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Everything you always wanted to know about a-posteriori error estimation in finite element methods, but were afraid to ask

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ABSTRACT
In this paper the basic concepts to obtain a posteriori error estimates for the finite element method are reviewed. *Explicit residual-based, implicit* (namely *subdomain* and *element*) *residual-based, hierarchical-based, recovery-base* and *functional-based* error estimators as well as *goal oriented* error estimators are presented for a test elliptic boundary value problem.

These notes are an introductory presentation, reviewing in a not-too-technical way the fundamental concepts involved in the subject and do not aim at being exhaustive or complete but rather simple and easy to follow. For more detailed explanations, we refer the interested reader to [3] and eventually to [4],[9],[15],[27],[30], chapter 4 of [37],[38],[39] – and the references therein – where most of the material contained in this report can be found.

2000 Mathematics Subject Classification: 74S05, 65N30
Keywords and Phrases: finite element method; a-posteriori error estimation; goal-oriented error estimation
Note: This work was carried out under project MAS3.2 - 'Scientific Computing for Systems Biology'.

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L. Ferracina*

Abstract

In this paper the basic concepts to obtain a posteriori error estimates for the finite element method are reviewed. *Explicit residual-based*, *implicit* (namely *subdomain* and *element*) *residual-based*, *hierarchical-based*, *recovery-base* and *functional-based* error estimators as well as *goal oriented* error estimators are presented for a test elliptic boundary value problem.

These notes are an introductory presentation, reviewing in a not-too-technical way the fundamental concepts involved in the subject and do not aim at being exhaustive or complete but rather simple and easy to follow. For more detailed explanations, we refer the interested reader to [3] and eventually to [4],[9],[15],[27],[30], chapter 4 of [37],[38],[39] – and the references therein – where most of the material contained in this report can be found.

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1 Introduction

1.1 The model problem and its discretization

For the purpose of this paper, it will be sufficient to consider the elliptic boundary value problem

$$-\nabla \cdot (D\nabla u) + cu = f \quad \text{on} \quad \Omega \tag{1.1.a}$$

$$D\frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma_N \tag{1.1.b}$$

$$u = 0 \quad \text{on} \quad \Gamma_D \tag{1.1.c}$$

in a two dimensional domain Ω with Lipschitz boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$, $\Gamma_N \cap \Gamma_D = \varnothing$, where Γ_D is close relative to $\partial\Omega$ and it has positive length. Moreover we assume that the data of the problem are sufficiently smooth, that is $f \in L_2(\Omega)$, $g \in L_2(\Gamma_N)$, coefficient c and matrix D are constant non negative.

Here, and in the following, given any open subset ω of Ω (with γ part of its Lipschitz boundary), $L_2(\omega)$, $H^1(\omega)$ and $L_2(\gamma)$ denote the standard Lebesgue and Sobolev spaces equipped respectively

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with the usual norms

$$\|\varphi\|_{L_2(\omega)} = \left\{ \int_\omega \varphi^2 \mathrm{d}x \right\}^{1/2}, \tag{1.2.a}$$

$$\|\varphi\|_{H^1(\omega)} = \left\{ \int_\omega \varphi^2 + |\nabla\varphi|^2 \mathrm{d}x \right\}^{1/2}, \tag{1.2.b}$$

$$\|\varphi\|_{L_2(\gamma)} = \left\{ \int_\gamma \varphi^2 \mathrm{d}x \right\}^{1/2}. \tag{1.2.c}$$

Moreover, we set $H^1_{0,\gamma}(\omega) = \{\varphi \in H^1(\omega) \,|\, \varphi = 0 \text{ on } \gamma\}$.

The standard weak formulation of problem (1.1) is then: find $u \in V \doteq H^1_{0,\Gamma_D}(\Omega)$ such that

$$B(u,v) = L(v) \quad \forall v \in V \tag{1.3.a}$$

where the bilinear form $B(.,.)$ and the functional $L(.)$ are defined by

$$B(u,v) = \int_\Omega D\nabla u \cdot \nabla v + cuv \,\mathrm{d}x, \tag{1.3.b}$$

$$L(v) = \int_\Omega fv + \int_{\Gamma_N} gv \,\mathrm{d}x. \tag{1.3.c}$$

For further reference we recall here that the energy norm of the problem – a weighted H^1 norm – is defined by

$$|||\varphi||| = \sqrt{B(\varphi,\varphi)}. \tag{1.4}$$

The finite element approximation of problem (1.3) is obtained by replacing the functional space V with a finite dimensional subspace $V^h \subseteq V$. In constructing V^h we consider a decomposition $\mathcal{T}^h$ of $\overline{\Omega}$ into triangles K with diameter h_K and we denote the size of the triangulation with $h = max_{K\in\mathcal{T}^h} h_K$. We assume that Ω has a polygon boundary and that the triangulation $\mathcal{T}^h$ has the standard properties used in the finite element method – i.e. any two triangles in $\mathcal{T}^h$ share at most a common edge or a common vertex and the minimal angle of all triangles in $\mathcal{T}^h$ is bounded away from zero, see, e.g., [22]. We introduce here also the following notation which will be used throughout the paper: for any triangle $K \in \mathcal{T}^h$ we denote with ∂K the set of its edges and with $\partial\mathcal{T}^h = \bigcup_{K\in\mathcal{T}^h} \partial K$ and we indicate with $\mathring{\partial}\mathcal{T}^h$ the internal edges of the triangulation, i.e., $\mathring{\partial}\mathcal{T}^h = \partial\mathcal{T}^h \cap \Omega$.

We then define the finite (dimensional) element space by:

$$V^h = V^h(\mathcal{T}^h) = \{v \in \mathscr{C}(\overline{\Omega}); \quad v|_K \in \Pi_q(K) \text{ for all } K \in \mathcal{T}^h\}. \tag{1.5}$$

Above $\mathscr{C}(\overline{\Omega})$ is the space of the continuous functions defined on $\overline{\Omega}$ and $\Pi_q(K)$ is the space of all polynomials of degree $\leq q$ defined on an element K.

The finite element approximation of problem (1.3) is then: find $u \in V^h$ such that

$$B(u^h, v^h) = L(v^h) \quad \forall v^h \in V^h. \tag{1.6}$$

1.2 The scope of this paper: a posteriori error estimators

The discretization process of transforming problem (1.3) into problem (1.6) naturally loses information, the result being that the numerical approximation u^h differs from the exact solution u. This approximation error can be so big to completely invalidate numerical predictions.

Over the last few decades, mathematical theories and computational procedure have been proposed for estimating this error. Clearly, if the error can be correctly estimated, then it should be possible to enhance the approximation so to reduce the error, i.e., error estimation is the heart of adaptive computational methods.

Basically there are two types of error estimation procedures. So called *a priori* error estimates aims at estimating the error to be expected in a computation to be done. Accordingly, these kind of estimators provide information on the asymptotic behavior of the discretization errors without involving the finite element solution u^h which is not at hand. On the contrary, *a posteriori* error estimators employ the finite element solution itself to derive estimates of the actual discretization error.

The focus of this report is on a posteriori error estimators. Accordingly, from now on, we will assume that u^h (solution of problem (1.6)), approximation to u (solution of problem (1.3)), is given and we will review several strategies to compute a-posteriori error estimators $\mathcal{E} \approx \|e\| = \|u - u^h\|$ with $\|.\|$ appropriate norm - for the finite element process itself and its properties we refer the reader to classical books, e.g., [5],[16],[20],[17],[19],[18],[22],[28],[39].

In surveying and comparing different approaches to error estimations, we will say that an error estimator $\mathcal{E}$:

- is *reliable* if there exist a positive constant C_r such that $\|e\| \leq C_r\, \mathcal{E}$;
- is *efficient* if there exist a positive constant C_e such that $\mathcal{E} \leq C_e\, \|e\|$;
- is *asymptotically correct* if it is reliable and efficient so that with increasing the size of the finite dimensional space V^h – either refining the underlying triangulation $\mathcal{T}^h$ or increasing the polynomial order p – the estimated error decreases to zero at the same rate of the actual error;
- is *asymptotically exact* if it is correct and with, increasing the size of the finite dimensional space V^h, $C_r = C_e = 1$ so that the estimated error $\mathcal{E}$ converge to the true error;
- should yield *guaranteed* and *sharp* upper and lower bound of the actual error;
- should be *simple* and *inexpensive* when compared with the rest of the computations;
- should be *robust* with regard to a wide range of applications;
- should be sufficiently accurate to steer an *adaptive process* when using the error indications to optimize the underling triangulation (and eventually the local polynomial degree) with respect to the goal of the computation.

Unfortunately, an ideal error estimator that meets all these requirements is not yet available. For the time being, each finite element user will have to decide, depending on the particular problem he is dealing with, the priorities he has (and his particular taste), which properties are indispensable and, accordingly, choose a specific error estimator.

There are several strategies which have evolved during the last decades to construct different error estimators (with different properties) for the finite element method. Few strategies employ specific properties of the finite element method namely the following two fundamental relations of the error $e = u - u^h$:

$$B(e, v) = B(u - u^h, v) = B(u, v) - B(u^h, v) = L(v) - B(u^h, v) \quad \forall v \in V \tag{1.7.a}$$

$$B(e, v^h) = L(v^h) - B(u^h, v^h) = 0 \quad \forall v^h \in V^h \tag{1.7.b}$$

The last relation is known as the orthogonality condition of the error e with respect to the finite element space V^h. *Explicit* and *implicit residual-based* and *hierarchical-based* error estimators are of this type.

Other strategies, on the other hand, are derived without making use of (1.7) and because of that they are suitable for more general methods than the finite element: *recovery-base* and *functional-based* error estimators belong to this second group.

Throughout this paper, the concepts are presented for the test problem (1.1) in a two dimensional domain. Most of the ideas can be easily extended to higher dimensions while the extension of the results to more general problems is less straightforward. We refer the reader, e.g., to [38] for a simple introduction to error estimation for quasilinear equations of second order; for time-dependent complications we refer, e.g., to [23],[24] where an adaptive space-time finite element method is presented for parabolic problems; in [6],[7],[12],[13] – see also [1],[2][35],[36] – the (adaptive) finite element method of lines approach was developed and analyzed for parabolic problems; finally we refer, e.g., to [14],[25],[34] for the error estimation of numerical solutions of systems of diffusion-reaction equations.

1.3 Outline of the rest of the paper

In this paper we review the five strategies mentioned above for computing a posteriori error estimations in finite element methods. Accordingly, in Section 2 we review explicit residual-based error estimators.

Implicit residual-based error estimator are the subject of Section 3. Here we distinguish two different methods: the subdomain residual method and the element residual methods. In the first approach, a series of the auxiliary local (boundary value) problems with *(homogeneous) Dirichlet boundary conditions* defined on small (overlapping) *patches* have to be solved. Similarly, in the element residual methods the boundary values problems are posed over a *single element* but, this time, with appropriate *Neumann boundary conditions*.

Section 4 is devoted to hierarchical-based error estimation.

In Section 5 recovery-based error estimators are discussed. Here the particular procedure introduced by Zienkiewicz and Zhu, the *superconvergent patch recovery* technique, is presented.

In Section 6 we give a short overview of the basic concepts to obtain functional-based error estimators.

Finally the last section, Section 7, is dedicated to summarize and compare the different error estimators exposed in previous sections.

In the appendix we shed some light on goal-oriented error estimation.

2 Explicit residual-based error estimation

Explicit residual-based error estimation were introduced by Babuška and Rheinboldt already in 1979 in [8]. In this approach two are the main ingredients: the basic relations (1.7), rewritten at element level, and bound properties of a special projection $\mathcal{P}_h$ from V to the finite elemtent space V^h.

We start with rewriting (1.7.a) element by element:

$$\begin{aligned}
B(e,v) &= L(v) - B(u^h,v) = \int_\Omega fv + \int_{\Gamma_N} gv\,\mathrm{d}x - \int_\Omega D\nabla u^h\cdot\nabla v + cu^h v\,\mathrm{d}x = \\
&= \sum_{K\in\mathcal{T}^h}\int_K fv + \int_{\partial K\cap\Gamma_N} gv\,\mathrm{d}x - \int_K \nabla Du^h\cdot\nabla v + cu^h v\,\mathrm{d}x = \\
&= \sum_{K\in\mathcal{T}^h}\int_K fv + \int_{\partial K\cap\Gamma_N} gv\,\mathrm{d}x + \int_K \nabla\cdot(\nabla Du^h)v - cu^h v\mathrm{d}x - \int_{\partial K} D\frac{\partial u^h}{\partial n_k}v\,\mathrm{d}s = \\
&= \sum_{K\in\mathcal{T}^h}\int_K \Big(\nabla\cdot(D\nabla u^h) - cu^h + f\Big)v + \int_{\partial K\cap\Gamma_N}\Big(g - D\frac{\partial u^h}{\partial n_k}\Big)v\,\mathrm{d}s - \int_{\partial K\cap\mathring{\partial}\mathcal{T}^h} D\frac{\partial u^h}{\partial n_k}v\,\mathrm{d}s = \\
&= \sum_{K\in\mathcal{T}^h}\int_K R_{int}(u^h)v + \int_{\partial K} R_{bou}(u^h)v\,\mathrm{d}s. \qquad (2.1)
\end{aligned}$$

Here $R_{int}(u^h)$ and $R_{bou}(u^h)$ are respectively the internal and the boundary residual (of u^h) defined on each triangle K and its boundary ∂K by

$$R_{int} = R_{int}(u^h) = D\nabla\cdot(\nabla u^h) - cu^h + f; \qquad (2.2.a)$$

$$R_{bou} = R_{bou}(u^h) = \begin{cases} g - D\frac{\partial u_h}{\partial n} & \text{for } \partial K\cap\Gamma_N, \\ -\frac{1}{2}[\![D\frac{\partial u^h}{\partial n}]\!]_\gamma & \text{for } \gamma\in\mathring{\partial}\mathcal{T}^h. \end{cases} \qquad (2.2.b)$$

In equation (2.2.b) the quantity $[\![\frac{\partial u^h}{\partial n}]\!]_\gamma$ denotes the jump of the gradient of u^h among two triangles of the partition $\mathcal{T}^h$ sharing the same internal edge γ – and the factor $\frac{1}{2}$ is to distribute the error equally onto the two elements.

Consider then the projector operator $\mathcal{P}_h : V \longrightarrow V^h$ with the following two properties:

$$\|v - \mathcal{P}_h(v)\|_{L_2(K)} \le c_{int}h_K\|v\|_{H^1(\omega_K)} \qquad (2.3.a)$$

$$\|v - \mathcal{P}_h(v)\|_{L_2(\gamma)} \le c_{bou}h_K^{1/2}\|v\|_{H^1(\omega_K)} \qquad (2.3.b)$$

where c_{int}, c_{bou} are positive constant, γ is any edge of the triangle K and ω_K denotes the patch of elements associate with K, i.e. $\omega_K = \bigcup_{K'\cap K\neq\varnothing} K'$. We note here that, if for one dimensional problems one can use as $\mathcal{P}_h$ the standard Lagrange interpolation operator, in higher dimension the existence of an operator $\mathcal{P}_h$ with properties (2.3) is not immediate – see for details, e.g, [21],[11],[4].

From (1.7.b) and (2.1) we obtain

$$B(e,\mathcal{P}_h(v)) = 0 = \sum_{K\in\mathcal{T}^h}\int_K R_{int}(u^h)\mathcal{P}_h(v) + \int_{\partial K} R_{bou}(u^h)\mathcal{P}_h(v)\,\mathrm{d}s \qquad (2.4)$$

Combining (2.4) with (2.1), we finally obtain

$$\begin{aligned}
B(e,v) &= B(e,v) - B(e,\mathcal{P}_h(v)) \qquad (2.5)\\
&= \sum_{K\in\mathcal{T}^h}\int_K R_{int}(u^h)(v-\mathcal{P}_h(v)) + \int_{\partial K} R_{bou}(u^h)(v-\mathcal{P}_h(v))\,\mathrm{d}s \qquad (2.6)\\
&\le \sum_{K\in\mathcal{T}^h}\|R_{int}(u^h)\|_{L_2(K)}\|v-\mathcal{P}_h(v)\|_{L_2(K)} + \|R_{bou}(u^h)\|_{L_2(\partial K)}\|v-\mathcal{P}_h(v)\|_{L_2(\partial K)} \qquad (2.7)
\end{aligned}$$

Inserting estimates (2.3) and applying the Cauchy-Schwarz inequality leads

$$B(e,v) \leq \left\{ \sum_{K\in\mathcal{T}^h} c_{int}^2 h_K^2 \|R_{int}\|_{L_2(K)}^2 + c_{bou}^2 h_K \|R_{bou}\|_{L_2(\partial K)}^2 \right\}^{1/2} \left\{ \sum_{K\in\mathcal{T}^h} \|v\|_{H^1(\omega_K)}^2 \right\}^{1/2} \tag{2.8}$$

$$\leq c_{T^h} \|v\|_{H^1(\Omega)} \left\{ \sum_{K\in\mathcal{T}^h} c_{int}^2 h_K^2 \|R_{int}\|_{L_2(K)}^2 + c_{bou}^2 h_K \|R_{bou}\|_{L_2(\partial K)}^2 \right\}^{1/2} \tag{2.9}$$

where the constant c_{T^h} depends only on the smallest angle in the triangulation $\mathcal{T}^h$. Finally, using the fact that $|||v||| \geq C_{c,D}\|v\|_{H^1(\Omega)}$ (with $C_{c,D}$ depending only on the coefficients D and c), we obtain, for $v = e$,

$$|||e||| \leq \frac{c_{T^h}}{C_{c,D}} \sum_{K\in\mathcal{T}^h} \left\{ c_{int}^2 h_k^2 \|R_{int}\|_{L_2(K)}^2 + \frac{1}{2} c_{bou}^2 h_k \|R_{bou}\|_{L_2(\partial K)}^2 \right\}^{1/2} \tag{2.10}$$

This clearly suggests to define an (explicit a posteriori) estimator $\mathcal{E}(u^h(t))$ (approximation of the energy norm of the error $|||e|||$) as

$$\mathcal{E}(u^h(t)) = \frac{c_{T^h}}{C_{c,D}} \left\{ \sum_{K\in\mathcal{T}^h} \eta_K^2(u^h(t)) \right\}^{1/2} \tag{2.11}$$

where $\eta_K(u^h(t))$ is the (local) error indicator defined on the triangle $K \in \mathcal{T}^h$ by

$$\eta_K^2(u^h(t)) = c_{int}^2 h_K^2 ||R_{int}(u^h(t))||_{L_2(K)}^2 + c_{bou}^2 h_K ||R_{bou}(u^h(t))||_{L_2(\partial K)}^2. \tag{2.12}$$

3 Implicit residual-based error estimation

Differently from explicit error estimations, that can be computed directly from the finite element solution (and the data of the problem), implicit error estimators involve the solution of auxiliary boundary value problems whose solution yields an approximation to the actual error.

The starting point of implicit residual-based error estimators is, once more, relation (1.7.a). Having at hand the finite element solution u^h, relation (1.7.a) can be seen as a problem of the type (1.3), now for the unknown e, with a modified right hand side. The idea is then to solve this new auxiliary problem, once more with the finite element method, to get approximation for the error e.

If u^h is the finite element solution on $V^h \subseteq V$, computing the approximation of e once more on V^h (or any subspace of V^h), will give only the trivial solution $e^h = 0$. On the other hand, it is clear that there is no reason for solving (1.7.a) on a space $\tilde{V}^h$ larger than V^h: it would be better to solve directly the original problem (1.3) more accurately on $\tilde{V}^h$ rather than simply estimating the error. An alternative is to replace the *global* problem (1.7.a) with a sequence of problems in *smaller local* domains. Depending on the choice of the *smaller local* domains we distinguish two different methods: the *subdomain residual method* and the *element residual methods*.

With the first approach, the global problem (1.7.a) is decomposed into local problems with *(homogeneous) Dirichlet boundary conditions* defined on small (overlapping) *patches*. Similarly, in the element residual methods the boundary values problems are local in that they are posed over a *single element* this time with appropriate *Neumann boundary conditions*.

3.1 Subdomain residual method

In deriving the subdomain residual method, an essential role is played by the so called partition of unity property, namely:

$$\sum_{i=1}^{N} \varphi_i(x) = 1 \quad \forall x \in \Omega \tag{3.1}$$

where $\varphi_i(x)$, $i = 1, 2, ..., N$ are the first order finite element shape functions corresponding to nodes of the triangulation ν_i (i.e. φ_i are linear on each triangle and characterized by the condition $\varphi_i(\nu_j) = \delta_{ij}$). Inserting (3.1) in (1.7.a), we obtain, for any $v \in V$:

$$B(e, v) = B(e, v \sum_{i=1}^{N} \varphi_i(x)) = \sum_{i=1}^{N} B(e, v\varphi_i(x)) = \sum_{i=1}^{N} L(v\varphi_i(x)) - B(u^h, v\varphi_i(x)). \tag{3.2}$$

The function $v\varphi_i(x)$ has support in the patch ω_i around node ν_i, i.e., $\omega_i = \bigcup_{\nu_i \in K'} K'$. Moreover we have $v\varphi_i = 0$ on $\Gamma_{D_i} \doteq \partial\omega_i \setminus \Gamma_N$. Accordingly, the subdomain residual method consist of finding, for each patch ω_i, $i = 1, 2, ..., N$, $\phi_i \in V_{\omega_i} \doteq H^1_{0,\Gamma_{D_i}}(\omega_i)$, solutions respectively of the problems

$$B_i(\phi_i, v) = L_i(v) - B_i(u^h, v) \quad \forall v \in V_{\omega_i} \tag{3.3.a}$$

where

$$B_i(u, v) = \int_{\omega_i} D\nabla u \cdot \nabla v + cuv \, \mathrm{d}x, \tag{3.3.b}$$

$$L_i(v) = \int_{\omega_i} fv + \int_{\partial\omega_i \cap \Gamma_N} gv \, \mathrm{d}x. \tag{3.3.c}$$

It is possible to prove, using once more results from approximation theory, that

$$|||e|||^2 \leq c_{\mathcal{T}^h} \sum_{i=1}^{N} B_i(\phi_i, \phi_i) \doteq c_{\mathcal{T}^h} \sum_{i=1}^{N} |||\phi_i|||^2_{\omega_i}$$

where $c_{\mathcal{T}^h}$ depends once more only on the regularity of the mesh – for details see, e.g., [4]. This suggests to define an estimator $\mathcal{E}(u^h(t))$ (approximation of the energy norm of the error $|||e|||$) as

$$\mathcal{E}(u^h) = c_{\mathcal{T}^h} \left\{ \sum_{i=1}^{N} \eta_i^2(u^h) \right\}^{1/2} \tag{3.4}$$

where $\eta_i(u^h)$ is the (local) error indicator associated with the patch ω_i defined by

$$\eta_i^2(u^h) = |||\phi_i|||^2_{\omega_i}, \quad \phi_i = \phi_i(u^h) \text{ solution of problem (3.3).} \tag{3.5}$$

Of course, the solution ϕ_i of problem (3.3) is not known exactly and has to be approximated by some finite element approximation ϕ_i^h on a finite dimensional space W^h based on a finer mesh (on ω_i) or higher order finite element method. However, even if one were satisfied with the approximation, in practical applications the method is seldom used because it is quite expensive due to the overlapping property. In fact, each element is considered several times according to the number of patches it belongs to.

We conclude this section by remarking that there exist a variant of the method described above, where, instead of patches ω_i around nodes ν_i, patches ω_K around triangles $K \in \mathcal{T}^h$ are considered – see, e.g., [38].

3.2 Element residual methods

A natural attempt to avoid the excessive computational costs of the subdomain residual method is to decompose the global problem (1.7.a) into single triangles. Let the error on a triangle K be denoted by e_K. The key observation here is that, being (1.7.a) valid for all $v \in V$, the error e_K satisfies, on the interior of each element K, the differential equation

$$-\nabla \cdot (D\nabla e_K) + ce_K = f + \nabla \cdot (D\nabla u^h) + cu^h. \tag{3.6}$$

The main difficulty is then to prescribe appropriate boundary conditions. Dealing with one single triangle, imposing homogeneous boundary conditions seems not appropriate since the approximation of the error would vanish on all the internal edges of $\mathring{\partial}\mathcal{T}^h$. In order to impose the correct boundary conditions one has to check whether the boundary ∂K intersects the boundary of the domain Ω. We then distinguish three different situations:

$$e_K = 0 \quad \text{on} \quad \Gamma_{D_K} \doteq \partial K \cap \Gamma_D \tag{3.7.a}$$

$$\frac{\partial e_K}{\partial n} = g - \frac{\partial u^h}{\partial n} \quad \text{on} \quad \partial K \cap \Gamma_N \tag{3.7.b}$$

and ideally one would like to impose $\dfrac{\partial e_K}{\partial n} = \dfrac{\partial u}{\partial n} - \dfrac{\partial u^h}{\partial n}$ on each internal edge. Obviously, the exact flux appearing in the last relation is not known. Therefore an approximation $F_K = F_K(u^n) \approx \dfrac{\partial u}{\partial n}$ has to be obtained from the finite element solution itself leading to the third kind of boundary condition

$$\frac{\partial e_K}{\partial n} = F_K(u^h) - \frac{\partial u^h}{\partial n} \quad \text{on } \partial K \cap \mathring{\partial}\mathcal{T}^h \tag{3.7.c}$$

With the help of (3.7), we can then obtain the appropriate weak formulation of problem (3.6) on each triangle K: find $\phi_K \in V_K = H^1_{0,\Gamma_{D_K}}$ such that:

$$B_K(\phi_K, v_K) = L_K(v_K) - B_K(u^h, v_K) \quad \forall v_K \in V_K \tag{3.8.a}$$

where

$$B_K(u, v) = \int_K D\nabla u \cdot \nabla v + cuv \,\mathrm{d}x, \tag{3.8.b}$$

$$L_K(v) = \int_K fv + \int_{\partial K \cap \Gamma_N} gv + \int_\gamma F_K(u^h) v \,\mathrm{d}s, \tag{3.8.c}$$

with $F_K(u^n) \approx \dfrac{\partial u}{\partial n}$ approximation of the true flux on the (internal) edge γ of the triangle K.

With the solution ϕ_K on each single triangle known, the error is then estimated using

$$\mathcal{E}(u^h) = \left\{ \sum_{K \in \mathcal{T}^h} \eta_K^2(u^h) \right\}^{1/2} \tag{3.9}$$

where $\eta_K(u^h)$ is the (local) error indicator associated with the triangle K defined by

$$\eta_K^2(u^h) = |||\phi_K|||_K^2 \doteq B_K(\phi_K, \phi_K), \quad \phi_K \text{ solution of problem (3.8).} \tag{3.10}$$

Unfortunately, the existence and the uniqueness of the weak problem (3.8) depends on the prescribed Neumann boundary conditions (which can be incompatible) – we note in fact, that for an internal triangle K, problem (3.8) will be completed with pure Neumann conditions on all boundary ∂K. Moreover, it is infeasible to require the exact solution of the problem (3.8). To overcome these drawbacks several techniques have been proposed. For instance a finite element approximation ϕ_K^h, to the solution ϕ_K, is sought in a subspace $V_K^h \subseteq V_K$ where the existence and uniqueness are guaranteed. Another approach employs special choices for the approximated flux $F_K(u^h)$ in (3.7.c) so to guarantee equilibrated boundary data and therefore the solvability of problem (3.8) – this approach is known as the *equilibrated residual method*, see, e.g., chapter 6 of [4] and references therein.

4 Hierarchical-based error estimation

While all the preceding estimates evaluate the error using one finite element solution (and eventually solutions of auxiliary problems), hierarchical-based error estimations employ (at least) two finite element solutions.

Let $u^h \in V^h$ solution of (1.6) and u^H a second solution on a "bigger" finite element space V^H of the problem:

$$B(u^H, v^H) = L(v^H) \quad \forall v^H \in V^H \tag{4.1}$$

Here $B(.,.)$ and $L(.)$ have the same meaning as in (1.3) and V^H is an enrichment of the original finite element space V^h which can be obtain adding to V^h higher-order basis functions or refining the underling triangulation $\mathcal{T}^h$.

If the solution $u^H \in V^H$ is a "much better" approximation of u than $u^h \in V^h$, then one can expects that

$$|||e||| = |||u - u^h||| \approx |||u^H - u^h||| \doteq |||e^H||| \tag{4.2}$$

It is important to note that e^H, define above as the difference between the solutions u^H and u^h, is actually the solution of the problem

$$B(e^H, v^H) = L(v^H) - B(u^h, v^H) \quad \forall v^H \in V^H \tag{4.3}$$

as one can obtain subtracting $B(h^h, v^H)$ from both sides of equation (4.1). Hence, e^H can be seen as an approximation, on the space V^H, to the solution e of equation (1.7.a).

Whether e^H is a good approximation of e depends on whether u^H really is a "much better" approximation of u than u^h. In this context an essential role is played by the so-called *saturation assumption*:

$$\text{there exists a constant } \beta \in [0,1) \text{ such that } \; ||| \, u - u^H ||| \leq \beta \, ||| \, u - u^h \, ||| \, . \tag{4.4}$$

Note that the existence of $\beta \in [0,1]$ is guaranteed (for symmetric problems, i.e. $B(u,v) = B(v,u)$), while it is not trivial to prove, and actually in general not true, that $\beta \in [0,1)$. Under the saturation assumption (4.4), it is easy to see that

$$|||e^H||| \leq |||e||| \leq \frac{1}{\sqrt{1-\beta^2}} \, ||| \, e^H|||$$

suggesting to use $|||e^H|||$ as an estimate for the error e.

The main drawback of this approach, *assuming* (4.4) holds, is the enormous extra effort that is needed to compute the global (in V^H) solution e^H of problem (4.3). Methods are then sought to avoid excessive computations. In the following we give an idea of one possible remedy.

Let $V^H = V^h \oplus V^{\mathcal{H}}$ be a decomposition of the enriched space ($V^h \cap V^{\mathcal{H}} = \varnothing$). Then also the error e^H can be splitted in $e^H = e^h + e^{\mathcal{H}}$ with $e^h \in V^h$, $e^{\mathcal{H}} \in V^{\mathcal{H}}$. Being $V^h \subset V^H$ from (4.3) and (1.6) we have immediately that:

$$B(e^h, v^h) + B(e^{\mathcal{H}}, v^h) = 0 \quad \forall v^h \in V^h.$$

On the other hand from (4.3) we also have:

$$B(e^h, v^{\mathcal{H}}) + B(e^{\mathcal{H}}, v^{\mathcal{H}}) = L(v^{\mathcal{H}}) - B(u^h, v^{\mathcal{H}}) \quad \forall v^{\mathcal{H}} \in V^{\mathcal{H}}.$$

The previous two relations can be seen as a system for e^h and $e^{\mathcal{H}}$. Obviously there is no advantage in solving such a system with respect to (4.3). A brutal simplification can be introduced negletting the mixed terms where functions of V^h and $V^{\mathcal{H}}$ are considered together – obviously such a reduction is justified if V^h and $V^{\mathcal{H}}$ are orthogonal with respect to the bilinear for $B(.,.)$, but for a complete explanation of the general case we refer the interested reader to [4].

The error e^H can then be approximated by: $e^H = e^h + e^{\mathcal{H}} \approx \phi^h + \phi^{\mathcal{H}}$ where $\phi^h \in V^h$ and $\phi^{\mathcal{H}} \in V^{\mathcal{H}}$ are the solution of the simplified system

$$B(\phi^h, v^h) = 0 \quad \forall v^h \in V^h \tag{4.5.a}$$

$$B(\phi^{\mathcal{H}}, v^{\mathcal{H}}) = L(v^{\mathcal{H}}) - B(u^h, v^{\mathcal{H}}) \quad \forall v^{\mathcal{H}} \in V^{\mathcal{H}} \tag{4.5.b}$$

Clearly, the first equation is satisfied by the trivial solution $\phi^h = 0$ so that the approximation of the error will be reconducted only to $\phi^{\mathcal{H}}$. In fact: $|||e||| \approx |||e^H||| = |||e^h + e^{\mathcal{H}}||| \approx |||\phi^h + \phi^{\mathcal{H}}||| = |||\phi^{\mathcal{H}}|||$. Having at hand the solution $\phi^{\mathcal{H}}$, the error is then estimated by

$$\mathcal{E}(u^h) = |||\phi^{\mathcal{H}}|||, \quad \phi^{\mathcal{H}} \text{ solution of (4.5.b)} \tag{4.6}$$

We note at the end of this section that after many simplifications, approximation and assumptions, the calculation of the solution $\phi^{\mathcal{H}}$ is still computationally prohibitive because of its global character, i.e., $\phi^{\mathcal{H}}$ requires the solution of a global problem on Ω. Further simplifications, e.g. localization of problem (4.5.b) at element level, are then necessary.

5 Recovery-based error estimation

In the recovery-based error estimation, also known as averaging error estimation, the idea is to post-process the gradient of the finite element solution and to obtain an estimate for the error measuring the difference between the post-processed gradient and the nonpost-processed one.

Let $G(u^h)$ be an approximation of ∇u^h. Suppose $G(u^h)$ is a better approximation of ∇u than ∇u^h itself, in the sense that there exist a constant $\beta \in [0, 1)$ such that

$$\|\nabla u - G(u^h)\|_{L_2(\Omega)} \leq \beta \|\nabla u - \nabla u^h\|_{L_2(\Omega)}. \tag{5.1}$$

A simple application of the triangle inequality shows that

$$\frac{1}{1+\beta}\|G(u^h) - \nabla u^h\|_{L_2(\Omega)} \leq \|\nabla u - \nabla u^h\|_{L_2(\Omega)} \leq \frac{1}{1-\beta}\|G(u^h) - \nabla u^h\|_{L_2(\Omega)}$$

suggesting therefore to consider $\|G(u^h) - \nabla u^h\|_{L_2(\Omega)}$ as an error estimator. More precisely, given $G(u^h)$ satisfying property (5.1) then the (global) error is estimated by

$$\mathcal{E}(u^h) = \left\{ \sum_{K \in \mathcal{T}^h} \eta_K^2(u^h) \right\}^{1/2} \tag{5.2}$$

where $\eta_K(u^h)$ is the (local) error indicator associated with the triangle K defined by

$$\eta_K(u^h) = \|G(u^h) - \nabla u^h\|_{L_2(K)}. \tag{5.3}$$

We note that if $c = 0$ in (1.1.a), then $\|\nabla u - \nabla u^h\|_{L_2(\Omega)}$ is, up to a multiplicative constant, the energy norm of e. The case $c \neq 0$ is dealt with by arguing that the dominant term in the error is the component containing the derivatives and so, it should be enough to estimate the dominant part only. In effect, the remaining terms are simply ignored see, e.g., [4].

To construct a post-processed gradient $G(u^h)$ is easier said than done: only under very special assumptions, e.g. in *superconvergence situations*, it is possible to guarantee property (5.1). In fact, the superconvergence condition is fulfilled for 1 dimensional problems and very special triangularization $\mathcal{T}^h$ for 2 dimensional problems but in general it is not satisfied. Nevertheless it is found that recovery-based error estimators perform astonishingly well even when applied to practical situations where most of the theoretical assumptions are not fulfilled. In the following a particular procedure to construct a post-processed gradient $G(u^h)$ is presented.

5.1 Superconvergent patch recovery (SPR)

Introduced by Zienkiewicz and Zhu in [40],[41],[42] the *superconvergent patch recovery (SPR)* – sometimes called ZZ-recovery – procedure reconstructs the fluxes at nodes as functions of the finite element approximation u^h. These values are then used to compute a global post-processed gradient $G(u^h)$ defined on each triangle K using the same interpolation employed for u^h. Note that if u^h is a finite element solution using a polynomial base of degree p, the gradient ∇u^h will be a piecewise polynomial of degree $p-1$ (usually discontinuous between elements) while the post-processed gradient $G(u^h)$ is a piecewise polynomial of degree p (continuous between elements). For the sake of simplicity, the construction process, to be given below, will be done for a linear finite element approximation u^h (on triangles).

For a vertex node ν_i, let ω_i be the patch of all triangles having ν_i as a vertex. Let $c_K = (x_K, y_K)$ be the coordinates of the centroid of each triangle $K \in \omega_i$. The values of the gradient ∇u^h sampled on the points c_K are employed to recover the value at the node ν_i using a discrete least square fitting process.

Writing the recovered (x-component) of the gradient on the patch as

$$p(x, y)\alpha_x$$

with $p(x, y) = [1, x, y]$ and $\alpha_x = [\alpha_{x,1}, \alpha_{x,2}, \alpha_{x,3}]^t$, we minimize the quadratic function

$$\sum_{K \in \omega_i} \left(\frac{\partial u^h}{\partial x}(c_K) - p(c_K)\alpha_x \right)^2. \tag{5.4}$$

It is easy to see that the minimizer of (5.4) is the solution of the system

$$M\alpha_x = b$$

where

$$M = \sum_{K \in \omega_i} p(c_K)^t p(c_K) \quad \text{and} \quad b = \sum_{K \in \omega_i} p(c_K)^t \frac{\partial u^h}{\partial x}(c_K).$$

Similar procedure has to be followed for the y-component of the gradient, α_y.

Finally the recovered approximation of the gradient at the node ν_i is defined to be

$$G(u^h)(\nu_i) = p(\nu_i)\alpha, \qquad \alpha = [\alpha_x, \alpha_y].$$

It should be noted that to perform the least square process (5.4) it is always necessary to have more data than the number of coefficients in α_x (and α_y). That means that on external boundaries the nodal values α_x (and α_y) should eventually be calculated from interior patches.

The SPR technique highlighted in this section bases on the superconvergence assumption. A different process of recovery, the so called *recovery by equilibrated patches* (REP), which does not need such an assumption, was introduced by Boroomand and Zienkiewicz see, e.g., [39] and reference therein.

6 Functional-based error estimation

Functional-based error estimators are derived on purely functional ground without making use of any specific property of the finite element method (e.g. orthogonality of the error with respect of the finite element space) or special feature of the solution (e.g. superconvergent effect). To simplify the exposition of this type of error estimators, in this section we will deal with problem (1.1) where we assume $D = 1$, $c = 0$ and $\Gamma_D = \partial\Omega$. For different and more general situations we refer the interested reader to Repin's publications, see, e.g., [30],[31],[29],[33] and [32] for a complete bibliography.

The weak formulation of problem (1.1) is then to find $u \in V \doteq H^1_{0,\Gamma_D}(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega f v \quad \forall v \in V. \tag{6.1}$$

It is well known that the solution to this problem can be characterized equivalently as the minimiser of the following variational problem:
Problem $\mathcal{P}$: find $u \in V$ such that $J(u) = \inf_{v \in V} J(v)$, where

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f v. \tag{6.2}$$

We introduce the dual variational problem:
Problem $\mathcal{P}^*$: find $p^* \in V_f^*$ such that $I^*(p^*) = \sup_{q \in V_f^*} I^*(q^*)$, where

$$I^*(q^*) = \int_\Omega \left(\nabla u_0 - \frac{1}{2}|q^*|^2 - f u_0 \right) \tag{6.3}$$

and $V_f^* = \{q^* \in L_2(\Omega) \,|\, \nabla \cdot q^* = -f\}$. It can be proved that both problems $\mathcal{P}$ and $\mathcal{P}^*$ have unique solutions u and p^* which satisfy the relation

$$J(u) = I^*(p^*), \quad \nabla u = p^*. \tag{6.4}$$

In view of (6.1), we have: $J(v) - J(u) = \frac{1}{2}\int_\Omega |\nabla(v-u)|^2 \mathrm{d}x$ for all $v \in V$. Using (6.4), one derives $\frac{1}{2}\|\nabla(v-u)\|^2_{L_2(\Omega)} = \inf_{q^* \in V_f^*} \{J(v) - I^*(q^*)\}$ and since $J(v) - I^*(q^*) = \frac{1}{2}\int_\Omega |\nabla v - q^*|^2 \mathrm{d}x$ for all $v \in V$ and $q \in V_f^*$, we obtain

$$\|\nabla (u-v)\|^2_{L_2(\Omega)} = \inf_{q^* \in V_f^*} \|\nabla v - q^*\|^2_{L_2(\Omega)} \quad \forall v \in V \tag{6.5}$$

In particular, for $v = u^h \in V^h \subseteq V$ finite element approximation of problem (6.1), we obtain an estimate for the error:

(6.6)

However, if q^* does not belong to V_f^*, the last term of the above inequality does not in general provide an upper bound for the error. This meas that any numerical approximation of q^* should satisfy the requirement $q^* \in V_f^*$ with very high accuracy in order to guarantee a reliable error estimation. We note that the approximation of the requirement $q^* \in V_f^*$ with the finite element method requires essentially the same – or even higher – effort which is needed in approximating the original problem (6.1); thus, estimate (6.5) is not very usefull for practical applications. One possible remedy is briefly explained in the following.

For a given $y^* \in V_\nabla = \{y^* \in L_2(\Omega) \,|\, \nabla \cdot y^* \in L_2\}$, consider, formally, the auxiliary problem

$$-\Delta w = \nabla \cdot y^* + f \quad \text{on } \Omega \tag{6.7.a}$$

$$u = 0 \qquad \text{on } \Gamma_D \tag{6.7.b}$$

Let $w \in V$ be the solution (which exists and is unique) and consider the function $q^* = y^* - \nabla w$. It is just a matter of few calculations to see that $q^* \in V_f$. Substituting $q^* = y^* - \nabla w$ into (6.5) and using Young's inequality, we obtain the estimates

$$\|\nabla(u-v)\|^2_{L_2(\Omega)} \leq (1+\beta)\|\nabla v - y^*\|^2_{L_2(\Omega)} + \left(1 + \frac{1}{\beta}\right)\|\nabla w\|^2_{L_2(\Omega)} \tag{6.8}$$

which is valid for any $\beta > 0$, any $v \in V$, $y^* \in V_\nabla$ and w solution of (6.7).

Appropriate estimation of $\|\nabla w\|_{L_2}$ – see, e.g, [29] – finally leads to the following estimate:

$$\|\nabla(u-v)\|^2_{L_2} \leq (1+\beta)\|\nabla v - y^*\|^2_{L_2(\Omega)} + \left(1 + \frac{1}{\beta}\right) C_\Omega^2 \|\nabla \cdot y^* + f\|^2_{L_2(\Omega)} \quad \forall \beta > 0,\ v \in V,\ y^* \in V_\nabla \tag{6.9}$$

Here C_Ω is the constant appearing in the Friedrichs-Poincaré inequality. Note that if $y^* \longrightarrow \nabla u$, then $\|\nabla \cdot y^* + f\|_{L_2(\Omega)} \longrightarrow 0$, so that β can be chosen very small and the whole right hand side of inequality (6.9) $\longrightarrow \|\nabla(v-u)\|_{L_2(\Omega)}$ – i.e. the estimation (6.9) is *asymptotically exact*.

Relation (6.9), for $v = u^h \in V^h \subseteq V$ gives an estimation of the error that depends on β and the particular function $y^* \in V_\nabla$. Note that the condition $y^* \in H_\nabla$ does *not* require the solution of a differential equation which, on the contrary, was necessary in (6.5).

If we denote the right hand side of inequality (6.9) by $M(v, \beta, y^*)$, then evidently we have

$$\|\nabla(u-v)\|^2_{L_2(\Omega)} \leq M(v, \beta, y^*) \quad \forall \beta > 0,\ \forall v \in V,\ \forall y^* \in V_\nabla \tag{6.10}$$

which again, for $v = u^h$, gives a computable estimation of the finite element error.

To get in a sense optimal upper bound of the error, we should minimize the majorant M over the variables β, y^*, i.e.,

$$M_{opt}(v) = \inf_{\substack{\beta > 0 \\ y^* \in V_\nabla}} M(v, \beta, y^*). \tag{6.11}$$

However, there is no need to solve this optimization exactly. In practice an upper bound of $M_{opt}(v)$ is obtained solving (6.11) where the condition $y^* \in V_\nabla$ is replaced with $y^* \in V_\nabla^h$ – here V_∇^h is a finite (element) dimensional space contained in V_∇, most of the times V^h. The choice of the finite dimensional space H_∇^h depends on the desired accuracy of the error estimation.

Concluding, the finite element error is estimated using

$$\mathcal{E}^2(u^h) = \inf_{\substack{\beta>0 \\ y^* \in V_\nabla^h}} M(u^h, \beta, y^*) \tag{6.12}$$

and the local error distribution is estimated by the elementwise value of the first term of the majorant $M(u^h, \beta, y^*)$, i.e.,

$$\eta_K(u^h) = \|\nabla(u^h - y^*)\|_{L_2(K)} \tag{6.13}$$

where y^* is the solution of (6.12).

7 Conclusions

In this paper we reviewed some basic concepts to obtain a posteriori error estimates for the finite element method. The crucial issues of any error estimator relate to questions of reliability, efficiency, accuracy and computational cost where the precise meaning of each one of these properties depends on the purpose of the computation.

As pointed out, *explicit residual-based* error estimators are build upon the fact that the approximate solution u^h does not satisfy the given partial differential equation. They are relatively simple, computationally inexpensive, and asymptotically exact. However, up to very rare exceptions, the last property is not really satisfied in the strict sense, since one has upper (and lower) bounds which depend on an unknown multiplicative constant – see, e.g., equations (2.11) and (2.12) where four different constants (c_{T^h}, $C_{c,D}$,c_{int} and c_{bou}) appear. In view of this last point these error estimators are, most of the times, not guaranteed. Nevertheless, to obtain an indication of the error or to steer an adaptive process, explicit residual-based error estimators might be sufficiently accurate.

Implicit residual-based error estimators remove much of the above drawbacks under the condition that auxiliary (local boundary value) problems can be solved exactly. This is of course hardly doable in practice and hence one has to approximate the solutions of these problems, leading to the loss of the guaranteed bounds and increased computational cost. On the other hand the increased computational costs are counterbalanced by much sharper error bounds that can be obtained by explicit methods.

Hierarchical-based error estimation employ not one but (at least) two finite element solutions. Therefore they are expensive and moreover are fully justified under saturation assumptions which in general fail.

The fourth kind of error estimators which was presented are the *recovery-based* error estimators. The main idea here is to smooth the gradients of the approximated solution u^h (obtain via a post-precessing a "smoother" gradient-function $G(u^h)$) and compare the (unsmoothed) gradient of the original approximation (∇u^h) and the smoothed gradient ($G(u^h)$) in order to asses the discretization error. Although it is found that recovery-based error estimators perform astonishingly well, even when applied to practical situations where most of the theoretical assumptions are not fulfilled (i.e robust), these kind of estimators are theoretically fully justified only under very special assumptions, namely in superconvergence situations.

Finally *functional-based* error estimators are derived on purely functional ground without making use of any specific property of the finite element method (e.g. orthogonality of the error with respect of the finite element space) or special feature of the solution (e.g. superconvergent effect). This last type of error estimators are asymptotically correct and are the only ones that yield guaranteed upper (and lower) bound. Unfortunately obtaining sharp estimates is a quite expensive task.

Appendix A: goal-oriented error estimation

It is frequently the case that the goal of a finite element computation is the determination of a specific quantity $Q(u)$ depending on the solution u. The functional $Q(u)$, known as the *quantity of interest*, is for example the average quantity of u in the neighborhood of a critical point or in the whole domain, the flux of u through the boundary and many others.

The most natural way to approximate the quantity of interest in the true solution u is, given the finite element approximation u^h, to simply take $Q(u^h) \approx Q(u)$. The question is then whether is possible to quantify the error in the quantity of interest, i.e., $Q(u) - Q(u^h)$. If we can find an estimate for the error in the, from now on, linear functional $Q : V \longrightarrow \mathbb{R}$ of the form

$$\mathcal{E}^Q_{low} \leq Q(e) = Q(u - u^h) = Q(u) - Q(u^h) \leq \mathcal{E}^Q_{up},$$

then immediately we also have an estimate for $Q(u)$, namely:

$$Q(u^h) + \mathcal{E}^Q_{low} \leq Q(u) \leq Q(u^h) + \mathcal{E}^Q_{up}.$$

In the following we will highlight the standard method for the estimation of $Q(e)$ based on duality techniques. For a deep exposition of this subject and for a general introduction to adjoint methods for a posteriori error estimates, see, e.g., [9],[10],[26]

To approximate $Q(e)$ one considers the solution $w \in V$ of the *dual* problem

$$B(v, w) = Q(v) \quad \forall v \in V \tag{7.1}$$

and its finite element approximation $w^h \in V^h \subseteq V$ solution of the problem

$$B(v^h, w^h) = Q(v^h) \quad \forall v^h \in V^h. \tag{7.2}$$

Here $B(.,.)$ is the bilinear form appearing in (1.3) and V and V^h are as in (1.3) and (1.6). Let e and e^Q be respectively the error associated with the original problem and the error of the approximation of the solution of the dual problem (7.1), i.e. $e = u - u^h$ and $e^Q = w - w^h$.

From (7.1) with $v = e \in V$, we obtain:

$$Q(e) = B(e, w). \tag{7.3}$$

On the other hand, the orthogonality condition (1.7.b) with $v^h = w^h$ solution of (7.2), implies

$$0 = B(e, w^h). \tag{7.4}$$

Subtracting (7.4) from (7.3) gives the fundamental relation:

$$Q(e) = B(e, e^Q). \tag{7.5}$$

An immediate consequence of (7.5) is the bound for the error in the quantity of interest:

$$|Q(e)| = |B(e, e^Q)| \leq |||e||| \; |||e^Q||| \,. \tag{7.6}$$

The estimate (7.6) shows that the rate of convergence of the error in the quantity of interest Q will be roughly twice the rate of the convergence of the errors e and e^Q measured in the energy norm.

Suppose now that $\mathcal{E}$ is a posteriori error estimator of the error e of the original problem (1.6), measured in the energy norm with one of the techniques described in one of the previous sections, i.e., $|||e||| \leq \mathcal{E}(u^h) = \left\{\sum_{i \in \mathcal{I}} \eta_i^2(u^h)\right\}^{1/2}$. In a similar way one can obtain an a posteriori estimator $\mathcal{E}^Q$

for the error in the dual problem (7.2), i.e., $|||e^Q||| \le \mathcal{E}^Q(w^h) = \left\{\sum_{i\in\mathcal{I}} \eta_i^2(w^h)\right\}^{1/2}$. Trivially from (7.6), we obtain an approximation of the error in the quantity of interest:

$$|Q(u) - Q(u^h)| = |Q(e)| \le \mathcal{E}(u^h)\mathcal{E}^Q(w^h) \tag{7.7}$$

and consequently

$$Q(u^h) - \mathcal{E}(u^h)\mathcal{E}^Q(w^h) \le Q(u) \le Q(u^h) + \mathcal{E}(u^h)\mathcal{E}^Q(w^h).$$

Concluding, we note that the strategy discussed in this section requires the solution of two problems: the original problem (1.6) and the dual problem (7.2). Nevertheless, in many situations the above approach is still very attractive because the amount of work can be drastically reduced using adaptive techniques based on the global error estimator $\tilde{\mathcal{E}}(u^h, w^h)$ and the local error indicators $\tilde{\eta}_i(u^h, w^h)$ defined by

$$\tilde{\mathcal{E}}(u^h, w^h) = \left\{\sum_{i\in\mathcal{I}} \tilde{\eta}_i^2(u^h, w^h)\right\}^{1/2} \doteq \left\{\sum_{i\in\mathcal{I}} \eta_i^2(u^h)\eta_i^2(w^h)\right\}^{1/2}. \tag{7.8}$$

AppendixB: a comparison table

In the following table we shortly (and roughly) summarize Section 7. The non-expert reader can, with a quick look, get an informal indication of the advantages and disadvantages of each estimator presented in the paper.

	Def. $\mathcal{E}$	Def. η	Correct	Exact	Guar./Sharp	Simple/Inexp.	Robust
Expl. res.	(2.11)	(2.12)	2	1	0	3	2
Subdomain	(3.4)*	(3.5)*	2	1	1	0	1
Element	(3.9)*	(3.10)*	2	2	1	0	1
Hierarchical	(4.6)*	-	2	1	1	0	0
SPR	(5.2)	(5.3)	2	1	1	3	3
Functional	(6.12)	(6.13)	3	3	3	1	1

Table 1: A rough visual comparison of different error estimators:
0 = essentially never fulfilled (except for simple test 1 dimensional problems);
1 = rarely fulfilled (under special conditions);
2 = often fulfilled (in standard test situations);
3 = essentially always fulfilled.
For the description of the characteristics (Correct, Exact, Guaranteed and Sharp, Simple and Inexpensive, Robust) see the introductory section. The symbol * indicates that the formula needs extra manipulation before being of any practical relevance.

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