenous increments and covariance function

$$\mathbb{E}X_tX_s = \frac{1}{2}\left(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}\right)$$

where $H \in (0,1)$ is called the Hurst index. Observe that for $H = 1/2$ it reduces to Lévy's Brownian motion considered in the preceding section. In the present case the structural function is $D(r) = r^{2H}$, so that we can argue like in the previous section to determine the corresponding spectral function. First, formula (1.2.42) is rewritten in the form (1.2.49) but with r^{2H} instead of r, and then the density of the form $\lambda^{1+2H}\Phi'(\lambda) = c_{HN}^2 \left|s^{N-1}\right|$ is sought, with a constant c_{HN}^2 to be determined by formula 6.561.14 of [29]. By straightforward calculations we arrive at

$$c_{HN}^2 = \frac{\Gamma(H+\frac{N}{2})\Gamma(1+H)\sin(\pi H)}{\pi^{\frac{N+2}{2}}2^{1-2H}}.$$

Thus by (1.2.3) we deal here with the spectral measure

$$\mu(d\lambda) = c_{HN}^2\lambda^{1-2H}d\lambda. \tag{1.2.50}$$

Chapter 2

Spectral theory of strings

In this chapter we recall the elements of the spectral theory of vibrating strings relevant to the study of stochastic processes. This theory was developed by M.G. Krein in the 1950's (see e.g. the historical remarks in [33]). It becomes particularly useful for our purposes when combined with the theory of reproducing kernel Hilbert spaces of entire functions of [9]. We essentially follow the presentation of this material given in [17], Chapters 5 and 6, with some additional implications - mainly in Sections 2.7 and 2.8. Omitted proofs and further details can be found there.

The main goal of this chapter is to describe how the association of a unique string with a spectral measure of the process is established. We are going to show how we can then analyze the reproducing kernel Hilbert space of the process by the means of such string and some associated functions. This is a preparatory chapter for the concrete results of Chapter 4.

2.1 Mathematical description of the string

Consider a number $l \leq \infty$ and a nonnegative, right-continuous, nondecreasing function m on the interval $[0, l)$. The number l is interpreted as the length of a string in equilibrium, $x \in [0, l]$ is thought of as a location on the string, $x = 0$ corresponding to the left endpoint and $x = l$ to the right endpoint, and $m(x)$ is interpreted as the mass of the piece of string from the left endpoint up to (and including) the point x. The jump of m at the point $x > 0$ is denoted by $m[x] = m(x) - m(x-)$, and $m(0-) = 0$.

From classical mechanics we know that the motion of the vibrating string is described by the solutions $u = u(t, x)$ of the wave equation

$$m' u_{tt} = u_{xx},$$

at least when m is a smooth function. The number $1/\sqrt{m'(x)}$ can be interpreted as the local propagation speed of the travelling wave, in the sense that it takes a wave

$$T(x) = \int_0^x \sqrt{m'(y)}\, dy \tag{2.1.1}$$

time units to travel from the point 0 to x.

One way of analyzing the vibrating string is to begin by looking at periodic solutions u of the form

$$u(t, x) = A(x, \lambda)e^{i\lambda t},$$

where λ is a fixed frequency and A describes the amplitude. For u of this form the wave equation reduces to the ordinary differential equation

$$A''(x, \lambda) = -\lambda^2 m'(x)A(x, \lambda) \tag{2.1.2}$$

for the function A. Note that the kinetic energy of the string which vibrates at the frequency λ is given by

$$\tfrac{1}{2} \int |u_t(t, x)|^2 dm(x) = \tfrac{1}{2}\lambda^2 \|A(\cdot, \lambda)\|_m^2. \tag{2.1.3}$$

Starting from this observation it is possible to associate with every string a unique symmetric measure μ on $\mathbb{R}$ which has the property that $\int (1+\lambda)^{-2}\, \mu(d\lambda) < \infty$, and such that $\lambda^2/\mu(d\lambda)$ can be interpreted as the kinetic energy of the string which vibrates at the frequency λ. In the remainder of this chapter we recall the construction of this so-called *principal spectral measure* of a string and some additional facts that we need below.

2.2 Differential equation

For general, not necessarily smooth mass distribution functions m, the eigenvalue problem (2.1.2) can be written as

$$\frac{dA^+}{dm} = -\lambda^2 A.$$

Here f^+ denotes the right-hand side derivative of the function f and, similarly, f^- is the left derivative. If we restrict the operator $A \mapsto dA^+/dm$ to an appropriate domain, it becomes a self-adjoint, negative definite, densely defined operator on the Hilbert space $L^2(m) = L^2([0,l],m)$. We will denote this operator by $\mathcal{G}$, and its domain by $\mathcal{D}(\mathcal{G})$.

The first requirement on $f \in \mathcal{D}(\mathcal{G})$ is of course that $\mathcal{G}f = df^+/dm$ exists, by which we mean that we can write

$$f(x) = f(0) + f^-(0)x + \int_0^x \Big(\int_{[0,y]} \mathcal{G}f(z)\, dm(z) \Big)\, dy \tag{2.2.1}$$

for $x \in [0,l)$. We think of functions in $\mathcal{D}(\mathcal{G})$ as being defined on the entire real line by putting m constant outside $[0,l]$, so that $f(x) = f(0) + xf^-(0)$ for $x \le 0$ and $f(x) = f(l) + (x-l)f^+(l)$ for $x \ge l$ if $l < \infty$.

Secondly, the functions in $\mathcal{D}(\mathcal{G})$ have to satisfy the appropriate boundary conditions. To make $\mathcal{G}$ self-adjoint we need to impose the boundary condition $f^-(0) = 0$ at the left endpoint of the string. For a *long* string, meaning that $l + m(l-) = \infty$, this is the only boundary condition, and the situation can be summarized as follows.

Theorem 2.2.1. *Suppose that $l + m(l-) = \infty$. Then there exists a dense subset $\mathcal{D}(\mathcal{G})$ of $L^2(m)$ such that every $f \in \mathcal{D}(\mathcal{G})$ has left and right derivatives, satisfies $f^-(0) = 0$, and the operator $\mathcal{G} : \mathcal{D}(\mathcal{G}) \to L^2(m)$ given by $\mathcal{G}f = df^+/dm$ is well defined, self-adjoint, and negative definite.*

If the string is *short*, i.e. $l + m(l-) < \infty$, we also have to prescribe how the string is tied down at the right end point. On this side of the string there is a continuum of possible boundary conditions, each leading to a self-adjoint, negative definite operator. The condition can be described by introducing an additional *tying constant* $k \in [0,\infty]$ and prescribing that $f(l+k) = 0$ for $f \in \mathcal{D}(\mathcal{G})$. Since $f(l+k) = f(l) + kf^+(l)$, this means that $f(l) + kf^+(l) = 0$. For $k = \infty$ this should be interpreted as $f^+(l) = 0$. The following theorem summarizes the situation for short strings.

Theorem 2.2.2. *Suppose that $l + m(l-) < \infty$ and $k \in [0,\infty]$. Then there exists a dense subset $\mathcal{D}(\mathcal{G})$ of $L^2(m)$ such that every $f \in \mathcal{D}(\mathcal{G})$ has left and right derivatives, satisfies $f^-(0) = 0$ and $f(l) + kf^+(l) = 0$, and the operator $\mathcal{G} : \mathcal{D}(\mathcal{G}) \to L^2(m)$ given by $\mathcal{G}f = df^+/dm$ is well defined, self-adjoint, and negative definite.*

Equation (2.2.1) shows that for $f \in \mathcal{D}(\mathcal{G})$, it holds that

$$f^+(x) - f^-(x) = \mathcal{G}f(x)m[x].$$

So if the mass function m is continuous, every function in $\mathcal{D}(\mathcal{G})$ is differentiable. Moreover, it holds that if m is absolutely continuous, with derivative m', then $\mathcal{G}f = f''/m'$ for all $f \in \mathcal{D}(\mathcal{G})$.

2.3 Solutions of the eigenvalue problem

Since the operator $\mathcal{G} : \mathcal{D}(\mathcal{G}) \to L^2(m)$ is self-adjoint and negative definite, its spectrum $\sigma(\mathcal{G})$ is contained in $(-\infty, 0]$. Hence, the second order ordinary differential equation $\mathcal{G}A = -\lambda^2 A$ can not have a solution in $\mathcal{D}(\mathcal{G})$ if λ^2 is not a real, nonnegative number. However, the equation does have solutions for every $\lambda^2 \in \mathbb{C}$. Throughout the text we denote by $A = A(\cdot, \lambda)$ the solution of

$$\begin{aligned} \mathcal{G}A(\cdot,\lambda) &= -\lambda^2 A(\cdot,\lambda), \\ A(0,\lambda) = 1, &\quad A^-(0,\lambda) = 0. \end{aligned} \tag{2.3.1}$$

The function A can be represented (cf. [17], p. 162, 171; [33], p. 29) as follows

$$A(x,\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^{2n} p_n(x), \tag{2.3.2}$$

where p_n's are defined recurrently according to $p_n(x) = \int_0^x \int_0^y p_{n-1}(z)dm(z)dy$ and $p_0(x) = 1$. Thus the function $A(x, \lambda)$ (and $A^+(x, \lambda)$) for any fixed $x \in [0, l]$ is an entire function of variable λ taking real values for real λ.

For λ^2 outside $[0, \infty)$, a complementary solution $D = D(\cdot, \lambda)$ of the equation can be constructed by setting

$$D(x,\lambda) = A(x,\lambda) \int_x^{l+k} \frac{1}{A^2(y,\lambda)}\, dy. \tag{2.3.3}$$

If the string is short the function $1/A^2$ is integrable, so D is well defined. In the case of a long string the integral is to be taken from x to ∞, and it exists as an improper integral. The function $D = D(\cdot, \lambda)$ satisfies

$$\mathcal{G}D(\cdot,\lambda) = -\lambda^2 D(\cdot,\lambda), \quad D^-(0,\lambda) = -1.$$

It is constructed in such a way that for the Wronskian of A and D, we have $A^+D - AD^+ = A^-D - AD^- = 1$.

A third function that will play an important role below is defined by

$$B(x,\lambda) = -\frac{1}{\lambda} A^+(x,\lambda).$$

It satisfies $dB = \lambda A\, dm$ and $B(0,\lambda) = \lambda m[0]$.

2.4 Resolvents of the string operator

For λ^2 outside $[0,\infty)$ the operator $-\lambda^2 I - \mathcal{G}$ is invertible, so that the resolvent

$$R_\lambda = (-\lambda^2 I - \mathcal{G})^{-1}$$

is well defined. For all λ^2 outside $[0,\infty)$ we have that $R_\lambda : L^2(m) \to L^2(m)$ is a bounded, self-adjoint operator. Its image, which is $\mathcal{D}(\mathcal{G})$, is dense in $L^2(m)$.

The resolvent R_λ is an integral operator with a kernel that can be expressed in terms of the "eigenfunctions" A and D of the operator $\mathcal{G}$ we introduced in the previous subsection. We define

$$r_\lambda(x,y) = \begin{cases} A(x,\lambda)D(y,\lambda), & \text{if } x \le y, \\ A(y,\lambda)D(x,\lambda), & \text{if } x \ge y. \end{cases} \tag{2.4.1}$$

Theorem 2.4.1. *For λ^2 outside $[0,\infty)$, it holds that*

$$R_\lambda f(x) = \int_{[0,l]} r_\lambda(x,y) f(y)\, dm(y).$$

2.5 Spectral measure of a string

If the string is short, so $l + m(l-) < \infty$, the spectrum of the operator $\mathcal{G}$ is $\{-\lambda_n^2 : n = 1, 2, \ldots\}$, where $\lambda_1, \lambda_2, \ldots$ are the nonnegative roots of the equation

$$kA^+(l,\lambda) + A(l,\lambda) = 0 \tag{2.5.1}$$

(as in Section 2.2, this should be read for $k = \infty$ as $A^+(l,\lambda) = 0$, or, equivalently, $B(l,\lambda) = 0$). The corresponding eigenfunctions are the functions $A(\cdot,\lambda_n)$. Now we construct a symmetric measure μ on the real line by putting mass

$$\frac{\pi}{2\|A(\cdot,\lambda_n)\|_m^2}$$

at the points $\pm\lambda_n$. We remark that μ has a clear physical interpretation. Up to a constant, the mass $\mu(\{\lambda_n\})$ is equal to $\lambda_n^2/K(\lambda_n)$, where $K(\lambda)$ is the kinetic energy of the string which vibrates at the frequency λ (cf. (2.1.3)). The measure μ is called the *principal spectral measure* of the string. For λ^2 outside $[0,\infty)$, the resolvent kernel r_λ can be expressed in terms of the measure μ and the eigenfunctions of the string:

$$r_\lambda(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{A(x,\omega)A(y,\omega)}{\omega^2-\lambda^2}\,\mu(d\omega). \tag{2.5.2}$$

Note that this implies in particular that we have the integrability property

$$\int_{\mathbb{R}} \frac{\mu(d\omega)}{1+\omega^2} = \pi r_i(0,0) < \infty.$$

For long strings the spectral measure can be constructed by first cutting the string to make it short, and then letting the cutting point tend to infinity. The resulting measure is then no longer discrete in general, but the spectral representation (2.5.2) of the resolvent kernel still holds. Moreover, there is only one measure with this property.

This uniqueness is in fact not hard to see. By definition (2.4.1) of the resolvent kernel and (2.5.2) we have

$$D(0,ib) = r_{ib}(0,0) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mu(d\lambda)}{b^2+\lambda^2}$$

for real-valued b. The symmetric measure μ is completely determined by the values of the integrals on the right-hand side for $b\in\mathbb{R}$ (see e.g. [17], p. 16). Hence, μ is completely determined by the values $D(0,ib)$ for $b\in\mathbb{R}$.

The complete theorem reads as follows.

Theorem 2.5.1. *For every given string, there exists a unique symmetric measure μ on $\mathbb{R}$ such that (2.5.2) holds. Conversely, given a symmetric measure μ on $\mathbb{R}$ such that $\int(1+\lambda^2)^{-1}\,\mu(d\lambda)<\infty$, there exists a unique string for which (2.5.2) holds true.*

In view of this theorem the following definition makes sense.

Definition 2.5.2. *The* principal spectral measure *of the string is the unique measure μ which satisfies (2.5.2).*

Example 2.5.3. ***Lebesgue measure***

In this example we will treat the simplest case of a spectral measure which is a Lebesgue measure

$$\mu(d\lambda) = d\lambda.$$

The goal is to show that it is a principal spectral function of a smooth, infinitely long string with linear mass function

$$m(x) = x.$$

Given the above mass function a differential equation for the function $A(x, \lambda)$ takes a form

$$A''(x, \lambda) = -\lambda^2 A(x, \lambda). \tag{2.5.3}$$

This, combined with the initial conditions, gives

$$A(x, \lambda) = \cos \lambda x, \tag{2.5.4}$$

from this we immediately get

$$B(x, \lambda) = \sin \lambda x. \tag{2.5.5}$$

To compute $D(x, ib)$ (we need only this to determine the measure) note that

$$A(x, ib) = \frac{1}{2}(e^{-bx} + e^{bx}) = \cosh bx.$$

According to formula (2.3.3) and since $\int \cosh^{-2} = \tanh$, we obtain

$$D(x, ib) = \cosh bx \int_x^\infty \frac{dy}{\cosh^2 by} = b^{-1} e^{-bx}. \tag{2.5.6}$$

Now it only suffices to show that the principal spectral measure of the string with mass $m(x) = x$ is the Lebesgue measure. This is apparent from the identity

$$D(0, ib) = b^{-1} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\lambda}{b^2 + \lambda^2}. \tag{2.5.7}$$

More examples can be found in Chapter 3.

2.6 Odd and even transforms

Now let μ be the principal spectral measure of the string and let $L^2_{\text{even}}(\mu)$, resp. $L^2_{\text{odd}}(\mu)$, be the space of even, resp. odd, functions in $L^2(\mu)$. The functions A and B defined in Section 2.3 give rise to integral transformations onto these function spaces.

Theorem 2.6.1. *The map* $\mathcal{T}_{\text{even}} : L^2(m) \to L^2_{\text{even}}(\mu)$ *defined by*

$$\mathcal{T}_{\text{even}} f(\lambda) = \hat{f}_{\text{even}}(\lambda) = \int_{[0,l]} A(x,\lambda) f(x)\, dm(x)$$

is one to one and onto. Its inverse is given by

$$\mathcal{T}^{-1}_{\text{even}} \psi(x) = \check{\psi}_{\text{even}}(x) = \frac{1}{\pi} \int_{\mathbb{R}} A(x,\lambda)\psi(\lambda)\, \mu(d\lambda).$$

It holds that $\|\mathcal{T}_{\text{even}} f\|^2_\mu = \pi \|f\|^2_m$.

So, up to a factor $\sqrt{\pi}$, the map $\mathcal{T}_{\text{even}}$ is a Hilbert space isometry between $L^2(m)$ and $L^2_{\text{even}}(\mu)$. As usual in this kind of setting, the integrals in the statement of the theorem should be interpreted in the wide sense. For short strings they converge as ordinary Lebesgue integrals, but for long strings they may only converge in an L^2-sense.

The domain of the corresponding map into $L^2_{\text{odd}}(\mu)$ is a subspace $\mathcal{X}$ of $L^2([0, l+k], dx)$, where k is the tying constant if the string is short, and $k = 0$ if it is long. For $k < \infty$, $\mathcal{X}$ is defined as the space of functions in $L^2([0, l+k], dx)$ which are constant on mass-free intervals. For $k = \infty$ we require in addition that the functions vanish on $[l, \infty]$.

Theorem 2.6.2. *The map* $\mathcal{T}_{\text{odd}} : \mathcal{X} \to L^2_{\text{odd}}(\mu)$ *defined by*

$$\mathcal{T}_{\text{odd}} f(\lambda) = \hat{f}_{\text{odd}}(\lambda) = \int_0^{l+k} B(x,\lambda) f(x)\, dx$$

is one to one and onto. Its inverse is given by

$$\mathcal{T}^{-1}_{\text{odd}} \psi(x) = \check{\psi}_{\text{odd}}(x) = \frac{1}{\pi} \int_{\mathbb{R}} B(x,\lambda)\psi(\lambda)\, \mu(d\lambda).$$

It holds that $\|\mathcal{T}_{\text{odd}} f\|^2_\mu = \pi \|f\|^2_2$.

If the spectral measure μ has an atom at ω (and hence also at $-\omega$), Theorem 2.6.1 states that

$$\mathcal{T}_{\text{even}}^{-1}(1_{\{-\omega\}} + 1_{\{\omega\}}) = \frac{2}{\pi} A(\cdot,\omega)\mu(\{\omega\}).$$

Hence, under the even transform the function $1_{\{-\omega\}} + 1_{\{\omega\}} \in L^2_{\text{even}}(\mu)$ corresponds to a function in $L^2(m)$ describing a vibration at frequency ω, with weight, or power, $2\mu(\{\omega\})/\pi$. In general, the theorem allows us to think of an arbitrary element $\psi \in L^2(\mu)$ as a frequency domain description of a vibration of the string, $2\psi(\lambda)\mu(d\lambda)/\pi$ being the contribution of frequency λ.

Note that if we apply the Parseval relation $\|\mathcal{T}_{\text{even}} f\|^2_\mu = \pi \|f\|^2_m$ for the even transform with $f = 1_{\{x\}}$ we see that for all x,

$$\int_{\mathbb{R}} A^2(x,\lambda)\,\mu(d\lambda) = \frac{\pi}{m[x]}. \tag{2.6.1}$$

In particular, it holds that the principal spectral measure μ has finite total mass if and only if $m[0] = 0$.

Remark 2.6.3. Throughout the whole text we will use both notations for the even transform $\mathcal{T}_{\text{even}} f$ and $\hat{f}_{\text{even}}$ interchangeably, according to which provides more clear display. The same holds for the odd transform. In some cases also the subscripts 'even' and 'odd' are omitted. It is always clear from the context what $\hat{f}$ and $\check{f}$ stand for and should not be confused with the ordinary Fourier transform.

2.7 Orthonormal basis

Let us deal for a while with the short string, assuming $l + m(l-) < \infty$ with the tying constant $k = \infty$. Consider the family of functions

$$x \mapsto A(x,\omega_n), \quad n = 1,2,\dots \tag{2.7.1}$$

where the ω_n's are the positive, real zeros of $B(l,\cdot)$ (we suppress the dependence of ω_n's on l, but the reader should keep it in mind; for more details regarding these roots see Lemma 2.8.7).

By definition of A and integration by parts we have

$$-\lambda^2 \int_0^l \overline{A(x,\omega)} A(x,\lambda) dm(x) = \int_0^l \overline{A(x,\omega)} dA^+(x,\lambda)$$
$$= \left[\overline{A(x,\omega)} A^+(x,\lambda) \right]_0^l - \int_0^l A^+(x,\lambda) \overline{A^+(x,\omega)} dx.$$

Reversing the roles of ω and λ gives

$$-\overline{\omega}^2 \int_0^l \overline{A(x,\omega)} A(x,\lambda) dm(x) = \left[A(x,\lambda) \overline{A^+(x,\omega)} \right]_0^l - \int_0^l \overline{A^+(x,\omega)} A^+(x,\lambda) dx.$$

Taking the difference of two above equalities results in

$$\int_0^l A(x,\lambda) \overline{A(x,\omega)} dm(x) = \frac{\overline{A(l,\omega)} A^+(l,\lambda) - A(l,\lambda) \overline{A^+(l,\omega)}}{\overline{\omega}^2 - \lambda^2}, \tag{2.7.2}$$

which is a so-called *Lagrange identity* ([33], Lemma 1.1; see also [17], p. 189, Exercise 3). Now we easily see that

$$\int_0^l A(x,\omega_n) A(x,\omega_j) dm(x) = \|A(\cdot,\omega_n)\|_m^2 \delta_n^j, \quad j,n = 1,2,\ldots,$$

where δ_n^j is Dirac's delta.

It is also true that the family (2.7.1) spans the function space $L^2(m)$. To show that, let us suppose that there exists $f \in L^2(m)$ such that for all $n \in \mathbb{N}$ we have $f \perp A(\cdot,\omega_n)$. It means that

$$\hat{f}_{\text{even}}(\omega_n) = \langle f, A(\cdot,\omega_n) \rangle_m = 0, \qquad n = 1,2,\ldots$$

Recall that in the present situation the principal spectral measure of the string has atoms only at the points $\pm\omega_n$ so that

$$\int_{\mathbb{R}} \left| \hat{f}_{\text{even}}(\lambda) \right|^2 \mu(d\lambda) = \sum_{n\in\mathbb{Z}} \left| \hat{f}_{\text{odd}}(\omega_n) \right|^2 \mu(\{\omega_n\}) = 0.$$

According to Theorem 2.6.1, $\|f\|_m^2 = 1/\pi \|\hat{f}_{\text{even}}\|_\mu^2 = 0$. Hence, $f = 0$ in $L^2(m)$. So, we have proved

Lemma 2.7.1. *If $l + m(l-) < \infty$, $k = \infty$ and ω_n's ($n = 1, 2, \dots$) are all positive, real zeros of $B(l, \cdot)$ then the family of functions*

$$\varphi_n(x) := \frac{A(x, \omega_n)}{\|A(\cdot, \omega_n)\|_m}, \quad x \in [0, l],\ n = 1, 2, \dots \tag{2.7.3}$$

form an orthonormal basis of the function space $L^2(m)$.

We would also like to have a basis of the corresponding space $\mathcal{X}$. To achieve this goal we use the Christoffel-Darboux-type relation (cf. [17], Section 6.3, p. 234)

$$\begin{aligned} &\int_0^l \overline{A(x,\omega)} A(x,\lambda) dm(x) + \int_0^l \overline{B(x,\omega)} B(x,\lambda) dx \\ = \quad &\frac{\overline{A(l,\omega)} B(l,\lambda) - \overline{B(l,\omega)} A(l,\lambda)}{\lambda - \overline{\omega}}. \end{aligned} \tag{2.7.4}$$

Combined with (2.7.2), it yields the corresponding relation for B, i.e.

$$\int_0^l B(x,\lambda) \overline{B(x,\omega)} dx = \frac{\overline{\omega} \overline{A(l,\omega)} B(l,\lambda) - \lambda A(l,\lambda) \overline{B(l,\omega)}}{\lambda^2 - \overline{\omega}^2}. \tag{2.7.5}$$

Now, we can prove the following

Lemma 2.7.2. *If $l + m(l-) < \infty$, $k = \infty$ and ω_n's ($n = 1, 2, \dots$) are all positive, real zeros of $B(l, \cdot)$, then the family of functions*

$$\psi_n(x) := \frac{B(x, \omega_n)}{\|B(\cdot, \omega_n)\|_2}, \quad x \in [0, l],\ n = 1, 2, \dots \tag{2.7.6}$$

form an orthonormal basis of the function space $\mathcal{X}$.

Proof. The orthonormality is self-evident by virtue of (2.7.5). The completeness is shown in the same manner as for (2.7.1) by using the odd transform instead of even one. □

As we will see further on, the norms appearing in the basis functions (2.7.3) and (2.7.6) will also appear in the series expansions. Therefore we will derive a simpler representation of these norms.

Lemma 2.7.3. *If $l+m(l-)<\infty$, $k=\infty$ and $\omega_1<\omega_2<\omega_3<\dots$ are positive real zeros of $B(l,\cdot)$, then the norms of the functions $A(\cdot,\omega_n)$ and $B(\cdot,\omega_n)$ in the spaces $L^2(m)$ and $L^2([0,l])$, respectively, simplify to*

$$\|A(\cdot,\omega_n)\|_m^2 = \|B(\cdot,\omega_n)\|_2^2 = \frac{1}{2}A(l,\omega_n)\dot{B}(l,\omega_n),$$

where $\dot{B}(x,\lambda)$ denotes the derivative of B with respect to λ.

Proof. We begin by showing the continuity of the function $A(\cdot,\lambda)$ in the space $L^2(m)$ in case of short string, i.e. $l+m(l-)<\infty$. In other words, we have to prove that $A(\cdot,\lambda)\to A(\cdot,\omega)$ in $L^2(m)$, as $\lambda\to\omega$ for $\lambda,\omega\in\mathbb{R}$. The mean value theorem ensures existence of such γ_0 between λ and ω that

$$\int_0^l |A(x,\lambda)-A(x,\omega)|^2\,dm(x) \le |\lambda-\omega|^2\int_0^l \left|\dot{A}(x,\gamma_0)\right|^2 dm(x).$$

Using the representation (2.3.2) of $A(x,\lambda)$ we can establish the upper bound

$$\int_0^l \left|\dot{A}(x,\gamma_0)\right|^2 dm(x) \le 4\sum_{n,j\ge 1} nj\gamma_0^{2(n+j)-2}\int_0^l p_n(x)p_j(x)dm(x).$$

In view of the property $p_n(x)\le (n!)^{-2}[xm(x)]^n$ (see [17], p. 162), we can bound the above integral using

$$\begin{aligned}&\sum_{n,j\ge 1}\frac{nj}{(n!j!)^2}\gamma_0^{2(n+j)-2}\int_0^l x^{n+j}m(x)^{n+j}dm(x)\\ \le\;&\sum_{n,j\ge 1}\frac{nj}{(n!j!)^2}\gamma_0^{2(n+j)-2}(l\,m(l))^{n+j+1}<\infty,\end{aligned}$$

since $l\,m(l)<\infty$ by assumption. Hence, we have proved that with some positive finite constant c

$$\int_0^l |A(x,\lambda)-A(x,\omega)|^2\,dm(x)\le c\,|\lambda-\omega|^2.$$

The same property holds for the function $B(\cdot,\lambda)$. Now, according to formulas (2.7.2) and (2.7.5) for $\lambda,\omega\in\mathbb{R}$, we can write

$$\begin{aligned}\|A(\cdot,\omega)\|_m^2 &= \lim_{\lambda\to\omega}\frac{\omega A(l,\lambda)B(l,\omega)-\lambda A(l,\omega)B(l,\lambda)}{\omega^2-\lambda^2},\\ \|B(\cdot,\omega)\|_2^2 &= \lim_{\lambda\to\omega}\frac{\omega A(l,\omega)B(l,\lambda)-\lambda A(l,\lambda)B(l,\omega)}{\lambda^2-\omega^2}.\end{aligned}$$

Since both limits are $\frac{0}{0}$, application of the *l'Hospital's rule* (knowing from (2.3.2) that involved functions are smooth enough) gives us, for $\omega \neq 0$,

$$\|A(\cdot,\omega)\|_m^2 = \frac{\omega\left[A(l,\omega)\dot{B}(l,\omega) - B(l,\omega)\dot{A}(l,\omega)\right] + A(l,\omega)B(l,\omega)}{2\omega} \quad (2.7.7)$$

$$\|B(\cdot,\omega)\|_2^2 = \frac{\omega\left[A(l,\omega)\dot{B}(l,\omega) - B(l,\omega)\dot{A}(l,\omega)\right] - A(l,\omega)B(l,\omega)}{2\omega} \quad (2.7.8)$$

Recall $B(l,\omega_n) = 0$ to complete the proof. □

So, we have not only found a simple expression for the norms (derivative instead of an integral) but also showed that they are, in fact, the same numbers for A and B.

2.8 RKHS of entire functions associated with a string

We now come to the description of the structure of $L^2(\mu)$. The central result is that for short strings, this space is a reproducing kernel Hilbert space (RKHS) of entire functions of the type studied by [9]. The reproducing kernel can be expressed in terms of the functions A and B introduced in Section 2.3.

Throughout this section, we consider a string with lenght l, mass distribution m and tying constant k.

Theorem 2.8.1. *Suppose that the string is short, so $l + m(l-) < \infty$. Then $L^2(\mu)$ is a RKHS of entire functions. The reproducing kernel is given by*

$$K(\omega,\lambda) = \frac{\overline{A(l+k',\omega)}B(l,\lambda) - \overline{B(l,\omega)}A(l+k',\lambda)}{\pi(\lambda - \overline{\omega})}, \quad (2.8.1)$$

where $k' = k$ if $k < \infty$ and $k' = 0$ if $k = \infty$.

Proof. This is proved in Section 6.3 of [17]. □

Some remarks are in order regarding this theorem. First of all, the elements of $L^2(\mu)$ are equivalence classes of functions. To say that $L^2(\mu)$ is a space of entire functions means that every element admits an entire version. If we consider such an element $\psi \in L^2(\mu)$ in the remainder of the text, we will always assume it is the smooth version. Secondly, the evaluation of the function A on the right of l should be interpreted as explained in Section 2.2, i.e.

$$A(l+k',\lambda) = A(l,\lambda) + k'A^+(l,\lambda) = A(l,\lambda) - k'\lambda B(l,\lambda).$$

The fact that $K(\omega,\lambda)$ is the reproducing kernel means that $K(\omega,\cdot) \in L^2(\mu)$ for all $\omega \in \mathbb{R}$ and for $\psi \in L^2(\mu)$,

$$\int \psi(\lambda)\overline{K(\omega,\lambda)}\,\mu(d\lambda) = \langle \psi, K(\omega,\cdot)\rangle_{L^2(\mu)} = \psi(\omega).$$

Finally, we note that expression (2.8.1) of course only makes sense for $\omega \neq \lambda$. Since $\lambda \mapsto K(\omega,\lambda)$ is smooth however, $K(\omega,\omega)$ is given by the limit for $\lambda \to \omega$ of the right-hand side of (2.8.1). Observe that $K(\omega,\omega) = \langle K(\omega,\cdot), K(\omega,\cdot)\rangle_\mu \geq 0$.

For long strings the situation is more complicated and we can only describe the analytic structure of certain subspaces of $L^2(\mu)$. For $T > 0$ we define $L^2_T(\mu)$ as

$$L^2_T(\mu) = \overline{\mathrm{sp}}\{\hat{1}_{(0,t]} : |t| \leq T\}. \tag{2.8.2}$$

For $T > 0$, we define $x(T)$ and $x(T+)$ as the smallest root $x \geq 0$ and the biggest root $x \leq l$ of the equation

$$T = \int_0^x \sqrt{m'(y)}\,dy, \tag{2.8.3}$$

respectively, where m' is the derivative of the absolute continuous part of m. Note that the function $x(T)$ is the inverse of the function $T(x)$ defined by (2.1.1). Hence, a wave starting at location $x = 0$ at time zero arrives at location $x(T)$ at time T.

Let us also describe here the concept of the *Krein spaces.* If $x \in (0,l)$ is a growth point of the string lmk then we define the space $\mathbb{K}^x$ of all functions $f \in L^2(\mu)$ that satisfy

$$\check{f}_{\text{even}}(y) = \check{f}_{\text{odd}}(y) = 0 \quad \text{for} \quad y > x,$$

and the space $\mathbb{K}^{x-}$ of all functions $f \in L^2(\mu)$ satisfying

$$\check{f}_{\text{even}}(y) = \check{f}_{\text{odd}}(y) = 0 \quad \text{for} \quad y \geq x.$$

Denote by I^T the set of all entire functions $f \in L^2(\mu)$ of exponential type less or equal T and by I^{T-} the set of all entire functions $f \in L^2(\mu)$ of exponential type strictly less than T. Then we can formulate the following Paley-Wiener-type theorems (for details and proofs see [17], Section 6.4).

Theorem 2.8.2. *Either $T < T(l)$ and I^T coincides with the Krein space $\mathbb{K}^{x(T+)}$ or else $T \geq T(l)$ and I^T spans $L^2(\mu)$.*

In other words, this theorem states that if the function is of finite exponential type, its inverse transforms are supported on a finite interval.

Theorem 2.8.3. *If $T \leq T(l)$ and $x(T) < l$ then I^{T-} spans the space $\mathbb{K}^{x(T)-}$. If $T = T(l)$ and $x(T) = l$ then either I^{T-} spans $\mathbb{K}^{x(T)-}$ if the string is short or I^{T-} spans $L^2(\mu)$ if the string is long. In case $T > T(l)$ either $I^{T-} = L^2(\mu)$ or I^{T-} spans $L^2(\mu)$ according as the string is short or long.*

Let us also introduce a related concept of *de Branges space*. An entire function E satisfying $|\overline{E(\overline{\omega})}| < |E(\omega)|$, for $\omega \in \mathbb{R}^{2+} := \{\omega = a + ib \ : \ a \in \mathbb{R}\,, b > 0\}$, is called a *de Branges function*. With such a function we associate the *de Branges space* $\mathbf{B}(E)$ of entire functions f satisfying

(a) $\|f\|^2 := \int_{\mathbb{R}} |f(\lambda)/E(\lambda)|^2 \, d\lambda < \infty$,

(b) $|f(\omega)/E(\omega)| < c_f/\sqrt{b}, \qquad b > 0$,

(c) $\left|f(\omega)/\overline{E(\overline{\omega})}\right| < c_f/\sqrt{-b}, \qquad b < 0$,

with the constant c_f not depending on $\omega = a + ib$. The space $\mathbf{B}(E)$ is called *short* if

$$\int_{\mathbb{R}} \frac{1}{\lambda^2+1} |E(\lambda)|^{-2} d\lambda < \infty \qquad \text{and} \qquad |(1 - i\omega)E(\omega)|^{-2} \leq c\, b^{-1} \tag{2.8.4}$$

for $\omega = a + ib \in \mathbb{R}^{2+}$ and some constant $c > 0$.

From the next theorem we will see that the functions in $L^2_T(\mu)$ correspond to motions of a finite piece of the string. We consider the string up to the point $x(T)$ and associate with it a new principal spectral measure μ_T.

Theorem 2.8.4. *Suppose that $0 < T < \int_0^l \sqrt{m'(y)}\, dy$. Then $L^2_T(\mu) = L^2(\mu_T)$, where μ_T is the principal spectral measure of the string with length $x(T)$, the* *same mass distribution m, and tying constant $k = \infty$. In particular, $L^2_T(\mu)$ is a RKHS of entire functions and its reproducing kernel is given by*

$$K_T(\omega, \lambda) = \frac{\overline{A(x(T), \omega)}B(x(T), \lambda) - \overline{B(x(T), \omega)}A(x(T), \lambda)}{\pi(\lambda - \overline{\omega})}. \tag{2.8.5}$$

Proof. The theorem follows from a combination of results of [17]. Firstly, it is stated on p. 241 that $L_T(\mu)$ (Z^T in the notation of the book) equals the span of the class I^{T-} of functions of exponential type strictly less than T. Under the condition of the theorem, by virtue of the result recalled in Theorem 2.8.3, the span of I^{T-} is identified as the Krein space $\mathbb{K}^{x(T)-}$. By the amplification on p. 236 this space is the de Brange space associated with the function $E(\lambda) = A(x(T), \lambda) - iB(x(T), \lambda)$. This is a RKHS of entire functions which has (2.8.5) as reproducing kernel. To conclude that $L^2_T(\mu) = L^2(\mu_T)$, we use the main result of Section 6.3. It states that for the short string lmk with principal spectral measure μ the space $L^2(\mu)$ can be identified with the short de Branges space based upon the function $E(\lambda) = A(l + k', \lambda) - iB(l + k', \lambda)$ with k' as in Theorem 2.8.1.

□

Using similar methods as in the proof of Lemma 2.7.3, we can prove the following.

Lemma 2.8.5. *Reproducing kernel K_T on the diagonal is given by*

$$\begin{aligned} K_T(\omega, \omega) &= \frac{1}{\pi}\Big(A(x(T), \omega)\dot{B}(x(T), \omega) - \dot{A}(x(T), \omega)B(x(T), \omega)\Big) \\ &= \frac{1}{\pi}\Big(\|A(\cdot, \omega)\|^2_m + \|B(\cdot, \omega)\|^2_2\Big), \end{aligned} \tag{2.8.6}$$

for $\omega \in \mathbb{R}$. Moreover, if $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ are the real-valued zeros of the function $\omega \mapsto B(x(T), \omega)$, we have that

$$K_T(\omega_n, \omega_n) = \frac{1}{\pi}A(x(T), \omega_n)\dot{B}(x(T), \omega_n). \tag{2.8.7}$$

By combining a number of the result recalled thus far, we can obtain an explicit orthonormal basis of the space $L^2_T(\mu) = \overline{\text{sp}}\{\hat{1}_{(0,t]} : |t| \leq T\}$. This idea is explained very briefly in Section 6.11 of [17]. For completeness, we include a proof of the following result.

Theorem 2.8.6. *Suppose that* $0 < T < \int_0^l \sqrt{m'(y)}\,dy$. *Let* $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ *be the real-valued zeros of the function* $\omega \mapsto B(x(T), \omega)$. *The family of functions* $\{K_T(\omega_n, \cdot) : n \in \mathbb{Z}\}$ *is an orthogonal basis of* $L^2_T(\mu)$. *Each function* $\psi \in L^2_T(\mu)$ *can be expanded as*

$$\psi(\lambda) = \sum_{n\in\mathbb{Z}} \psi(\omega_n) \frac{K_T(\omega_n, \lambda)}{K_T(\omega_n, \omega_n)},$$

the convergence taking place in $L^2(\mu)$.

Proof. For $n, m \in \mathbb{Z}$ we have, by the reproducing property of K_T,

$$\langle K_T(\omega_n, \cdot), K_T(\omega_m, \cdot)\rangle_\mu = K_T(\omega_n, \omega_m).$$

Hence $\|K_T(\omega_n, \cdot)\|^2_\mu = K_T(\omega_n, \omega_n)$, and expression (2.8.5) implies that the functions are orthogonal if $n \neq m$.

We have that $L^2_T(\mu) = L^2(\mu_T)$, where μ_T is the spectral measure of the string with mass m, length $x(T)$ and tying constant $k = \infty$. As explained in Section 2.5, this means that μ_T is a measure which only has isolated masses, located at the zeros ω_n of $B(x(T), \cdot)$. Hence, a function $\psi \in L^2_T(\mu) = L^2(\mu_T)$ vanishes if it vanishes at every zero ω_n. Now suppose that ψ is orthogonal to every $K_T(\omega_n, \cdot)$. Then by the reproducing property, $\psi(\omega_n) = 0$ for every $n \in \mathbb{Z}$, hence ψ vanishes in $L^2_T(\mu)$. This shows that the functions $K_T(\omega_n, \cdot)$ form a complete system.

Since the $K_T(\omega_n, \cdot)$ form a complete orthogonal system, we have

$$\psi(\lambda) = \sum_n \frac{\langle \psi, K_T(\omega_n, \cdot)\rangle_\mu}{\|K_T(\omega_n, \cdot)\|^2_\mu} K_T(\omega_n, \lambda)$$

for $\psi \in L^2_T(\mu)$. By the reproducing property the inner product appearing in the sum equals $\psi(\omega_n)$. The squared norm was just shown to be $K_T(\omega_n, \omega_n)$. □

Let us remark here that it is also possible to produce different orthogonal bases of $L^2_T(\mu)$, using the zeros of equation (2.5.1) for different $k \in [0, \infty)$. See Section 6.11 of [17]. Theorem 2.8.6 deals with the case $k = \infty$.

We will need the following facts concerning the functions A and B, cf. Exercises 6-8 on p. 232 of [17].

Lemma 2.8.7. *(i) The functions A and B satisfy the condition*

$$\lim_{R\nearrow\infty} R^{-1}\log\left|f(Re^{i\theta})\right| = T\sin\theta, \qquad \theta\in(0,\pi) \tag{2.8.8}$$

where $f = A(x(T),\cdot)$ or $B(x(T),\cdot)$.

(ii) The roots of the functions $A(x(T),\cdot)$ and $B(x(T),\cdot)$ are real, simple and interlacing. Denoted by $\cdots<\omega_{-1}<\omega_0=0<\omega_1<\cdots$, they behave according to $\omega_n = (|n|\pi/T)(1+o(1))$ as $|n|\to\infty$.

Before we step to the actual proof let us introduce some additional notions.
The function h, which is analytic in the open upper half-plane $\mathbb{R}^{2+} := \{\omega = a+ib \,:\, a\in\mathbb{R}\,, b>0\}$ and continuous in $\overline{\mathbb{R}^{2+}}$, belongs to the *Hardy class* $\mathbb{H}^{2+}$ if

$$\|h\|_{2+} = \sup_{b>0}\|h_b\|_2 = \sup_{b>0}\left(\int_{\mathbb{R}}|h(a+ib)|^2 da\right)^{\frac{1}{2}} < \infty,$$

where $h_b(a) := h(a+ib)$. We put $h_{0+} := \lim_{b\searrow 0} h_b$. For any function $h\in\mathbb{H}^{2+}$ it holds that (cf. [17], Section 2.6)

$$\log|h(\omega)| \le \left(p_b * \log|h_{0+}(\cdot)|\right)(a), \qquad \omega = a+ib\in\mathbb{R}^{2+}, \tag{2.8.9}$$

where $p_b(a) = b/\pi(a^2+b^2)^{-1}$ is the so-called *Poisson kernel* and

$$(f_1 * f_2)(x) = \int_{\mathbb{R}} f_1(x-y)f_2(y)dy.$$

If a nontrivial function $h\in\mathbb{H}^{2+}$ satisfies condition (2.8.9) with equality it is called an *outer function.*

Proof. (i) Define a function $E(\omega) := A(x(T),\omega) - iB(x(T),\omega)$. According to Section 6.3 of [17] the function

$$\omega \longmapsto \left[e^{i\omega\left[\int_0^{x(T)}\sqrt{m'(y)}dy\right]}(1-i\omega)E(\omega)\right]^{-1} \tag{2.8.10}$$

is an outer function of the Hardy class $\mathbb{H}^{2+}$. The fact that $\int_0^{x(u)}\sqrt{m'(y)}dy = u$ and definition of the outer class imply

$$bT + \log|(1-i\omega)|^{-1} + \log|E(\omega)|^{-1} = \left(p_b * \left[\log|(1-i\,\cdot)|^{-1} + \log|E(\cdot)|^{-1}\right]\right)(a).$$

And since $\omega \mapsto (1-i\omega)^{-1}$ is an outer function, we obtain

$$bT + \log|E(\omega)|^{-1} = \left(p_b * \log|E(\cdot)|^{-1}\right)(a).$$

For $\omega = Re^{i\theta}$, $\theta \in (0,\pi)$ it gives

$$\log|E(Re^{i\theta})| = TR\sin\theta + \frac{R\sin\theta}{\pi}\int_{\mathbb{R}} \frac{\log|E(\lambda)|d\lambda}{|\lambda - Re^{i\theta}|^2}, \tag{2.8.11}$$

We can bound the integral on the right-hand side

$$\begin{aligned} \frac{R\sin\theta}{\pi}\int_{\mathbb{R}} \frac{\log|E(\lambda)|d\lambda}{|\lambda - Re^{i\theta}|^2} &\le \frac{R\sin\theta}{1-|\cos\theta|}\frac{2}{1+|\cos\theta|}\frac{1}{\pi}\int_{\mathbb{R}} \frac{\log|E(\lambda)|d\lambda}{\lambda^2+R^2} \\ = \frac{2R}{\pi\sin\theta}\int_{\mathbb{R}} \frac{\log|E(\lambda)|d\lambda}{\lambda^2+R^2} &= R\,o(1) \end{aligned}$$

as $R \nearrow \infty$. If we combine this with (2.8.11) we obtain

$$\lim_{R\nearrow\infty} R^{-1}\log\left|E(Re^{i\theta})\right| = T\sin\theta.$$

By the definition of E it holds that

$$|A(x(T),\omega)|^2 + |B(x(T),\omega)|^2 \le |E(\omega)|^2,$$

hence

$$|A(x(T),\omega)| \le |E(\omega)| \quad \text{and} \quad |B(x(T),\omega)| \le |E(\omega)|. \tag{2.8.12}$$

This gives

$$\lim_{R\nearrow\infty} R^{-1}\log\left|A(x(T),Re^{i\theta})\right| \le T\sin\theta, \tag{2.8.13}$$

$$\lim_{R\nearrow\infty} R^{-1}\log\left|B(x(T),Re^{i\theta})\right| \le T\sin\theta. \tag{2.8.14}$$

To show the reverse inequality note that

$$-i\,\Im\left[\overline{A(x(T),\omega)}B(x(T),\omega)\right] = \pi b K_{x(T)}(\omega,\omega) \tag{2.8.15}$$

for $\omega = a+ib$, $b>0$ and

$$K_x(\omega,\lambda) = \frac{\overline{A(x,\omega)}B(x,\lambda) - \overline{B(x,\omega)}A(x,\lambda)}{\pi(\lambda-\overline{\omega})}$$

(cf. (2.7.4)). For arbitrary $x \geq y \geq 0$ we have that

$$K_x(\omega,\omega) \geq e^{2b(T(x)-T(y))} K_y(\omega,\omega) \tag{2.8.16}$$

where $T(x) = \int_0^x \sqrt{m'(u)}du$. To show this we argue as in Lemma 2.2 of [16]. Consider a function

$$f(u) := e^{-2bT(u)} K_u(\omega,\omega). \tag{2.8.17}$$

Using the integral representation (2.7.4) we have that

$$\begin{aligned} f'(u) &= \pi^{-1} e^{-2bT(u)} \Big[i\sqrt{m'(u)} \left(\overline{A(u,\omega)} B(u,\omega) - \overline{B(u,\omega)} A(u,\omega) \right) \\ &+ \overline{A(u,\omega)} A(u,\omega) m'(u) + \overline{B(u,\omega)} B(u,\omega) \Big] \\ &= \pi^{-1} e^{-2bT(u)} \left| A(u,\omega)\sqrt{m'(u)} + iB(u,\omega) \right|^2 \geq 0. \end{aligned}$$

Hence,

$$e^{-2bT(x)} K_x(\omega,\omega) - e^{-2bT(y)} K_y(\omega,\omega) = \int_y^x f'(u)du \geq 0,$$

that proves (2.8.16). Using the latter and (2.8.15), we obtain

$$-i\,\Im\Big[\overline{A(x(T),\omega)} B(x(T),\omega)\Big] \geq e^{2b(T-S)} K_{x(S)}(\omega,\omega)$$

for any $0 \leq S \leq T$, since $T(x(u)) = u$. Since $|z| \geq |\Im(z)|$, we have

$$\left| \overline{A(x(T),\omega)} B(x(T),\omega) \right| \geq e^{2b(T-S)} |K_{x(S)}(\omega,\omega)|.$$

As we know, $K_x(\omega,\omega) \geq 0$ for any x and ω, therefore we have

$$\log|A(x(T),\omega)| + \log|B(x(T),\omega)| \geq 2b(T-S),$$

for every $S \geq 0$. For $\omega = Rr^{i\theta}$, $\theta \in (0,\pi)$, this implies

$$\log|A(x(T),Re^{i\theta})| + \log|B(x(T),Re^{i\theta})| \geq 2RT\sin\theta.$$

Hence,

$$\lim_{R \nearrow \infty} R^{-1} \log\left|A(x(T),Re^{i\theta})\right| + \lim_{R \nearrow \infty} R^{-1} \log\left|B(x(T),Re^{i\theta})\right| \geq 2T\sin\theta$$

which combined with (2.8.13) and (2.8.14) completes the proof of (i).

(ii) To prove reality of the roots consider formula (2.7.2) with $\lambda = \omega$. Then it takes the form

$$-4i\,\Re(\omega)\Im(\omega)\int_0^l |A(x,\omega)|^2 dm(x) = 2i\,\Im\Big[\overline{A(l,\omega)}A^+(l,\omega)\Big]. \tag{2.8.18}$$

From the latter display it is apparent that zeros of $A(l,\cdot)$ and $A^+(l,\cdot)$ are either real or pure imaginary. Take some $b \in \mathbb{R}$ and note that $A(l,ib) = 0$ is excluded by virtue of formula (2.3.2), since $p_0(x) = 1$.

To show that zeros of A and B are interlacing, consider the function

$$\zeta(\omega) := \frac{B(l,\omega)}{A(l,\omega)}.$$

It follows from (2.7.7) and (2.7.8) that for real ω

$$\|A(\cdot,\omega)\|_m^2 + \|B(\cdot,\omega)\|_2^2 = A(l,\omega)\dot{B}(l,\omega) - \dot{A}(l,\omega)B(l,\omega) \tag{2.8.19}$$

therefore, the derivative

$$\dot{\zeta}(\omega) = \frac{A(l,\omega)\dot{B}(l,\omega) - \dot{A}(l,\omega)B(l,\omega)}{(A(l,\omega))^2}$$

is positive. Hence, function ζ is monotone, so its zeros are interlaced with its poles.

Now we will show that the zeros are simple i.e. they are zeros of multiplicity one. Let $\omega_0 \neq 0$ be a real zero of $A(l,\cdot)$. Formula (2.8.19) takes the form

$$\|A(\cdot,\omega_0)\|_m^2 + \|B(\cdot,\omega_0)\|_2^2 = -\dot{A}(l,\omega_0)B(l,\omega_0).$$

If ω_0 were a zero of multiplicity at least two, we would have $\dot{A}(l,\omega_0) = 0$ which would contradict the fact that the left-hand side of the latter display is positive.

In order to prove the asymptotic behavior of the zeros let us recall the so-called *Jensen's formula* (cf. [17]): If f is analytic in the closed disc $|\lambda| \le R$ we have

$$\log|f(0)| = -\int_0^R \sharp(r)\frac{dr}{r} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right| d\theta, \tag{2.8.20}$$

where $\sharp(r)$ denotes the number of roots of f of modulus less or equal r. Since $A(0) = 1$ (for clarity we omit here the first coordinate $x(T)$), we can rewrite this as follows

$$\int_0^R \sharp(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|A(Re^{i\theta})\right| d\theta. \tag{2.8.21}$$

By virtue of statement (i) we have

$$\begin{aligned} & \frac{1}{\pi}\int_0^{\pi} \log\left|A(Re^{i\theta})\right| d\theta = \frac{R}{\pi}\int_0^{\pi} R^{-1}\log\left|A(Re^{i\theta})\right| d\theta \\ = \;& R\left[\frac{T}{\pi}\int_0^{\pi}\sin\theta d\theta + o(1)\right] = R\left[\frac{2T}{\pi} + o(1)\right] \end{aligned} \tag{2.8.22}$$

as $R \nearrow \infty$. Hence,

$$\int_0^R \sharp(r)\frac{dr}{r} = R\left[\frac{2T}{\pi} + o(1)\right]. \tag{2.8.23}$$

By virtue of an elementary Tauberian theorem (see for instance [89], Corollary 4.4b, p.194) we have that $\sharp(R) = R\left[2T/\pi + o(1)\right]$. From this we can conclude that a new root appears as soon as the radius exceeds another multiple of π/T. Hence, the n-th positive root of $A(x(T),\cdot)$ behaves approximately like $n\pi/T$ as $n \to \infty$. The fact that $\omega_{-n} = -\omega_n$ completes the statement for the zeros of A. The same argument works for the function B. $\square$

Chapter 3

Computation of strings

We know so far that with the spectral measure of a given Gaussian si-process we can associate a certain string. Unfortunately Theorem 2.5.1 ensures only the existence of the string. Even though the two notions are coupled via formula (2.5.2), there is no general method for the computation of the string. That is why we know the shape of the string only in a number of special cases. However, we can handle a number of interesting examples. Examples can be generated by so-called 'rules'. The basic idea is that we slightly modify the spectral measure for which we already know the associated string and obtain a new string expressed in terms of the one we started with. The first rules concerning scaling, shifting and multiplication by first order polynomial appeared in [38]. The collected set of known rules can be found in [17], Chapter 6. We also know how to handle the case of a spectral measure with a number of finite moments - this is due to Stieltjes [81]. We present this case in detail (following 5.8 and 5.9 of [17]) in Section 3.2. The last section of this chapter is devoted to power spectral measures. We present there recent developments that were first published in [22]. In particular, we exhibit the string associated to fractional Brownian motion.

3.1 Constant spectra

In this section we handle the simplest case of a constant spectral density. We have already seen in Example 2.5.3, that Lebesgue measure $\mu(d\lambda) = d\lambda$ is the principal spectral measure of the infinitely long string with mass distribution

$$m(x) = x. \tag{3.1.1}$$

The associated functions are given by

$$A(x,\lambda) = \cos \lambda x, \qquad B(x,\lambda) = \sin \lambda x,$$
$$D(x, ib) = b^{-1} \exp(-bx).$$

According to Example 1.1.1 the spectral measure of a standard Brownian motion is a multiple of Lebesgue measure. To handle this example we will need the following

Lemma 3.1.1. *Let $\mu^\bullet$ be a measure defined as*

$$\mu^\bullet(d\lambda) = c\mu(d\lambda), \tag{3.1.2}$$

where $c > 0$ and μ is the principal spectral measure of a given string lmk with associated functions A, B and D. Then the characteristics of the string $m^\bullet l^\bullet k^\bullet$ associated with $\mu^\bullet$ are as follows

μ	$\mu^\bullet = c\mu$
$m(x)$	$m^\bullet(x) = c^{-1}m(c^{-1}x)$
l	$l^\bullet = cl$
k	$k^\bullet = ck$
$A(x,\lambda)$	$A^\bullet(x,\lambda) = A(c^{-1}x,\lambda)$
$B(x,\lambda)$	$B^\bullet(x,\lambda) = c^{-1}B(c^{-1}x,\lambda)$
$D(x,\lambda)$	$D^\bullet(x,\lambda) = cD(c^{-1}x,\lambda)$

This is Rule 6.9.1 of [17], p. 265.

According to the above Lemma we obtain the following result for standard Brownian motion.

Example 3.1.2. ***String of the Brownian motion***

The measure

$$\mu(d\lambda) = \frac{1}{2\pi} d\lambda \tag{3.1.3}$$

is the principal spectral measure of a long string with mass distribution

$$m(x) = 4\pi^2 x, \tag{3.1.4}$$

and the associated functions are given by

$$A(x,\lambda) = \cos 2\pi\lambda x, \qquad B(x,\lambda) = 2\pi \sin 2\pi\lambda x,$$
$$D(x,ib) = (2\pi b)^{-1} \exp(-2\pi b x).$$

Remark 3.1.3. The computation of the string associated with the fractional Brownian motion requires a different method and is more demanding. This example is presented in Section 3.3.

3.2 Spectra with finite moments

In this section we will present a classical way of computing the string associated to a spectral measure. It goes back to [81] and [82]. It allows to find the string associated to a spectral measure μ possessing a number of moments, i.e. such that

$$\int_{\mathbb{R}} \lambda^{2k} \mu(d\lambda) < \infty \tag{3.2.1}$$

for $k = 0, 1, 2, ..., d$ and some $d \in \mathbb{N}$. We will restrict our considerations only to spectral measures of the form

$$\mu(d\lambda) = \frac{d\lambda}{(\lambda^2+1)^n} \qquad n \in \mathbb{N}. \tag{3.2.2}$$

The string connected with (3.2.2) starts with an alternating series of jumps and mass-free intervals and starting from a certain point continues linearly as $m(x) = x + c$ (the constant will be determined later). The first jump of m is always placed at $x_0 = 0$, further placements are denoted by $x_1, x_2, ...$ and m_k denotes the mass placed at $x = x_k$. To justify this we will make use of the formulas

$$\|A(x,\cdot)\|_\mu^2 = \pi/m[x] \tag{3.2.3}$$

for x a growth point of m, and

$$\|B(x,\cdot)\|_\mu^2 = \pi/(x^* - x_*) \tag{3.2.4}$$

in which x_* is the biggest growth point $y \leq x$ and x^* is the smallest growth point $y > x$. The norm $\|\cdot\|_\mu$ in this case is given by

$$\|g(\cdot)\|_\mu^2 = \int_{\mathbb{R}} \frac{|g(\lambda)|^2}{(\lambda^2+1)^n} d\lambda.$$

Both formulas are straightforward consequences of the norm equivalences from Theorems 2.6.1 and 2.6.2, respectively, applied to the function $f(x) = 1_{\{x\}}$ (cf. (2.6.1)). The number of jumps and mass-free intervals is determined by the number of moments of μ, i.e. the largest number $d \in \mathbb{N}$ such that

$$\int_{\mathbb{R}} \frac{\lambda^{2k} d\lambda}{(\lambda^2+1)^n} < \infty \tag{3.2.5}$$

for $k = 0, 1, ..., d$.

To see the direct link consider $x_0 = 0$. If $\|A(0,\cdot)\|_\mu$ is finite, which provided that $A(0,\lambda) = 1$ is equivalent to

$$\mu(\mathbb{R}) < \infty,$$

we have $m_0 = m[0] > 0$. Hence, we have a jump of m at $x_0 = 0$. To investigate what happens further we have to compute the form of $B(0,\lambda)$. Recall the definition

$$B(x,\lambda) = -A^+(x,\lambda)/\lambda. \tag{3.2.6}$$

Differential equation (2.3.1) implies that

$$A^+(x,\lambda) - A^-(x,\lambda) = -\lambda^2 A(x,\lambda) m[x], \tag{3.2.7}$$

and given that $A^-(0,\lambda) = 0$ we obtain $B(0,\lambda) = \lambda m_0$. If

$$\int_{\mathbb{R}} \frac{\lambda^2 d\lambda}{(\lambda^2+1)^n} < \infty,$$

we have that $\|B(0,\lambda)\|_\mu < \infty$, hence, according to (3.2.4), there exists $x_1 > x_0$ such that on the interval $[x_0, x_1)$ the mass function of the string is constant. For $x \in (x_0, x_1)$ equation (3.2.7) implies $A^+(x,\lambda) = A^-(x,\lambda)$, so function A is a solution of $A''(x,\lambda) = 0$, i.e.

$$A(x,\lambda) = ax + b, \qquad x \in [x_0, x_1].$$

The constants are given by $a = A^+(0,\lambda) =$ and $b = A(0,\lambda)$. Using (3.2.6) we obtain

$$A(x_1,\lambda) = A(0,\lambda) - \lambda B(0,\lambda) x_1. \tag{3.2.8}$$

Next, we check what happens in the point x_1 by computing the $L^2(\mu)$-norm of $A(x_1,\cdot)$. If the norm is finite we continue the procedure, otherwise we end up with a string whose 'jump-part' ends with a mass-free interval and then has to be continuous, according to $m[x] = 0$ for $x > x_1$.

In general we distinguish two cases: $n = 2d + 1$ and $n = 2d + 2$, for $d = 0, 1, 2, ...$ If $n = 2d + 1$ we have

$$\int_{\mathbb{R}} \frac{\lambda^{2k} d\lambda}{(\lambda^2 + 1)^n} < \infty \tag{3.2.9}$$

for $k = 0, 1, ..., 2d$. It implies that

$$\|A(x_d, \cdot)\|_\mu < \infty \qquad \text{and} \qquad \|B(x_d, \cdot)\|_\mu = \infty. \tag{3.2.10}$$

From this we conclude that the non-continuous part of the string ends with a jump placed at x_d and that then the mass function increases continuously. If $n = 2d + 2$ we have

$$\int_{\mathbb{R}} \frac{\lambda^{2k} d\lambda}{(\lambda^2 + 1)^n} < \infty \tag{3.2.11}$$

for $k = 0, 1, ..., 2d + 1$. It results in

$$\|B(x_d, \cdot)\|_\mu < \infty \qquad \text{and} \qquad \|A(x_{d+1}, \cdot)\|_\mu = \infty. \tag{3.2.12}$$

From this we see that the 'jump-part' of the string ends with a mass-free interval $[x_d, x_{d+1})$.

To determine what happens with the mass function after the point when it becomes continuous we use Rule 6.9.4 of [17], which is recalled in the following lemma.

Lemma 3.2.1. *Let mlk be a long string and let μ be its principal spectral function. Define the new measure by putting*

$$\mu^\bullet(d\lambda) = \frac{\mu(d\lambda)}{\lambda^2 + 1}. \tag{3.2.13}$$

Then the string associated with $\mu^\bullet$ is a long string $m^\bullet l^\bullet k^\bullet$ and at any point x such that $m'(x) > 0$ we have

$$m^\bullet(y(x)) = \int_0^x K^{-2}(z) dm(z), \tag{3.2.14}$$

where

$$K(x) = -\frac{D(x,i)}{D^+(x,i)} \quad \textit{and} \quad y(x) = \int_{[0,x]} K^2(z)dz. \tag{3.2.15}$$

Moreover, the function $D^\bullet$ is given by

$$D^\bullet(y(x),\omega) = \frac{D(x,\omega) + K(x)D^+(x,\omega)}{\omega^2+1}. \tag{3.2.16}$$

Let us now show how it applies to our situation. Consider the Lebesgue spectral measure and the associated string $m(x) = x$ (see Example 2.5.3). The function D is given by $D(x, ib) = b^{-1}e^{-bx}$, hence $K(x) = 1$. Since function K is constant, Lemma 3.2.1 implies that the mass function $m^\bullet$ associated with the measure $\mu^\bullet(d\lambda) = (\lambda^2+1)^{-1}d\lambda$, is of the form $m^\bullet(x) = x + c_1$, for some constant c_1.

Note that the $D^\bullet$ function is of the form

$$D^\bullet(y(x), ib) = \frac{e^{-bx}}{b(b+1)}. \tag{3.2.17}$$

Knowing this, we can apply Lemma 3.2.1 to the measure

$$\mu^{\bullet\bullet}(d\lambda) = \frac{\mu^\bullet(d\lambda)}{\lambda^2+1}. \tag{3.2.18}$$

where $\mu^\bullet(d\lambda) = (\lambda^2+1)^{-1}d\lambda$. Given the underlying mass function $m^\bullet(x) = x + c_1$ and $D^\bullet$ as in (3.2.17), we have

$$K^\bullet(y) = -\frac{D^\bullet(y(x),i)}{D^{\bullet+}(y(x),i)} = -\frac{D^\bullet(y(x),i)y'(x)}{(D^\bullet(y(x),i))^+} = 1. \tag{3.2.19}$$

So, again the mass function of the string associated with $\mu^{\bullet\bullet}$ is of the form $m^{\bullet\bullet}(x) = x + c_2$, where c_2 is a certain constant.

Now, we can iterate this procedure. Observe that the function $D^{n\bullet}$ associated with the measure $\mu^{n\bullet}(d\lambda) = (\lambda^2+1)^{-n}d\lambda$ will be of the form

$$\frac{e^{-bx}}{b(b+1)^n}, \tag{3.2.20}$$

and that for any $n \in \mathbb{N}$ we have $K^{n\bullet} \equiv 1$ and $\frac{d}{dx}y^{n\bullet}(x) = 1$. From this we conclude that regardless of n the continuous part of the mass function associated with the measure $\mu^{n\bullet}$ is of the form $m^{n\bullet}(x) = x + c_n$, for some $c_n \in \mathbb{R}$. The constants will be determined later.

To summarize, let the measure

$$\mu(d\lambda) = \frac{d\lambda}{(\lambda^2+1)^n} \qquad n \in \mathbb{N}. \tag{3.2.21}$$

be the principal spectral measure of the string lmk. We set $B(x_{-1}, \lambda) = 0$, $A(x_0, \lambda) = 1$, $x_0 = 0$ and use the following recurrence formulas to compute consecutive m_k's, x_k's, $A(x_k, \lambda)$'s and $B(x_k, \lambda)$'s

$$m_k = \frac{\pi}{\|A(x_k, \cdot)\|_\mu^2}, \qquad 2k < n, \tag{3.2.22}$$

$$B(x_k, \lambda) = m_k \lambda A(x_k, \lambda) + B(x_{k-1}, \lambda), \qquad 1 \le 2k+1 < n, \tag{3.2.23}$$

$$x_{k+1} = \frac{\pi}{\|B(x_k, \cdot)\|_\mu^2} + x_k, \qquad 2k+1 < n, \tag{3.2.24}$$

$$A(x_{k+1}, \lambda) = -\lambda B(x_k, \lambda)(x_{k+1} - x_k) + A(x_k, \lambda), \quad 2 \le 2k+2 < n. \tag{3.2.25}$$

The norms can be calculated by use of the formula

$$\int_{\mathbb{R}} \frac{\lambda^{2m} d\lambda}{(\lambda^2+1)^n} = \frac{\pi(2m-1)!!(2n-2m-3)!!}{(2n-2)!!}, \qquad n > m+1. \tag{3.2.26}$$

(see [29], 3.251.4). Observe that $A(x_k, \lambda)$ and $B(x_k, \lambda)$, considered as a polynomials in λ, are of degree $2k$ and $2k+1$, respectively.

The associated eigenfunction $A(x, \lambda)$ for x's in the 'jump area' will be piecewise linear interpolating between the polynomials $A(x_k, \lambda)$ at points of jump, i.e.

$$A(x, \lambda) = A(x_k, \lambda)(\lambda) - (x - x_k)\lambda B(x_k, \lambda) \tag{3.2.27}$$

for $x \in [x_k, x_{k+1})$.

To find the shape of the function $A(x, \lambda)$ for x's from the 'linear area' of the mass function, i.e. $x \ge x_d$ or $x \ge x_{d+1}$, if $n = 2d+1$ or $n = 2d+2$, respectively, we have to solve the differential equation from Section 2.3,

$$dA^+(x, \lambda) = -\lambda^2 A(x, \lambda) dm(x), \tag{3.2.28}$$

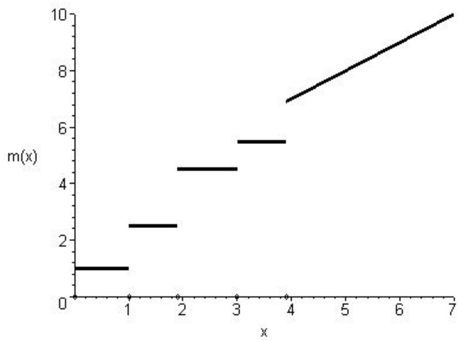
with suitable initial conditions implied by continuity and derivative left to the point of interest (x_d or x_{d+1}, according to the situation). Let us investigate the odd and even case separately.

Subcase $n = 2d + 1$. Formalizing the discussion above we have the mass function of the form

$$m(x) = \begin{cases} m_0 & x \in [x_0, x_1) \\ m_0 + m_1 & x \in [x_1, x_2) \\ ... & ... \\ m_0 + m_1 + ... + m_{d-1} & x \in [x_{d-1}, x_d) \\ (x - x_d) + m_0 + m_1 + ... + m_d & x \geq x_d \end{cases}$$

So the non-linear part of the string ends with a jump of magnitude m_d placed

Figure 3.1: Mass function (n=2d+1)



at the point x_d and we know that the function $A(x, \lambda)$ on the interval $[x_{d-1}, x_d)$ is given by

$$A(x, \lambda) = A(x_{d-1}, \lambda) - (x - x_{d-1})\lambda B(x_{d-1}, \lambda). \tag{3.2.29}$$

At the point $x = x_d$ we have

$$A^+(x_d, \lambda) - A^-(x_d, \lambda) = -\lambda^2 A(x_d, \lambda) m_d. \tag{3.2.30}$$

From (3.2.29) we obtain $A^-(x_d, \lambda) = -\lambda B(x_{d-1}, \lambda)$, so

$$A^+(x_d, \lambda) = \lambda(\lambda^2 m_d(x_d - x_{d-1}) - 1)B(x_{d-1}, \lambda) - \lambda^2 m_d A(x_{d-1}, \lambda). \tag{3.2.31}$$

For $x > x_d$, according to $m[x] = 0$ and hence $A^+(x, \lambda) = A^-(x, \lambda)$, the differential equation takes a form

$$A''(x, \lambda) = -\lambda^2 A(x, \lambda). \tag{3.2.32}$$

The solution of the above is

$$A(x,\lambda) = p_n(\lambda)\cos\lambda x + q_n(\lambda)\sin\lambda x, \tag{3.2.33}$$

where the functions p_n and q_n are determined by the right-hand derivative of A in x_d given by (3.2.31) and

$$A(x_d,\lambda) = A(x_{d-1},\lambda) - (x_d - x_{d-1})\lambda B(x_{d-1},\lambda).$$

So we would have

$$\begin{aligned} p_n(\lambda) &= A(x_d,\lambda)\cos\lambda x_d - \frac{1}{\lambda}A^+(x_d,\lambda)\sin\lambda x_d \\ q_n(\lambda) &= A(x_d,\lambda)\sin\lambda x_d + \frac{1}{\lambda}A^+(x_d,\lambda)\cos\lambda x_d \end{aligned}$$

and incorporating the initial conditions we end up with

$$\begin{aligned} p_n(\lambda) &= [A(x_{d-1},\lambda) - (x_d - x_{d-1})\lambda B(x_{d-1},\lambda)]\cos\lambda x_d \qquad (3.2.34) \\ &+ \big[m_d\lambda A(x_{d-1},\lambda) + (1-\lambda^2 m_d(x_d - x_{d-1}))B(x_{d-1},\lambda)\big]\sin\lambda x_d, \\ q_n(\lambda) &= [A(x_{d-1},\lambda) - (x_d - x_{d-1})\lambda B(x_{d-1},\lambda)]\sin\lambda x_d \qquad (3.2.35) \\ &- \big[m_d\lambda A(x_{d-1},\lambda) + (1-\lambda^2 m_d(x_d - x_{d-1}))B(x_{d-1},\lambda)\big]\cos\lambda x_d, \end{aligned}$$

for $n = 2d+1$. To make the formulas consistent for all d's we have to assume $x_{-1} = 0$.

Remark 3.2.2. Observe that the highest power of λ in both $p_n(\lambda)$ and $q_n(\lambda)$ is equal to $n = 2d+1$. So $|p_n(\lambda)|$ and $|q_n(\lambda)|$ behave like $|\lambda|^n$ for big values of λ.

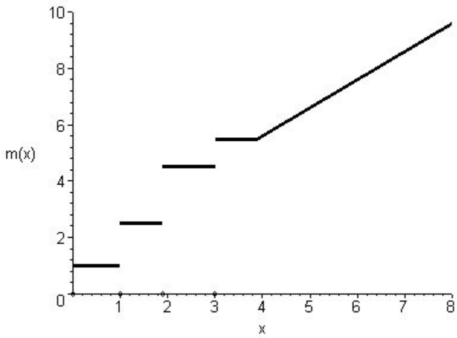
Subcase $n = 2d+2$. In this case the shape of the string will be

$$m(x) = \begin{cases} m_0 & x \in [x_0, x_1) \\ m_0 + m_1 & x \in [x_1, x_2) \\ ... & ... \\ m_0 + m_1 + ... + m_d & x \in [x_d, x_{d+1}) \\ (x - x_{d+1}) + m_0 + m_1 + ... + m_d & x \geq x_{d+1} \end{cases}$$

Here, the non-linear part of the string ends with a mass-free interval $[x_d, x_{d+1})$ and $m_{d+1} = 0$. Function $A(x,\lambda)$ for $x \in [x_d, x_{d+1})$ takes a form

$$A(x,\lambda) = A(x_d,\lambda) - (x - x_d)\lambda B(x_d,\lambda). \tag{3.2.36}$$

Figure 3.2: Mass function (n=2d+2)



At the endpoint $x = x_{d+1}$ the differential equation gives

$$A^+(x_{d+1}, \lambda) - A^-(x_{d+1}, \lambda) = -\lambda^2 A(x_{d+1}, \lambda) m_{d+1} = 0, \tag{3.2.37}$$

hence

$$A^+(x_{d+1}, \lambda) = A^-(x_{d+1}, \lambda) = -\lambda B(x_d, \lambda). \tag{3.2.38}$$

Similar to the odd case, to find the form of the function $A(x, \lambda)$ for $x \geq x_{d+1}$, we have to solve

$$A''(x, \lambda) = -\lambda^2 A(x, \lambda),$$

but now with the initial conditions

(i) $A(x_{d+1}, \lambda) = A(x_d, \lambda) - (x_{d+1} - x_d)\lambda B(x_d, \lambda)$,

(ii) $A^+(x_{d+1}, \lambda) = -\lambda B(x_d, \lambda)$.

We obtain the solution

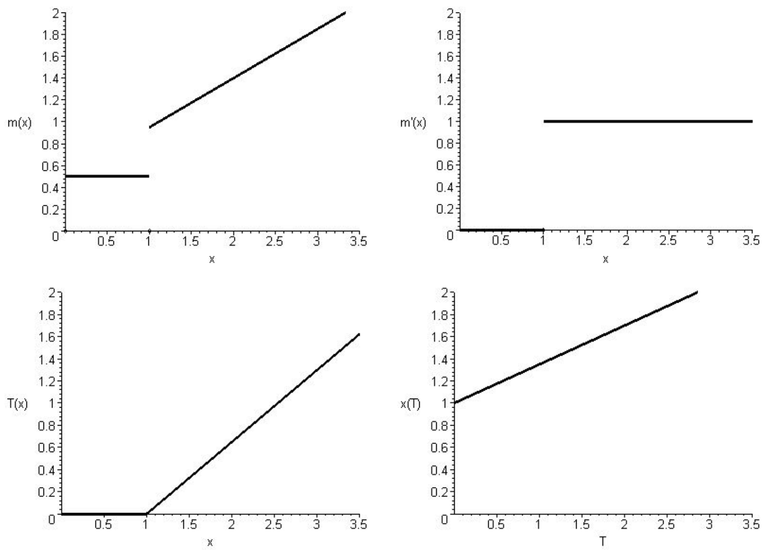
$$A(x, \lambda) = p_n(\lambda) \cos \lambda x + q_n(\lambda) \sin \lambda x, \tag{3.2.39}$$

with

$$\begin{aligned} p_n(\lambda) &= [A(x_d, \lambda) - (x_{d+1} - x_d)\lambda B(x_d, \lambda)] \cos \lambda x_{d+1} + B(x_d, \lambda) \sin \lambda x_{d+1}, \\ q_n(\lambda) &= [A(x_d, \lambda) - (x_{d+1} - x_d)\lambda B(x_d, \lambda)] \sin \lambda x_{d+1} - B(x_d, \lambda) \cos \lambda x_{d+1}, \end{aligned}$$

for $n = 2d + 2$.

Figure 3.3: Illustration of $x(T)$ computation



Remark 3.2.3. Observe that also in this case the highest power of λ in both $p_n(\lambda)$ and $q_n(\lambda)$ is equal to $n = 2d+2$. Hence, $|p_n(\lambda)|$ and $|q_n(\lambda)|$ behave like $|\lambda|^n$ for big values of λ.

In both cases the function $B(x,\lambda)$ can be computed according to the formula $B(x,\lambda) = -A^+(x,\lambda)/\lambda$. Let us also mention here that the function $x(T)$ (see (2.8.3)) will be given by

$$x(T) = \begin{cases} T + x_d & n = 2d+1, \\ T + x_{d+1} & n = 2d+2, \end{cases} \tag{3.2.40}$$

for $T \in [0,\infty)$. Knowing this, we can write a general form of the functions

$$A(x(T),\lambda) = p_n(\lambda)\cos\lambda(T + x(0)) + q_n(\lambda)\sin\lambda(T + x(0)), \tag{3.2.41}$$

$$B(x(T),\lambda) = p_n(\lambda)\sin\lambda(T + x(0)) - q_n(\lambda)\cos\lambda(T + x(0)), \tag{3.2.42}$$

for $T \geq 0$.

3.2.1 Examples

Example 3.2.4. *String of Ornstein-Uhlenbeck process*

We can now apply the above consideration to the spectral measure of the Ornstein-Uhlenbeck process. As we have seen in Section 1.1.3 we deal with the spectral density of the form

$$\mu(d\lambda) = \frac{d\lambda}{\lambda^2 + 1}. \tag{3.2.43}$$

Hence, we have $n = 1$. We see that the mass function of the string associated with those measure will have one jump at $x_0 = 0$ whose magnitude can easily be computed using (3.2.22) and

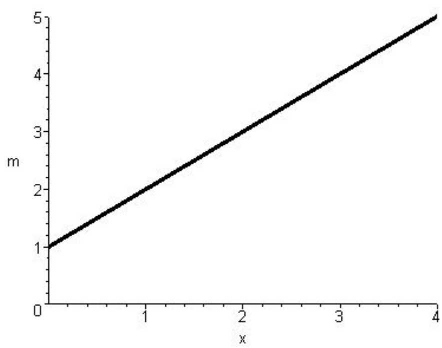
$$m_0 = 1, \tag{3.2.44}$$

hence, the mass function is

$$m(x) = x + 1, \tag{3.2.45}$$

for $x \in [0, \infty)$. With $B(x_0, \lambda) = \lambda$ and using appropriate formulas for functions

Figure 3.4: Ornstein-Uhlenbeck string



$p_1(\lambda)$ and $q_1(\lambda)$ we obtain

$$A(x,\lambda) = \cos\lambda x - \lambda\sin\lambda x, \tag{3.2.46}$$

$$B(x,\lambda) = \lambda\cos\lambda x + \sin\lambda x, \tag{3.2.47}$$

for $x \in [0,\infty)$. Let us also note for future use that the function $x(T)$ defined by (2.8.3) will be

$$x(T) = T, \quad T > 0, \tag{3.2.48}$$

hence, we have

$$A(x(T),\lambda) = \cos\lambda T - \lambda\sin\lambda T, \tag{3.2.49}$$

$$B(x(T),\lambda) = \lambda\cos\lambda T + \sin\lambda T. \tag{3.2.50}$$

Example 3.2.5. *Matérn density with $n = 2$*

To make the general results more clear we present the details of another example, namely the process with spectral density of Matérn type with $n = 2$. The spectral measure is of the form

$$\mu(d\lambda) = \frac{d\lambda}{(\lambda^2+1)^2}. \tag{3.2.51}$$

According to the general theory we will have here jump at $x_0 = 0$, a mass-free interval $[0, x_1)$ and from x_1 on the mass function will be linear. Using formulas (3.2.22)-(3.2.25) we have

$$m_0 = 2,\ B(x_0,\lambda) = 2\lambda,\ x_1 = \frac{1}{2}. \tag{3.2.52}$$

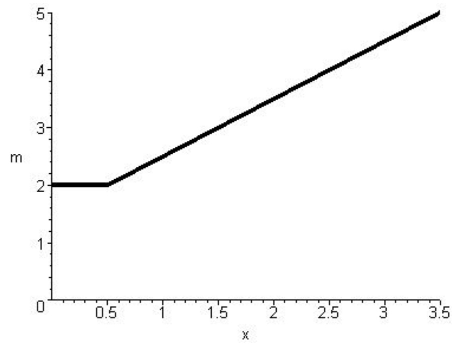
The string is given by

$$m(x) = \begin{cases} 2 & x \in [0, \frac{1}{2}), \\ x + \frac{3}{2} & x \geq \frac{1}{2}. \end{cases} \tag{3.2.53}$$

The eigenfunctions satisfy

$$A(x,\lambda) = \begin{cases} 1 - 2\lambda^2 x & x \in [0, \frac{1}{2}), \\ \left[(1+\lambda^2)\cos\lambda/2 + 2\lambda\sin\lambda/2\right]\cos\lambda x & \\ + \left[(1+\lambda^2)\sin\lambda/2 - 2\lambda\cos\lambda/2\right]\sin\lambda x & x \geq \frac{1}{2}, \end{cases} \tag{3.2.54}$$

Figure 3.5: String mass of a Matérn process



$$B(x,\lambda)=\begin{cases}2\lambda & x\in[0,\frac{1}{2}),\\ \left[(1+\lambda^2)\cos\lambda/2+2\lambda\sin\lambda/2\right]\sin\lambda x & \\ -\left[(1+\lambda^2)\sin\lambda/2-2\lambda\cos\lambda/2\right]\cos\lambda x & x\geq\frac{1}{2}.\end{cases} \tag{3.2.55}$$

In this case we have $x(T)=T+\frac{1}{2}$, for $T\geq 0$, hence

$$\begin{aligned} A(x(T),\lambda) &= \left[(1+\lambda^2)\cos\lambda/2+2\lambda\sin\lambda/2\right]\cos\lambda(T+1/2)\\ &+ \left[(1+\lambda^2)\sin\lambda/2-2\lambda\cos\lambda/2\right]\sin\lambda(T+1/2), \end{aligned} \tag{3.2.56}$$

$$\begin{aligned} B(x(T),\lambda) &= \left[(1+\lambda^2)\cos\lambda/2+2\lambda\sin\lambda/2\right]\sin\lambda(T+1/2)\\ &- \left[(1+\lambda^2)\sin\lambda/2-2\lambda\cos\lambda/2\right]\cos\lambda(T+1/2). \end{aligned} \tag{3.2.57}$$

Remark 3.2.6. Note that next to the 'Rules' cited above in Lemmas 3.1.1 and 3.2.1 there are several more already established. They can be found in [38] or [17]. In general, they allow to compute the string associated with the measure

$$\mu^\bullet(d\lambda)=\frac{P(\lambda^2)}{Q(\lambda^2)}\mu(d\lambda) \tag{3.2.58}$$

where P and Q are polynomials and μ is the principal spectral measure of a string of a known shape.

3.3 Power spectra

In this section we determine the principal spectral measure of the string with power mass distribution, namely $m(x) = cx^p$. It will turn out that it corresponds to a power spectral density and hence, that it can be used to handle the fractional Brownian motion case (Section 3.3.1).

We begin with the observation that just from scaling properties, it follows that the principal spectral measure of an infinitely long string with power mass function is an absolutely continuous measure with a power density, and the two powers can be related explicitly.

Theorem 3.3.1. *The principal spectral measure of an infinitely long string with mass distribution function $m(x) = cx^p$ for $c, p > 0$ is absolutely continuous, and its density is given by $\lambda \mapsto C|\lambda|^{(p-1)/(p+1)}$ for some $C > 0$.*

Proof. The eigenfunction A of the string solves the eigenvalue problem

$$A''(x, \lambda) = -\lambda^2 m'(x) A(x, \lambda)$$

(see Section 2.3). Now fix $a > 0$ and define the function $F(x, \lambda) = A(ax, \lambda)$. Since $m'(ax) = a^{p-1} m'(x)$, this function satisfies

$$F''(x, \lambda) = -\left(\lambda a^{\frac{p+1}{2}}\right)^2 m'(x) F(x, \lambda).$$

We see that F satisfies the same equation as $A(x, a^{(p+1)/2}\lambda)$. Hence, since it also satisfies the same initial conditions $F(0, \lambda) = 1$ and $F'(0, \lambda) = 0$, we have $F(x, \lambda) = A(x, a^{(p+1)/2}\lambda)$. In other words, the eigenfunctions have the property that $A(ax, \lambda) = A(x, a^{(p+1)/2}\lambda)$, or $A(x, a\lambda) = A(a^{2/(p+1)}x, \lambda)$. Differentiation of the latter identity yields the relation $B(x, a\lambda) = a^{(1-p)/(1+p)} B(a^{2/(1+p)}x, \lambda)$ for the function B defined in Section 2.3.

These scaling properties of the functions A and B carry over to the odd and even transforms introduced in Section 2.6. For $f \in L^2(m)$ we have the relation

$$\begin{aligned}
\mathcal{T}_{\text{even}} f(a\lambda) &= \int_0^\infty A(x, a\lambda) f(x)\, dm(x) \\
&= \int_0^\infty A(a^{\frac{2}{1+p}} x, \lambda) f(x)\, dm(x) \\
&= a^{\frac{-2p}{1+p}} \int_0^\infty A(x, \lambda) f(a^{-\frac{2}{1+p}} x)\, dm(x) \\
&= a^{\frac{-2p}{1+p}} \mathcal{T}_{\text{even}} \tilde{f}(\lambda),
\end{aligned}$$

where $\tilde{f}(x) = f(a^{-2/(1+p)}x)$. Since $\|\tilde{f}\|^2_{L^2(m)} = a^{2p/(1+p)}\|f\|^2_{L^2(m)}$, it follows that

$$\begin{aligned}
\|\mathcal{T}_{\text{even}} f(a\cdot)\|^2_{L^2(\mu)} &= a^{\frac{-4p}{1+p}} \|\mathcal{T}_{\text{even}} \tilde{f}\|^2_{L^2(\mu)} \\
&= \pi a^{\frac{-4p}{1+p}} \|\tilde{f}\|^2_{L^2(m)} \\
&= \pi a^{\frac{-2p}{1+p}} \|f\|^2_{L^2(m)} \\
&= a^{\frac{-2p}{1+p}} \|\mathcal{T}_{\text{even}} f\|^2_{L^2(\mu)}.
\end{aligned}$$

Similar calculations for the odd transform imply that for all $f \in L^2[0, \infty)$,

$$\|\mathcal{T}_{\text{odd}} f(a\cdot)\|^2_{L^2(\mu)} = a^{\frac{-2p}{1+p}} \|\mathcal{T}_{\text{odd}} f\|^2_{L^2(\mu)}.$$

Consequently, we have the scaling property

$$\|\psi(a\cdot)\|_\mu = a^{\frac{-p}{1+p}} \|\psi\|_\mu$$

for every $\psi \in L^2(\mu)$.

If we take $\lambda \in \mathbb{R}$ and $\psi = 1_{\{\lambda\}}$ we find that for $a > 0$,

$$\mu(\{\lambda/a\}) = \|\psi(a\cdot)\|^2_{L^2(\mu)} = a^{\frac{-2p}{1+p}} \|\psi\|^2_{L^2(\mu)} = a^{\frac{-2p}{1+p}} \mu(\{\lambda\}),$$

which implies that μ does not have atoms. Similarly, taking $\psi = 1_{(0,1]}$ implies that

$$\mu(0, \lambda] = \lambda^{\frac{2p}{1+p}} \mu(0, 1]$$

for all $\lambda > 0$. This completes the proof. □

3.3.1 String of the Fractional Brownian motion

In this section we identify the string associated with the fBm with Hurst index $H \in (0,1)$. We prove that the spectral measure μ_H of the fBm is the principal spectral measure of an infinitely long string with mass distribution $C_H x^{(1-H)/H}$, for an explicitly given constant C_H, and we give explicit expressions for the eigenfunctions A, B and D introduced in Section 2.3.

Theorem 3.3.1 shows that for the mass distribution $m(x) = x^{(1-H)/H}$ for $H \in (0,1)$, the principle spectral measure is given by $\mu(d\lambda) = C|\lambda|^{1-2H}\,d\lambda$ for some constant $C > 0$. The following theorem provides the value of the constant and gives explicit expressions for the functions A, B and D introduced in Section 2.3. The proof of the theorem relies heavily on properties of Bessel functions. We refer to Appendix A for background.

Theorem 3.3.2. *Let $H \in (0,1)$ be given. The measure*

$$\mu(d\lambda) = \frac{\pi H 4^H}{\Gamma^2(1-H)}|\lambda|^{1-2H}\,d\lambda$$

is the principal spectral measure of the infinitely long string with mass distribution

$$m(x) = \frac{x^{\frac{1-H}{H}}}{4H(1-H)}.$$

The corresponding eigenfunctions are given by

$$A(x,\lambda) = \Gamma(1-H)\Big(\frac{\lambda}{2}\Big)^H \sqrt{x} J_{-H}(\lambda x^{\frac{1}{2H}}),$$

and

$$B(x,\lambda) = -\frac{1}{\lambda}A'(x,\lambda) = \frac{\Gamma(1-H)}{2H}\Big(\frac{\lambda}{2}\Big)^H x^{\frac{1-H}{2H}} J_{1-H}(\lambda x^{\frac{1}{2H}}).$$

For $b \in \mathbb{R}$ we have

$$D(x,ib) = \frac{2H}{\Gamma(1-H)}\Big(\frac{2}{b}\Big)^H \sqrt{x} K_{-H}(bx^{\frac{1}{2H}}).$$

Here J_ν is the Bessel function of the first of order ν and K_ν is the modified Bessel function of the second kind of order ν.

Proof. If $m(x) = x^{(1-H)/H}/(4H(1-H))$, then

$$m'(x) = \frac{1}{4H^2}x^{(1-2H)/H}$$

and hence the equations for the function $A = A(\cdot, \lambda)$ are given by

$$A'' + \frac{\lambda^2}{4H^2}x^{\frac{1-2H}{H}}A = 0, \quad A(0) = 1, \quad A'(0) = 0. \tag{3.3.1}$$

According to formula 4 of table 8.491 on p. 971 of [29] (applied with $\beta = \lambda, \gamma = 1/(2H)$ and $\nu = -H$) the differential equation is solved by the function

$$A(x) = c\sqrt{x}J_{-H}(\lambda x^{\frac{1}{2H}}),$$

where c is an arbitrary constant. Using the power series expansion of the Bessel function it is easily verified that this function also satisfies $A'(0) = 0$, as required. Moreover, from the power series we see that

$$A(0) = c\frac{2^H}{\lambda^H\Gamma(1-H)}.$$

Hence, the requirement $A(0) = 1$ leads to the choice $c = \lambda^H\Gamma(1-H)/2^H$ and we find that for this string, the eigenfunctions are given by

$$A(x, \lambda) = \Gamma(1-H)\Big(\frac{\lambda}{2}\Big)^H\sqrt{x}J_{-H}(\lambda x^{\frac{1}{2H}}).$$

The expression for B is obtained by using the formula

$$\frac{d}{dz}J_\nu(z) = \frac{\nu}{z}J_\nu(z) - J_{\nu+1}(z)$$

(cf. (A.0.14)) for the Bessel function of the first kind.

To derive the expression for D, observe first that for $\lambda = ib$, $b \in \mathbb{R}$, the solution $A(\cdot, ib)$ of (3.3.1) is given by

$$\begin{aligned} A(x, ib) &= \Gamma(1-H)\Big(\frac{b}{2}\Big)^H\sqrt{x}e^{i\frac{\pi H}{2}}J_{-H}(ibx^{\frac{1}{2H}}) \\ &= \Gamma(1-H)\Big(\frac{b}{2}\Big)^H\sqrt{x}I_{-H}(bx^{\frac{1}{2H}}), \end{aligned}$$

where I_ν is the modified Bessel function of the first kind of order ν. Denoting the modified Bessel function of the third kind of order ν by K_ν, we have the

Wronskian relation $I_\nu(z)K'_\nu(z) - K_\nu(z)I'_\nu(z) = -1/z$ (cf. (A.0.27)). It follows that for $x \leq y$,

$$\int_x^y \frac{dz}{z^2 I_\nu^2(z)} = \frac{K_\nu(x)}{I_\nu(x)} - \frac{K_\nu(y)}{I_\nu(y)}$$

and hence, by the asymptotic properties of K_ν and I_ν (see (A.0.25) and (A.0.26)),

$$\int_x^\infty \frac{dz}{z^2 I_\nu^2(z)} = \frac{K_\nu(x)}{I_\nu(x)}.$$

A straightforward computation now shows that

$$D(x, ib) = A(x, ib)\int_x^\infty \frac{dy}{A^2(y, ib)} = \frac{2H}{\Gamma(1-H)}\left(\frac{2}{b}\right)^H \sqrt{x} K_{-H}(bx^{\frac{1}{2H}}).$$

It is argued in Section 2.5 that for the proof of the formula for μ, it suffices to show that

$$D(0, ib) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mu(d\lambda)}{b^2 + \lambda^2}$$

for every $b \in \mathbb{R}$. By formula 3.251.2 on p. 295 of [29] (applied with $\nu = 0$ and $\mu = 2 - 2H$) the right-hand side equals

$$\frac{H 4^H \Gamma(H)}{b^{2H}\Gamma(1-H)}.$$

Using formulas (A.0.24) and (A.0.22) we see that this indeed coincides with $D(0, ib)$. □

Recall from Example 1.1.2 that the spectral measure of the fractional Brownian motion X with Hurst index $H \in (0,1)$ is given by $\mu_H(d\lambda) = c_H|\lambda|^{1-2H}\,d\lambda$, with

$$c_H = \frac{\Gamma(1+2H)\sin \pi H}{2\pi}.$$

The preceding theorem yields the following result for μ_H.

Corollary 3.3.3. *The measure μ_H is the principle spectral measure of the infinitely long string with mass distribution*

$$m(x) = \frac{\kappa_H^{1/H}}{4H(1-H)} x^{\frac{1-H}{H}},$$

where

$$\kappa_H = \frac{2\pi^{3/2}}{\Gamma(1-H)\Gamma(1/2+H)} = \frac{2\sqrt{\pi}\Gamma(H)\sin \pi H}{\Gamma(1/2+H)}.$$

The associated eigenfunctions are given by

$$A(x,\lambda) = \Gamma(1-H)\Big(\frac{\lambda}{2}\Big)^H \sqrt{\kappa_H x} J_{-H}\Big(\lambda(\kappa_H x)^{\frac{1}{2H}}\Big)$$

and

$$B(x,\lambda) = \frac{\kappa_H \Gamma(1-H)}{2H}\Big(\frac{\lambda}{2}\Big)^H (\kappa_H x)^{\frac{1-H}{2H}} J_{1-H}\Big(\lambda(\kappa_H x)^{\frac{1}{2H}}\Big).$$

For $b \in \mathbb{R}$ it holds that

$$D(x, ib) = \frac{2H}{\Gamma(1-H)\kappa_H}\Big(\frac{2}{b}\Big)^H \sqrt{\kappa_H x} K_{-H}\Big(b(\kappa_H x)^{\frac{1}{2H}}\Big).$$

Proof. The proof is a straightforward application of Lemma 3.1.1. □

Observe that in the standard Brownian case $H = 1/2$ we have $\kappa_{1/2} = 2\pi$. So indeed, the string associated with ordinary Brownian motion is the infinitely long string with mass distribution $m(x) = 4\pi^2 x$, cf. Example 3.1.2.

Since we know the exact shape of $m(x)$ we see that the function T defined by (2.1.1) is now given by

$$T(x) = (\kappa_H x)^{\frac{1}{2H}},$$

hence, $\kappa_H x(T) = T^{2H}$. So, we have the following

$$A(x(T),\lambda) = \Gamma(1-H)\left(\frac{\lambda T}{2}\right)^H J_{-H}(\lambda T), \tag{3.3.2}$$

$$B(x(T),\lambda) = \frac{1}{x'(T)}\Gamma(1-H)\left(\frac{\lambda T}{2}\right)^H J_{1-H}(\lambda T). \tag{3.3.3}$$

Chapter 4

Representations of processes and fields

With this section we begin to use the string theory, described in details in Chapter 2, to obtain results for stochastic processes and random fields. We will proceed by using the spectral isometry to translate the results obtained for the linear space $\mathscr{L}_T$. Our main goal is to obtain a series representation of the process X of the form

$$X_t = \sum_{k \in \mathbb{Z}} \phi_k(t) Z_k,$$

with Z_k independent standard Gaussian random vectors and ϕ_k certain explicit, deterministic functions. We will also investigate the rates at which

$$X_t^n = \sum_{|k| \leq n} \phi_k(t) Z_k$$

converges as $n \to \infty$. Such results are obviously useful for the purpose of simulation and are also closely related to the so-called small deviation behavior of the processes, which is the main subject of the next chapter.

Another representation of our interest is the so-called moving average representation

$$X_t = \int_0^t w_t(u) dM_u \tag{4.0.1}$$

of X as a stochastic integral of the deterministic kernel w_t with respect to Gaussian martingale M.

4.1 Gaussian processes

All the background results for this chapter were already obtained in Section 2.8. Given a spectral measure μ, Theorem 2.8.4 describes the RKHS structure of the space

$$L^2_T(\mu) = \overline{\mathrm{sp}}\{\hat{1}_{(0,t]} : t \in [-T, T]\} \subseteq L^2(\mu).$$

In connection with our study of stochastic processes however, we are more interested in the space

$$\mathscr{L}_T = \overline{sp}\{\lambda \mapsto \hat{1}_{(0,t]}(\lambda) \;:\; t \in [0, T]\} \subset L^2(\mu) \tag{4.1.1}$$

which, provided that $\mu(\mathbb{R}) < \infty$, can be equivalently defined as

$$\mathscr{L}_T = \overline{sp}\{\lambda \mapsto e^{i\lambda t} \;:\; t \in [0, T]\} \subset L^2(\mu), \tag{4.1.2}$$

for $T > 0$ (cf. (1.1.11), (1.1.17)). From the simple observation that if $\psi \in \mathscr{L}_{2T}$, then $\lambda \mapsto \exp(-i\lambda T)\psi(\lambda)$ belongs to $L^2_T(\mu)$, it easily follows that the reproducing kernel on $\mathscr{L}_{2T}$ is given by

$$S_{2T}(\omega, \lambda) = e^{i(\lambda-\omega)T} K_T(\omega, \lambda). \tag{4.1.3}$$

According to this formula we can easily prove the following result. Let measure μ is the principal spectral measure of a given string lmk.

Theorem 4.1.1. *Suppose* $0 < T < \int_0^l \sqrt{m'(y)}\, dy$.

(i) The space $\mathscr{L}_T$ *is a reproducing kernel Hilbert space of entire functions. Its reproducing kernel is given by*

$$S_T(\omega, \lambda) = e^{\frac{iT(\lambda-\omega)}{2}} \frac{A(x(T/2), \omega)B(x(T/2), \lambda) - B(x(T/2), \omega)A(x(T/2), \lambda)}{\pi(\lambda - \omega)}.$$

(ii) With $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ *the real-valued zeros of the function* $\omega \mapsto B(x(T/2), \omega)$*, the functions* $S_T(\omega_k, \cdot)$ *form an orthogonal basis of* $\mathscr{L}_T$*. Each function* $\psi \in \mathscr{L}_T$ *can be expanded as*

$$\psi(\lambda) = \sum_{k \in \mathbb{Z}} \psi(\omega_k) \frac{S_T(\omega_k, \lambda)}{S_T(\omega_k, \omega_k)}.$$

Proof. Both statements are results of combining Theorems 2.8.4 and 2.8.6 with formula (4.1.3). □

The first assertion of the theorem says that every $\psi \in \mathscr{L}_T$ has a version that can be extended to an entire function on the complex plane. Moreover, it holds that $S_T(\omega, \cdot) \in \mathscr{L}_T$ for every $\omega \in \mathbb{R}$ and for $\psi \in \mathscr{L}_T$, we have the reproducing property

$$\psi(\omega) = \langle \psi, S_T(\omega, \cdot) \rangle_\mu .$$

Remark 4.1.2. Clearly the condition of the theorem is fulfilled for all $T > 0$ if we have the finite propagation speed property $\int_0^l \sqrt{m'(y)}\, dy = \infty$. Using the results of [17] it can be shown that the latter holds if and only if $\mathscr{L}_s \subsetneqq \mathscr{L}_t$ for all $0 \leq s < t$. In terms of the Gaussian process X, this is a condition saying the process is non-deterministic in an appropriate sense. By Exercise 5 on p. 247 of [17], a sufficient condition on the spectral measure is that $\mu(d\lambda) = f(\lambda)\, d\lambda$, and $\int \log f(\lambda)/(1+\lambda^2)\, d\lambda > -\infty$.

4.1.1 Series representations of si-processes

4.1.1.1 Series expansion

Let $X = (X_t)_{t\in[0,T]}$ be a Gaussian si-process with spectral measure μ. Let lmk be the associated string. Moreover, let the functions A, B, m' and x associated with the string be defined as in Chapter 2.

By virtue of the spectral isometry (see (1.1.18)), part (ii) of Theorem 4.1.1 yields a series representation for the process X. We use the notation $\dot{B}$ for the derivative of B relative to the variable ω (which always exists).

Theorem 4.1.3. *Suppose* $0 < T < \int_0^l \sqrt{m'(y)}\, dy$. *Let* $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ *be the real-valued zeros of the function* $\omega \mapsto B(x(T/2), \omega)$. *We have the representation*

$$X_t = \sum_{k\in\mathbb{Z}} c_k \hat{1}_{(0,t]}(\omega_k) Z_k, \qquad t \in [0,T], \tag{4.1.4}$$

where

$$c_k = \Big(\frac{1}{\pi}A(x(T/2),\omega_k)\dot{B}(x(T/2),\omega_k)\Big)^{-1/2}$$

and the Z_k *are independent standard Gaussian random variables. The series converges in mean-square sense for every fixed* $t \in [0,T]$*. If the process* X *admits a continuous version then with probability* 1*, the series converges uniformly in* $t \in [0,T]$.

Proof. The mean-square convergence follows from Theorem 4.1.1 by expanding $\hat{1}_{(0,t]} \in \mathscr{L}_T$ in the orthogonal basis of part (ii) of the theorem and combining $c_k = (S_T(\omega_k,\omega_k))^{-1/2}$ with Lemma 2.8.5. The uniform convergence will be provided by the proof of Theorem 4.2.5 in a multi-dimensional setting. □

4.1.1.2 Rates of convergence

Our next aim is to quantify the rate of convergence of the series in (4.1.4). For a positive integer n we introduce the partial sum process

$$X_t^n = \sum_{|k|\leq n} c_k \hat{1}_{(0,t]}(\omega_k) Z_k, \qquad t \in [0,T],$$

and we study the behavior of approximation error $\mathbb{E}\|X - X^n\|$ for $n \to \infty$, for several choices of the norm $\|\cdot\|$. Note that here, X is the process defined by (4.1.4).

The convergence rate is determined by the tail behavior of the reproducing kernel S_T of the space $\mathscr{L}_T$, as defined in Theorem 4.1.1. To make this precise we will assume that there exists a positive constant c and a number $\alpha > -1$ such that

$$S_T(\lambda,\lambda) \geq c|\lambda|^\alpha \tag{4.1.5}$$

for $|\lambda|$ large enough. The upper bounds that we find for the rate of convergence in (4.1.4) depends on the exponent α and on the specific norm that is considered. The examples will show that in many cases, the upper bounds are in fact sharp.

We use the notation $\|\cdot\|_p$ for the norm on $L^p[0,T]$, i.e. $\|f\|_p^p = \int_0^T |f(t)|^p\,dt$. The Orlicz norm relative to the function $\psi_p(t) = \exp(t^p) - 1$ is denoted by $\|\cdot\|_{\psi_p}$, i.e.

$$\|f\|_{\psi_p} = \inf\Big\{c > 0 \;:\; \int_0^T \psi_p(|f(t)|/c)dt \leq 1\Big\},$$

and $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)|$. The notation $a \lesssim b$ means there exists a fixed constant $C > 0$ such that $a \le Cb$.

Theorem 4.1.4. *Suppose* $0 < T < \int_0^l \sqrt{m'(y)}\,dy$ *and assume that (4.1.5) holds for* $\alpha > -1$. *Then we have, for* n *large enough,*

(i)
$$\sup_{t\in[0,T]} \mathbb{E}\,|X_t - X_t^n| \lesssim n^{-\frac{1+\alpha}{2}},$$

(ii) for $p \ge 1$,
$$\mathbb{E}\|X - X^n\|_p \lesssim n^{-\frac{1+\alpha}{2}},$$

(iii)
$$\mathbb{E}\|X - X^n\|_\infty \lesssim n^{-\frac{1+\alpha}{2}} \sqrt{\log n},$$

(iv) for $p \ge 2$,
$$\mathbb{E}\|X - X^n\|_{\psi_p} \lesssim n^{-\frac{1+\alpha}{2}} \left(\sqrt{\log n}\right)^{1-2/p}.$$

Below we will determine the exponent α and hence the convergence rate in several concrete cases. We typically see that α is connected to the "smoothness" of the process X. The smoother the sample paths of a process, the larger α can typically be chosen, and hence the better the convergence rate.

Proof. *Proof of (i).* Recall from the proof of Theorem 4.1.3 that the constant c_k in the series expansion (4.1.4) can be written as
$$c_k = \frac{1}{\sqrt{S_T(\omega_k,\omega_k)}}.$$
By the independence of the variables in the series (4.1.4) we have
$$\mathbb{E}\left|\sum_{|k|>n} \frac{\hat{1}_{(0,t]}(\omega_k)}{\sqrt{S_T(\omega_k,\omega_k)}} Z_k\right|^2 \lesssim \sum_{|k|>n} \frac{1}{\omega_k^2 S_T(\omega_k,\omega_k)} \lesssim \sum_{k>n} \frac{1}{\omega_k^2 S_T(\omega_k,\omega_k)},$$
the last inequality being justified by the fact that $\omega_{-k} = -\omega_k$. Now, using condition (4.1.5) and Lemma 2.8.7, the series on the right is, up to a constant, bounded by
$$\sum_{k>n} \frac{1}{\omega_k^2 \omega_k^\alpha} \sim \sum_{k>n} \frac{1}{k^{2+\alpha}} \sim \int_n^\infty x^{-(2+\alpha)}dx = \frac{1}{1+\alpha} n^{-(1+\alpha)}.$$

This completes the proof of part (i).

Proof of (iii). We argue as in [21].
Fix a positive integer M and consider

$$S_M(t) = \sum_{2^{M-1}<k\le 2^M} \frac{\hat{1}_{(0,t]}(\omega_k)}{\sqrt{S_T(\omega_k,\omega_k)}} Z_k. \tag{4.1.6}$$

For a given $\varepsilon > 0$ to be specified later, cover interval $[0,T]$ with $N \lesssim \varepsilon^{-1}$ intervals $B_i = (t^{(i)} - \varepsilon/2, t^{(i)} + \varepsilon/2)$, $i = 1, ..., N$. Then we have the inequality

$$\mathbb{E} \sup_{t\in[0,T]} |S_M(t)| \le \mathbb{E} \sup_{i=1,\dots,N} |S_M(t^{(i)})| + \mathbb{E} \sup_{i=1,\dots,N} \sup_{t,s\in B_i} |S_M(t) - S_M(s)|\,. \tag{4.1.7}$$

We will start by considering the first term on the right-hand side. By virtue of a standard maximal inequality for Gaussian sequences (cf. [83], Lemma 2.2.2)

$$\mathbb{E} \sup_{i=1,\dots,N} |S_M(t^{(i)})| \lesssim \sqrt{1+\log N} \sup_{i=1,\dots,N} \sqrt{\mathbb{E}|S_M(t^{(i)})|^2}.$$

Using the independence of the sequence Z_k we have

$$\mathbb{E}|S_M(t^{(i)})|^2 = \sum_{2^{M-1}<k\le 2^M} \frac{|\hat{1}_{(0,t]}(\omega_k)|^2}{S_T(\omega_k,\omega_k)} \lesssim \sum_{2^{M-1}<k\le 2^M} \frac{1}{\omega_k^2 S_T(\omega_k,\omega_k)}.$$

Applying condition (4.1.5), Lemma 2.8.7 and bounding the number of terms by 2^{M-1} yields

$$\sum_{2^{M-1}<k\le 2^M} \frac{1}{\omega_k^2 S_T(\omega_k,\omega_k)} \lesssim \sum_{2^{M-1}<k\le 2^M} \frac{1}{\omega_k^{2+\alpha}} \lesssim \sum_{2^{M-1}<k\le 2^M} \frac{1}{k^{2+\alpha}} \le 2^{-(M-1)(1+\alpha)}.$$

Hence,

$$\mathbb{E} \sup_{i=1,\dots,N} |S_M(t^{(i)})| \lesssim 2^{-M(1+\alpha)/2}\sqrt{1+\log N}. \tag{4.1.8}$$

Now we move on to the second term on the right of (4.1.7). It is bounded by

$$\mathbb{E} \sup_{i=1,\dots,N} \sup_{t,s\in B_i} \sum_{2^{M-1}<k\le 2^M} |Z_k| \frac{\left|\hat{1}_{(0,t]}(\omega_k) - \hat{1}_{(0,s]}(\omega_k)\right|}{\sqrt{S_T(\omega_k,\omega_k)}}.$$

Note that

$$\left|\hat{1}_{(0,t]}(\omega_k) - \hat{1}_{(0,s]}(\omega_k)\right| = \left|\int_s^t e^{i\omega_k u} du\right| \le |t-s| \le \varepsilon \tag{4.1.9}$$

for $s, t \in B_i$. Since $\sqrt{S_T(\omega_k, \omega_k)}$ is bounded below by a multiple of $k^{\alpha/2}$ we end up with

$$\mathbb{E} \sup_{i=1,\dots,N} \sup_{t,s\in B_i} |S_M(t) - S_M(s)| \lesssim \varepsilon \sum_{2^{M-1}<k\le 2^M} \frac{1}{k^{\alpha/2}} \le \varepsilon 2^{-(M-1)(\alpha/2-1)}. \tag{4.1.10}$$

Combining the latter with (4.1.8) we obtain the bound

$$\mathbb{E} \sup_{t\in[0,T]} |S_M(t)| \lesssim 2^{-M(1+\alpha)/2}\sqrt{1+\log N} + \varepsilon 2^{-M(\alpha/2-1)}.$$

Recall that $N \lesssim 1/\varepsilon$. Taking $\varepsilon = 2^{-2M}$ makes the second term on the right of lower order than the first one. Hence, we obtain

$$\mathbb{E} \sup_{t\in[0,T]} |S_M(t)| \lesssim \sqrt{M} 2^{-M(1+\alpha)/2}. \tag{4.1.11}$$

To complete the proof we fix a positive integer n. Take such M that $2^{M-1} < n \le 2^M$. Decompose the sum of interest to obtain

$$\left|\sum_{k>n} \frac{\hat{1}_{(0,t]}(\omega_k)}{\sqrt{S_T(\omega_k,\omega_k)}} Z_k\right| \le \left|\sum_{2^M\ge k>n} \frac{\hat{1}_{(0,t]}(\omega_k)}{\sqrt{S_T(\omega_k,\omega_k)}} Z_k\right| + \sum_{j>M} |S_j(t)|. \tag{4.1.12}$$

To bound the second term we use $\sum_{k\ge m} k^p a^{-k} \lesssim m^p a^{-m}$ in combination with (4.1.11) to get

$$\begin{aligned}\sum_{j>M} \mathbb{E} \sup_{t\in[0,T]} |S_j(t)| &\lesssim \sum_{j>M} \sqrt{j} 2^{-j(1+\alpha)/2} \\ &\lesssim \sqrt{(M+1)} 2^{-(M+1)(1+\alpha)/2} \lesssim n^{-(1+\alpha)/2}\sqrt{\log n},\end{aligned}$$

since $2^{M+1} < 4n$. Note that we can obtain the same bound as (4.1.11) if in the sum (4.1.6) we replace 2^{M-1} by some $l \ge 2^{M-1}$. Hence for the first term on the right of (4.1.12) we have

$$\mathbb{E} \sup_{t\in[0,T]} \left|\sum_{2^M\ge k>n} \frac{\hat{1}_{(0,t]}(\omega_k)}{\sqrt{S_T(\omega_k,\omega_k)}} \xi_k\right| \lesssim \sqrt{M} 2^{-M(1+\alpha)/2} \lesssim n^{-(1+\alpha)/2}\sqrt{\log n}.$$

This completes the proof of (iii).

Proof of (iv). The argument is similar to the one in the proof of Proposition 4 of [13].

First fix $t \in [0, T]$ and note that the random variable

$$f(t) = X_t - X_t^n = \sum_{|k|>n} \frac{\hat{1}_{(0,t]}(\omega_k)}{\sqrt{S_T(\omega_k, \omega_k)}} Z_k$$

has the same distribution as $\sigma(t)Z$, where Z is standard normal random variable and

$$\sigma^2(t) = \sum_{|k|>n} \frac{\left|\hat{1}_{(0,t]}(\omega_k)\right|^2}{S_T(\omega_k, \omega_k)}.$$

Moreover, we have the bound

$$\sigma(t) \leq \sigma := C \sqrt{\sum_{k>n} \frac{1}{\omega_k^2 S_T(\omega_k, \omega_k)}}$$

for some constant $C > 0$.

For every $\theta \geq 1$ we have

$$\begin{aligned} \mathbb{E}\, \psi_2\left(\frac{|f(t)|}{2\sigma\theta}\right) &= \mathbb{E} \exp\left(\frac{\xi^2 \sigma^2(t)}{4\sigma^2\theta^2}\right) - 1 \leq \mathbb{E} \exp\left(\frac{\xi^2}{4\theta^2}\right) - 1 \\ &= \left(1 - \frac{1}{2\theta^2}\right)^{-1/2} - 1 = \sqrt{\frac{2\theta^2}{2\theta^2 - 1}} - 1 \end{aligned}$$

Observe that

$$\frac{2\theta^2}{2\theta^2 - 1} \leq \frac{2\theta^2 + \theta^4 + 1}{\theta^4} = \left(1 + \frac{1}{\theta^2}\right)^2,$$

the inequality being justified by $0 \leq 3\theta^4 - 1 = (2\theta^2 - 1)(2\theta^2 + \theta^4 + 1) - 2\theta^6$. It follows that $\mathbb{E}\, \psi_2(|f(t)|/(2\sigma\theta)) \leq \theta^{-2}$, so that

$$\mathbb{E} \int_0^T \psi_2\left(\frac{|f(t)|}{2\sigma\theta}\right) dt \leq \frac{T}{\theta^2}.$$

According to the definition of the Orlicz norm, if $\|f\|_{\psi_2} > 2\sigma\theta$ then

$$\int_0^T \psi_2\left(\frac{|f(t)|}{2\sigma\theta}\right) dt > 1.$$

Hence, due to Markov's inequality we have, for $\theta \geq 1$,

$$\mathbb{P}(\|f\|_{\psi_2} > 2\sigma\theta) \leq \mathbb{P}\left(\int_0^T \psi_2\left(\frac{|f(t)|}{2\sigma\theta}\right) dt > 1\right) \leq \mathbb{E}\int_0^T \psi_2\left(\frac{|f(t)|}{2\sigma\theta}\right) dt \leq \frac{T}{\theta^2}.$$

It follows that

$$\begin{aligned}\mathbb{E}\|f\|_{\psi_2} &= \int_0^\infty \mathbb{P}(\|f\|_{\psi_2} > x)dx = 2\sigma\int_0^\infty \mathbb{P}(\|f\|_{\psi_2} > 2\sigma\theta)d\theta \\ &\leq 2\sigma\left(1 + T\int_1^\infty \frac{d\theta}{\theta^2}\right) = (2+2T)\sigma.\end{aligned}$$

We can bound σ as in the proof of part (i), obtaining

$$\mathbb{E}\|X - X^n\|_{\psi_2} \lesssim n^{-\frac{1+\alpha}{2}}. \tag{4.1.13}$$

For the case $2 < p < \infty$ we use the fact that with $1/p = \beta/2$,

$$\|X - X^n\|_{\psi_p} \leq \left(\sup_{t\in[0,T]} |X - X^n|\right)^{1-\beta} \|X - X^n\|_{\psi_2}^{\beta}.$$

Using Hölder's inequality, the result of part (iii) and (4.1.13) we get

$$\begin{aligned}\mathbb{E}\|X - X^n\|_{\psi_p} &\leq \mathbb{E}\left[\left(\sup_{t\in[0,T]} |X_t - X_t^n|\right)^{1-\beta} \|f\|_{\psi_2}^{\beta}\right] \\ &\leq \left(\mathbb{E}\sup_{t\in[0,T]} |X_t - X_t^n|\right)^{1-\beta} \left(\mathbb{E}\|X - X^n\|_{\psi_2}\right)^{\beta} \\ &\lesssim n^{-(1+\alpha)/2}\left(\sqrt{\log n}\right)^{1-2/p}\end{aligned}$$

Proof of (ii). This statement follows directly from the inequality (cf. [83], section 2.2) $\|f\|_p \leq \sqrt{\log 2}\, p!\, \|f\|_{\psi_2}$ for $p \geq 1$. □

4.1.1.3 Examples

Example 4.1.5. *Fractional Brownian motion*

By virtue of the results from Section 3.3.1, Theorem 4.1.3 implies that for $T > 0$ we have the explicit series expansion

$$X_t = \sum_{k\in\mathbb{Z}} c_k \frac{e^{it2\omega_k/T} - 1}{i2\omega_k/T} Z_k, \qquad t \in [0, T], \tag{4.1.14}$$

where the ω_k are the zeros of J_{1-H}, the Z_k are independent standard Gaussians, and

$$c_k^2 = \frac{4H\Gamma(1/2 + H)}{\sqrt{\pi}\Gamma(1 - H)T^{2-2H}\omega_k^2 J_{-H}^2(\omega_k)}.$$

This result was first obtained using different methods in [20].

According to Theorem 4.1.1 and formulas (3.3.2) and (3.3.3), the reproducing kernel S_T in this case is of the form

$$\begin{aligned} &\frac{S_T(\omega, \lambda)\, x'(\frac{T}{2})}{(2 - 2H)\Gamma^2(1 - H)T} \qquad (4.1.15) \\ &= e^{i\frac{(\lambda-\omega)T}{2}} \left(\frac{\omega\lambda T^2}{16}\right)^H \frac{J_{-H}\left(\frac{\omega T}{2}\right) J_{1-H}\left(\frac{\lambda T}{2}\right) - J_{1-H}\left(\frac{\omega T}{2}\right) J_{-H}\left(\frac{\lambda T}{2}\right)}{\pi(\lambda - \omega)T}. \end{aligned}$$

where

$$x(T) = \frac{T^{2H}\Gamma(1/2 + H)}{2\sqrt{\pi}\Gamma(H)\sin \pi H}.$$

If we let $\lambda \to \omega$ in the preceding expression for the reproducing kernel S_T and use the recurrence formulae for the Bessel function J_ν (cf. Appendix A) we see that we have

$$\begin{aligned} &\frac{S_T(\omega, \omega)\, \pi\, x'(\frac{T}{2})}{(1 - H)\Gamma^2(1 - H)T} \\ &= \left(\frac{\omega T}{4}\right)^{2H} \frac{2}{T} \left(J_{-H}\left(\frac{\omega T}{2}\right) \frac{d}{d\omega} J_{1-H}\left(\frac{\omega T}{2}\right) - J_{1-H}\left(\frac{\omega T}{2}\right) \frac{d}{d\omega} J_{-H}\left(\frac{\omega T}{2}\right)\right) \\ &= \left(\frac{\omega T}{4}\right)^{2H} \left(J_{1-H}^2\left(\frac{\omega T}{2}\right) + \frac{4H - 2}{\omega T} J_{-H}\left(\frac{\omega T}{2}\right) J_{1-H}\left(\frac{\omega T}{2}\right) + J_{-H}^2\left(\frac{\omega T}{2}\right)\right) \end{aligned}$$

on the diagonal. Using well-known asymptotic properties of the Bessel function one finds that for large $|\lambda|$, $S_T(\lambda,\lambda)$ behaves as a multiple of $|\lambda|^{2H-1}$ in this case (cf. [95], Lemma 6.1), i.e. condition (4.1.5) is satisfied with $\alpha = 2H-1$.

Theorem 4.1.4 then implies the following rates of convergence of the series expansion:

norm	rate
pointwise	n^{-H}
L^p	n^{-H}
supremum	$n^{-H}\sqrt{\log n}$
Orlicz	$n^{-H}\left(\sqrt{\log n}\right)^{1-2/p}$

According to the results of Kühn and Linde [39] the pointwise and supremum norm rates that we obtain are in fact optimal. We will show below that L^2 rate is optimal as well. For the L^p-norm with $p \neq 2$ and the Orlicz norm, the optimal rate seems to be unknown.

Optimality of the rate of convergence in L^2. Before starting the actual proof we introduce some additional notation. Given a linear bounded operator $\mathbb{T}$ from a separable Hilbert space $\mathbf{H}$ into a Banach space $(E, \|\cdot\|)$ we define the l-numbers of the operator by

$$l_n(\mathbb{T}) := \inf\left\{\left(\mathbb{E}\left\|\sum_{k>n}\xi_k\mathbb{T}e_k\right\|^2\right)^{1/2} : \{e_k\}_{k=1}^{\infty} \text{ orthonormal basis of } \mathbf{H}\right\},$$

$n \in \mathbb{N}$, where $(\xi_k)_{k\geq 1}$ is a sequence of standard normal random variables and the infimum is taken over all orthonormal bases of $\mathbf{H}$. If $\mathbb{T}$ maps $\mathbf{H}$ into itself, the $l_n(\mathbb{T})$ can be computed according to

$$l_n(\mathbb{T}) = \left(\sum_{k>n}\mu_k^2\right)^{1/2} \tag{4.1.16}$$

where the numbers $\mu_1 \geq \mu_2 \geq ...$ are the eigenvalues of the operator $\mathbb{T}\mathbb{T}^*$.

Now, we will mimic the proof of the optimality of the rate in sup-norm case that can be found in Kühn and Linde ([39]). The crucial fact is that the Riemann-Liouville operator

$$(R_{H+1/2}f)(t) := \int_0^t (t-s)^{H-1/2}f(s)ds, \qquad t < T \tag{4.1.17}$$

regarded as an operator from $L^2([0,T])$ into itself has the l-numbers of the form

$$l_n(R_{H+1/2}) \approx \left(\sum_{k>n} k^{-2H-1}\right)^{1/2} \approx n^{-H}. \tag{4.1.18}$$

It is a consequence of (4.15) of [39] and formula (4.1.16). Now let us define the operator

$$T_H := R_{H+1/2} \oplus Q_{2H} \tag{4.1.19}$$

from the Hilbert space $\mathbf{H} := L^2([0,T]) \oplus L^2((-\infty,0])$ to $L^2([0,T])$, where

$$(Q_{2H}f)(t) := \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{0} \left((t-s)^{H-1/2} - (-s)^{H-1/2}\right) f(s)ds.$$

Consider a projection

$$P := \mathbf{H} \to L^2([0,T]), \quad Pf = f1_{[0,T]}.$$

According to this definition we have

$$T_H \circ P = R_{H+1/2}$$

(formally P is considered as an operator on $\mathbf{H}$ and into $\mathbf{H}$ with $P(f \oplus g) = f \oplus 0$ and $R_{H+1/2}$ as an operator on H with $R_{H+1/2}(f \oplus g) = R_H(f)$). Using a property of l-numbers (see [39], p.672) we obtain

$$l_n(R_{H+1/2}) = l_n(T_H \circ P) \leq l_n(T_H) a_1(P) \tag{4.1.20}$$

where $a_1(P)$ is a first approximation number of the projection operator. The only thing that we need to know about it is that it is finite and does not depend on n. So, by virtue of (4.1.18), we have that

$$n^{-H} \lesssim l_n(T_H). \tag{4.1.21}$$

It turns out that the operator T_H generates a multiple of fBm X (cf. [55]), in a sense that

$$X_t = c\sum_{j=1}^{\infty} \xi_j (T_H e_j)(t), \qquad t \in [0,T],\ c > 0, \tag{4.1.22}$$

for any orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of the space $L^2([-\infty, T])$ and the convergence taking place almost surely in $L^2([0,T])$.

As is explained in Chapter 5 (Lemma 5.1.1) the combination of (4.1.22) and (4.1.21) ensures the optimality of the rate n^{-H} of convergence in L^2 for the series expansion (4.1.14) of fractional Brownian motion. □

Example 4.1.6. *Brownian motion*

The results for standard Brownian motion can be easily obtained from the above by considering $H = 1/2$. Let us just remark that since $\sqrt{z}J_{1/2}(z) = \sqrt{2/\pi}\sin z$, the series representation (4.1.14) reduces to

$$\sum_{k\in\mathbb{N}} \frac{e^{2ik\pi t}-1}{2ik\pi} Z_k \tag{4.1.23}$$

with the Z_k independent, standard Gaussian variables. This is the expression that Paley and Wiener [71] used as a definition of the standard Brownian motion.

4.1.2 Series representations of stationary processes

4.1.2.1 Series expansion

The methodology applied to stationary increments processes in the preceding section can be easily adapted to stationary processes as well. This is basically due to the fact that the linear spaces $\mathscr{L}_T$, defined by (4.1.1) and (4.1.2), coincide.

So, let now Y be a stationary Gaussian process with spectral measure μ. Let lmk be the associated string and let the functions A, B, m' and x associated with the string be defined as in Chapter 2. Straightforward adaptation of Theorem 4.1.3 gives the following result.

Theorem 4.1.7. *Suppose $0 < T < \int_0^l \sqrt{m'(y)}\,dy$. Let $\cdots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \cdots$ be the real-valued zeros of the function $\omega \mapsto B(x(T/2), \omega)$. We have the representation*

$$Y_t = \sum_{k\in\mathbb{Z}} c_k e^{i\omega_k t} Z_k, \qquad t \in [0,T], \tag{4.1.24}$$

where

$$c_k = \Big(\frac{1}{\pi} A(x(T/2), \omega_k)\dot{B}(x(T/2), \omega_k)\Big)^{-1/2}$$

and the Z_k are independent standard Gaussian random variables. The series converges in mean-square sense for every fixed $t \in [0,T]$. If the process Y admits a continuous version then with probability 1, the series converges uniformly in $t \in [0,T]$.

4.1.2.2 Rates of convergence

For Y as in (4.1.24), define the partial sum process by

$$Y_t^n = \sum_{|k| \leq n} c_k e^{i\omega_k t} Z_k, \qquad t \in [0,T].$$

The rate of convergence result takes the following form in the stationary case. Note the different exponents.

Theorem 4.1.8. *Suppose $0 < T < \int_0^l \sqrt{m'(y)}\, dy$ and assume that condition (4.1.5) holds for $\alpha > 1$. Then we have, for n large enough,*

(i)

$$\sup_{t \in [0,T]} \mathbb{E}\, |Y_t - Y_t^n| \lesssim n^{-\frac{\alpha-1}{2}},$$

(ii) for $p \geq 1$,

$$\mathbb{E}\|Y - Y^n\|_p \lesssim n^{-\frac{\alpha-1}{2}},$$

(iii)

$$\mathbb{E}\|Y - Y^n\|_\infty \lesssim n^{-\frac{\alpha-1}{2}} \sqrt{\log n},$$

(iv) for $p \geq 2$,

$$\mathbb{E}\|Y - Y^n\|_{\psi_p} \lesssim n^{-\frac{\alpha-1}{2}} \left(\sqrt{\log n}\right)^{1-2/p}.$$

Proof. This theorem can be proved by straightforward adaptation of the arguments used in the proof of Theorem 4.1.4. □

4.1.2.3 Examples

Example 4.1.9. *Ornstein-Uhlenbeck process*

In view of Example 3.2.4, Theorem 4.1.7 implies that the Ornstein-Uhlenbeck process $(Y_t)_{t\in[0,T]}$ can be represented as a series

$$Y_t = \sum_{k\in\mathbb{Z}} c_k e^{i\omega_k t} Z_k, \qquad t \in [0,T],$$

where the ω_k are the real roots of the equation $\tan((\omega T)/2) = -\omega$, the Z_k are independent standard Gaussian variables and

$$c_k^2 = \frac{2\pi}{2+T+T\omega_k^2}. \tag{4.1.25}$$

This expansion is related but not identical to the Karhunen-Loève expansion of the Ornstein-Uhlenbeck process (cf. [45]).

Given that $x'(T) = 1$ the reproducing kernel can be computed explicitly and takes the form

$$\begin{aligned} S_T(\omega,\lambda)e^{-i(\lambda-\omega)T/2} &= \frac{(\lambda-\omega)(\cos\frac{\omega T}{2}\cos\frac{\lambda T}{2} + \sin\frac{\omega T}{2}\sin\frac{\lambda T}{2})}{\pi(\lambda-\omega)} \\ &+ \frac{(1+\omega\lambda)(\cos\frac{\omega T}{2}\sin\frac{\lambda T}{2} - \sin\frac{\omega T}{2}\cos\frac{\lambda T}{2})}{\pi(\lambda-\omega)} \\ &= \frac{(\lambda-\omega)\cos\frac{(\omega-\lambda)T}{2} + (1+\omega\lambda)\sin\frac{(\lambda-\omega)T}{2}}{\pi(\lambda-\omega)}. \end{aligned}$$

Let $\lambda \to \omega$. Using de l'Hospital rule we obtain that on the diagonal

$$S_T(\omega,\omega) = \frac{1+(1+\omega^2)T/2}{\pi} \gtrsim \omega^2. \tag{4.1.26}$$

Observe that resemblance of (4.1.25) is not accidental since $S_T(\omega_k,\omega_k) = c_k^{-2}$ (cf. Lemma 2.8.5). Hence, condition (4.1.5) is satisfied with $\alpha = 2$. By Theorem 4.1.8, we have the following convergence rates for the series:

norm	rate
pointwise	$n^{-1/2}$
L^p	$n^{-1/2}$
supremum	$n^{-1/2}\sqrt{\log n}$
Orlicz	$n^{-1/2}\left(\sqrt{\log n}\right)^{1-2/p}$

We will show below that the rate $n^{-1/2}$ is in fact optimal for the L^2-norm. For other norms no optimality results seem to be known, but in view of results for Brownian motion it seems to be plausible that the rate $n^{-1/2}\sqrt{\log n}$ for the supremum norm is optimal as well.

Optimality of the series rate of convergence in L^2. To prove the optimality of the rate $n^{-1/2}$ of convergence in L^2 we will use the same method as in the fBm case (Example 4.1.5).

Consider the well-known Karhunen-Loève expansion of the Ornstein-Uhlenbeck process (cf. [45], Theorem 5)

$$X_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\phi_k(t)\xi_k \tag{4.1.27}$$

where $\{\xi_k\}$ are independent standard normal variables, the eigenvalues λ_k behave as

$$\lambda_k \approx k^{-2} \tag{4.1.28}$$

for big k's and the family ϕ_k is a orthonormal basis of $L^2([0,T])$. We define the operator $\mathbb{T} : L^2([0,T]) \to L^2([0,T])$ on the basis $\{\phi_k\}$ by setting

$$\mathbb{T}\phi_k := \sqrt{\lambda_k}\phi_k, \quad k = 1, 2, ... \tag{4.1.29}$$

Since such operator is self adjoint, the set of eigenvalues of $\mathbb{T}\mathbb{T}^*$ is $\{\lambda_k,\ k = 1, 2, ...\}$, so according to (4.1.16)

$$l_n(\mathbb{T}) = \left(\sum_{k>n} \lambda_k\right)^{1/2} \approx \left(\sum_{k>n} k^{-2}\right)^{1/2} \approx n^{-1/2}. \tag{4.1.30}$$

By virtue of Lemma 5.1.1 this completes the proof.

□

Example 4.1.10. *Matérn processes*

If $Y = (Y_t)_{t\in[0,T]}$ is a process with Matérn-type density

$$f(\lambda) = \frac{1}{(1+\lambda^2)^2},$$

as in Example 1.1.4, Theorem 4.1.7 then yields the series expansion

$$Y_t = \sum c_k e^{i\omega_k t} Z_k,$$

where the Z_k are independent standard Gaussian variables and the ω_k are the real roots of the equation $B(x(T), \lambda) = 0$ where B is of the form (3.2.57). The c_k can be computed explicitly using expressions (3.2.56) and (3.2.57) for $A(x(T), \lambda)$ and $B(x(T), \lambda)$, respectively. By virtue of these formulas and the relation

$$S_T(\lambda, \lambda) = \Big(\frac{1}{\pi} A(x(T/2), \lambda) \dot{B}(x(T/2), \lambda)\Big) \tag{4.1.31}$$

we observe that

$$S_T(\lambda, \lambda) \geq c|\lambda|^4 \tag{4.1.32}$$

for $|\lambda|$ large enough. Hence, by Theorem 4.1.8 we obtain the following rates of convergence for the series:

norm	rate
pointwise	$n^{-3/2}$
L^p	$n^{-3/2}$
supremum	$n^{-3/2}\sqrt{\log n}$
Orlicz	$n^{-3/2}\left(\sqrt{\log n}\right)^{1-2/p}$

No earlier results of this kind are known to us.

In view of the general form of the functions $A(x(T), \lambda)$ and $B(x(T), \lambda)$ (see (3.2.41) and (3.2.42)) for the string of the process with spectral density

$$f(\lambda) = \frac{1}{(1 + \lambda^2)^r},$$

for $r \in \mathbb{N}$, in particular the fact that the functions $p_r(\lambda)$ and $q_r(\lambda)$ are of order $|\lambda|^r$ for large $|\lambda|$ (see Remark 3.2.2 and 3.2.3), we see that condition (4.1.5) is satisfied with $\alpha = 2r$. According to this, the truncation of the series expansion

$$Y_t = \sum c_k e^{i\omega_k t} Z_k,$$

with appropriate c_k, converges according to the rates

norm	rate
pointwise	$n^{-(2r-1)/2}$
L^p	$n^{-(2r-1)/2}$
supremum	$n^{-(2r-1)/2}\sqrt{\log n}$
Orlicz	$n^{-(2r-1)/2}\left(\sqrt{\log n}\right)^{1-2/p}$

4.1.3 Space-domain moving average representation

In this section we will derive a space-domain moving average representation of the Gaussian process. We use the term 'space-domain' to distinguish it from the usual 'time-domain' moving average representation. This section shows what can be derived without any additional assumptions imposed on the distribution function of the string.

Let us start with a process with stationary increments. Let X be a centered Gaussian process with stationary increments, with spectral measure μ and associated string lmk. The representation (1.1.15)

$$\mathbb{E}X_tX_s = \int_{\mathbb{R}} \frac{(e^{i\lambda t}-1)(e^{-i\lambda s}-1)}{\lambda^2}\mu(d\lambda) \tag{4.1.33}$$

can be rewritten as

$$\mathbb{E}X_tX_s = \int_{\mathbb{R}} \frac{\sin\lambda t}{\lambda}\frac{\sin\lambda s}{\lambda}\mu(d\lambda) + \int_{\mathbb{R}} \frac{(\cos\lambda t-1)}{\lambda}\frac{(\cos\lambda s-1)}{\lambda}\mu(d\lambda). \tag{4.1.34}$$

The functions $\sin\lambda t/\lambda$ and $(\cos\lambda s-1)/\lambda$ are even and odd, respectively, as a functions of λ. Hence, we can apply the transforms of Section 2.6 and define

$$f_0(t,y) := \int_{\mathbb{R}} \frac{\sin\lambda t}{\lambda}A(y,\lambda)\mu(d\lambda), \tag{4.1.35}$$

$$f_1(t,y) := \int_{\mathbb{R}} \frac{\cos\lambda t-1}{\lambda}B(y,\lambda)\mu(d\lambda). \tag{4.1.36}$$

We can prove the following

Theorem 4.1.11. *Let $X=(X_t)_{t\in\mathbb{R}}$ be a centered, mean-square continuous Gaussian process with stationary increments. Its spectral measure μ is a principal spectral measure of the string lmk. We have a representation*

$$\mathbb{E}X_tX_s = \pi\int_0^{x(t+)\wedge x(s+)} f_0(t,y)f_0(s,y)dm(y) \tag{4.1.37}$$

$$+ \pi\int_0^{x(t+)\wedge x(s+)} f_1(t,y)f_1(s,y)dy, \tag{4.1.38}$$

where f_0 and f_1 are defined by (4.1.35) and (4.1.36), respectively.

Proof. By virtue of Theorems 2.6.1 and 2.6.2 we have that

$$\int_{\mathbb{R}} \frac{\sin\lambda t}{\lambda}\frac{\sin\lambda s}{\lambda}\mu(d\lambda) = \pi\int_0^l f_0(t,y)f_0(s,y)dm(y),$$
$$\int_{\mathbb{R}} \frac{(\cos\lambda t-1)}{\lambda}\frac{(\cos\lambda s-1)}{\lambda}\mu(d\lambda) = \pi\int_0^{l+k} f_1(t,y)f_1(s,y)dy.$$

Note that both functions $\sin\lambda t/\lambda$ and $(\cos\lambda s - 1)/\lambda$ are of type at most t. Hence, Theorem 2.8.2 implies that their inverse transforms $f_0(t,y)$ and $f_1(t,y)$ are supported on the finite interval $[0, x(t+)]$. This completes the proof. □

Corollary 4.1.12. *Under assumptions of Theorem 4.1.11 process X can be represented as*

$$X_t = \sqrt{\pi}\sum_{l=0,1}\int_0^{x(t+)} f_l(t,y)dM_l(y), \tag{4.1.39}$$

where M_0, M_1 are independent Gaussian processes with independent increments, with mean zero and variance

$$\mathbb{E}|M_l(y)|^2 = \begin{cases} m(y) & l=0, \\ y & l=1. \end{cases}$$

Due to the fact that we can write the covariance representation (1.1.9) of a stationary process Y in the form

$$\mathbb{E}Y_tY_s = \int_{\mathbb{R}} \cos\lambda t\cos\lambda s\,\mu(d\lambda) + \int_{\mathbb{R}} \sin\lambda t\sin\lambda s\,\mu(d\lambda). \tag{4.1.40}$$

we can prove the result similar to Corollary 4.1.12.

Theorem 4.1.13. *Let $Y = (Y_t)_{t\in\mathbb{R}}$ be a centered, mean-square continuous stationary Gaussian process with spectral measure μ satisfying $\int(\lambda^2+1)^{-1}\mu(d\lambda) < \infty$. Let the measure μ be a principal spectral measure of the string lmk. We can represent Y as follows*

$$Y_t = \sqrt{\pi}\sum_{l=0,1}\int_0^{x(t+)} g_l(t,y)dM_l(y), \tag{4.1.41}$$

where

$$g_0(t,y) := \int_{\mathbb{R}} \cos \lambda t A(y,\lambda)\mu(d\lambda), \tag{4.1.42}$$

$$g_1(t,y) := \int_{\mathbb{R}} \sin \lambda t B(y,\lambda)\mu(d\lambda), \tag{4.1.43}$$

and M_0*,* M_1 *are independent Gaussian processes with independent increments, with mean zero and variance*

$$\mathbb{E}|M_l(y)|^2 = \begin{cases} m(y) & l = 0, \\ y & l = 1. \end{cases}$$

Proof. We apply here the same reasoning as in the proof of Theorem 4.1.11. □

Formulas (4.1.39) and (4.1.41) can be viewed as a sort of moving-average representations of a Gaussian process. However, our aim is to obtain a 'time-domain' representation of the form

$$X_t = \int_0^t l_t(s)dM(s),$$

where the integration is with respect to time over an interval $[0,t]$. This, compared to the representations of the present section, can be obtained upon introducing some extra conditions. Namely, an invertibility of the function x which is closely related to smoothness of the mass distribution m. In the following section we present the details and derive a proper moving average representation of fractional Brownian motion.

4.1.4 Representations in the case of a smooth string

In previous sections we did not impose any additional conditions on the string of the processes of interest. Hence, the results presented above can be applied to any process with spectral measure satisfying $\int(1+\lambda^2)^{-1}d\mu < \infty$. In order to obtain some additional results, in this section we will restrict our investigation to the case of smooth mass distribution function m. We will study the structure of the RKHS $\mathscr{L}_T$ by deriving an integral representation of the reproducing kernel and use it to define a Fourier-type transform on $\mathscr{L}_T$.

Throughout this section we suppose that the mass distribution function m is continuously differentiable, its derivative m' is strictly positive and we have the finite propagation speed property

$$\int_0^l \sqrt{m'(y)}\,dy = \infty.$$

Observe that the map $t \mapsto x(t)$ is then differentiable, whence we can define the functions a and b by

$$a(t,\lambda) = A(x(t),\lambda), \quad b(t,\lambda) = B(x(t),\lambda)x'(t). \tag{4.1.44}$$

Next, the functions ϕ and V are defined by

$$\phi(2t,\lambda) = e^{i\lambda t}\Big(a(t,\lambda) + ib(t,\lambda)\Big) \tag{4.1.45}$$

and

$$V(2t) = \frac{1}{\pi}m(x(t)). \tag{4.1.46}$$

Note that the assumptions on m imply that V is smooth, strictly increasing, and $V(0) = 0$. Differentiation of the identity

$$t = \int_0^{x(t)} \sqrt{m'(y)}\,dy$$

shows that

$$2V'(2t) = \frac{d}{dt}V(2t) = \frac{1}{\pi}m'(x(t))x'(t) = \frac{1}{\pi x'(t)}. \tag{4.1.47}$$

Knowing this, we can rewrite the general form of the reproducing kernel of Theorem 4.1.1 as

$$S_T(\omega,\lambda) = 2V'(T)\exp\Big(i\frac{(\lambda-\omega)T}{2}\Big)\frac{a\Big(\frac{T}{2},\omega\Big)b\Big(\frac{T}{2},\lambda\Big) - b\Big(\frac{T}{2},\omega\Big)a\Big(\frac{T}{2},\lambda\Big)}{\lambda-\omega}.$$

The following theorem gives an integral representation for the reproducing kernel of $\mathscr{L}_T$.

Theorem 4.1.14. *The reproducing kernel of $\mathscr{L}_T$ can be written as*

$$S_T(\omega,\lambda) = \int_0^T \phi(t,\lambda)\overline{\phi(t,\omega)}\,dV(t). \tag{4.1.48}$$

Proof. Note that (4.1.3) implies that we have

$$\frac{d}{dt}S_{2t}(\omega,\lambda) = e^{i(\lambda-\omega)t}\Big(\frac{d}{dt}K_t(\omega,\lambda) + i(\lambda-\omega)K_t(\omega,\lambda)\Big).$$

By Theorem 2.8.4, it follows from straightforward calculation that

$$\pi\frac{d}{dt}S_{2t}(\omega,\lambda) = \phi(t,\lambda)\overline{\phi(t,\omega)}\frac{d}{dt}m(x(t)).$$

Integration of this identity completes the proof. □

Observe that since $\phi(0,\lambda)=1$, it follows from (4.1.48) that

$$S_t(0,\lambda) = \int_0^t \phi(u,\lambda)\,dV(u). \tag{4.1.49}$$

In particular we have $S_t(0,0) = V_t$.

Corollary 4.1.15. *The subspaces* $\mathrm{sp}\{S_T(\omega,\cdot) : \omega\in\mathbb{R}\}$ *and* $\mathrm{sp}\{S_t(0,\cdot) : t\in[0,T]\}$ *are dense in* $\mathscr{L}_T$.

Proof. Suppose that $\psi\in\mathscr{L}_T$ is orthogonal to all functions $S_T(\omega,\cdot)$. Then for every $\omega\in\mathbb{R}$

$$\psi(\omega) = \int \psi(\lambda)\overline{S_T(\omega,\lambda)}\,\mu(d\lambda) = 0,$$

hence ψ vanishes. This shows that the first subspace is dense. To prove that the second space is dense it now suffices to show that for every $\omega\in\mathbb{R}$, the function $S_T(\omega,\cdot)$ belongs to $\overline{\mathrm{sp}}\{S_t(0,\cdot) : t\in[0,T]\}$. It follows from (4.1.49) that

$$S_T(\omega,\lambda) = \int_0^T \phi(t,\lambda)\overline{\phi(t,\omega)}\,dV(t) = \int_0^T \overline{\phi(t,\omega)}\,dS_t(0,\lambda).$$

This completes the proof. □

Using the result of the preceding theorem we can now introduce a Fourier-type transform between the spaces $L^2([0,T],V)$ and $\mathscr{L}_T$, the function $\phi(t,\lambda)$ acting as the Fourier kernel. By putting $\omega=\lambda$ in (4.1.48) we see that $\phi(\cdot,\lambda)\in L^2([0,T],V)$, so for $f\in L^2([0,T],dV)$ the function

$$\mathcal{U}f(\lambda) = \int_0^T f(t)\varphi(t,\lambda)dV(t)$$

is well defined, by Cauchy-Schwarz. We have the following result regarding the linear operator $\mathcal{U}$.

Theorem 4.1.16. *We have that* $\mathcal{U} : L^2([0,T], V) \to \mathscr{L}_T$ *is an isometry and*

$$\mathcal{U}^{-1}\psi(t) = \frac{d}{dV(t)} \int \psi(\lambda)\Big(\int_0^t \overline{\phi(u,\lambda)}\, dV(u)\Big)\, \mu(d\lambda) \tag{4.1.50}$$

for $\psi \in \mathscr{L}_T$.

Proof. By (4.1.48) the operator $\mathcal{U}$ maps the indicator function $1_{(0,t]}$ to $S_t(0,\cdot) \in \mathscr{L}_t \subseteq \mathscr{L}_T$. Hence, by linearity, the simple functions in $L^2([0,T], V)$ are mapped to $\text{sp}\{S_t(0,\cdot) : t \in [0,T]\} \subseteq \mathscr{L}_T$. Moreover, using the preceding theorem it is straightforward to verify that $\mathcal{U}$ is an isometry between these two subspaces. Since the simple functions are dense in $L^2([0,T], V)$ and $\text{sp}\{S_t(0,\cdot) : t \in [0,T]\}$ is dense in $\mathscr{L}_T$, we see that $\mathcal{U}$ is indeed an isometry between $L^2([0,T], V)$ and $\mathscr{L}_T$.

Since $\mathcal{U}$ is an isometry, its inverse $\mathcal{U}^{-1}$ coincides with its adjoint $\mathcal{U}^*$. For the adjoint we have

$$\begin{aligned}\int_0^t \mathcal{U}^*\psi(u)\, dV(u) &= \big\langle \mathcal{U}^*\psi, 1_{(0,t]}\big\rangle_V \\ &= \big\langle \psi, \mathcal{U}1_{(0,t]}\big\rangle_\mu \\ &= \int \psi(\lambda)\overline{S_t(0,\lambda)}\, \mu(d\lambda)\end{aligned}$$

for every $t \geq 0$. By differentiating with respect to t and using (4.1.49) we obtain the formula for $\mathcal{U}^{-1}$. □

Observe that if the function $\psi \in \mathscr{L}_T$ is such that the order of the differentiation and integration operations in (4.1.50) may be reversed, the inversion formula reduces to

$$\mathcal{U}^{-1}\psi(t) = \int \psi(\lambda)\overline{\phi(t,\lambda)}\, \mu(d\lambda). \tag{4.1.51}$$

In general however this integral on the right-hand side does not converge as a Lebesgue integral, but only in an L^2-sense.

4.1.5 Representations of fractional Brownian motion

As we have shown in Section 3.3.1, the mass function of the string associated to the fractional Brownian motion is smooth with

$$m'(x) = \frac{\kappa_H^{1/H}}{4H^2} x^{\frac{1-2H}{H}}. \tag{4.1.52}$$

So, we can apply to this example the results of Section 4.1.4. We obtain, in particular, the so-called (finite past) moving average representation of the fractional Brownian motion.

Observe that (4.1.52) implies in particular that we have the finite propagation speed property

$$\int_0^l \sqrt{m'(y)}\, dy = \infty,$$

so all conditions posed in Section 4.1.4 are satisfied.

Before we proceed further we need to define a notion of stochastic integral with respect to fBm. So, let $X = (X_t)_{t\geq 0}$ be a fractional Brownian motion with Hurst index $H \in (0,1)$. The spectral representation (see Section 1.1.1) can be used to define a stochastic integral with respect to X of a large class of deterministic integrands. Consider the class of functions $\mathcal{I}_T = \{f \in L^2[0,T] : \hat{f} \in L^2(\mu_H)\}$ and endow it with the inner product $\langle f, g\rangle_{\mathcal{I}_T} = \langle \hat{f}, \hat{g}\rangle_{\mu_H}$, where $\hat{f}$ denotes the ordinary Fourier transform. Then the spectral representation can be written as $\mathbb{E}X_sX_t = \langle 1_{(0,s]}, 1_{(0,t]}\rangle_{\mathcal{I}_T}$. In particular, the mapping $1_{(0,t]} \to X_t$ extends to a linear map $I : \mathcal{I}_T \to \mathscr{H}_T$ with the property that $I(1_{(0,t]}) = X_t$ and for $f, g \in \mathcal{I}_T$,

$$\mathbb{E}I(f)\overline{I(g)} = \langle \hat{f}, \hat{g}\rangle_{\mu_H}.$$

We denote the random variable $I(f)$ by $\int f\, dX$ or $\int_0^T f(t)\, dX_t$, and call it the Wiener integral of f with respect to X. (We note that in general not every element of $\mathscr{H}_T$ can be represented as such an integral, since for $H > 1/2$ the space $\mathcal{I}_T$ is not complete, see [73].)

To compute the functions a, b, ϕ and V defined by (4.1.44), (4.1.45) and (4.1.46) in the present case, recall from Section 3.3.1 that $x(T) = T^{2H}/\kappa_H$. Hence, by virtue of (3.3.2) and (3.3.2) we obtain

$$a(t,\lambda) = \Gamma(1-H)\Big(\frac{\lambda t}{2}\Big)^H J_{-H}(\lambda t),$$

$$b(t,\lambda) = \Gamma(1-H)\Big(\frac{\lambda t}{2}\Big)^H J_{1-H}(\lambda t),$$
$$\phi(t,\lambda) = \Gamma(1-H)\Big(\frac{\lambda t}{4}\Big)^H e^{i\frac{\lambda t}{2}}\Big(J_{-H}\Big(\frac{\lambda t}{2}\Big) + iJ_{1-H}\Big(\frac{\lambda t}{2}\Big)\Big),$$
$$V(t) = \frac{\sqrt{\pi}}{2^{3-2H}H\Gamma(2-H)\Gamma(1/2+H)}t^{2-2H}.$$

The theorem below is now immediate consequence of the results for general smooth strings obtained in Section 4.1.4.

The isometry $\mathcal{U}$ takes the following form in the fBm case.

Theorem 4.1.17. *We have an isometry* $\mathcal{U} : L^2([0,T],V) \to \mathscr{L}_T$*, given by*

$$\mathcal{U}f(\lambda) = \Gamma(1-H)\int_0^T f(t)\Big(\frac{\lambda t}{4}\Big)^H e^{i\frac{\lambda t}{2}}\Big(J_{-H}\Big(\frac{\lambda t}{2}\Big) + iJ_{1-H}\Big(\frac{\lambda t}{2}\Big)\Big)\,dV(t).$$

Proof. Follows from Theorem 4.1.16. □

Observe that for $H = 1/2$ we have $\phi(t,\lambda) = \exp(i\lambda t)$, $V(t) = t$ and $\Gamma(1/2) = \sqrt{\pi}$, so $\mathcal{U}$ is simply the ordinary Fourier transform in this case.

4.1.5.1 Fundamental martingale

For every $t \geq 0$, define the random variable $M_t \in \mathscr{H}_t$ as the image under the spectral isometry of the function $S_t(0,\cdot) \in \mathscr{L}_t$. Then in particular, M is adapted to the filtration generated by the fBm X. From the reproducing property of S_t it immediately follows that M has uncorrelated increments, i.e. it is a martingale. Indeed, for $s \leq t \leq u$ we have

$$\begin{aligned}\mathbb{E}X_s(M_u - M_t) &= \big\langle \hat{1}_{(0,s]}, S_u(0,\cdot)\big\rangle_{\mu_H} - \big\langle \hat{1}_{(0,s]}, S_t(0,\cdot)\big\rangle_{\mu_H} \\ &= \hat{1}_{(0,s]}(0) - \hat{1}_{(0,s]}(0) = 0.\end{aligned}$$

For the variance function of M we have

$$\mathbb{E}M_t^2 = \|S_t(0,\cdot)\|^2_{\mu_H} = S_t(0,0) = V_t.$$

We will see below that M is the martingale considered for instance in [68]. according to their terminology, we call M the *fundamental martingale.*

We want to show that M_t is in fact a Wiener integral of some deterministic kernel $k_t \in \mathcal{I}_t$ with respect to the fBm X. In spectral terms, this means we have to show that $S_t(0, \lambda)$ is the Fourier transform of some square integrable kernel. By substituting $\omega = 0$ in the formula (4.1.15) and using the basic property

$$z^{-\nu} J_\nu(z) \to \frac{1}{2^\nu \Gamma(\nu + 1)} \quad \text{as} \quad z \to 0 \tag{4.1.53}$$

(cf. A.0.16) we see that

$$S_t(0, \lambda) = \frac{\sqrt{\pi}}{2H\Gamma(H + 1/2)} \left(\frac{t}{\lambda}\right)^{1-H} e^{\frac{i\lambda t}{2}} J_{1-H}\left(\frac{\lambda t}{2}\right).$$

To write this as a Fourier transform we use Poisson's integral representation of the Bessel function (see (A.0.15)). Rather straightforward computations now show that

$$S_t(0, \lambda) = \hat{k}_t(\lambda),$$

where

$$k_t(u) = \frac{u^{1/2-H}(t-u)^{1/2-H}}{2H\Gamma(1/2 + H)\Gamma(3/2 - H)}, \quad u \le t. \tag{4.1.54}$$

See for instance the proof of Proposition 2.2 of [18], where the computation is carried out in the reverse direction.

Hence, we have arrived at the following well-known theorem (cf. e.g. [63], [68], [70]).

Theorem 4.1.18. *For $t \ge 0$, let the kernel k_t be defined by (4.1.54). We have that $k_t \in \mathcal{I}_t$, and the process M defined by*

$$M_t = \int_0^t k_t(u)\, dX_u, \quad t \ge 0,$$

is a martingale with variance function $\mathbb{E}M_t^2 = V_t$.

Let us remark (as in [20]) that the Poisson formula also shows that $S_t(0, \cdot)$ can be expressed in terms of fractional integrals, namely

$$S_t(0, \lambda) = \frac{1}{\Gamma(1/2 + H)} I_{0+}^{3/2-H} \left(u^{1/2-H} e^{i\lambda u}\right)(t). \tag{4.1.55}$$

This establishes the well-known connection between the fBm and the fractional calculus. (See [76] for background on fractional calculus.)

4.1.5.2 Moving average representation

It follows from Corollary 4.1.15 that $\mathscr{H}_t = \overline{\mathrm{sp}}\{M_u : u \leq t\}$. In particular, this means that X_t is a linear functional of $(M_u)_{u\leq t}$. The aim is now to write

$$X_t = \int_0^t l_t(u)\, dM_u \tag{4.1.56}$$

for some explicit kernel $l_t \in L^2([0,t],V)$. This moving average representation of the fBm is the converse of Theorem 4.1.18.

In spectral terms, we have to determine l_t such that

$$\hat{1}_{(0,t]}(\lambda) = \int_0^t l_t(u)\, dS_u(0,\lambda).$$

By (4.1.49) we have $dS_u(0,\lambda) = \phi(u,\lambda)\, dV(u)$, so that the latter equation is equivalent to $\hat{1}_{(0,t]} = \mathcal{U} l_t$. It follows that (4.1.56) holds with $l_t = \mathcal{U}^{-1}\hat{1}_{(0,t]}$.

An explicit expression for l_t is most easily obtained by using some fractional calculus. Indeed, it follows from (4.1.49), (4.1.55) and fractional integration by parts that

$$\Gamma(1/2+H)\mathcal{U}f(\lambda) = \int_0^T e^{i\lambda t} t^{1/2-H} (I_{T-}^{1/2-H} f)(t)\, dt, \tag{4.1.57}$$

for each $f \in L^2([0,T],V)$ for which the factional integral is well defined. Taking $f = \Gamma(1/2+H) I_{T-}^{H-1/2}(u^{H-1/2} 1_{(0,T]}(u))$ we obtain $\mathcal{U}f(\lambda) = \hat{1}_{(0,T]}(\lambda)$. Hence, we have proved the following theorem (cf. e.g. [63], [10], [68], [70], [73]).

Theorem 4.1.19. *For $t \geq 0$, let the kernel l_t be defined by*

$$l_t(u) = \Gamma(1/2+H) I_{t-}^{H-1/2}(v^{H-1/2} 1_{(0,t]}(v))(u), \quad u \leq t.$$

We have that $l_t \in L^2([0,t],V)$ and (4.1.56) holds for all $t \geq 0$.

Evaluation of the fractional integral yields the more explicit expression

$$l_t(u) = t^{H-1/2}(t-u)^{H-1/2} - \int_u^t (t-v)^{H-1/2} dv^{H-1/2}, \quad u \leq t.$$

Since M is a continuous Gaussian martingale with bracket $\langle M \rangle_t = V_t$, we have $dM_u = \sqrt{V_u'}\, dW_u$, where W is a standard Brownian motion. Hence, the moving average representation can be rephrased as follows.

Corollary 4.1.20. *For $t \geq 0$, let the kernel w_t be defined by $w_t(u) = l_t(u)\sqrt{V'_u}$ for $u \leq t$. Then*

$$X_t = \int_0^t w_t(u) dW_u \tag{4.1.58}$$

holds for $t \geq 0$, with W a standard Brownian motion.

Also observe that the computations above can easily be generalized to more general integration kernels. For $r_t \in \mathcal{I}_t$ it holds that

$$\begin{aligned}\int_0^t r_t(u)\, dX_u &= \int_0^t \mathcal{U}^{-1}\hat{r}_t(u)\, dM_u \\ &= \Gamma(1/2+H) \int_0^t I_{t-}^{H-1/2}(v^{H-1/2} r_t(v))(u)\, dM_u,\end{aligned}$$

provided the fractional integral is well defined. Conversely, (4.1.57) shows that if $f \in L^2([0,t],V)$ and $I_{t-}^{1/2-H} f$ exists and also belongs to $L^2([0,t],V)$, then

$$\Gamma(1/2+H) \int_0^t f(u)\, dM_u = \int_0^t u^{1/2-H} (I_{t-}^{1/2-H} f)(u)\, dX_u.$$

4.2 Isotropic random fields

So far we have presented two types of representations of a stochastic process: series expansion and moving average representation, the latter being investigated according to whether the associated string is smooth or not. In this section we will obtain similar results for isotropic random fields.

In this section one can also find the proof of uniform convergence of the series in Theorem 4.1.3 (see Theorem 4.2.5).

4.2.1 Covariance representation

Consider first an isotropic Gaussian random field X with homogenous increments (as defined in Section 1.2.1). The spectral measure μ defined by

$$\mu(d\lambda) = \frac{\lambda^2 d\Phi(\lambda)}{|s^{N-1}|} \tag{4.2.1}$$

(cf. (1.2.3)) will be viewed as the principal spectral measure of a unique string lmk in the sense of Theorem 2.5.1, which can be applied in view of the condition (1.2.5). We can prove the following corollary of Theorem 1.2.3.

Corollary 4.2.1. *Let S_l^m, $l = 0, 1, ..., m = 1, ..., h(l, N)$, where*

$$h(l,N) = \frac{(2l+N-2)(l+N-3)!}{(N-2)!l!},$$

be the complete set of orthonormal spherical harmonics, as defined in Section 1.2.2, and let the surface area of the unit sphere be given by $\left|s^{N-1}\right| = 2\pi^{N/2}/\Gamma(N/2)$. The covariance function of the centered, mean-square continuous isotropic Gaussian random field X with homogeneous increments can be represented as follows:

$$\begin{aligned} &\mathbb{E}X_s X_t \qquad (4.2.2)\\ &= \pi|s^{N-1}|^2 \sum_{l=0,2,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \int_0^{l+k} \check{G}_l(\|t\|,x)\check{G}_l(\|s\|,x)dx \\ &+ \pi|s^{N-1}|^2 \sum_{l=1,3,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \int_0^{l} \check{G}_l(\|t\|,x)\check{G}_l(\|s\|,x)dm(x), \end{aligned}$$

where

$$\check{G}_l(r,x) = \frac{1}{\pi}\int_{\mathbb{R}} G_l(r,\lambda)A(x,\lambda)\mu(d\lambda), \quad l = 1,3,\dots \qquad (4.2.3)$$

$$\check{G}_l(r,x) = \frac{1}{\pi}\int_{\mathbb{R}} G_l(r,\lambda)B(x,\lambda)\mu(d\lambda), \quad l = 0,2,\dots \qquad (4.2.4)$$

and the functions $A(x, \lambda)$ and $B(x, \lambda)$ are the functions associated with the string lmk whose principal spectral measure μ is given by (1.2.3). The function G_l is as defined by (1.2.35).

Proof. As we already mentioned, condition (1.2.5) ensures that the measure μ satisfies the assumptions of Theorem 2.5.1. By virtue of this theorem there exists an unique associated string with mass m, length $l \le \infty$ and tying constant $k \in [0, \infty]$. Note that the function $\check{G}_l(r, x)$ is defined as the even or odd (for appropriate l's) inverse transform of the function $G_l(r, \lambda)$. Since transforms are isometries, we have

$$\langle G_l(r_1,\cdot), G_l(r_2,\cdot)\rangle_\mu = \pi\langle \check{G}_l(r_1,\cdot), \check{G}_l(r_2,\cdot)\rangle_m, \quad l = 1,3,\dots$$
$$\langle G_l(r_1,\cdot), G_l(r_2,\cdot)\rangle_\mu = \pi\langle \check{G}_l(r_1,\cdot), \check{G}_l(r_2,\cdot)\rangle_2, \quad l = 0,2,\dots$$

The proof is completed by applying this to representation (1.2.44). □

Remark 4.2.2. Recall the assertion of Lemma 1.2.2 that the function $G_l(r, \cdot)$ is of finite exponential type at most r. Combined with Theorem 2.8.2, this implies that the inverse transforms of such functions are supported on the finite interval $[0, x(r+)]$ and that the representation (4.2.2) is in fact of the form

$$\begin{aligned} &\mathbb{E}X_sX_t \qquad (4.2.5)\\ &= \pi|s^{N-1}|^2 \sum_{l=0,2,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \int_0^{n(s,t)} \check{G}_l(\|t\|, y)\check{G}_l(\|s\|, y)dy \\ &+\pi|s^{N-1}|^2 \sum_{l=1,3,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \int_0^{n(s,t)} \check{G}_l(\|t\|, y)\check{G}_l(\|s\|, y)dm(y)\end{aligned}$$

with $n(s,t) := x(\|t\|+) \wedge x(\|s\|+)$. This immediately allows us to write down the following moving average-type representation of the random field X:

$$X_t = \sqrt{\pi}|s^{N-1}| \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) \int_0^{x(\|t\|+)} \check{G}_l(\|t\|, y)dM_l^m(y) \qquad (4.2.6)$$

where for $l = 0, 1, \dots$ the sets $\{M_l^m, m = 1, \dots, h(l, N)\}$ consist of $h(l, N)$ independent copies of Gaussian processes M_l with independent increments, whose variances are given by

$$\mathbb{E}\,|M_l(y)|^2 = \begin{cases} y & l = 0, 2, \dots \\ m(y) & l = 1, 3, \dots \end{cases}$$

Note that this is a generalization of the representation obtained in Section 4.1.3 for processes with stationary increments. In the Section 4.2.3 we will return to this subject.

Now, consider a homogenous isotropic random field $Y = (Y_t)_{t\in\mathbb{R}}$. Recall that $\Phi(\lambda) = \int_{\|v\|<\lambda} \varrho(dv)$, where ϱ is the spectral measure

$$\mathbb{E}Y_tY_s = \int_{\mathbb{R}^N} e^{i\langle v,t-s\rangle}\varrho(dv) \qquad (4.2.7)$$

(see Section (1.2.1)). Theorem 1.2.4 allows us to derive the following

Corollary 4.2.3. *Let S_l^m, $l = 0, 1, ...,$ $m = 1, ..., h(l, N)$, where*

$$h(l, N) = \frac{(2l + N - 2)(l + N - 3)!}{(N - 2)!l!},$$

be the complete set of orthonormal spherical harmonics, as defined in Section 1.2.2, and let the surface area of the unit sphere be given by $\left|s^{N-1}\right| = 2\pi^{N/2}/\Gamma(N/2)$. *If the function Φ associated with a centered, mean-square continuous homogenous isotropic Gaussian random field Y satisfies $\int(1+\lambda^2)^{-1}d\Phi(\lambda) < \infty$, the covariance function of Y can be represented as follows:*

$$\begin{aligned} &\mathbb{E}Y_sY_t \qquad (4.2.8) \\ &= \pi|s^{N-1}| \sum_{l=0,2,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \int_0^l \check{j}_l(x, \|t\|)\check{j}_l(x, \|s\|)dm(x) \\ &+ \pi|s^{N-1}| \sum_{l=1,3,\dots} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \int_0^{l+k} \check{j}_l(x, \|t\|)\check{j}_l(x, \|s\|)dx \end{aligned}$$

where

$$\check{j}_l(x, r) = \frac{1}{\pi} \int_{\mathbb{R}} j_l(\lambda r)A(x, \lambda)d\Phi(\lambda), \quad l = 0, 2, \dots \qquad (4.2.9)$$

$$\check{j}_l(x, r) = \frac{1}{\pi} \int_{\mathbb{R}} j_l(\lambda r)B(x, \lambda)d\Phi(\lambda), \quad l = 1, 3, \dots \qquad (4.2.10)$$

and the functions $A(x, \lambda)$ and $B(x, \lambda)$ are the functions associated with the string lmk whose principal spectral function is Φ. The function j_l is the spherical Bessel function defined by (1.2.28).

Proof. Note that according to the property of Bessel functions, function j_l is even for $l = 0, 2, \dots$ and odd for $l = 1, 3, \dots$. Now, the proof is similar to the proof of Corollary 4.2.1. □

Remark 4.2.4. The function $j_l(\cdot\, r)$ is of finite exponential type at most r. Theorem 2.8.2 implies that its inverse transforms are supported on the finite interval $[0, x(r+)]$. Hence, similar to the case of homogenous increments, we can write the moving average-type representation of the random field Y:

$$Y_t = \sqrt{\pi|s^{N-1}|} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) \int_0^{x(\|t\|+)} \check{j}_l(y, \|t\|)dM_l^m(y) \qquad (4.2.11)$$

where for $l = 0, 1, \ldots$ the set $\{M_l^m, m = 1, \ldots, h(l, N)\}$ consist of $h(l, N)$ independent copies of Gaussian processes M_l with independent increments whose variances are given by

$$\mathbb{E}\left|M_l(y)\right|^2 = \begin{cases} m(y) & l = 0, 2, \ldots \\ y & l = 1, 3, \ldots \end{cases}$$

Observe that this generalizes representation obtained in Theorem 4.1.13 for stationary processes.

4.2.2 Series expansion

Fix a positive $T \in \mathbb{R}$ and define the ball in R^N as

$$\mathcal{B}_T := \left\{u \in \mathbb{R}^N \ : \ \|u\| \leq T\right\}. \tag{4.2.12}$$

Consider a isotropic Gaussian random field $X = (X_t)_{t \in \mathcal{B}_T}$ with homogenous increments. Let m be the mass function of string associated with the field X via measure (4.2.1). Define a new string by cutting the string associated with X at the point $l := x(T+)$ (which we assume to be finite) and assigning a new tying constant $k = \infty$. Then $m(l-) < \infty$ and the new string is short.

We will use representation (4.2.2) to obtain the series expansion. Since it consists of two components we first concentrate on the odd l's. Since $\check{G}_l(\|t\|, \cdot)$ belongs to the space $L^2(m)$, we can expand it in the basis (2.7.3) so that

$$\check{G}_l(\|t\|, x) = \sum_{n=0}^{\infty} \langle \check{G}_l(\|t\|, \cdot), \varphi_n \rangle_m \varphi_n(x).$$

Having this, we can write

$$\begin{aligned} & \int_0^l \check{G}_l(\|t\|, x)\check{G}_l(\|s\|, x) dm(x) \\ = \ & \sum_{n=0}^{\infty} \left(\int_0^l \check{G}_l(\|t\|, x)\varphi_n(x) dm(x)\right)\left(\int_0^l \check{G}_l(\|s\|, x)\varphi_n(x) dm(x)\right), \end{aligned}$$

which is same as

$$\int_0^l \check{G}_l(\|t\|, x)\check{G}_l(\|s\|, x) dm(x) = \sum_{n=0}^{\infty} \frac{G_l(\|t\|, \omega_n) G_l(\|s\|, \omega_n)}{\|A(\cdot, \omega_n)\|_m^2},$$

since

$$\int_0^l \check{G}_l(\|t\|, x)\varphi_n(x)dm(x) = \frac{G_l(\|t\|, \omega_n)}{\|A(\cdot, \omega_n)\|_m}.$$

Exactly the same argument for even l's results in corresponding formula

$$\int_0^l \check{G}_l(\|t\|, x)\check{G}_l(\|s\|, x)dx = \sum_{n=0}^{\infty} \frac{G_l(\|t\|, \omega_n)G_l(\|s\|, \omega_n)}{\|B(\cdot, \omega_n)\|_2^2}.$$

Then, keeping in mind Lemma 2.7.3, we can rewrite representation (4.2.2) as follows

$$\mathbb{E}X_s X_t = \pi|s^{N-1}|^2 \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) S_l^m\left(\frac{s}{\|s\|}\right) \sum_{n=0}^{\infty} \frac{G_l(\|t\|, \omega_n)G_l(\|s\|, \omega_n)}{\|A(\cdot, \omega_n)\|_m^2}. \tag{4.2.13}$$

We finally can prove the following

Theorem 4.2.5. *Let X be a centered, mean-square continuous Gaussian isotropic random field with homogenous increments on $\mathbb{R}^N$. If the mass function associated with μ (cf. (1.2.3)) is such that $x(T+) + m(x(T+)-) < \infty$ for $T > 0$, then we have the following representation:*

$$X_t = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) G_l(\|t\|, \omega_n)\xi_{l,n}^m, \qquad t \in \mathcal{B}_T, \tag{4.2.14}$$

where $\xi_{l,n}^m$ are independent, mean-zero Gaussian random variables with variances

$$\sigma_n^2 = \frac{2\pi|s^{N-1}|^2}{A(x(T+), \omega_n)\dot{B}(x(T+), \omega_n)} \tag{4.2.15}$$

and the ω_n's are the zeros of $B(x(T+), \cdot)$. This series converges in mean-square sense for any fixed $t \in \mathcal{B}_T$. Moreover, if the process $(X_t)_{\|t\|<T}$ is continuous, the series converges with probability one in the space of continuous functions $C(\mathcal{B}_T)$ endowed with the supremum norm.

Proof. For $M \in \mathbb{N}$, consider the partial sum of the series defined by

$$X^M(t) = \sum_{n,l=0}^{M} \sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) G_l(\|t\|, \omega_n)\xi_{l,n}^m.$$

The covariance representation (4.2.13) ensures, as $M \to \infty$, mean-square convergence of $X^M(t)$ to process X_t for every t. The L^2-convergence for every t implies that the finite dimensional distributions of X^M converge weakly to those of X. The summands

$$t \mapsto S_l^m\left(\frac{t}{\|t\|}\right) G_l(\|t\|, \omega_n)\xi_{l,n}^m$$

are symmetric random elements of $C(\mathcal{B}_T)$. The Lévy-Ito-Nisio theorem then implies that with probability one, the convergence of X^M to X is uniform on $\mathcal{B}_T$ (cf. [41], Theorem 2.4). □

Remark 4.2.6. Our expansion (4.2.14) is of a different form than the one derived recently in [54], Theorem 1. The conditions of the latter theorem seem difficult to verify, except in the case of Lévy's fractional Brownian motion.

The same reasoning allows us to prove the version of Theorem 4.2.5 for homogenous random fields.

Theorem 4.2.7. *Let Y be a centered, mean-square continuous homogenous Gaussian isotropic random field on $\mathbb{R}^N$. Let ϱ be the measure satisfying (4.2.7). If the mass distribution m of the string associated with the function*

$$\Phi(\lambda) = \int_{\|v\|<\lambda} \varrho(dv), \qquad \int_{\mathbb{R}} \frac{d\Phi(\lambda)}{\lambda^2+1} < \infty,$$

is such that $x(T+) + m(x(T+)-) < \infty$ for $T > 0$, then we have the following representation:

$$Y_t = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right) j_l(\omega_n\|t\|)\xi_{l,n}^m, \qquad t \in \mathcal{B}_T, \quad (4.2.16)$$

where $\xi_{l,n}^m$ are independent, mean-zero Gaussian random variables with variances

$$\sigma_n^2 = \frac{2\pi|s^{N-1}|}{A(x(T+),\omega_n)\dot{B}(x(T+),\omega_n)} \quad (4.2.17)$$

and the ω_n's are the zeros of $B(x(T+),\cdot)$. This series converges in mean-square sense for any fixed $t \in \mathcal{B}_T$. Moreover, if the process $(Y_t)_{\|t\|<T}$ is continuous, the series converges with probability one in the space of continuous functions $C(\mathcal{B}_T)$ endowed with the supremum norm.

4.2.2.1 Examples

Example 4.2.8. *Lévy's Brownian motion*

As we computed in Example 1.2.5, the measure μ in the case of Levy's Brownian motion is given by

$$\mu(d\lambda) = d\lambda/|s^N|, \qquad |s^N| = \frac{2\pi^{(N+1)/2}}{\Gamma((N+1)/2)}.$$

By virtue of Lemma 3.1.1 we see that the mass function associated with Lévy's Brownian motion is $m(x) = |s^N|^2 x$, while $A(x,\lambda) = \cos(|s^N|\lambda x)$ and $B(x,\lambda) = |s^N|\sin(|s^N|\lambda x)$. But since in this case $x(t) = t/|s^N|$, the constant disappears:

$$A(x(t),\lambda) = \cos(\lambda t), \quad B(x(t),\lambda)x'(t) = \sin(\lambda t).$$

We can apply Theorem 4.2.5 to the present case and obtain the following.

Theorem 4.2.9. *Let X be Lévy's Brownian motion on $\mathbb{R}^N$. It can be represented on the ball $\mathcal{B}_T$ of radius T (cf. (4.2.12)) as follows*

$$X_t = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) G_l(\|t\|,\omega_n)\xi_{l,n}^m, \qquad t \in \mathcal{B}_T, \quad (4.2.18)$$

where

$$\omega_n = \frac{n\pi}{T}$$

and the $\xi_{l,n}^m$ are independent mean zero Gaussian random variables with variances

$$\sigma_n^2 = \frac{4\pi^{(N+1)/2}\Gamma((N+1)/2)}{T\Gamma^2(N/2)}.$$

This series converges with probability one in the space of continuous functions on $\mathcal{B}_T$.

Remark 4.2.10. In the scalar case $N = 1$ we obtain a series representation of standard Brownian motion on $[0, 1]$

$$W(t) = \sqrt{2}\sum_{n=0}^{\infty}\frac{1-\cos(tn\pi)}{n\pi}\xi_n^0 - \sqrt{2}\sum_{n=0}^{\infty}\frac{\sin(tn\pi)}{n\pi}\xi_n^1$$

where $\{\xi_n^0\}$ and $\{\xi_n^1\}$ are independent sequences of centered Gaussian independent random variables with unit variance, so that (4.2.18) can be viewed as a multivariate version of the classical Paley–Wiener expansion.

Example 4.2.11. *Lévy's fractional Brownian motion*

Recall from Example 1.2.6 that we deal here with the spectral measure

$$\mu(d\lambda) = c_{HN}^2 \lambda^{1-2H} d\lambda, \qquad c_{HN}^2 = \frac{\Gamma(H+\frac{N}{2})\Gamma(1+H)\sin(\pi H)}{\pi^{\frac{N+2}{2}} 2^{1-2H}},$$

that differs only by a constant factor from the spectral measure of one-dimensional fractional Brownian motion (Section 3.3.1). Therefore the expressions for the mass function m and the functions A and B can be easily obtaineed with the help of Lemma 3.1.1. We get

$$\begin{aligned} m(x) &= \frac{\kappa_{HN}^{1/H}}{4H(1-H)} x^{\frac{1-H}{H}}, \\ A(x,\lambda) &= \Gamma(1-H)\Big(\frac{\lambda}{2}\Big)^H \sqrt{\kappa_{HN} x} J_{-H}\Big(\lambda(\kappa_{HN}x)^{\frac{1}{2H}}\Big), \\ B(x,\lambda) &= \frac{\kappa_{HN}\Gamma(1-H)}{2H}\Big(\frac{\lambda}{2}\Big)^H (\kappa_{HN}x)^{\frac{1-H}{2H}} J_{1-H}\Big(\lambda(\kappa_{HN}x)^{\frac{1}{2H}}\Big). \end{aligned}$$

The new constant is

$$\kappa_{HN} = \frac{2\pi^{(N+2)/2}}{\Gamma(H+N/2)\Gamma(1-H)} \tag{4.2.19}$$

(it in fact extends the constant κ_{H1} appearing in Section 3.3.1, to the multidimensional case). After the necessary substitution $x(t) = t^{2H}/\kappa_{HN}$ this constant does not occur in the functions

$$A(x(t),\lambda) = \Gamma(1-H)\Big(\frac{\lambda t}{2}\Big)^H J_{-H}(\lambda t), \tag{4.2.20}$$

$$B(x(t),\lambda)x'(t) = \Gamma(1-H)\Big(\frac{\lambda t}{2}\Big)^H J_{1-H}(\lambda t). \tag{4.2.21}$$

We specify our general series expansion of Theorem 4.2.5 to the case of Lévy's fractional Brownian motion.

Theorem 4.2.12. *Let $\omega_0 < \omega_1 < \omega_2 < \cdots$ be the non-negative real-valued zeros of the Bessel function J_{1-H}. Then Lévy's fractional Brownian motion X with Hurst index H restricted to the ball $\mathcal{B}_T$ of radius T (cf. (4.2.12)) can be represented as follows*

$$X_t \quad = \quad \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} \sum_{n=0}^{\infty} S_l^m\left(\tfrac{t}{\|t\|}\right) G_l\left(\|t\|, \frac{\omega_n}{T}\right) \xi_{l,n}^m, \qquad t \in \mathcal{B}_T,$$

where the $\xi_{l,n}^m$ are independent mean zero Gaussian random variables with variances

$$\sigma_n^2 = \frac{8\pi^{N/2} H T^{2H-2} \Gamma\left(H + \frac{N}{2}\right)}{\Gamma^2(N/2)\Gamma(1-H)\left(\frac{\omega_n}{2}\right)^{2H} J_{-H}^2(\omega_n)}.$$

This series converges with probability 1 in the space of continuous functions on $\mathcal{B}_T$.

Proof. By (4.2.21) we have that $B(x(t), \lambda) = 0$ if and only if $\lambda = \omega_n/T$ and

$$\dot{B}(x(T), \omega_n/T) = \frac{\Gamma(1-H)\kappa_{HN}}{2HT^{2H-2}} \left(\frac{\omega_n}{2}\right)^H J_{-H}(\omega_n).$$

By (4.2.20)

$$A(x(T), \omega_n/T) = \Gamma(1-H)\left(\frac{\omega_n}{2}\right)^H J_{-H}(\omega_n).$$

The required expression for σ_n^2 is now verified by (4.2.15). The assertion of the present theorem thus follows from Theorem 4.2.5. $\square$

4.2.3 Representations in the case of a smooth string

In this section we will show how the representation (4.2.6) (and similarly (4.2.11)) simplifies when the string associated with the random field has a smooth mass function. We will obtain an integral representation in the time domain, which can be viewed as a multivariate moving average representation.

To this end, we have to invert the function $t(x) = \int_0^x \sqrt{m'(y)}dy$ defined by (2.1.1). Therefore we need to require that the mass function is continuously differentiable with a positive derivative. This then yields the following representation of the covariance function in the time domain.

Theorem 4.2.13. *If the mass function m associated with the centered, mean-square continuous isotropic random field X with homogenous increments is continuously differentiable and $m'>0$, then for every $s,t\in\mathbb{R}^N$ we have*

$$\begin{aligned}\mathbb{E}X_sX_t &= 2\pi^2|s^{N-1}|^2\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right)S_l^m\left(\frac{s}{\|s\|}\right)\\ &\times \int_0^{\|s\|\wedge\|t\|} k_l(\|t\|,u)k_l(\|s\|,u)dV(2u), \qquad (4.2.22)\end{aligned}$$

where $V(2u)=\pi^{-1}m(x(u))$ and the kernels are given by

$$k_l(\|t\|,u) = \check{G}_l(\|t\|,x(u))x'(u), \qquad l=0,2,\dots, \qquad (4.2.23)$$

$$k_l(\|t\|,u) = \check{G}_l(\|t\|,x(u)), \qquad l=1,3,\dots. \qquad (4.2.24)$$

for $u\le\|t\|$.

Proof. Recall that by virtue of (4.1.47) we have

$$2\pi\, V'(2t)\, x'(t)=1. \qquad (4.2.25)$$

To prove the representation (4.2.22) we apply the change of variable $y=x(u)$ to both terms on the right side of (4.2.5). Due to (4.2.25), the measure dy in the integral of the first term becomes

$$x'(u)du = 2\pi x'(u)^2 dV(2u).$$

Hence,

$$\begin{aligned}&\int_0^{n(s,t)} \check{G}_l(\|t\|,y)\check{G}_l(\|s\|,y)dy \qquad (4.2.26)\\ =\;& 2\pi\int_0^{\|s\|\wedge\|t\|}\check{G}_l(\|t\|,x(u))\check{G}_l(\|s\|,x(u))x'(u)^2dV(2u)\,.\end{aligned}$$

The same change of variables allows us to write the integral of the second term in (4.2.5) in the following manner:

$$\begin{aligned}&\int_0^{n(s,t)} \check{G}_l(\|t\|,y)\check{G}_l(\|s\|,y)dm(y) \qquad (4.2.27)\\ =\;& 2\pi\int_0^{\|s\|\wedge\|t\|}\check{G}_l(\|t\|,x(u))\check{G}_l(\|s\|,x(u))dV(2u),\end{aligned}$$

since the measure $dm(y) = m'(y)dy$ turns into $m'(x(u))x'(u)du = 2\pi dV(2u)$. Due to (4.2.26) and (4.2.27) the representation (4.2.5) turns into (4.2.22). □

Corollary 4.2.14. *Under assumptions of Theorem 4.2.13 we have*

$$X_t = \sqrt{2}\pi\,|s^{N-1}|\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right)\int_0^{\|t\|} k_l(\|t\|,u)dM_l^m(u), \qquad (4.2.28)$$

where $\{M_l^m\}$ *are independent copies of the Gaussian martingale* M *with zero mean and variance function* $E|M(u)|^2 = V(2u)$.

Remark 4.2.15. The representation (4.2.22) may be compared with a similar result by Malyarenko [54], that is derived under a number of conditions on the spectral measure, listed in his Theorem 1.

Similar argument allows us to derive from Corollary 4.2.3 the following result.

Corollary 4.2.16. *Let* Y *be a centered, mean-square continuous homogenous isotropic Gaussian random field. Let the mass distribution of the associated string be continuously differentiable with* $m' > 0$. *We have that*

$$Y_t = \pi\sqrt{2|s^{N-1}|}\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,N)} S_l^m\left(\frac{t}{\|t\|}\right)\int_0^{\|t\|} k_l(\|t\|,u)dM_l^m(u), \qquad (4.2.29)$$

where $\{M_l^m\}$ *are independent copies of the Gaussian martingale* M *with zero mean and variance function* $E|M(u)|^2 = V(2u)$ *and*

$$k_l(\|t\|,u) = \check{j}_l(x(u),\|t\|)x'(u), \qquad l = 1,3,\dots, \qquad (4.2.30)$$

$$k_l(\|t\|,u) = \check{j}_l(x(u),\|t\|), \qquad l = 0,2,\dots. \qquad (4.2.31)$$

4.2.3.1 Examples

Example 4.2.17. *Lévy's Brownian motion*

In this example we will provide the representation of type (4.2.28) for Levy's Brownian motion. The setup is as in Examples 1.2.5 and 4.2.8.

To this end we need to compute the explicit forms of the kernels k_l. Observe that in this case the transforms in Section 2.6 are the Fourier cosine and sine transforms. Equations (4.2.3) and (4.2.4), in conjunction with (4.2.23) and (4.2.24), become

$$k_{2n+1}(r,u) = \check{G}_{2n+1}(r,x(u)) = \frac{2}{\pi|s^N|}\int_0^\infty G_{2n+1}(r,\lambda)\cos(u\lambda)\,d\lambda,$$

$$k_{2n}(r,u) = \check{G}_{2n}(r,x(u))x'(u) = \frac{2}{\pi|s^N|}\int_0^\infty G_{2n}(r,\lambda)\sin(u\lambda)\,d\lambda$$

for $n = 0, 1, 2, \ldots$. By definition (1.2.35), we deal here with the cosine transform of the function $J_{2n+N/2}(r\lambda)/\lambda^{N/2}$ and for $n > 0$ with the sine transform of the function $J_{2n-1+N/2}(r\lambda)/\lambda^{N/2}$ to be found in the tables [26], Vol. I; see formulas 1.12.10 or 1.12.13 for the cosine transform and formulas 2.12.10 or 2.12.11 for the sine transform. We get, with F denoting Gauss's hypergeometric function,

$$\begin{aligned}\pi|s^{N-1}|\,k_{2n+1}(r,u) &= \frac{(-1)^n\Gamma(N)\Gamma(2n+1)}{\Gamma(2n+N)}\left(1-\frac{u^2}{r^2}\right)^{\frac{N-1}{2}} C_{2n}^{N/2}\left(\frac{u}{r}\right)\\ &= -\frac{\Gamma(\frac{N+1}{2})\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+\frac{N}{2})}\left(1-\frac{u^2}{r^2}\right)^{\frac{N-1}{2}} F\left(-n, n+\tfrac{N}{2}; \tfrac{1}{2}; \tfrac{u^2}{r^2}\right),\end{aligned} \tag{4.2.32}$$

and for $n > 0$

$$\begin{aligned}\pi|s^{N-1}|\,k_{2n}(r,u) &= \frac{(-1)^n\Gamma(N)\Gamma(2n)}{\Gamma(2n-1+N)}\left(1-\frac{u^2}{r^2}\right)^{\frac{N-1}{2}} C_{2n-1}^{N/2}\left(\frac{u}{r}\right)\\ &= -\frac{\Gamma(\frac{N+1}{2})\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+\frac{N-1}{2})}\left(1-\frac{u^2}{r^2}\right)^{\frac{N-1}{2}} \tfrac{2u}{r} F\left(1-n, n+\tfrac{N}{2}; \tfrac{3}{2}; \tfrac{u^2}{r^2}\right)\end{aligned} \tag{4.2.33}$$

(for the relationship between Gegenbauer's polynomials and the Gauss hypergeometric functions see e.g. [29], formulas 8.932). Note that the expressions involving Gegenbauer's polynomials can also be obtained by inverting the Fourier transform (1.2.37) mentioned above. The remaining k_0 is obtained by integrating (1.2.36) with respect to $2\sin(\lambda u)d\lambda/\pi|s^N|$ over $\mathbb{R}_+$. Since

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda w)}{\lambda}\sin(\lambda u)d\lambda = 1_{(u,r)}(w)$$

(see [29], formulas 3.721.1 and 3.741.2) we obtain

$$\pi|s^{N-1}|\,k_0(r,u) = (N-1)\int_{u/r}^1 \left(1-y^2\right)^{\frac{N-3}{2}} dy. \tag{4.2.34}$$

Given this and the fact that

$$V(2t) = \frac{m(x(t))}{\pi} = \frac{t}{\pi},$$

Corollary 4.2.14 yields the following

Theorem 4.2.18. *Let X be Lévy's Brownian motion on $\mathbb{R}^N$. It can be represented as*

$$X_t = \sqrt{\frac{2}{\pi}} \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m \left(\tfrac{t}{\|t\|}\right) \int_0^{\|t\|} \pi\, |s^{N-1}| k_l(\|t\|, u) dM_l^m(u)$$

where the kernels $\pi\, |s^{N-1}| k_l$ are given by (4.2.32)–(4.2.34), while $\{M_l^m\}$ are independent copies of a standard Brownian motion.

Remark 4.2.19. The kernels (4.2.32)–(4.2.34) occurred already in the paper [57], in which McKean has pointed out that these kernels are in fact singular in the sense that a nontrivial square integrable function can be found that is orthogonal to k_l when $l > 2$. He has shown how to replace them by more convenient nonsingular kernels that admitted him to confirm Lévy's conjecture that the Brownian motions in odd-dimensional spaces are Markov, but not in even-dimensional spaces. Obviously, the transition from singular to nonsingular kernels is highly desirable in the present setting as well. It is however not clear how to obtain such kernels in general.

Remark 4.2.20. A representation of Levy's Brownian motion similar to Theorem 4.2.18 was presented in [92] as an example of general representation of isotropic random field of the form

$$X_t = \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m \left(\tfrac{t}{\|t\|}\right) \xi_l^m(\|t\|)$$

where ξ_l^m is a sequence of random variables satisfying

$$\mathbb{E}\xi_l^m(r)\xi_{l'}^{m'}(s) = b(r,s)\delta_m^{m'}\delta_l^{l'},$$

where

$$b(r,s) = |s^{N-1}| \int_{-1}^{1} B(r,s,u) \frac{C_l^{\frac{N-2}{2}}(u)}{C_l^{\frac{N-2}{2}}(1)} (1-u^2)^{\frac{N-3}{2}} du$$

and $\mathbb{E}X_t X_s = B(\|t\|, \|s\|, \cos\angle(t,s))$.

Example 4.2.21. *Lévy's fractional Brownian motion*

Here, we will try to extend the above to Lévy's fractional Brownian motion. Based on Example 4.2.11 we have that

$$V(2t) = \frac{m(x(t))}{\pi} = \frac{\kappa_{HN}\, t^{2-2H}}{4H(1-H)\pi} = \frac{\pi^{N/2}\, t^{2-2H}}{2H\Gamma(2-H)\Gamma(H+N/2)}. \qquad (4.2.35)$$

The assertion of Theorem 4.2.18 is extended to the present fractional case as follows.

Theorem 4.2.22. *Let X be Lévy's fractional Brownian motion on $\mathbb{R}^N$ with Hurst index $H \in (0,1)$. Then it is represented as follows*

$$X_t = \sqrt{2}|s^{N-1}| \sum_{l=0}^{\infty} \sum_{m=1}^{h(l,N)} S_l^m\left(\tfrac{t}{\|t\|}\right) \int_0^{\|t\|} \pi k_l(\|t\|, u) dM_l^m(u)$$

where $\{M_l^m\}$ are independent copies of a Gaussian martingale M with mean zero and the variance function $\mathbb{E}|M(u)|^2 = V(2u)$ given by (4.2.35), with the kernels k_l defined by

$$\frac{\pi k_0(r,u)}{c_{HN}^2 \Gamma(\frac{N}{2})} = \frac{2\Gamma(1-H)}{\Gamma(H-1+\frac{N}{2})} \left(\frac{2}{u}\right)^{1-2H} \int_{u/r}^{1} y^{1-2H}(1-y^2)^{H-2+\frac{N}{2}} dy,$$

for $n = 0, 1, 2, \ldots$

$$\begin{aligned} \frac{\pi k_{2n+1}(r,u)}{c_{HN}^2 \Gamma(\frac{N}{2})} &= -\frac{\Gamma\left(n+1-H\right)}{\Gamma\left(n+H+\frac{N}{2}\right)} \left(\frac{2}{r}\right)^{1-2H} \left(1-\tfrac{u^2}{r^2}\right)^{H-1+\frac{N}{2}} \\ &\times \; F\left(-n, n+\frac{N}{2}; 1-H; \frac{u^2}{r^2}\right) \end{aligned}$$

and for $n = 1, 2, \ldots$

$$\begin{aligned} \frac{\pi k_{2n}(r,u)}{c_{HN}^2 \Gamma(\frac{N}{2})} &= -\frac{\Gamma\left(n+1-H\right)}{(1-H)\Gamma\left(n+H-1+\frac{N}{2}\right)} \left(\frac{2}{r}\right)^{1-2H} \left(1-\frac{u^2}{r^2}\right)^{H-1+\frac{N}{2}} \\ &\times \; \frac{u}{r} F\left(1-n, n+\frac{N}{2}; 2-H; \frac{u^2}{r^2}\right). \end{aligned}$$

Proof. We need the inverse transforms of function G_l with respect to the measure (1.2.50), as defined by the formulas (4.2.3) and (4.2.4). Since the functions A and B are given by (4.2.20) and (4.2.21), respectively, it follows from (4.2.23) and (4.2.24) that for $l > 0$ the kernels k_l are evaluated as Hankel transforms of the following form

$$\begin{aligned} k_{2n+1}(r,u) &= \check{G}_{2n+1}(r,x(u)) \\ &= \frac{2^{1-H}c_{HN}^2\Gamma(1-H)\,u^H}{\pi}\int_0^\infty G_{2n+1}(r,\lambda)J_{-H}(u\lambda)\,\lambda^{1-H}d\lambda \end{aligned}$$

and

$$\begin{aligned} k_{2n}(r,u) &= \check{G}_{2n}(r,x(u))x'(u) \\ &= \frac{2^{1-H}c_{HN}^2\Gamma(1-H)\,u^H}{\pi}\int_0^\infty G_{2n}(r,\lambda)J_{1-H}(u\lambda)\,\lambda^{1-H}d\lambda. \end{aligned}$$

The required results are then found in the tables [26], Vol II, formula 8.11.9. To complete the proof we will show

$$k_0(r,u) = \frac{c_{HN}^2\,\Gamma^2(1-H)}{\pi}\left(\frac{u}{2}\right)^{2H-1}\left(1 - \frac{B_{u^2/r^2}(1-H, H-1+\frac{N}{2})}{B(1-H, H-1+\frac{N}{2})}\right)$$

where $B_x(\alpha,\beta)$ is the *incomplete beta function* (see [29], formula 8.391). Indeed, the kernel k_0 is computed as the sum of the following two terms. The first term is

$$c_{HN}^2\Gamma(1-H)\left(\frac{u}{2}\right)^H\frac{2}{\pi}\int_0^\infty J_{1-H}(u\lambda)\,\frac{d\lambda}{\lambda^H} = \frac{c_{HN}^2}{\pi}\Gamma^2(1-H)\left(\frac{u}{2}\right)^{2H-1}$$

(the integral is taken by means of formula 6.561.14 in [29]). The second term has the same expression as k_{2n} given above, but evaluated at $n = 0$ (for the relationship between the incomplete Beta function and the Gauss hyperbolic function, see [29], formula 8.391). □

Remark 4.2.23. It can be shown in the present fractional case too that the kernels k_l with $l > 2$ are singular in the same sense as in the special case $H = 1/2$ already mentioned in Remark 4.2.19. To see this, observe first that the Gauss hypergeometric functions that occur in the expressions for k_l are classical orthogonal polynomials, known in the literature as *generalized Gegenbauer polynomials* (see e.g. [15], section 1.5.2). It is then straightforward to follow McKean's arguments in [57], however we do not dwell upon this here and note only that the analogue of McKean's nonsingular kernels to the fractional case is known for the general Hurst index H as well, see Malyarenko [54]. For general isotropic fields the question of finding non-singular moving average representations remains open at this time.

Chapter 5

Small deviation results

The aim of this chapter is to study the *small deviation probabilities* also known as *small ball probabilities* of a given stationary or stationary increments Gaussian process. Namely, we want to study the behavior of the quantity

$$-\log \mathbb{P}\left(\|X\| < \varepsilon\right) \tag{5.0.1}$$

as $\varepsilon \to 0$. We view here the stochastic process X as a random element with values in an appropriate Banach space $(E, \|\cdot\|)$. We will present results for various norms.

The small ball results have numerous applications. They are an important tool in the study of Gaussian processes. They are proven to be closely related with various approximation quantities of compact sets and operators, such as metric entropy or l-approximation numbers. A variety of applications includes laws of iterated logarithms, Hausdorff dimensions or rates of escape, to mention just a few. For more details see the review paper [50].

It is well known that it is difficult to determine the asymptotic behavior of (5.0.1) and there exist complete results only for a few types of processes. Results for particular cases are recalled and discussed later, by the concrete examples. For an extensive list of references to the small deviations literature, see [53].

It turns out that small deviations results are mathematically closely related to the convergence results of Section 4.1. We will use the results of the latter section to formulate general theorems for si- and stationary processes. The chapter will be completed with the calculation of bounds for small ball probabilities in several particular cases. We will argue that several bounds are in fact sharp.

5.1 Preliminaries and known results

Let X be a Borel measurable random element with values in the separable Banach space $(E, \|\cdot\|)$, i.e. a map

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (E, \|\cdot\|).$$

such that for any Borel set $B \subset E$ we have that $X^{-1}(B) \in \mathcal{F}$. The random element X is called Gaussian if for any element e^* from the dual space E^* of E the random variable e^*X is Gaussian. We call X a centered random element if for any $e^* \in E^*$ it holds that $\mathbb{E}\,(e^*X) = 0$.

The particular Banach spaces of our interest are going to be the function spaces $L^p([0,T])$ with the norm

$$\|f\|_p := \left(\int_0^T |f(t)|^p dt\right)^{\frac{1}{p}}, \qquad p \geq 1,\, T > 0,$$

the space of continuous functions $C([0,T])$ with $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)|$ and the Orlicz space of integrable functions with the norm

$$\|f\|_{\psi_p} = \inf\left\{c > 0 \;:\; \int_0^T \psi_p(|f(t)|/c)dt \leq 1\right\}.$$

Given a centered Gaussian random element X with values in a separable Banach space $(E, \|\cdot\|)$, there exist a sequence of elements $u_k \in E$ and a sequence of independent standard normal random variables $(\xi_k)_{k\geq 1}$ such that

$$X = \sum_{k=1}^{\infty} \xi_k u_k, \tag{5.1.1}$$

and the series converges almost surely in the norm of E (cf. [52], [84]). According to this fact, we can define for $n \in \mathbb{N}$ the so-called l-number by

$$l_n(X) := \inf\left\{\left(\mathbb{E}\left\|\sum_{k>n} \xi_k u_k\right\|^2\right)^{1/2} \;:\; X = \sum_{k=1}^{\infty} \xi_k u_k \text{ a.s.}\right\} \tag{5.1.2}$$

where the infimum is taken over all representations of type (5.1.1).

If $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis of some separable Hilbert space H and u_k is as in (5.1.1), we can define an operator $\mathbb{T} : H \to E$ by setting

$$\mathbb{T}(f_k) := u_k, \quad k = 1, 2, \dots . \tag{5.1.3}$$

Such an operator is well defined and bounded. To see the latter, take an arbitrary $f \in H$, expand it in the basis and apply the operator $\mathbb{T}$ to obtain

$$\mathbb{T}f = \sum_{k=1}^{\infty} \langle f, f_k \rangle u_k$$

Applying $e^* \in E^*$ to both sides of the above gives

$$e^*\mathbb{T}f = \sum_{k=1}^{\infty} \langle f, f_k \rangle e^* u_k.$$

By virtue of Cauchy-Schwarz inequality we have

$$|e^*\mathbb{T}f|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \sum_{k=1}^{\infty} |e^* u_k|^2 = \|f\|^2 \sum_{k=1}^{\infty} |e^* u_k|^2. \tag{5.1.4}$$

Given (5.1.1) we have that $\mathbb{E}|e^*X|^2 = \sum |e^*u_k|^2$, and since $|e^*X| \leq \|e^*\| \, \|X\|$, it implies that

$$\sup_{\|e^*\|=1} \sum |e^* u_k|^2 \leq \mathbb{E}\|X\|^2. \tag{5.1.5}$$

Recall that by Hahn-Banach,

$$\|x\| = \sup\{|e^*x| \ : \ \|e^*\| = 1, \, e^* \in E^*\}, \tag{5.1.6}$$

for $x \in E$. It follows that

$$\|\mathbb{T}f\| \leq \|f\| \sqrt{\mathbb{E}\|X\|^2},$$

where $\mathbb{E}\|X\|^2 < \infty$ (see e.g. [83], Proposition A.2.1 and A.2.3).

Now, the representation (5.1.1) can be rewritten as

$$X = \sum_{k=1}^{\infty} \xi_k \mathbb{T} f_k. \tag{5.1.7}$$

The l-numbers of the operator $\mathbb{T}$ are defined as

$$l_n(\mathbb{T}) := \inf\left\{ \left(\mathbb{E}\left\|\sum_{k>n}\xi_k\mathbb{T}e_k\right\|^2\right)^{1/2} : \{e_k\}_{k=1}^{\infty} \text{ orthonormal basis of } H \right\}.$$

We have the following relation (see [75], [39]).

Lemma 5.1.1. *Let X be a centered Gaussian random variable with values in separable Banach space E. If the operator $\mathbb{T}$ is as defined by (5.1.3) it holds that*

$$l_n(X) = l_n(\mathbb{T}). \tag{5.1.8}$$

Let us now move to the result indicated in the beginning of the chapter on the connection between the rates of convergence of series expansions of a process and small ball probabilities. We cite below Proposition 4.1 of [48]. First we have to define some additional notions.

For $p \in [1,\infty]$ the space ℓ_p^n is the space of $x = (x_1, ..., x_n) \in \mathbb{R}^n$ satisfying

$$\|x\|_{\ell_p^n} := \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} < \infty.$$

For given two Banach spaces E and F the *Banach-Mazur distance* $d(E,F)$ is defined as

$$d(E,F) = \inf\left\{\|\mathbb{T}\|\|\mathbb{T}^{-1}\| \ : \ \mathbb{T} : E \to F, \mathbb{T} \text{ is an isomorphism}\right\}.$$

We say that a Banach space E contains ℓ_p^n's uniformly if there exists a sequence of subspaces $E_n \subset E$ such that for some $\eta > 1$ there is

$$d(E_n, \ell_p^n) \le \eta \qquad \text{for all } n. \tag{5.1.9}$$

The Banach space E is said to be K-convex if it does not contain ℓ_1^n's uniformly. Let us remark that the space $L^p([0,T])$, $1 < p < \infty$ is K-convex. For further details see [75], [74].

Lemma 5.1.2. *Let X be a centered Gaussian random variable with values in separable Banach space E.*

(i) If

$$l_n(X) \lesssim n^{-1/A}(1+\log n)^B \tag{5.1.10}$$

for some $A > 0$ and $B \in \mathbb{R}$, then

$$-\log \mathbb{P}(\|X\| \leq \varepsilon) \lesssim \varepsilon^{-A}(\log 1/\varepsilon)^{AB}. \tag{5.1.11}$$

(ii) Conversely, if (5.1.11) holds for some $A > 0$ and $B \in \mathbb{R}$, this implies

$$l_n(X) \lesssim n^{-1/A}(1+\log n)^{B+1}. \tag{5.1.12}$$

Moreover, if E is K-convex then (5.1.10) and (5.1.11) are equivalent.

(iii) If

$$-\log \mathbb{P}(\|X\| \leq 2\varepsilon) \gtrsim -\log \mathbb{P}(\|X\| \leq \varepsilon) \gtrsim \varepsilon^{-A}(\log 1/\varepsilon)^{AB} \tag{5.1.13}$$

for some $A > 0$ and $B \in \mathbb{R}$, then

$$l_n(X) \gtrsim n^{-1/A}(1+\log n)^{B-1/A}. \tag{5.1.14}$$

Hence, given some series expansion of the random element X, say

$$X = \sum_k \xi_k u_k$$

with $u_k \in E$, and knowing its rate of convergence, i.e.

$$\left(\mathbb{E} \left\| \sum_{k>n} \xi_k u_k(t) \right\|^2 \right)^{1/2} \lesssim n^{-A}(1+\log n)^B$$

for some $A > 0$ and $B \in \mathbb{R}$, by definition of $l_n(X)$ we have

$$l_n(X) \lesssim n^{-A}(1+\log n)^B. \tag{5.1.15}$$

Having this, we can apply Lemma 5.1.2 to obtain bounds for small ball probabilities. That is the plan of the next section.

5.2 Bounds for small ball probabilities

We will use Theorems 4.1.4 and 4.1.8, for si and stationary processes, respectively. According to this the small ball results will also depend on the tail behavior of the reproducing kernel S_T of the space $\mathscr{L}_T$, defined by (4.1.1), on the diagonal, i.e. condition (4.1.5):

$$\exists\, K > 0 \;:\; S_T(\lambda, \lambda) \geq c|\lambda|^\alpha \quad \text{for} \quad |\lambda| > K. \tag{5.2.1}$$

Let $X = (X_t)_{t\in[0,T]}$ be a centered Gaussian si-process with spectral measure μ and associated string lmk. Theorem 4.1.4 allows us to derive the following result.

Theorem 5.2.1. *Suppose* $0 < T < \int_0^l \sqrt{m'(y)}\, dy$ *and assume that (5.2.1) holds for* $\alpha > -1$. *Then we have, for* $\varepsilon > 0$ *small enough,*

(i) for $p \geq 1$,
$$-\log \mathbb{P}\left(\|X\|_p < \varepsilon\right) \lesssim \varepsilon^{-\frac{2}{1+\alpha}},$$

(ii)
$$-\log \mathbb{P}\left(\|X\|_\infty < \varepsilon\right) \lesssim \varepsilon^{-\frac{2}{1+\alpha}} \left(\log 1/\varepsilon\right)^{\frac{1}{1+\alpha}},$$

(iii) for $p \geq 2$,
$$-\log \mathbb{P}\left(\|X\|_{\psi_p} < \varepsilon\right) \lesssim \varepsilon^{-\frac{2}{1+\alpha}} \left(\log 1/\varepsilon\right)^{\frac{1-2/p}{1+\alpha}}.$$

An interesting aspect of this result is that it gives small deviation bounds for the process X determined by its frequency domain reproducing kernel, without referring explicitly to the associated string. In particular, the result is also useful if the reproducing kernel is determined by some other means. To illustrate this point, let us mention that if X is a standard Brownian motion, then using the Fourier inversion formula it is not difficult to see that the reproducing kernel is given by

$$S_T(\omega, \lambda) = \frac{1}{2\pi} \frac{e^{i(\omega-\lambda)T} - 1}{i(\omega - \lambda)}.$$

This means we have $\alpha = 0$ in (5.2.1) in this case. It is well-known (e.g. [50]) that for Brownian motion it holds that

$$-\log \mathbb{P}\left(\|X\|_2 < \varepsilon\right) \approx -\log \mathbb{P}\left(\|X\|_\infty < \varepsilon\right) \approx \varepsilon^{-2}.$$

The notation $a \approx b$ stands for $a \lesssim b$ and $b \lesssim a$. By inserting $\alpha = 0$ in Theorem 5.2.1 we see that the theorem gives a sharp bound for the L^2-norm, but for the uniform norm the bound is only correct up to a logarithmic factor. This is a typical phenomenon that occurs when small deviations results are derived via series expansions. In Section 5.2.1 we give additional explicit examples.

Similarly, Theorem 4.1.8 combined with Lemma 5.1.2 gives the following small deviations results in the stationary case. Let $Y = (Y_t)_{t\in[0,T]}$ be a centered stationary Gaussian process with spectral measure μ and associated string mlk.

Theorem 5.2.2. *Suppose $0 < T < \int_0^l \sqrt{m'(y)}\,dy$ and assume that condition (5.2.1) holds for $\alpha > 1$. Then we have, for $\varepsilon > 0$ small enough,*

(i) for $p \geq 1$,

$$-\log \mathbb{P}\left(\|Y\|_p < \varepsilon\right) \lesssim \varepsilon^{-\frac{2}{\alpha-1}},$$

(ii)

$$-\log \mathbb{P}\left(\|Y\|_\infty < \varepsilon\right) \lesssim \varepsilon^{-\frac{2}{\alpha-1}} \left(\log 1/\varepsilon\right)^{\frac{1}{\alpha-1}},$$

(iii) for $p \geq 2$,

$$-\log \mathbb{P}\left(\|Y\|_{\psi_p} < \varepsilon\right) \lesssim \varepsilon^{-\frac{2}{\alpha-1}} \left(\log 1/\varepsilon\right)^{\frac{1-2/p}{\alpha-1}}.$$

5.2.1 Examples

Example 5.2.3. *Fractional Brownian motion*

Recall from Example 4.1.5, that in the case of fractional Brownian motion with Hurst parameter H, the condition (5.2.1) is satisfied with $\alpha = 2H - 1$. Hence, Theorem 5.2.1 gives us the following small ball results summarized in the table.

norm	bound
L^p	$\varepsilon^{-1/H}$
supremum	$\varepsilon^{-1/H} \left(\log 1/\varepsilon\right)^{1/(2H)}$
Orlicz	$\varepsilon^{-1/H} \left(\log 1/\varepsilon\right)^{(1-2/p)/(2H)}$

According to well-known results (cf. [78], [64], [50]) we have that

$$-\log \mathbb{P}\left(\|X\|_\infty < \varepsilon\right) \approx \varepsilon^{-1/H}. \tag{5.2.2}$$

Hence, Theorem 5.2.1 does not provide a sharp small ball estimates for the sup-norm. The bounds are sharp up to a logarithmic factor.

On the contrary, the L^p estimate is a sharp one. According to Li and Shao ([49], Corollary 1)

$$-\log \mathbb{P}\left(\|X\|_p < \varepsilon\right) \approx \varepsilon^{-1/H}. \tag{5.2.3}$$

Our upper bound for the Orlicz norm coincides with the result obtained in Dunker [14], who also studies the fractional Brownian sheet on higher dimensional spaces. Any sharpness results appear to be unknown in the Orlicz case.

Example 5.2.4. *Ornstein-Uhlenbeck process*

Here, the condition 5.2.1 is satisfied with $\alpha = 2$ (cf. Example 4.1.9). So, the bounds for the small deviation probability $-\log \mathbb{P}(\|Y\| < \varepsilon)$ given by Theorem 5.2.2 are as follows:

norm type	bound
L^p	ε^{-2}
supremum	$\varepsilon^{-2}\left(\log 1/\varepsilon\right)^{1/3}$
Orlicz	$\varepsilon^{-2}\left(\log 1/\varepsilon\right)^{(1-2/p)/3}$

The bound ε^{-2} for the L^p-norm is sharp (cf. [46]). Again, the bound for the supremum norm is only sharp up to a logarithmic factor, the correct bound is ε^{-2} as well (see [47]). No results concerning the Orlicz case are known to us.

Example 5.2.5. *Matérn processes*

Let us begin with the special case of the spectral density, i.e.

$$f(\lambda) = \frac{1}{(\lambda^2+1)^2}. \tag{5.2.4}$$

From the consideration in Example 4.1.10 we have that this process fulfills condition (5.2.1) with $\alpha = 4$. Hence, by virtue of Theorem 5.2.2 we obtain

norm	estimate
L^p	$\varepsilon^{-2/3}$
supremum	$\varepsilon^{-2/3}\left(\log 1/\varepsilon\right)^{1/3}$
Orlicz	$\varepsilon^{-2/3}\left(\log 1/\varepsilon\right)^{(1-2/p)/3}$

For arbitrary $r \in \mathbb{N}$ we have $\alpha = 2r$ and the bounds implied by Theorem 5.2.2 for the process with spectral density

$$f(\lambda) = \frac{1}{(\lambda^2 + 1)^r} \tag{5.2.5}$$

are given by

norm	estimate
L^p	$\varepsilon^{-2/(2r-1)}$
supremum	$\varepsilon^{-2/(2r-1)} (\log 1/\varepsilon)^{1/(2r-1)}$
Orlicz	$\varepsilon^{-2/(2r-1)} (\log 1/\varepsilon)^{(1-2/p)/(2r-1)}$

In view of the results of [51] the bound for the supremum norm is sharp up to a logarithmic factor. We are not aware of any other results on the small deviations behavior of the Matérn process. However, in view of our previous results for fractional Brownian motion and the Ornstein-Uhlenbeck process it seems reasonable to expect that the bounds for the L^p-norm for $p \geq 1$ and Orlicz norm with $p = 2$ are sharp, while the bounds for the Orlicz norm for $p > 2$ are off by a logarithmic factor.

Appendix A

Special functions

In this appendix we gather some general facts concerning several special function appearing throughout the whole text. This presentation is based on Chapter 8-9 of [29] and Chapter 5 of [40], where much more information can be found.

The *Gamma function* (a.k.a. Euler's integral of the second kind) is defined by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \qquad z \in \mathbb{C}, \Re(z) > 0. \tag{A.0.1}$$

It holds that $\Gamma(x+1) = x\Gamma(x)$, hence, for $n \in \mathbb{N}$ we have that $\Gamma(n) = (n-1)!$. Note that $\Gamma(1/2) = \sqrt{\pi}$.

The *Gauss hypergeometric function* $F(\alpha, \beta; \gamma; z)$ is defined by the series

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 1 \cdot 2} z^2 + ..., \quad |z| \leq 1. \tag{A.0.2}$$

Given that $\Re(\alpha + \beta - \gamma) < 0$ the series converges absolutely in the whole unit ball $|z| \leq 1$, if $\Re(\alpha + \beta - \gamma) \in [0, 1)$ it converges in $|z| < 1$, otherwise diverges. It admits the integral representation of the form

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} (1-xz)^{-\alpha} dx, \tag{A.0.3}$$

provided $\Re(\gamma) > \Re(\beta) > 0$. The *incomplete beta function* defined by the formula

$$B_x(\alpha, \beta) = \int_0^x s^{\alpha-1} (1-s)^{\beta-1} ds \tag{A.0.4}$$

can be related to the Gauss hypergeometric function via the formula

$$B_x(\alpha,\beta) = \frac{x^\alpha}{\alpha} F(\alpha, 1-\beta; \alpha+1; x). \tag{A.0.5}$$

The Gegenbauer polynomials C_n^α of degree $n \in \mathbb{N}$ are defined as the coefficients of x^n in the power series expansion of the function

$$(1-2tx+x^2)^{-\alpha} = \sum_{n=0}^{\infty} C_n^\alpha(t) x^n. \tag{A.0.6}$$

We can represent it as an integral

$$C_n^\alpha(t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2\alpha+n)}{n!\Gamma(2\alpha)\Gamma(\alpha)} \Gamma\left(\frac{2\alpha+1}{2}\right) \int_0^\pi (t+\sqrt{t^2-1}\cos\varphi)^n \sin^{2\alpha-1}\varphi \, d\varphi$$

The Gegenbauer polynomials can be related to the Gauss hypergeometric function using the expressions

$$\begin{aligned} C_n^\alpha(t) &= \frac{\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma(n+1)} F\left(2\alpha+n, -n; \alpha+\frac{1}{2}; \frac{1-t}{2}\right), & \text{(A.0.7)} \\ &= \frac{2^n\Gamma(\alpha+n)}{n!\Gamma(\alpha)} t^n F\left(-\frac{n}{2}, \frac{1-n}{2}; 1-\alpha-n; \frac{1}{t^2}\right). & \text{(A.0.8)} \end{aligned}$$

Note that the above formula can be used as a definition of the *generalized Gegenbauer polynomials*, where n can be an arbitrary number. For even and odd degrees we have the following

$$C_{2n}^\alpha(t) = \frac{(-1)^n\Gamma(\alpha)\Gamma(n+1)}{(\alpha+n)\Gamma(\alpha+n+1)} F\left(-n, n+\alpha; \frac{1}{2}; t^2\right), \tag{A.0.9}$$

$$C_{2n+1}^\alpha(t) = \frac{(-1)^n 2t\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} F\left(-n, n+\alpha+1; \frac{3}{2}; t^2\right). \tag{A.0.10}$$

The function $J_\nu(z)$ of a complex variable z defined by

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |\arg z| < \pi, \tag{A.0.11}$$

is called the *Bessel function of the first kind of order* ν. This definition implies the following properties

$$\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z), \quad \frac{d}{dz}[z^{-\nu} J_\nu(z)] = -z^{-\nu} J_{\nu+1}(z), \quad \text{(A.0.12)}$$

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z), \quad J_{\nu-1}(z) - J_{\nu+1}(z) = 2\frac{d}{dz} J_\nu(z), \quad \text{(A.0.13)}$$

$$\frac{d}{dz} J_\nu(z) = -\frac{\nu}{z} J_\nu(z) + J_{\nu-1}(z), \quad \frac{d}{dz} J_\nu(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z). \quad \text{(A.0.14)}$$

Bessel function of the first kind admits the integral representation

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} \cos(zx)(1-x^2)^{\nu-1/2} dx\,, \quad \nu > -\frac{1}{2}, \quad \text{(A.0.15)}$$

often referred to as a *Poisson's integral representation.* For $x \geq 0$ and $\nu \geq 0$ the behavior of function $J_\nu(x)$ is described by

$$J_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(1+\nu)}, \qquad x \to 0, \qquad \text{(A.0.16)}$$

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \nu\pi/2 - \pi/4), \qquad x \to \infty. \qquad \text{(A.0.17)}$$

We also often use a modified Bessel functions. The *modified Bessel function of the first kind of order* ν is defined as

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad z \in \mathbb{C}, \ |\arg z| < \pi. \qquad \text{(A.0.18)}$$

We define the *modified Bessel function of the second kind of order* ν by

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \qquad |\arg z| < \pi, \ \nu \neq 0, \pm 1, \pm 2, ... \quad \text{(A.0.19)}$$

$$K_n(z) = \lim_{\nu \to n} K_\nu(z), \qquad n \in \mathbb{Z} \qquad \text{(A.0.20)}$$

We have that

$$I_{-n}(z) = I_n(z), \qquad n \in \mathbb{Z}, \qquad \text{(A.0.21)}$$

$$K_{-\nu}(z) = K_\nu(z). \qquad \text{(A.0.22)}$$

and for $x > 0$ and $\nu \geq 0$ we have the asymptotic formulas

$$I_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(1+\nu)}, \qquad x \to 0, \tag{A.0.23}$$

$$K_\nu(x) \sim \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu}, \qquad x \to 0, \tag{A.0.24}$$

and as $x \to \infty$

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \tag{A.0.25}$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}. \tag{A.0.26}$$

It holds that the Wronskian is given by

$$\begin{vmatrix} I_\nu(z) & K_\nu(z) \\ \frac{d}{dz} I_\nu(z) & \frac{d}{dz} K_\nu(z) \end{vmatrix} = -\frac{1}{z}. \tag{A.0.27}$$

The so-called addition formula for Bessel functions is given by

$$\frac{J_\nu(\lambda R)}{(\lambda R)^\nu} = 2^\nu \Gamma(\nu) \sum_{l=0}^{\infty} (\nu + l) \frac{J_{\nu+l}(\lambda r_1) J_{\nu+l}(\lambda r_2)}{(\lambda r_1)^\nu (\lambda r_2)^\nu} C_l^\nu(\cos\theta), \tag{A.0.28}$$

where R, r_1, r_2 are sides of an arbitrary triangle and θ is the angle opposite the side R, so that

$$R = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta.} \tag{A.0.29}$$

Appendix B

Fractional calculus

Given a function $f \in L^1([a,b])$ we define respectively right and left fractional integral of the order $\alpha > 0$ by the formulas

$$(I^\alpha_{a+}f)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t f(s)(t-s)^{\alpha-1}ds, \qquad t \geq a,$$
$$(I^\alpha_{b-}f)(t) := \frac{1}{\Gamma(\alpha)}\int_t^b f(s)(t-s)^{\alpha-1}ds, \qquad t \leq b.$$

For any $\alpha \geq$, any $f \in L^p([a,b])$ and $g \in L^q([a,b])$ where $p^{-1} + q^{-1} \leq \alpha$ it holds that

$$\int_0^t f(s)(I^\alpha_{0+}g)(s)ds = \int_0^t (I^\alpha_{t-}f)(s)g(s)ds.$$

This is the so-called fractional integration by parts formula (cf. [76], (5.16)).

We can also define the fractional derivative of a function. For f defined on the interval $[a,b]$, the formulas

$$(\mathcal{D}^\alpha_{a+}f)(t) := \left(\frac{d}{dt}\right)^{[\alpha]+1} I^{1-\{\alpha\}}_{a+}f(t),$$
$$(\mathcal{D}^\alpha_{b-}f)(t) := \left(-\frac{d}{dt}\right)^{[\alpha]+1} I^{1-\{\alpha\}}_{b-}f(t),$$

respectively define the right and left fractional derivative of the function f (provided it exists), where $[\alpha]$ denotes the biggest integer smaller than α and

$\{\alpha\} = \alpha - [\alpha]$. A sufficient condition for the function f to be a.e. α-differentiable is that f is continuously k-differentiable for any integer $k < [\alpha]$ and that $[\alpha]$-th derivative of f is absolutely continuous. The following relations are true for $\alpha > 0$

$$\begin{aligned} \mathcal{D}^{\alpha}_{a+} I^{\alpha}_{a+} f &= f, &\qquad f &\in L^1([a,b]), \\ I^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} f &= f, &\qquad f &\in I^{\alpha}_{a+}\left(L^1([a,b])\right). \end{aligned}$$

According to the above, the derivative $\mathcal{D}^{\alpha}_{a+}$ is often denoted as $I^{-\alpha}_{a+}$.

Bibliography

[1] Addie, R., Mannersalo, P. and Norros, I. (1999). *Performance formulae for queues with Gaussian input.* ITC **16**.

[2] Alòs, E., Mazet, O. and Nualart, D. (2001). *Stochastic calculus with respect to Gaussian processes.* Ann. Probab. **29**(2), 766–801.

[3] Andrews, G. E., Askey, R. and Roy, R. (1999). *Special Functions.* Cambridge University Press.

[4] Barton, R. and Poor, H. (1988). *Signal detection in fractional Gaussian noise.* IEEE Trans. Inform. Theory 34, 943–959.

[5] Baudoin, F. and Nualart, D. (2003). *Equivalence of Volterra processes.* Stochastic Process. Appl. **107**(2), 327–350.

[6] Chentsov, N. N. (1957). *Brownian motion for several parameters and generalized white noise.* Theory Prob. Appl. **2**, p. 265-266.

[7] Cheridito, P. (2004). *Gaussian moving averages, semimartingales and option pricing.* Stochastic Process. Appl. **109**, 47–68.

[8] Boas, R. P. (Jr.) (1954). *Entire functions.* New York, N.Y. : Academic Press Inc.

[9] de Branges, L. (1968). *Hilbert spaces of entire functions*. Prentice-Hall Inc., Englewood Cliffs.

[10] Decreusefond, L. and Üstünel, A.S. (1999). *Stochastic analysis of the fractional Brownian motion.* Potential Anal. **10**(2), 177–214.

[11] Doob, J.L. (1953). *Stochastic processes*. John Wiley & Sons Inc., New York.

[12] Doukhan, P., Oppenheim, G. and Taqqu, M. S. (2002). *Theory and Applications of Long-Range Dependence.* Birkhauser, Boston.

[13] Dunker, T., Linde, W., Kühn, T. and Lifshits, M. A. (1999). *Metric Entropy of Integration Operators and Small Ball Probabilities for the Brownian Sheet.* Journal of Approximation Theory 101, 63-77.

[14] Dunker, T. (2000). *Estimates for the small ball probabilities of the fractional Brownian sheet.* J. Theoret. Probab. 13, no. 2, 357–382.

[15] Dunkl, C. F. and Xu, Y. (2001). *Orthogonal Polynomials of Several Variables.* Cambridge University Press.

[16] Dym, H. (1971). *An Introduction to de Branges Spaces of Entire Functions with Applications to Differential Equations of the Sturm-Liouville Type*. Advances in Mathematics, **5**, 395–471.

[17] Dym, H. and McKean, H.P. (1976). *Gaussian processes, function theory, and the inverse spectral problem*. Academic Press, New York.

[18] Dzhaparidze, K. and Ferreira, J.A. (2002). *A frequency domain approach to some results on fractional Brownian motion.* Statist. Probab. Lett. **60**, 229–235.

[19] Dzhaparidze, K. and Van Zanten, J.H. (2004). *A series expansion of fractional Brownian motion.* Probab. Theory Related Fields **130**(1), 39–55.

[20] Dzhaparidze, K., van Zanten, J. H. (2005). *Krein's spectral theory and the Paley-Wiener expansion for fractional Brownian motion.* Ann. Probab. 33, no. 2, 620–644.

[21] Dzhaparidze, K., van Zanten, J. H. (2005). *Optimality of an explicit series expansion of fractional Brownian sheet.* Statist. Prob. Lett. 71, 295-301.

[22] Dzhaparidze, K., van Zanten, H. and Zareba, P. (2005). *Representations of fractional Brownian motion using vibrating strings.* Stochastic Processes and their Applications 115, 1928-1953.

[23] Dzhaparidze, K., van Zanten, H. and Zareba, P. (2006) *Representations of isotropic Gaussian random fields with homogeneous increments.* J. Appl. Math. Stoch. Anal. 2006, Art. ID 72731.

[24] Egorov, Yu. V. and Shubin, M. A. (1998). *Foundations of the Classical Theory of Partial Differential Equations.* Springer-Verlag Brerlin Heidelberg.

[25] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953). *Higher transcendental functions. Vol. II.* McGraw-Hill Book Company, Inc., New York-Toronto-London.

[26] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. (1954). *Tables of Integral Transforms* Vol. I and II, McGraw-Hill, New York.

[27] Feyel, D. and de la Pradelle, A. (1999). *Fractional Integrals and Brownian Processes.* Potential Analysis, vol.10, p.273-288

[28] Golosov, J. and Molchan, G. (1969). *Gaussian stationary processes with asymptotic power spectrum.* Soviet Mathematics Doklady **10**(1), 134-137.

[29] Gradshteyn, I.S. and Ryzhik, I.M. (1980). *Table of integrals, series, and products*. Academic Press, New York. Corrected and enlarged ed.

[30] Hochstadt, H. (1971). *The Functions of Mathematical Physics.* John Wiley & Sons, Inc.

[31] Hult, H. (2003). *Approximating some Volterra type stochastic integrals with applications to parameter estimation.* Stochastic Process. Appl. **105**(1), 1–32.

[32] Ibragimov, I.A. and Rozanov, Y.A. (1978). *Gaussian stochastic processes.* Springer-Verlag.

[33] Kac, I.S. and Krein, M.G. (1974). *On the spectral functions of the string.* Amer. Math. Soc. Transl. **103**(2), 19–102.

[34] Karhunen, K. (1947) *Über linear Methoden in der Wahrscheinlichkeitsrechung.* Ann. Acad. Scient. Fennicae, **37**, pp. 1-79.

[35] Kilpi, J. and Norros, I. (2002). *Testing the Gaussian approximation of aggregate traffic.* 2nd Internet measurement workshop, Marseille.

[36] Kolmogorov, A. N. (1940). *Curves in Hilbert space which are invariant with respect to one-parameter group motion.* Dokl. Akad. Nauk SSSR, **26**(1), 6–9.

[37] Kolmogorov, A. N. (1940). *Wiener's spiral and some interesting curves in Hilbert space.* Dokl. Akad. Nauk SSSR, **26**(2), 9–13.

[38] Krein, M.G. (1953). *On some cases of the effective determination of the density of a non-homogeneous string by its spectral function.* Dokl. Akad. Nauk SSSR 94 **93**, 617–620

[39] Kühn, T. and Linde, W. (2002). *Optimal series representation of fractional Brownian sheet.* Bernoulli 8(5), 669-696.

[40] Lebedev, N.N. (1972). *Special functions and their applications.* Dover Publications Inc., New York.

[41] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach spaces. Isoperimetry and processes.* Springer-Verlag, Berlin.

[42] Leland, W. E., Taqqu, M. S., Willinger, W. and Wilson, D. V. (1994). *On the self-similar nature of ethernet traffic (extended version).* IEEE/ACM Transactions on Networking 2, 115.

[43] Leonenko, N. N. (1999). *Limit Theorems for Random Fields with Singular Spectrum.* Kluwer Academic Publishers, Dordrecht/Boston/London.

[44] Lévy, P. (1948). *Processus Stochastique et Mouvement Brownien.* Paris, Gauthier-Villars.

[45] Li, W. V. (1992). *Comparison results for the lower tail of Gaussian seminorms.* J. Theoret. Probab. 5, no. 1, 1–31.

[46] Li, W. V. (2001). *Small ball probabilities for Gaussian Markov processes under the L_p-norm.* Stochastic Process. Appl. 92, no. 1, 87–102.

[47] Li, W. V. (1999). *Small deviations for Gaussian Markov processes under the sup-norm.* J. Theoret. Probab. 12, no. 4, 971–984.

[48] Li, W. V. and Linde, W. (1999). *Approximation, Metric Entropy and Small Ball Estimates for Gaussian Measures.* The Annals of Probability 27, No. 3, 1556-1578.

[49] Li, W. V. and Shao, Q.-M. (1999). *Small ball estimates for Gaussian processes under Sobolev type norms.* J. Theoret. Probab. 12, no. 3, 699–720.

[50] Li, W. V. and Shao, Q.-M. (2001). *Gaussian processes: inequalities, small ball probabilities and applications.* Stochastic processes: theory and methods, 533–597, Handbook of Statist., 19, North-Holland, Amsterdam.

[51] Lifshits, M.A. and Tsirelson, B.S. (1986). *Small deviations of Gaussian fields.* Theory Probab. Appl. 31:3, 557-558.

[52] Lifshits, M.A. (1995). *Gaussian Random Functions.* Dordrecht: Kluwer.

[53] Lifshits, M.A. (2007). *Bibliography of small deviation probabilities.* Available at `www.proba.jussieu.fr/pageperso/smalldev/biblio.pdf`

[54] Malyarenko, A. (2005). *A series expansion of a certain class of isotropic gaussian random fields with homogeneous increments.* Preprint. `http://arxiv.org/PS_cache/math/pdf/0411/0411539v3.pdf`

[55] Mandelbrot, B. B. and van Ness, J. W. (1968). *Fractional Brownian motions, fractional noises and applications.* SIAM Rev. 10, 422–437.

[56] Mannersalo, P. and Norros, I. (2002). *A most probable path approach to queueing systems with general Gaussian input.* Comp. Networks **40**(3), 399–412.

[57] McKean, H. P. (1963). *Brownian motion with a several dimensional time.* Theory Prob. Appl. **8**, p. 335-354.

[58] Mémin, J., Mishura, Y. and Valkeila, E. (2001). *Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion.* Statist. Probab. Lett. **51**(2), 197–206.

[59] Mercer, T. (1909) *Functions of positive and negative type and their connection with the theory of integral equation.* Trans. London Phil. Soc., (A), **29**, pp. 415–446.

[60] Molchan, G. M. (1967). *On some problems concerning Brownian motion in Lévy's sense*, Theory Probab. Appl., 12 , pp. 682-695.

[61] Molchan, G. M. (1969). *Gaussian processes with asymptotic power spectrum.* Summaries of papers presented at the sessions of the probability and statistic section of the Moscow Mathematical Society (February - December 1968) Theory Probab. Appl. **14**(3), 556–549.

[62] Molchan, G.M. (2002). *Linear problems for fractional Brownian motion: a group approach.* Teor. Veroyatnost. i Primenen. **47**(1), 59–70.

[63] Molchan, G.M. (2003). *Historical comments related to fractional Brownian motion.* In Theory and applications of long-range dependence, pp. 39–42. Birkhäuser, Boston.

[64] Monrad, D. and Rootzn, H. (1995). *Small values of Gaussian processes and functional laws of the iterated logarithm.* Probab. Theory Related Fields 101 , no. 2, 173–192.

[65] Montanari, A. (2003). *Long-range dependence in hydrology.* Theory and applications of long-range dependence, 461–472. Birkhuser Boston, Boston, MA.

[66] Nédélec, A. (2001). *Acoustic and Electromagnetic Equations.* Springer.

[67] Norros, I. (1995). *On the use of fractional Brownian motion in the theory of connectionless networks.* IEEE J. Sel. Ar. Commun. **13**(6), 1995.

[68] Norros, I., Valkeila, E. and Virtamo, J. (1999). *An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions.* Bernoulli **5**(4), 571–587.

[69] Novikov, A. and Valkeila, E. (1999). *On some maximal inequalities for fractional Brownian motions.* Statist. Probab. Lett. **44**(1), 47–54.

[70] Nuzman, C. J. and Poor, H. V. (2000). *Linear estimation of self-similar processes via Lamperti's transformation.* J. Appl. Probab. **37**(2), 429–452.

[71] Paley, R.E. and Wiener, N. (1934). *Fourier transforms in the complex domain.* Am. MAth Soc. VIII, New York.

[72] Pinsker, M.S. and Yaglom, A.M. (1954). *On linear extraploation of random processes with stationary n-th inctements.* Dokl. Akad. Nauk SSSR **100**(3), 546–549 (reprinted in Soviet Math. Dokl. **10**(1), 134-137)

[73] Pipiras, V. and Taqqu, M.S. (2001). *Are classes of deterministic integrands for fractional Brownian motion on an interval complete?* Bernoulli **7**(6), 873–897.

[74] Pisier, G. (1986). *Factorization of linear operators and geometry of Banach spaces.* CBMS Regional Conference Series in Mathematics, 60. Published for the Conference Board of the Mathematical Sciences, Washington, DC.

[75] Pisier, G. (1989). *The volume of convex bodies and Banach space geometry.* Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge.

[76] Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993). *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon.

[77] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable non-Gaussian random processes*. Chapman & Hall, New York.

[78] Shao, Q. M. (1993). *A note on small ball probability of a Gaussian process with stationary increments.* J. Theoret. Probab. 6, no. 3, 595–602.

[79] Sottinen, T. and Valkeila, E. (2001). *Fractional Brownian motion as a model in finance.* University of Helsinki, Department of Mathematics, Preprint 302.

[80] Sottinen, T. (2004). *On Gaussian processes equivalent in law to fractional Brownian motion.* J. Theoret. Probab. **17**(2), 309–325.

[81] Stieltjes, T. J. *Recherches sur les fractions continues.* Ann. Fac. Sci. Toulouse **8**(1894), 1–122; **9**(1895), 1–47. Reprinted in: Oeuvres Completes de Thomas Jan Stieltjes, Vol. 2, Nijhoff, The Hague, 1918.

[82] Szegö, G. (1959). Orthogonal polynomials. *American Mathematical Society Colloquium Publications, Vol. 23. Revised ed. American Mathematical Society, Providence, R.I.*

[83] van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Processes.* Springer-Verlag.

[84] van der Vaart, A.W. and van Zanten, J. H. (2007). *Reproducing Kernel Hilbert Spaces of Gaussian Priors.* To appear in IMS Lecture Notes – Monograph Series.

[85] Vilenkin, N. J. (1968). *Special Functions and the Theory of Group Representations.* American Mathematical Society, Providence, Rhode Island.

[86] Vladimirov, V. S. (1984). *Equations of Mathematical Physics.* Mir Publishers, Moscow.

[87] Watson, G.N. (1944). *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Cambridge, England.

[88] Whitt, W. (2002). *Stochastic-process limits.* Springer-Verlag, New York.

[89] Widder, D. (1946). *The Laplace Transform.* Princeton, New Jersey.

[90] Willinger, W., Taqqu, M. S. and Teverovsky, V. (1999). *Stock market prices and long-range dependence.* Finance Stoch. **3**(1), 113.

[91] Wolpert, R. L. and Taqqu, M. S. (2005). *Fractional Ornstein-Uhlenbeck Lévy processes and the Telecom process: upstairs and downstairs.* Signal Processing, Volume 85, Issue 8, 1523 - 1545.

[92] Yadrenko, M. I. (1980). *Spectral Theory of Random Fields.* Optimization Softwere, Inc.

[93] Yaglom, A.M. (1957). *Some classes of random fields in n-dimensional space, related to stationary random processes.* Theory Prob. Appl. **2**, p. 273-320.

[94] Yaglom, A.M. (1987). *Correlation theory of stationary and related random functions. Vol. I.* Springer-Verlag, New York.

[95] van Zanten, J. H. (2007). *When is a linear combination of independent fBm's equivalent to a single fBm?* Stochastic Processes and Their Applications 117, no. 1, 57–70.

Samenvatting (Summary in Dutch)

Representaties van Gaussische processen met stationaire incrementen

Dit proefschrift beschrijft een nieuwe methode om een reeks- en een 'moving average'-representatie van een bepaald Gaussisch proces te vinden. De representatie wordt geintroduceerd voor de klassen van stationaire processen en processen met stationaire incrementen. De methode wordt ook toegepast op Gaussische isotrope velden. In het geval van de reeksontwikkelings wordt de convergentiesnelheid bepaald, met behulp waarvan een aantal 'small ball'-resultaten verkregen worden.

Dit onderzoek werd gemotiveerd door resultaten voor de fractionele Brownse beweging (fBm), het belangrijkste voorbeeld van een Gaussisch proces met stationaire incrementen. Deze processen hebben talrijke toepassingen in bijvoorbeeld de wachtrijtheorie, financiële wiskunde en telecommunicatienetwerken.

De analyse van deze processen is gecompliceerd, aangezien ze in het algemeen noch een Markov proces, noch een semimartingaal zijn, zodat veel klassieke methodes niet kunnen worden gebruikt. Daarom is een representatie in termen van eenvoudigere, beter begrepen processen zeer wenselijk.

Voor het vinden van de representaties maken we gebruik van de Krein-de Branges theorie. In de context van dit proefschrift is het meeste relevante resultaat uit deze theorie het bestaan van een 1-1 relatie tussen de spectraal maat van een bepaald proces en een snaar met bepaalde massaverdeling. We combineren deze relatie met de theorie van 'reproducing kernel Hilbert spaces' van analytische functies.

Hoofdstuk 1 begint met een introductie van de in de tekst gebruikte kanstheoretische begrippen, zoals de spectrale representatie van stationaire Gaussische processen en processen met stationaire incrementen. Vervolgens worden isotrope velden en hun spectrale representatie gedefinieerd. In dit hoofdstuk geven we voorbeelden van processen en velden die we later gebruiken om de algemene resultaten te illustreren.

Hoofdstuk 2 is volledig gewijd aan de Krein-de Branges theorie en haar toepassing op onze vraagstelling. Eerst behandelen we het bekende resultaat dat er 1-1 relatie bestaat tussen snaren en symmetrische Borel maten op de reële rechte. Vervolgens beschrijven we de met deze snaren geassocieerde 'reproducing kernel Hilbert spaces'.

Hoofdstuk 3 behandelt de snaren die zijn gerelateerd aan de spectrale maten van specifieke processen, zoals de Brownse beweging, het Ornstein-Uhlenbeck proces, of de fBm.

De belangrijkste resultaten van dit proefschrift, namelijk de representatie van processen en velden, zijn verzameld in Hoofdstuk 4. We gebruiken de algemene resultaten van Hoofdstuk 2 voor het afleiden van reeks- en 'moving average'-representaties van Gaussisch processen (stationair en met stationaire incrementen) en isotrope velden (homogeen en met homogene incrementen). De in hoofdstuk 3 verkregen snaren worden gebruikt voor het vinden van de representatie voor bepaalde voorbeelden.

In hoofdstuk 5 worden de convergentiesnelheden van de reeksontwikkelingen gebruikt om de zogeheten 'small ball probabilities' te bepalen voor diverse processen en normen.