Two-Point Taylor Expansions of Analytic Functions

By José L. López and Nico M. Temme

Taylor expansions of analytic functions are considered with respect to two points. Cauchy-type formulas are given for coefficients and remainders in the expansions, and the regions of convergence are indicated. It is explained how these expansions can be used in deriving uniform asymptotic expansions of integrals. The method is also used for obtaining Laurent expansions in two points.

1. Introduction

In deriving uniform asymptotic expansions of a certain class of integrals one encounters the problem of expanding a function that is analytic in some domain Ω of the complex plane in two points. The first mention of the use of such expansions in asymptotics is given in [1], where Airy-type expansions are derived for integrals having two nearby (or coalescing) saddle points. This reference does not give further details about two-point Taylor expansions, because the coefficients in the Airy-type asymptotic expansion are derived in a different way.

To demonstrate the application in asymptotics we consider the integral

$$F_b(\omega) = \frac{1}{2\pi i}\int_C e^{\omega(\frac{1}{3}z^3-b^2z)} f(z)\, dz, \tag{1}$$

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where ω is a large positive parameter and b is a parameter that may assume small values. The contour starts at $\infty e^{-i\pi/3}$, terminates at $\infty e^{i\pi/3}$, and lies in a domain where the function f is analytic. In particular, f is analytic in a domain that contains the saddle points $\pm b$ of the exponent in the integrand. One method for obtaining an asymptotic expansion of $F_b(\omega)$ that holds uniformly for small values of b is based on expanding f at the two saddle points:

$$f(z) = \sum_{n=0}^{\infty} A_n(z^2 - b^2)^n + z \sum_{n=0}^{\infty} B_n(z^2 - b^2)^n, \tag{2}$$

and substituting this expansion into (1). When interchanging summation and integration, the result is a formal expansion in two series in terms of functions related with Airy functions. A Maple algorithm for obtaining the coefficients A_n and B_n, with applications to Airy-type expansions of parabolic cylinder functions, is given in [2].

In a future paper we shall use expansions like (2) to derive convergent expansions for orthogonal polynomials and hypergeometric functions that also have an asymptotic nature. The purpose of the present article is to give details on the two-point Taylor expansion (2), in particular on the region of convergence and on representations in terms of Cauchy-type integrals of coefficients and remainders of these expansions. Some information on this type of expansions is also given in [3, p. 149, Exercise 24].

Without referring to the applications in asymptotic analysis we include analogous properties of the two-point Laurent expansions and of another related type, the two-point Taylor–Laurent expansion.

2. Two-point Taylor expansions

We consider the expansion (2) in a more symmetric form and give information on the coefficients and the remainder in the expansion.

THEOREM 1. *Let $f(z)$ be an analytic function on an open set $\Omega \subset \mathbb{C}$ and z_1, $z_2 \in \Omega$ with $z_1 \neq z_2$. Then, $f(z)$ admits the two-point Taylor expansion*

$$\begin{aligned} f(z) = \sum_{n=0}^{N-1} &[a_n(z_1, z_2)(z - z_1) + a_n(z_2, z_1)(z - z_2)]\,(z - z_1)^n(z - z_2)^n \\ &+ r_N(z_1, z_2; z), \end{aligned} \tag{3}$$

where the coefficients $a_n(z_1, z_2)$ and $a_n(z_2, z_1)$ of the expansion are given by the Cauchy integral

$$a_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_C \frac{f(w)\, dw}{(w - z_1)^n(w - z_2)^{n+1}}. \tag{4}$$

Figure 1. (a) Contour C in the integrals (3)–(5). (b) For $z \in O_{z_1,z_2}$, we can take a contour C in Ω which contains O_{z_1,z_2} inside and therefore, $|(z-z_1)(z-z_2)| < |(w-z_1)(w-z_2)| \forall\, w \in C$.

The remainder term $r_N(z_1, z_2; z)$ is given by the Cauchy integral

$$r_N(z_1, z_2; z) \equiv \frac{1}{2\pi i}\int_C \frac{f(w)\, dw}{(w-z_1)^N(w-z_2)^N(w-z)}(z-z_1)^N(z-z_2)^N. \quad (5)$$

The contour of integration C is a simple closed loop which encircles the points z_1 and z_2 (for a_n) and z, z_1 and z_2 (for r_N) in the counterclockwise direction and is contained in Ω (see Figure 1(a)).

The expansion (3) is convergent for z inside the Cassini oval (see Figure 2)

$$O_{z_1,z_2} \equiv \{z \in \Omega,\ |(z-z_1)(z-z_2)| < r\}$$

where

$$r \equiv \mathrm{Inf}_{w\in\mathbb{C}\setminus\Omega}\{|(w-z_1)(w-z_2)|\}.$$

In particular, if $f(z)$ is an entire function ($\Omega = \mathbb{C}$), then the expansion (3) converges $\forall\, z \in \mathbb{C}$.

Proof: By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\, dw}{w-z}, \quad (6)$$

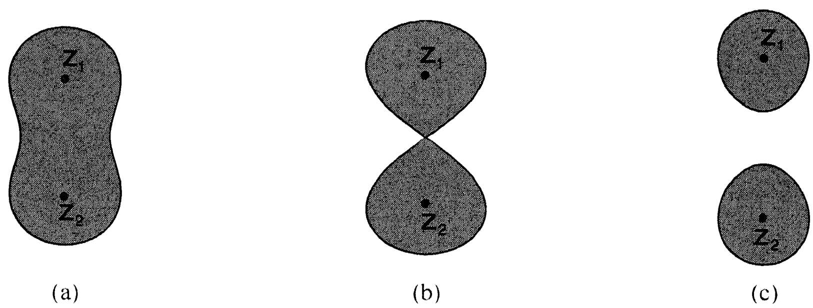


Figure 2. Shape of the Cassini oval depending on the relative size of the parameter r and the focal distance $|z_1 - z_2|$. (a) $4r > |z_1 - z_2|^2$; (b) $4r = |z_1 - z_2|^2$; (c) $4r < |z_1 - z_2|^2$.

where C is the contour defined above (Figure 1(a)). Now we write

$$\frac{1}{w-z}=\frac{z+w-z_1-z_2}{(w-z_1)(w-z_2)}\frac{1}{1-u}, \tag{7}$$

where

$$u\equiv\frac{(z-z_1)(z-z_2)}{(w-z_1)(w-z_2)}. \tag{8}$$

Now we introduce the expansion

$$\frac{1}{1-u}=\sum_{n=0}^{N-1}u^n+\frac{u^N}{1-u} \tag{9}$$

in (7) and this in (6). After straightforward calculations we obtain (3)–(5).

For any $z\in O_{z_1,z_2}$, we can take a contour C in Ω such that $|(z-z_1)(z-z_2)|<|(w-z_1)(w-z_2)|\ \forall\ w\in C$ (see Figure 1 (b)). In this contour $|f(w)|$ is bounded by some constant C: $|f(w)|\le C$. Introducing these two bounds in (5) we see that $\lim_{N\to\infty}r_N(z_1,z_2;z)=0$ and the proof follows. □

2.1. An alternative form of the expansion

The present expansion of $f(z)$ in the form (3) stresses the symmetry of the expansion with respect to z_1 and z_2. In this representation it is not possible, however, to let z_1 and z_2 coincide, which causes a little inconvenience (the coefficients $a_n(z_1,z_2)$ become infinitely large as $z_1\to z_2$; the remainder $r_N(z_1,z_2;z)$ remains well-defined). An alternative way is the representation (cf. (2))

$$f(z)=\sum_{n=0}^{\infty}[A_n(z_1,z_2)+B_n(z_1,z_2)z](z-z_1)^n(z-z_2)^n,$$

and we have the relations

$$A_n(z_1,z_2)=-z_1a_n(z_1,z_2)-z_2a_n(z_2,z_1),$$

$$B_n(z_1,z_2)=a_n(z_1,z_2)+a_n(z_2,z_1),$$

which are regular when $z_1\to z_2$. In fact we have

$$A_n(z_1,z_2)=\frac{1}{2\pi i}\int_C\frac{w-z_1-z_2}{[(w-z_1)(w-z_2)]^{n+1}}f(w)\,dw,$$

$$B_n(z_1,z_2)=\frac{1}{2\pi i}\int_C\frac{f(w)\,dw}{[(w-z_1)(w-z_2)]^{n+1}}.$$

Letting $z_1\to 0$ and $z_2\to 0$, we obtain the standard Maclaurin series of $f(z)$ with even part (the A_n series) and odd part (the B_n series).

2.2. *Explicit forms of the coefficients*

Equation (4) is not appropriate for numerical computations. A more practical formula to compute the coefficients of the above two-point Taylor expansion is given in the following proposition.

PROPOSITION 1. *Coefficients $a_n(z_1, z_2)$ in the expansion* (3) *are also given by the formulas*

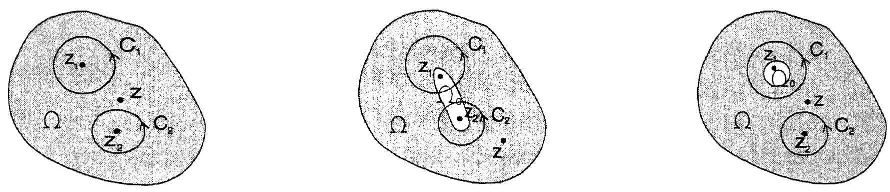
$$a_0(z_1, z_2) = \frac{f(z_2)}{z_2 - z_1} \tag{10}$$

and, for $n = 1, 2, 3, \ldots,$

$$a_n(z_1, z_2) = \sum_{k=0}^{n} \frac{(n+k-1)!}{k!(n-k)!} \frac{(-1)^{n+1} n f^{(n-k)}(z_2) + (-1)^k k f^{(n-k)}(z_1)}{n!(z_1 - z_2)^{n+k+1}}. \tag{11}$$

Proof: We deform the contour of integration $\mathcal{C}$ in equation (4) to any contour of the form $\mathcal{C}_1 \cup \mathcal{C}_2$ also contained in Ω, where $\mathcal{C}_1$ ($\mathcal{C}_2$) is a simple closed loop which encircles the point z_1 (z_2) in the counterclockwise direction and does not contain the point z_2 (z_1) inside (see Figure 3(a)). Then,

$$\begin{aligned} a_n(z_1, z_2) &= \frac{1}{2\pi i(z_2 - z_1)} \left\{ \int_{\mathcal{C}_1} \frac{f(w)}{(w - z_2)^{n+1}} \frac{dw}{(w - z_1)^n} \right. \\ &\quad \left. + \int_{\mathcal{C}_2} \frac{f(w)}{(w - z_1)^n} \frac{dw}{(w - z_2)^{n+1}} \right\} \\ &= \frac{1}{(z_2 - z_1)} \left\{ \frac{1}{(n-1)!} \frac{d^{n-1}}{dw^{n-1}} \frac{f(w)}{(w - z_2)^{n+1}} \bigg|_{w=z_1} \right. \\ &\quad \left. + \frac{1}{n!} \frac{d^n}{dw^n} \frac{f(w)}{(w - z_1)^n} \bigg|_{w=z_2} \right\}. \end{aligned}$$



(a) (b) (c)

Figure 3. (a) The function $(w - z_2)^{-n-1} f(w)$ is analytic inside $\mathcal{C}_1$, whereas $(w - z_1)^{-n} f(w)$ is analytic inside $\mathcal{C}_2$. (b) The function $(w - z_2)^{-n-1} g_1(w)$ is analytic inside $\mathcal{C}_1$, whereas $(w - z_1)^{-n} g_2(w)$ is analytic inside $\mathcal{C}_2$. (c) The function $(w - z_2)^{-n-1} g(w)$ is analytic inside $\mathcal{C}_1$, whereas $(w - z_1)^{-n} f(w)$ is analytic inside $\mathcal{C}_2$.

From here, equations (10) and (11) follow after straightforward computations. □

2.3. *Two-point Taylor polynomials*

Next we can define the two-point Taylor polynomial of the function $f(z)$ in the following way:

DEFINITION 1. *Let z be a real or complex variable and z_1 and z_2 ($z_1 \neq z_2$) be any two real or complex numbers. If $f(z)$ is $n-1$ times differentiable at those two points, we define the two-point Taylor polynomial of $f(z)$ at z_1 and z_2 and degree $2n-1$ as*

$$P_n(z_1, z_2; z) \equiv \sum_{k=0}^{n-1}[a_k(z_1, z_2)(z-z_1) + a_k(z_2, z_1)(z-z_2)](z-z_1)^k(z-z_2)^k,$$

where the coefficients $a_k(z_1, z_2)$ are given in (10) *and* (11).

PROPOSITION 2. *In the conditions of the above definition, the remainder of the approximation of $f(z)$ by $P_n(z_1, z_2; z)$ at z_1 and z_2 is defined as*

$$r_n(z_1, z_2; z) \equiv f(z) - P_n(z_1, z_2; z).$$

Then, (i) *$r_n(z_1, z_2; z) = o(z-z_1)^{n-1}$ as $z \to z_1$ and $r_n(z_1, z_2; z) = o(z-z_2)^{n-1}$ as $z \to z_2$.* (ii) *If $f(z)$ is n times differentiable at z_1 and/or z_2, then $r_n(z_1, z_2; z) = \mathcal{O}(z-z_1)^n$ as $z \to z_1$ and/or $r_n(z_1, z_2; z) = \mathcal{O}(z-z_2)^n$ as $z \to z_2$.*

Proof: The proof is trivial if $f(z)$ is analytic at z_1 and z_2 by using (5). In any case, for real or complex variable, the proof follows after straightforward computations by using l'Hôpital's rule and equations (10) and (11). □

Remark 1: Observe that the Taylor polynomial of $f(z)$ at z_1 and z_2 and degree $2n-1$ is the same as the Hermite's interpolation polynomial of $f(z)$ at z_1 and z_2 with data $f(z_i)$, $f'(z_i), \ldots, f^{(n-1)}(z_i)$, $i = 1, 2$.

3. Two-point Laurent expansions

In the standard theory for Taylor and Laurent expansions much analogy exists between the two types of expansions. For two-point expansions, we have a similar agreement in the representations of coefficients and remainders.

THEOREM 2. *Let Ω_0 and Ω be closed and open sets, respectively, of the complex plane, and $\Omega_0 \subset \Omega \subset \mathbb{C}$. Let $f(z)$ be an analytic function on $\Omega \backslash \Omega_0$ and $z_1, z_2 \in \Omega_0$ with $z_1 \neq z_2$. Then, for any $z \in \Omega \backslash \Omega_0$, $f(z)$ admits the two-point Laurent expansion*

$$f(z) = \sum_{n=0}^{N-1} [b_n(z_1, z_2)(z - z_1) + b_n(z_2, z_1)(z - z_2)](z - z_1)^n (z - z_2)^n$$
$$+ \sum_{n=0}^{N-1} [c_n(z_1, z_2)(z - z_1) + c_n(z_2, z_1)(z - z_2)](z - z_1)^{-n-1} (z - z_2)^{-n-1}$$
$$+ r_N(z_1, z_2; z), \tag{12}$$

where the coefficients $b_n(z_1, z_2), b_n(z_2, z_1), c_n(z_1, z_2),$ *and* $c_n(z_2, z_1)$ *of the expansion are given, respectively, by the Cauchy integrals*

$$b_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_{\Gamma_1} \frac{f(w)\, dw}{(w - z_1)^n (w - z_2)^{n+1}} \tag{13}$$

and

$$c_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_{\Gamma_2} (w - z_1)^{n+1} (w - z_2)^n f(w)\, dw. \tag{14}$$

The remainder term $r_N(z_1, z_2; z)$ *is given by the Cauchy integrals*

$$r_N(z_1, z_2; z) \equiv \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)\, dw}{(w - z_1)^N (w - z_2)^N (w - z)} (z - z_1)^N (z - z_2)^N$$
$$- \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(w - z_1)^N (w - z_2)^N f(w)\, dw}{w - z} \frac{1}{(z - z_1)^N (z - z_2)^N}. \tag{15}$$

In these integrals, the contours of integration Γ_1 *and* Γ_2 *are simple closed loops contained in* $\Omega \backslash \Omega_0$ *which encircle the points* z_1 *and* z_2 *in the counterclockwise direction. Moreover,* Γ_2 *does not contain the point* z *inside, whereas* Γ_1 *encircles* Γ_2 *and the point* z *(see Figure* 4(*a*)).

The expansion (12) *is convergent for* z *inside the Cassini annulus (see Figure* 5)

$$A_{z_1, z_2} \equiv \{z \in \Omega \backslash \Omega_0, \quad r_2 < |(z - z_1)(z - z_2)| < r_1\} \tag{16}$$

where

$$r_1 \equiv \mathrm{Inf}_{w \in \mathbb{C} \backslash \Omega} \{|(w - z_1)(w - z_2)|\}, \quad r_2 \equiv \mathrm{Sup}_{w \in \Omega_0} \{|(w - z_1)(w - z_2)|\}.$$

Proof: By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)\, dw}{w - z} - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)\, dw}{w - z}, \tag{17}$$

where Γ_1 and Γ_2 are the contours defined above. We substitute (7) and (8) into the first integral above and

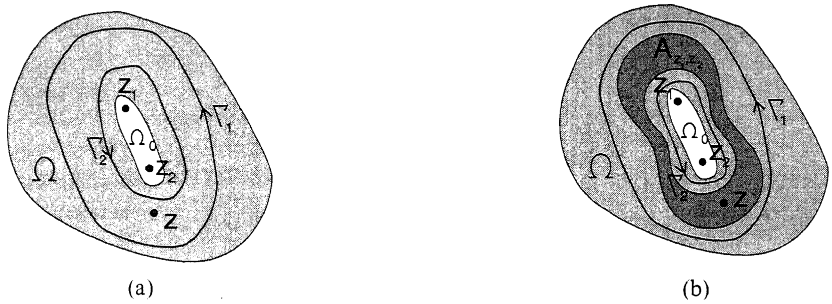


Figure 4. (a) Contours Γ_1 and Γ_2 in the integrals (12)–(15). (b) For $z \in A_{z_1,z_2}$, we can take a contour Γ_2 in Ω situated between Ω_0 and A_{z_1,z_2} and a contour Γ_1 in Ω which contains A_{z_1,z_2} inside. Therefore, $|(z-z_1)(z-z_2)| < |(w-z_1)(w-z_2)|\ \forall\ w \in \Gamma_1$ and $|(w-z_1)(w-z_2)| < |(z-z_1)(z-z_2)|\ \forall\ w \in \Gamma_2$.

$$\frac{1}{w-z} = \frac{z_1+z_2-z-w}{(z-z_1)(z-z_2)}\frac{1}{1-u}, \qquad u \equiv \frac{(w-z_1)(w-z_2)}{(z-z_1)(z-z_2)},$$

into the second one. Now we introduce the expansion (9) of the factor $(1-u)^{-1}$ in both integrals in (17). After straightforward calculations we obtain (12)–(15).

For any z verifying (16), we can take simple closed loops Γ_1 and Γ_2 in $\Omega\backslash\Omega_0$ such that $|(z-z_1)(z-z_2)| < |(w-z_1)(w-z_2)|\ \forall\ w \in \Gamma_1$ and $|(z-z_1)(z-z_2)| > |(w-z_1)(w-z_2)|\ \forall\ w \in \Gamma_2$ (see Figure 4(b)). On these contours $|f(w)|$ is bounded by some constant C: $|f(w)| \le C$. Introducing these bounds in (15) we see that $\lim_{N\to\infty} r_N(z_1, z_2; z) = 0$ and the proof follows. □

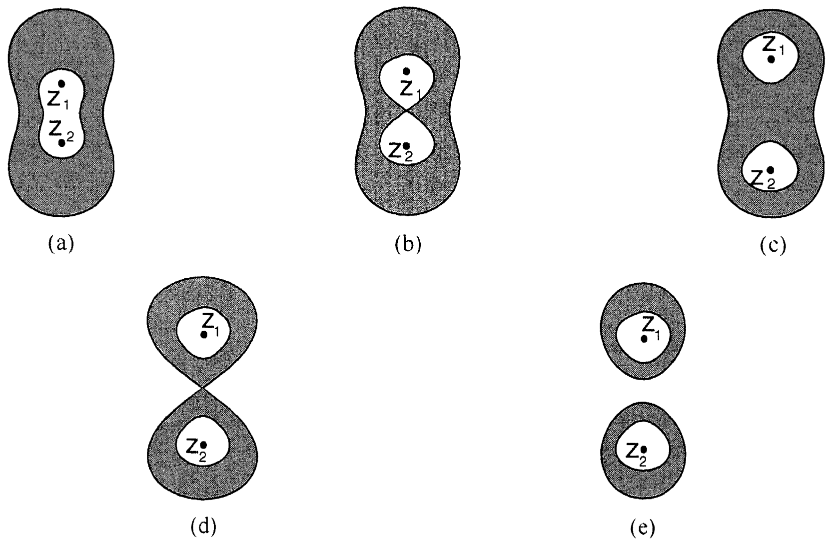


Figure 5. Shape of the Cassini annulus depending on the relative size of the parameters r_1, r_2 and the focal distance $|z_1-z_2|$. (a) $4r_1 > 4r_2 > |z_1-z_2|^2$; (b) $4r_1 > |z_1-z_2|^2 = 4r_2$; (c) $4r_1 > |z_1-z_2|^2 > 4r_2$; (d) $4r_1 = |z_1-z_2|^2 > 4r_2$; (e) $|z_1-z_2|^2 > 4r_1 > 4r_2$.

If the only singularities of $f(z)$ inside Ω_0 are just poles at z_1 and/or z_2, then an alternative formula to (13) and (14) to compute the coefficients of the above two-point Laurent expansion is given in the following proposition.

PROPOSITION 3. *Suppose that* $g_1(z) \equiv (z-z_1)^{m_1} f(z)$ *and* $g_2(z) \equiv (z-z_2)^{m_2} f(z)$ *are analytic functions in* Ω *for certain* $m_1, m_2 \in \mathbb{N}$. *Then, for* $n = 0, 1, 2, \ldots$, *coefficients* $b_n(z_1, z_2)$ *and* $c_n(z_1, z_2)$ *in the expansion* (12) *are also given by the formulas*:

$$b_n(z_1,z_2) = \sum_{k=0}^{n+m_1-1} \binom{n+m_1-1}{k} \frac{(-1)^{k+1}(n+1)_k g_1^{(n+m_1-k-1)}(z_1)}{(n+m_1-1)!(z_1-z_2)^{n+k+2}} + \sum_{k=0}^{n+m_2} \binom{n+m_2}{k} \frac{(-1)^k (n)_k g_2^{(n+m_2-k)}(z_2)}{(n+m_2)!(z_2-z_1)^{n+k+1}}, \tag{18}$$

where $(n)_k$ *denotes the Pochhammer symbol and*

$$c_n(z_1,z_2) = -\sum_{k=0}^{m_1-n-2} k! \binom{m_1-n-2}{k} \binom{n}{k} \frac{(z_1-z_2)^{n-k-1} g_1^{(m_1-n-k-2)}(z_1)}{(m_1-n-2)!} + \sum_{k=0}^{m_2-n-1} k! \binom{m_2-n-1}{k} \binom{n+1}{k} \frac{(z_2-z_1)^{n-k} g_2^{(m_2-n-k-1)}(z_2)}{(m_2-n-1)!}. \tag{19}$$

In these formulas, empty sums must be understood as zero. Coefficients $b_n(z_2, z_1)$ *and* $c_n(z_2, z_1)$ *are given, respectively, by* (18) *and* (19) *interchanging* z_1, g_1, *and* m_1 *by* z_2, g_2, *and* m_2 *respectively.*

Proof: We deform both the contour Γ_1 in equation (13) and Γ_2 in equation (14) to any contour of the form $C_1 \cup C_2$ contained in Ω, where C_1 (C_2) is a simple closed loop which encircles the point z_1 (z_2) in the counterclockwise direction and does not contain the point z_2 (z_1) inside (see Figure 3(b)). Then,

$$\begin{aligned} b_n(z_1,z_2) &= \frac{1}{2\pi i(z_2-z_1)} \left\{ \int_{C_1} \frac{g_1(w)}{(w-z_2)^{n+1}} \frac{dw}{(w-z_1)^{n+m_1}} + \int_{C_2} \frac{g_2(w)}{(w-z_1)^n} \frac{dw}{(w-z_2)^{n+m_2+1}} \right\} \\ &= \frac{1}{z_2-z_1} \left\{ \frac{1}{(n+m_1-1)!} \frac{d^{n+m_1-1}}{dw^{n+m_1-1}} \frac{g_1(w)}{(w-z_2)^{n+1}} \bigg|_{w=z_1} + \frac{1}{(n+m_2)!} \frac{d^{n+m_2}}{dw^{n+m_2}} \frac{g_2(w)}{(w-z_1)^n} \bigg|_{w=z_2} \right\} \end{aligned}$$

and

$$c_n(z_1, z_2) = \frac{1}{2\pi i(z_2 - z_1)} \left\{ \int_{C_1} \frac{(w - z_2)^n g_1(w)}{(w - z_1)^{m_1 - n - 1}} dw \right.$$

$$\left. + \int_{C_2} \frac{(w - z_1)^{n+1} g_2(w)}{(w - z_2)^{m_2 - n}} dw \right\}$$

$$= \frac{1}{z_2 - z_1} \left\{ \frac{d^{m_1 - n - 2}}{dw^{m_1 - n - 2}} \left[\frac{(w - z_2)^n g_1(w)}{(m_1 - n - 2)!} \right] \Bigg|_{w = z_1} \right.$$

$$\left. + \frac{d^{m_2 - n - 1}}{dw^{m_2 - n - 1}} \left[\frac{(w - z_1)^{n+1} g_2(w)}{(m_2 - n - 1)!} \right] \Bigg|_{w = z_2} \right\}.$$

From here, equations (18) and (19) follow after straightforward computations. □

Remark 2: Let z be a real or complex variable and z_1, z_2 $(z_1 \neq z_2)$ be any two real or complex numbers. Suppose that $g_1(z) \equiv (z - z_1)^{m_1} f(z)$ is n times differentiable at z_1 and $g_2(z) \equiv (z - z_2)^{m_2} f(z)$ is n times differentiable at z_2. Define

$$g(z) \equiv f(z) - \sum_{n=0}^{M-1} [c_n(z_1, z_2)(z - z_1)$$

$$+ c_n(z_2, z_1)(z - z_2)](z - z_1)^{-n-1}(z - z_2)^{-n-1},$$

where $M \equiv \max\{m_1, m_2\}$. Then, the thesis of Proposition 2 holds for $f(z)$ replaced by $g(z)$. Moreover, if $(z - z_1)^{m_1}(z - z_2)^{m_2} f(z)$ is an analytic function in Ω, then the thesis of Theorem 1 applies to $g(z)$.

4. Two-point Taylor–Laurent expansions

THEOREM 3. *Let Ω_0 and Ω be closed and open sets, respectively, of the complex plane, and $\Omega_0 \subset \Omega \subset \mathcal{C}$. Let $f(z)$ be an analytic function on $\Omega \backslash \Omega_0$, $z_1 \in \Omega_0$ and $z_2 \in \Omega \backslash \Omega_0$. Then, for $z \in \Omega \backslash \Omega_0$, $f(z)$ admits the Taylor–Laurent expansion*

$$f(z) = \sum_{n=0}^{N-1} [d_n(z_1, z_2)(z - z_1) + d_n(z_2, z_1)(z - z_2)](z - z_1)^n (z - z_2)^n$$

$$+ \sum_{n=0}^{N-1} e_n(z_1, z_2)(z - z_2)^n (z - z_1)^{-n-1} + r_N(z_1, z_2; z), \tag{20}$$

where the coefficients $d_n(z_1, z_2), d_n(z_2, z_1)$ *and* $e_n(z_1, z_2)$ *of the expansion are given by the Cauchy integrals*

$$d_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_{\Gamma_1} \frac{f(w)\, dw}{(w - z_1)^n (w - z_2)^{n+1}} \tag{21}$$

and

$$e_n(z_1, z_2) \equiv \frac{z_1 - z_2}{2\pi i} \int_{\Gamma_2} \frac{(w - z_1)^n}{(w - z_2)^{n+1}} f(w)\, dw. \tag{22}$$

The remainder term $r_N(z_1, z_2; z)$ *is given by the Cauchy integrals*

$$\begin{aligned} r_N(z_1, z_2; z) \equiv {} & \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)\, dw}{(w - z_1)^N (w - z_2)^N (w - z)} (z - z_1)^N (z - z_2)^N \\ & - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(w - z_1)^N f(w)\, dw}{(w - z_2)^N (w - z)} \frac{(z - z_2)^N}{(z - z_1)^N}. \end{aligned} \tag{23}$$

In these integrals, the contours of integration Γ_1 *and* Γ_2 *are simple closed loops contained in* $\Omega\backslash\Omega_0$ *which encircle* Ω_0 *in the counterclockwise direction. Moreover,* Γ_2 *does not contain the points* z *and* z_2 *inside, whereas* Γ_1 *encircles* Γ_2 *and the points* z *and* z_2 *(see Figure* 6(*a*)).

The expansion (20) *is convergent in the region* (*Figure* 7)

$$D_{z_1,z_2} \equiv \{z \in \Omega\backslash\Omega_0, |(z - z_1)(z - z_2)| < r_1 \quad \text{and} \quad |z - z_2| < r_2|z - z_1|\} \tag{24}$$

where $r_1 \equiv \text{Inf}_{w\in\mathcal{C}\backslash\Omega}\{|(w - z_1)(w - z_2)|\}$ *and* $r_2 \equiv \text{Inf}_{w\in\Omega_0}\{|(w - z_2)(w - z_1)^{-1}|\}$.

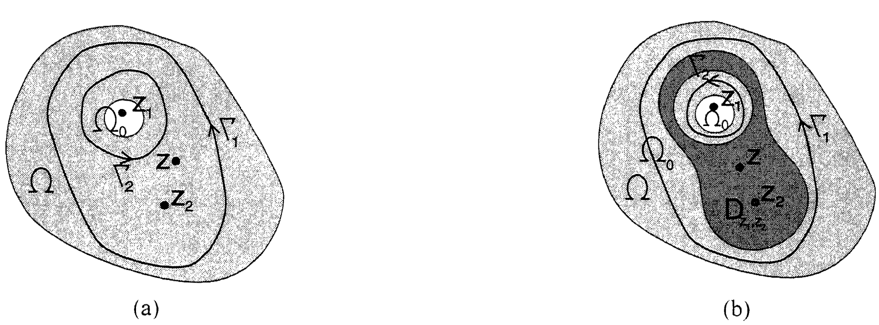


Figure 6. (a) Contours Γ_1 and Γ_2 in the integrals (20)–(23). (b) For $z \in D_{z_1,z_2}$, we can take a contour Γ_2 situated between Ω_0 and D_{z_1,z_2} and a contour Γ_1 in Ω which contains D_{z_1,z_2} inside. Therefore, $|(z - z_1)(z - z_2)| < |(w - z_1)(w - z_2)|\ \forall\ w \in \Gamma_1$ and $|(w - z_1)(z - z_2)| < |(z - z_1)(w - z_2)|\ \forall\ w \in \Gamma_2$.

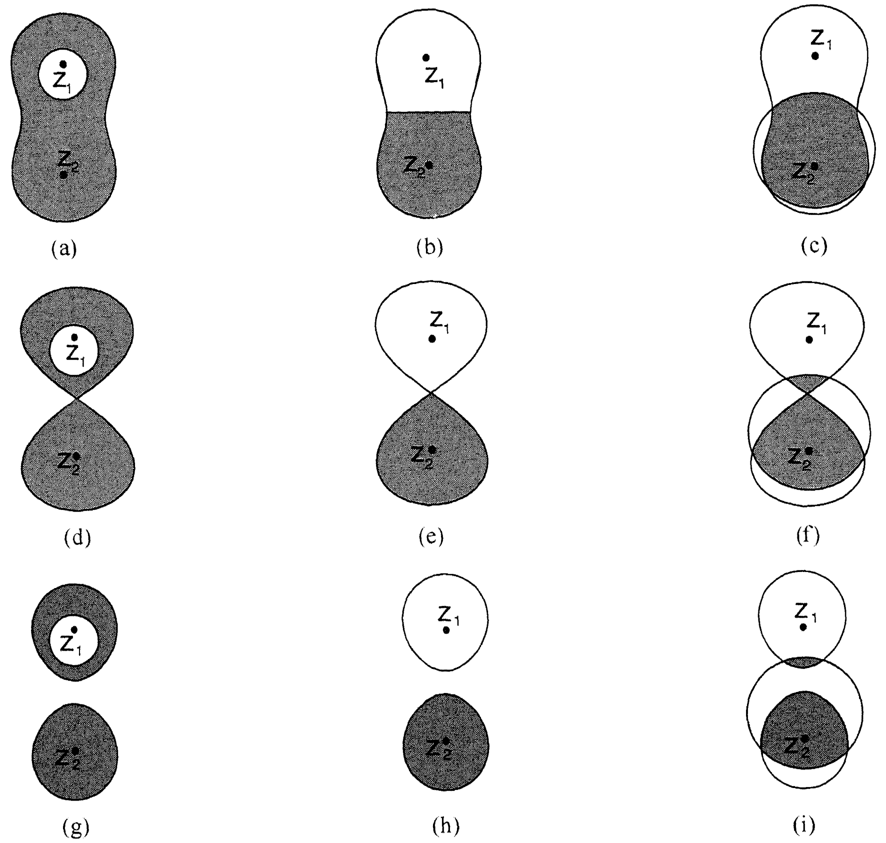


Figure 7. The region D_{z_1,z_2} defined in Theorem 3 is given by $D_{z_1,z_2} = D_1 \bigcap D_2$, where D_1 is the Cassini oval of focus z_1 and z_2 and parameter r_1. On the other hand, for $r_2 < 1$ ($r_2 > 1$), D_2 is the interior (exterior) of the circle of center $z_1 + (1 - r_2^2)^{-1}(z_2 - z_1) = z_2 + r_2^2(r_2^2 - 1)^{-1}(z_1 - z_2)$ and radius $|z_1 - z_2|r_2/|r_2^2 - 1|$. For $r_2 = 1$, D_2 is just the half plane $|z - z_2| < |z - z_1|$. The shape of the Cassini annulus depends on the relative size of the parameters $\sqrt{r_1}$, $\sqrt{r_2}$ and the focal distance $|z_1 - z_2|$. (a) $4r_1 > |z_1 - z_2|^2, r_2 > 1$; (b) $4r_1 > |z_1 - z_2|^2, r_2 = 1$; (c) $4r_1 > |z_1 - z_2|^2, r_2 < 1$; (d) $4r_1 = |z_1 - z_2|^2, r_2 > 1$; (e) $4r_1 = |z_1 - z_2|^2, r_2 = 1$; (f) $4r_1 = |z_1 - z_2|^2, r_2 < 1$; (g) $4r_1 < |z_1 - z_2|^2, r_2 > 1$; (h) $4r_1 < |z_1 - z_2|^2, r_2 = 1$; (i) $4r_1 < |z_1 - z_2|^2, r_2 < 1$.

Proof: By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)\,dw}{w - z} - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(w)\,dw}{w - z}, \tag{25}$$

where Γ_1 and Γ_2 are the contours defined above. We substitute (7) and (8) into the first integral above and

$$\frac{1}{w - z} = \frac{z_2 - z_1}{(z - z_1)(w - z_2)}\frac{1}{1 - u}, \quad u \equiv \frac{(w - z_1)(z - z_2)}{(z - z_1)(w - z_2)} \tag{26}$$

into the second one. Now we introduce the expansion (9) of the factor $(1 - u)^{-1}$ in both integrals in (25). After straightforward calculations we obtain (20)–(23).

For any z verifying (24), we can take simple closed loops Γ_1 and Γ_2 in $\Omega\backslash\Omega_0$ such that $|(z-z_1)(z-z_2)| < |(w-z_1)(w-z_2)|\, \forall\, w \in \Gamma_1$ and $|(z-z_1)(w-z_2)| > |(w-z_1)(z-z_2)|\, \forall\, w \in \Gamma_2$ (see Figure 6(b)). On these contours $|f(w)|$ is bounded by some constant C: $|f(w)| \le C$. Introducing these bounds in (23) we see that $\lim_{N\to\infty} r_N(z_1, z_2; z) = 0$ and the proof follows. □

If the only singularities of $f(z)$ inside Ω_0 are just poles at z_1, then an alternative formula to (21) and (22) to compute the coefficients of the above two-point Taylor–Laurent expansion is given in the following proposition.

PROPOSITION 4. *Suppose that $g(z) \equiv (z-z_1)^m f(z)$ is an analytic function in Ω for certain $m \in \mathbb{N}$. Then, coefficients $d_n(z_1, z_2)$ and $d_n(z_2, z_1)$ in the expansion* (20) *are also given by the formulas*:

$$d_0(z_1, z_2) = \frac{f(z_2)}{z_2 - z_1} - \sum_{k=0}^{m-1} \frac{1}{(m-k-1)!} \frac{g^{(m-k-1)}(z_1)}{(z_2-z_1)^{k+2}}, \tag{27}$$

$$d_0(z_2, z_1) = \frac{1}{m!} \frac{g^{(m)}(z_1)}{z_1 - z_2},$$

and, for $n = 1, 2, 3, \ldots,$

$$d_n(z_1, z_2) = -\frac{(-1)^n}{n!} \left\{ \sum_{k=0}^{m+n-1} \frac{(n+k)!}{k!(m+n-k-1)!} \frac{g^{(m+n-k-1)}(z_1)}{(z_2-z_1)^{n+k+2}} \right.$$

$$\left. + n \sum_{k=0}^{n} \frac{(n+k-1)!}{k!(n-k)!} \frac{f^{(n-k)}(z_2)}{(z_1-z_2)^{n+k+1}} \right\}, \tag{28}$$

$$d_n(z_2, z_1) = -\frac{(-1)^n}{n!} \left\{ n \sum_{k=0}^{m+n} \frac{(n+k-1)!}{k!(m+n-k)!} \frac{g^{(m+n-k)}(z_1)}{(z_2-z_1)^{n+k+1}} \right.$$

$$\left. + \sum_{k=0}^{n-1} \frac{(n+k)!}{k!(n-k-1)!} \frac{f^{(n-k-1)}(z_2)}{(z_1-z_2)^{n+k+2}} \right\}. \tag{29}$$

For $n = 0, 1, 2, \ldots,$ *coefficients* $e_n(z_1, z_2)$ *are given by*

$$e_n(z_1, z_2) = \frac{(-1)^n}{n!} \sum_{k=0}^{m-n-1} \frac{(n+k)!}{k!(m-n-k-1)!} \frac{g^{(m-n-k-1)}(z_1)}{(z_2-z_1)^{n+k}}. \tag{30}$$

Proof: We deform both the contour Γ_1 in equation (21) and the contour Γ_2 in equation (22) to any contour of the form $C_1 \cup C_2$ contained in Ω, where C_1 (C_2) is a simple closed loop which encircles the point z_1 (z_2) in the counterclockwise direction and does not contain the point z_2 (z_1) inside (see Figure 3(c)). Then,

$$d_n(z_1, z_2) = \frac{1}{2\pi i(z_2 - z_1)}\left\{\int_{C_1} \frac{g(w)}{(w - z_2)^{n+1}}\frac{dw}{(w - z_1)^{n+m}}\right.$$
$$\left.+ \int_{C_2} \frac{f(w)}{(w - z_1)^n}\frac{dw}{(w - z_2)^{n+1}}\right\}$$
$$= \frac{1}{(z_2 - z_1)}\left\{\frac{1}{(n + m - 1)!}\frac{d^{n+m-1}}{dw^{n+m-1}}\frac{g_1(w)}{(w - z_2)^{n+1}}\right|_{w=z_1}$$
$$\left.+ \frac{1}{n!}\frac{d^n}{dw^n}\frac{f(w)}{(w - z_1)^n}\right|_{w=z_2}\Bigg\},$$

an analog formula for $d_n(z_2, z_1)$, and

$$e_n(z_1, z_2) = \frac{z_1 - z_2}{2\pi i}\int_{C_1} \frac{g(w)}{(w - z_2)^{n+1}}\frac{dw}{(w - z_1)^{m-n}}$$
$$= (z_1 - z_2)\frac{1}{(m - n - 1)!}\frac{d^{m-n-1}}{dw^{m-n-1}}\frac{g(w)}{(w - z_2)^{n+1}}\Bigg|_{w=z_1}.$$

From here, equations (27)–(30) follow after straightforward computations. □

Remark 3: Let z be a real or complex variable and z_1 and z_2 ($z_1 \neq z_2$) two real or complex numbers. Suppose that $(z - z_1)^m f(z)$ is n times differentiable at z_1 for certain $m \in \mathbb{N}$ and $f(z)$ is n times differentiable at z_2. Define

$$g(z) \equiv f(z) - \sum_{n=0}^{m-1} e_n(z_1, z_2)(z - z_1)^{-n-1}(z - z_2)^n.$$

Then, the thesis of Proposition 2 holds for $g(z)$. If moreover, $(z - z_1)^m f(z)$ is an analytic function in Ω, then the thesis of Theorem 1 applies to $g(z)$.

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