-294-

On Likelihoods for Markov Random Sets and Boolean Models

M N M van LIESHOUT

A paper submitted to the Symposium on Advances in Theory and Applications of Random Sets, Fontainebleau.

July 1996

On likelihoods for Markov random sets and Boolean models [1]

M N M van Lieshout

Department of Statistics, University of Warwick
Coventry CV4 7AL, United Kingdom

1 Introduction

Markov point processes as introduced in the statistical literature by Ripley and Kelly [23] provide a useful class of models for a range of applications in image analysis, spatial statistics and mathematical physics (see eg. [15]). They can be defined by local interaction functions that are easy to interpret and can be exploited in statistical inference. Most attention has been paid to pairwise interaction processes, with the interaction between two points defined in terms of the distance between them.

In the last few years though, models of a more set-geometric nature have been proposed. Baddeley and Møller [3, 21] suggested a realisation-dependent interaction, where two points interact if they can be joined by a chain of close points. Equivalently, if we place a ball around each point, points on the same connected component interact. Furthermore, it was realised that many cluster processes [7] are Markovian in this sense [5].

Baddeley and Van Lieshout [1, 15, 12, 13] proposed to use Markov object processes where two objects interact if they - or their influence zones - intersect as a prior distribution in image interpretation. For further developments see Stryhn et al. [28] and Mardia et al. [16].

Widom and Rowlinson's penetrable sphere model for liquid-vapour equilibrium [29], defined in terms of the volume occupied by the union of molecular influence zones, has recently received renewed attention. Baddeley and Van Lieshout [2] studied the model from a statistical point of view and generalised to allow for repulsion as well as attraction between molecules; Van Lieshout and Molchanov [14] took into account the coverage function of the molecular influence zones, while in [4] the replacement of the volume by other geometrical functionals such as the Euler-Poincaré characteristic is studied. The phase transition behaviour of the penetrable sphere model proved by Ruelle [24] was investigated in Chayes et al. [6] and Georgii and Häggström [8] using percolation arguments. See also [9].

Models of the types described above, force one to think of the random set formed by a *union*, eg. of objects, influence zones or individual clusters. Yet, explicit random set models defined in likelihood terms seem to be scarce [19]. In his 1975 book, Matheron [17] considered the Boolean model. This is a germ-grain model associating to each of the points in a Poisson process (the germs) a random set called the grain. In particular, a Poisson cluster process

[1] Submitted to "Symposium on Advances in Theory and Applications of Random Sets", Fontainebleau, 1996

[7] can be seen as a Boolean model. In this paper, we build random set models by specifying a Radon-Nikodym density with respect to an underlying Boolean model similar to the point process models described above. We ask when the models are well-defined and study their Markov properties.

The paper is organised as follows. In Section 2 we review the theory of Markov marked point processes. Basic results about Boolean models are collected in Section 3 and in Section 4 we investigate when a Boolean models has a likelihood with respect to another Boolean model on the same space. The results are applied to the special cases of marked point processes, Poisson cluster processes and Boolean models with deterministic grains. Finally, Section 5 discusses local and global Markov properties.

2 Markov marked point processes

Here we give an overview of the theory of Markov marked point processes, following definitions and results in Ripley and Kelly [23] and Baddeley and Møller [3].

Heuristically, a marked point process assigns to each point of a 'location' process a 'mark' giving additional information. For instance in forestry, the location points may denote tree positions and the mark could be the stem diameter [22]. Graphically we can depict such a process by drawing a circle of stem radius around each tree position, see Figure 1.

Formally, we will define a marked point process as a point process on the product space of locations and marks with the additional property that the marginal location process is itself a well-defined point process (see [7, page 204]).

Definition 1 *Let $(S, \mathcal{B}, \lambda)$ be a measure space such that $\lambda(S) < \infty$ and the σ-algebra $\mathcal{B}$ contains all singletons. Let $(K, \mathcal{K}, \nu)$ be a measure space such that $\nu(K) = 1$ and the σ-algebra $\mathcal{K}$ contains all singletons. Then a marked point process on S with marks in K is a (finite) point process on the exponential space $\Omega_{S\times K}$ of configurations $\{(s_1, t_1), \ldots, (s_n, t_n)\}, n \geq 0$.*

We are interested in defining a marked point process by its density. A suitable reference process is a Poisson process on $S \times K$, that is a Poisson process on S with intensity $\lambda(\cdot)$, labelled by iid marks with probability distribution $\nu(\cdot)$. The absence of interaction between the marked points makes it an appropriate benchmark process and we can define new models by specifying their interaction functions.

Definition 2 *A density $p(\cdot)$ is a measurable, integrable mapping*

$$p : (\Omega_{S\times K}, \mathcal{F}_{S\times K}) \to (\mathbb{R}^+, \mathcal{B}or)$$

Here $\mathcal{F}_{S\times K}$ is the smallest σ-algebra making

$$N_{B\times L} = \text{ number of points in } B \text{ with marks in } L$$

a random variable for all $B \in \mathcal{B}, L \in \mathcal{K}$.

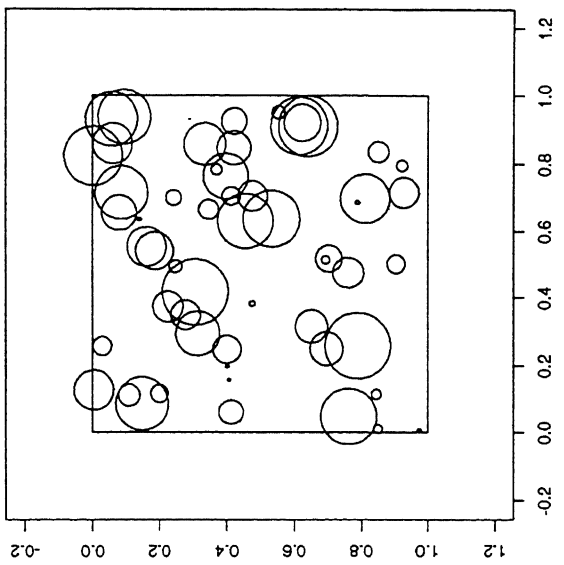

Figure 1: Poisson process of discs with intensity 50 and uniform radius in (0, .1)

Due to the high dimension of $\Omega_{S\times K}$, it is desirable to impose further restrictions. A common choice is a *Markov* density, which has only 'local' interactions: given a symmetric *neighbourhood relation* $\sim$ on $S \times K$ that may or may not depend on the mark, the likelihood ratio (conditional intensity) for adding a new point (s,k) to a configuration $\mathbf{y} = \{(s_1,k_1),\ldots,(s_n,k_n)\}$ depends only on the neighbours of the added point.

Definition 3 *A density $p(\cdot)$ is Markov if*

(M) *for all* $\mathbf{y}$ *such that* $p(\mathbf{y}) > 0$, $\frac{p(\mathbf{y}\cup\{s,k\})}{p(\mathbf{y})}$ *depends only on* $(s_i,k_i) \in \mathbf{y} : (s,k) \sim (s_i,k_i)$;

(H) *if* $p(\mathbf{y}) > 0$ *then also* $p(\mathbf{z}) > 0$ *for all configurations* $\mathbf{z} \subseteq \mathbf{y}$.

By the Hammersley-Clifford theorem [3, 23] a density $p(\cdot)$ is Markov if and only if it can be factorised into a product of clique interaction functions

$$p(\mathbf{y}) = \prod_{\mathbf{z}\subseteq\mathbf{y},\text{cliques}} \phi(\mathbf{z})$$

where $\phi(\cdot) \geq 0$ is measurable and a clique is a configuration $\mathbf{y}$ such that for all $(s,k),(s',k') \in \mathbf{y} : (s,k) \sim (s',k')$. This characterisation can be used to define new models by defining the interaction functions. However, for each choice we must check that the model is well-defined, i.e. measurable and integrable.

As an example, consider the random disc process of [3]. Here S is a bounded Borel set in $\mathbb{R}^d$, λ is Lebesgue measure and $\mathcal{B}$ the Borel σ-algebra on S. Let $K = \mathbb{R}^+$ and ν any probability measure on $\mathbb{R}^+$; the mark $k \in K$ is interpreted as the disc radius.

Taking interaction function $\phi \equiv 1$ on cliques with three or more members, a pairwise interaction density could take the form

$$p(\{(s_1,k_1),\ldots,(s_n,k_n)\}) = \alpha\beta^n \prod_{i<j} g(||s_i - s_j||, k_i, k_j) \tag{1}$$

for some measurable, integrable $g : [0,\infty)^3 \to [0,\infty)$. Taking

$$g(d,k_1,k_2) = \begin{cases} \gamma & \text{if } d \leq k_1 + k_2 \\ 1 & \text{else} \end{cases}$$

for $0 \leq \gamma \leq 1$ yields an analogue of the Strauss model [10, 27]. It is easily seen that this model is Markov with respect to the neighbourhood relation

$$(s_1,k_1) \sim (s_2,k_2) \Leftrightarrow ||s_1 - s_2|| \leq k_1 + k_2 \Leftrightarrow B(s_1,k_1) \cap B(s_2,k_2) \neq \emptyset.$$

Here $B(s,k)$ denotes the closed ball of radius k centred at s. For $\gamma = 0$, no balls are allowed to overlap, $0 < \gamma < 1$ yields repulsion and for $\gamma = 1$, (1) is a Poisson process. Another example, allowing interactions of order higher than two, is the *area-interaction process* [2, 14]

$$p(\{(s_1,k_1),\ldots,(s_n,k_n)\}) = \alpha\beta^n\gamma^{-\lambda(\cup_i B(s_i,k_i)\cap S)}, \tag{2}$$

a generalisation of the penetrable sphere model introduced by Widom and Rowlinson [29]. The model is well-defined for all $\beta, \gamma > 0$.

Returning to Definition 2, a density can be interpreted in an infinitesimal sense as follows: $e^{-\lambda(S)}p(\{(s_1,k_1),\ldots,(s_n,k_n)\})\lambda(ds_1)\nu(dk_1)\ldots\lambda(ds_n)\nu(dk_n)$ is the probability of having exactly n points, one at each of $(s_i + \lambda(ds_i), k_i + \nu(dk_i))$, $i = 1,\ldots,n$ [7, page 122]. Thus, for any $F \in \mathcal{F}_{S\times K}$, $\mathbb{P}(X \in F)$ equals

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda(S)}}{n!} \int \cdots \int_{(S\times K)^n} p(\{(s_1,t_1),\ldots,(s_n,t_n)\}\mathbf{1}_F(\{(s_1,t_1),\cdots,(s_n,t_n)\}$$

$$d\lambda(s_1)d\nu(t_1)\cdots d\lambda(s_n)d\nu(t_n). \tag{3}$$

Generalisations where the neighbourhood relation depends on the configuration were proposed in [3]. We are particularly interested in their *connected component relation*: two points $(s_i,k_i),(s_j,k_j) \in \mathbf{y} = \{(s_1,k_1),\ldots,(s_n,k_n)\}$ are neighbours if there exists a path $(s_{l_1},k_{l_1}),\ldots,(s_{l_m},k_{l_m}) \in \mathbf{y}$ such that

$$(s_i,k_i) \sim (s_{l_1},k_{l_1}) \sim \cdots \sim (s_{l_m},k_{l_m}) \sim (s_j,k_j)$$

for some fixed relation $\sim$. We write $(s_i,k_i) \sim_{\mathbf{y}} (s_j,k_j)$. For instance in the random disc example above, $(s_i,k_i) \sim_{\mathbf{y}} (s_j,k_j)$ iff s_i, s_j belong to the same connected component of $\cup_{i=1}^n B(s_i,k_i)$.

An analogue of the Strauss process could be Møller's continuum random cluster model [21]

$$p(\mathbf{y}) = \alpha\beta^n\gamma^{-c(\mathbf{y})} \tag{4}$$

where $\gamma > 0$ and $c(\mathbf{y})$ is the number of connected components. Note that, in contrast to (1), models (2) and (4) allow for both attraction ($\gamma > 1$) and repulsion ($\gamma < 1$) between the points.

3 Boolean models

The analogue of the Poisson point process for random sets is the Boolean model introduced by Matheron [17]. It can be defined by a Poisson point process marked by a random set, then taking the union of all marks. It is important to note that **only** the union set is observable, **not** the mark sets that constitute the union (Figure 2). Models of this kind are useful to produce sets of a complicated shape from simple building blocks. However, due to the lack of identifiability of the components, statistical inference is difficult [20, 18].

Here, we will concentrate on the Euclidean case, taking for $S \subset \mathbb{R}^d$ a compact Borel set and a homogenous Poisson process with intensity $\lambda > 0$ for the locations. The mark distribution is a probability measure $\nu(\cdot)$ on $(K,\mathcal{K})$ as before. We will denote the distribution of the Boolean model thus specified by $Q_{\lambda,\nu}$. Throughout we will assume that the union set can be observed fully, avoiding edge effects.

Definition 4 *A density $p(\cdot)$ is a measurable, integrable mapping*

$$p : (\mathcal{F}_S, \mathcal{A}_S) \to (\mathbb{R}^+, \mathcal{B}or)$$

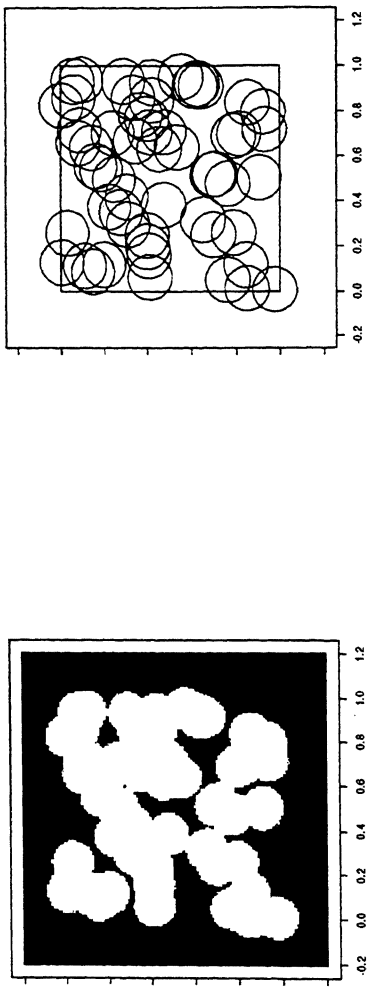

Figure 2: Poisson disc process of intensity 50 interpreted as point process (left) and Boolean model (right)

where $\mathcal{F}_S$ is the collection of closed sets in S, $\mathcal{A}_S$ is the hitting σ-algebra [17, Chapter 1], [26, page 194].

When specifying densities with respect to the Boolean law $Q_{\lambda,\nu}$, without loss of generality $p(\cdot)$ assigns mass zero to any set that cannot be obtained as a finite union $\cup_{i=1}^n (s_i \oplus K_i)$. Here we write $s_i \oplus K_i$ for the translation of set $K_i \in K$ over vector $s_i \in S$, or in other words the set described by location s_i and mark K_i.

Analogously to (3), for any $B \in \mathcal{A}_S$, the probability that the random set X with density $p(\cdot)$ falls in B equals

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!} \int \cdots \int_{(S\times K)^n} \lambda^n p(\cup_{i=1}^n (s_i \oplus K_i)) 1_B(\cup_{i=1}^n (s_i \oplus K_i)) ds_1 d\nu(K_1) \cdots ds_n d\nu(K_n), \quad (5)$$

writing $|S|$ for the Lebesgue measure of S.

In particular, for $p \equiv 1$ and $B = \{X \cap L \neq \emptyset\}$ for some compact set L, we obtain the capacity functional $T(L)$ of the Boolean model parametrised by λ and ν:

$$\begin{aligned} 1 - T(L) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!} \int \cdots \int_{(S\times K)^n} \lambda^n \prod_{i=1}^n 1\{s_i \oplus K_i) \cap L = \emptyset\} ds_1 d\nu(K_1) \cdots ds_n d\nu(K_n) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!} \left[\lambda \int \int_{(S\times K)} 1\{s_1 \oplus K_1) \cap L = \emptyset\} ds_1 d\nu(K_1) \right]^n \end{aligned}$$

Now $(s_1 \oplus K_1) \cap L = \emptyset \Leftrightarrow s_1 \in L \oplus \check{K}_1$, the dilation of L by K_1 [17, 25]. Hence

$$\lambda \int \int_{S\times K} 1\{(s_1 \oplus K_1) \cap L = \emptyset\} ds_1 d\nu(K_1) = \lambda \int_K (|S| - |L \oplus \check{K}_1|) d\nu(K_1) = \lambda[|S| - \mathbb{E}^\nu |L \oplus \check{K}_1|]$$

and thus

$$T(L) = 1 - \exp[-\lambda \mathbb{E}^\nu |L \oplus \check{K}_1|]. \quad (6)$$

By the Choquet-Matheron theorem [17], a random closed set distribution is determined completely by its capacity functional. The latter can be expressed in the density $p(\cdot)$, λ and $\nu(\cdot)$ by equation (5).

Turning attention to the form a density might take, the simple pairwise interaction density (1) is no longer valid, since distances between marked points are no longer observable, nor is the number of points. The latter observation rules out terms β^n, but the role of the parameter β is effectively taken over by the intensity parameter λ in $Q_{\lambda,\nu}$. In general, coverage models [14] of the form

$$p(X) = \alpha \exp[-\log \gamma \int_S f(c_X(a)) da]$$

do not generalise to random sets, since the coverage function $c_X(\cdot)$ is not known. However, special cases such as the area-interaction model (2) may yield a valid density

$$p(X) = \alpha \gamma^{-\tau(X)} \quad (7)$$

for some finite measure τ, $0 < \gamma < \infty$. Similarly the continuum random cluster model (4) suggests a random set density

$$p(X) = \alpha\gamma^{-c(X)}. \tag{8}$$

Both (7) and (8) can be used to encourage many disjoint components ($\gamma < 1$), few components ($\gamma > 1$), or indeed a Boolean model ($\gamma = 1$). For exact realisations see [9].

Sometimes the measure $\tau(\cdot)$ in (7) can be replaced by other Minkowski functionals [17]. For details see [4].

4 Boolean likelihoods

It is well known that the density of a Poisson process on a bounded subset $S \subset \mathbb{R}^d$ with intensity $\beta > 0$ with respect to a unit rate Poisson process is

$$p(\mathbf{x}) = e^{(1-\beta)|S|}\beta^{n(\mathbf{x})}, \tag{9}$$

$n(\mathbf{x})$ the number of points in configuration $\mathbf{x}$. Hence all homogeneous Poisson processes are absolutely continuous with respect to each other.

For Boolean models, the situation is more complicated due to the occlusion of individual mark sets. A further complication is caused by the mark distribution $\nu(\cdot)$.

Theorem 1 *If $\tilde{\nu} << \nu$ are probability measures on $(K, \mathcal{K})$ and $\lambda, \tilde{\lambda} > 0$ then $\tilde{Q} := Q_{\tilde{\lambda},\tilde{\nu}} << Q_{\lambda,\nu} := Q$.*

Proof : Suppose $Q(B) = 0$, that is

$$0 = \sum_{n=0}^{\infty} \frac{e^{-\lambda(S)}}{n!} \int \cdots \int_{(S\times K)^n} \lambda^n \mathbf{1}_B(\cup_{i=1}^n (s_i \oplus K_i)) ds_1 d\nu(K_1) \ldots ds_n d\nu(K_n).$$

Hence all terms must be zero, that is for all $n \in \mathbf{N}_0$:

$$\lambda^n \int \cdots \int_{(S\times K)^n} \mathbf{1}_B(\cup_{i=1}^n (s_i \oplus K_i)) ds_1 d\nu(K_1) \ldots ds_n d\nu(K_n) = 0. \tag{10}$$

Since $\lambda, \tilde{\lambda} \neq 0$ by general assumption and $\tilde{\nu} << \nu$:

$$\tilde{\lambda}^n \int \cdots \int_{(S\times K)^n} \mathbf{1}_B(\cup_{i=1}^n (s_i \oplus K_i)) ds_1 d\tilde{\nu}(K_1) \ldots ds_n d\tilde{\nu}(K_n) =$$

$$(\frac{\tilde{\lambda}}{\lambda})^n \lambda^n \int \cdots \int_{(S\times K)^n} \mathbf{1}_B(\cup_{i=1}^n (s_i \oplus K_i)) \prod_{i=1}^n f(K_i) ds_1 d\nu(K_1) \ldots ds_n d\nu(K_n)$$

where f denotes the density of $\tilde{\nu}$ with respect to ν.

Now by (10), $\mathbf{1}_B(\cup_{i=1}^n (s_i \oplus K_i)) = 0$ (leb$\otimes\nu)^n$ - a.e., hence $\mathbf{1}_B(\cup_{i=1}^n (s_i \oplus K_i)) \prod_{i=1}^n f(K_i) = 0$ almost everywhere and we obtain, for all $n \in \mathbf{N}_0$

$$\tilde{\lambda}^n \int \cdots \int_{(S\times K)^n} \mathbf{1}_B(\cup_{i=1}^n (s_i \oplus K_i)) ds_1 d\tilde{\nu}(K_1) \ldots ds_n d\tilde{\nu}(K_n) = 0$$

yielding $\tilde{Q}(B) = 0$. Since B was arbitrary, $\tilde{Q} << Q$. □

By the Radon-Nikodym theorem, $\tilde{Q}$ has a derivative with respect to Q. By (5), we obtain

$$\frac{d\tilde{Q}}{dQ}(X) = \frac{\sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}|S|}}{n!}\tilde{\lambda}^n \int \cdots \int_{(S\times K)^n} \mathbf{1}\{\cup_{i=1}^n (s_i \oplus K_i) = X\} ds_1 d\tilde{\nu}(K_1) \cdots ds_n d\tilde{\nu}(K_n)}{\sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!}\lambda^n \int \cdots \int_{(S\times K)^n} \mathbf{1}\{\cup_{i=1}^n (s_i \oplus K_i) = X\} ds_1 d\nu(K_1) \cdots ds_n d\nu(K_n)}. \tag{11}$$

Below we investigate some special cases.

4.1 Point processes

A Poisson point process can be seen as a Boolean model with a degenerate mark, i.e. $\nu(\{0\}) = 1$. Thus by (11), the density of $Q_{\tilde{\lambda},\nu}$ with respect to $Q_{\lambda,\nu}$ is

$$\begin{aligned} \frac{dQ_{\tilde{\lambda},\nu}}{dQ_{\lambda,\nu}}(X) &= \frac{\sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}|S|}}{n!}\tilde{\lambda}^n \int \cdots \int_{S^n} 1_X(\{s_1,\ldots,s_n\}) ds_1 \cdots ds_n}{\sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!}\lambda^n \int \cdots \int_{S^n} 1_X(\{s_1,\ldots,s_n\}) ds_1 \cdots ds_n} \\ &= \frac{e^{-\tilde{\lambda}|S|}\tilde{\lambda}^n(X)}{e^{-\lambda|S|}\lambda^n(X)} = e^{(\lambda-\tilde{\lambda})|S|}(\frac{\tilde{\lambda}}{\lambda})^n(X) \end{aligned}$$

agreeing with (9).

4.2 Wire frame processes

Consider a Boolean disc process as in Figure 1. Since an homogeneous Poisson process almost surely does not have multiple points, there is a one-to-one correspondence between the union sets

$$\cup_{i=1}^n S(s_i,k_i), \qquad S(s_i,k_i) = \partial B(s_i,k_i)$$

the topological boundary of the ball at s_i with radius k_i, and configurations $\{(s_i,k_i) : i = 1,\ldots,n\}$. In particular, the number of circles is an observable functional for both the random set and the marked point process representation.

Suppose we have two radius distributions $\nu(\cdot)$ and $\tilde{\nu}(\cdot)$ on $(\mathbb{R}^+, \mathcal{B}or)$ with $\tilde{\nu} << \nu$ and intensities $\lambda, \tilde{\lambda} > 0$. Then

$$\frac{dQ_{\tilde{\lambda},\tilde{\nu}}}{dQ_{\lambda,n\nu}} = e^{(\lambda-\tilde{\lambda})|S|}(\frac{\tilde{\lambda}}{\lambda})^{n(X)} \prod_{i=1}^{n(X)} f(t_i) \tag{12}$$

where $f(\cdot)$ is the Radon-Nikodym derivative $\frac{d\tilde{\nu}}{d\nu}$.

To verify (12), note that for any $F \in \mathcal{F}_{S\times K}$

$$\begin{aligned} Q_{\tilde{\lambda},\tilde{\nu}}(F) &= \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}|S|}}{n!}\tilde{\lambda}^n \int \cdots \int_{(S\times K)^n} 1_F(X) ds_1 d\tilde{\nu}(k_1) \ldots ds_n d\tilde{\nu}(k_n) \\ &= \sum_{n=0}^{\infty} e^{(\lambda-\tilde{\lambda})|S|}(\frac{\tilde{\lambda}}{\lambda})^n \frac{e^{-\lambda|S|}}{n!}\lambda^n \int \cdots \int_{(S\times K)^n} 1_F(X) \prod_{i=1}^{n} f(k_i) ds_1 d\nu(k_1) \ldots ds_n d\nu(k_n) \end{aligned}$$

or alternatively, use (11) as in 4.1.

The expression (12) also holds for other 'wire frame' processes with transparent marks, where the individual component sets can be uniquely identified from their union.

4.3 Poisson cluster processes

In a Poisson cluster process [7], each of the points in a 'parent' Poisson process is marked by a finite, bounded point process of 'daughters'; a realisation consists of the total offspring.

Thus, as before, let $S \subseteq \mathbb{R}^d$ be bounded Borel, equipped with Lebesgue measure, and take for K the exponential space Ω_T for some large enough bounded Borel set $T \supseteq S$. The mark distribution $\nu(\cdot)$ could be specified by its density $g(\cdot)$ with respect to a unit rate Poisson process on T. Then, conditional on parent configuration $\mathbf{s} = \{s_1, \ldots, s_n\}$, the offspring $X = \cup_{i=1}^n (s_i \oplus K_i)$ is absolutely continuous with respect to a unit rate Poisson process on T, with conditional density

$$p(X|\mathbf{s}) = e^{|T|} \sum_{\varphi} \prod_{i=1}^{n} \left[g(X_{\varphi^{-1}(i)} - s_i) e^{-|T|} \right]$$

where the sum is over all ordered partitions φ of X.

The unconditional density can be found by taking the expectation over $\mathbf{s}$:

$$p(X) = \sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!} \lambda^n e^{|T|(1-n)} \int \cdots \int_{S^n} \sum_{\varphi} \prod_{i=1}^{n} g(X_{\varphi^{-1}(i)} - s_i) ds_1 \ldots ds_n \tag{13}$$

Next consider two Poisson cluster processes, such that the daughter distributions satisfy $\tilde{\nu} << \nu$, i.e. if $g \equiv 0$ on a non-null set, then the same is true for $\tilde{g}$. Then, by (11) or (13),

$$\frac{dQ_{\tilde{\lambda},\tilde{\nu}}}{dQ_{\lambda,\nu}}(X) = \frac{\sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}|S|}}{n!} \tilde{\lambda}^n e^{|T|(1-n)} \sum_{\varphi} \int \cdots \int_{S^n} \prod_{i=1}^{n} \tilde{g}(X_{\varphi^{-1}(i)} - s_i) ds_1 \ldots ds_n}{\sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!} \lambda^n e^{|T|(1-n)} \sum_{\varphi} \int \cdots \int_{S^n} \prod_{i=1}^{n} g(X_{\varphi^{-1}(i)} - s_i) ds_1 \ldots ds_n}. \tag{14}$$

The expression (14) can be simplified if we make further model assumptions. For instance, let each parent have a single daughter, with displacement densities $h(\cdot), \tilde{h}(\cdot)$ concentrated on a ball of radius R. Then the process is nearest-neighbour Markov at range $2R$, and (14) reduces to

$$e^{(\lambda-\tilde{\lambda})|S|} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{n(X)} \prod_{i=1}^{n(X)} \left(\frac{\int_S \tilde{h}(y_i - s) ds}{\int_S h(y_i - s) ds} \right).$$

4.4 Boolean models with deterministic grains

Consider a homogeneous Poisson process of locations and place at each of the points in a realisation a copy of the fixed set K_0. Then for every $\lambda, \tilde{\lambda} > 0$, $Q_{\tilde{\lambda}} << Q_{\lambda}$ and

$$\frac{dQ_{\tilde{\lambda}}}{dQ_{\lambda}}(X) = \frac{\sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}|S|}}{n!} \tilde{\lambda}^n \int \cdots \int_{S^n} 1\{\cup_{i=1}^n (s_i \oplus K_0) = X\} ds_1 \cdots ds_n}{\sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!} \lambda^n \int \cdots \int_{S^n} 1\{\cup_{i=1}^n (s_i \oplus K_0) = X\} ds_1 \cdots ds_n}$$

Note that the term for $n < m$ in both numerator and denominator vanish, where m is the number of non-empty connected components in X.

5 Markov properties for random sets

In this Section, we focus on finding an analogue to Definition 3 for random sets. We will exploit the Hammersley-Clifford theorem, giving an equivalence between Markov densities satisfying (M) and (H) in Definition 3 and factorisation of the clique interaction functions. Intuitively, in the random set case, it seems reasonable to expect a factorisation into disjoint connected components. Indeed, by a result in [5], Poisson cluster processes (simple Boolean models!) are connected component Markov. For this family of neighbourhood relations, the Hammersley-Clifford factorisation is

$$(M') \quad p(\Xi) = p(\emptyset) \prod_{j=1}^{m} \Phi(\Xi_j)$$

where $\Xi_1, \ldots, \Xi_m$ are the maximal connected components in Ξ and the interaction function $\Phi(\cdot) \geq 0$ is such that if Y is connected and $Z \subseteq Y$ is connected too, then $\Phi(Y) > 0 \Rightarrow \Phi(Z) > 0$ [5].

Hence we can define a random set density $p(\cdot)$ to be Markovian if it satisfies (M').

Theorem 2 *The Boolean model (11), the continuum random cluster random set (8) and the area-interaction random set (7) satisfy the factorisation property (M').*

Proof : For the Boolean model density (11), consider the denominator

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda|S|}}{n!} \lambda^n \int \cdots \int_{(S \times K)^n} \mathbf{1}\{\cup_{i=1}^{n}(s_i \oplus K_i) = \Xi\} ds_1 d\nu(K_1) \cdots ds_n d\nu(K_n). \qquad (15)$$

Write $\Xi = \Xi_1 \cup \cdots \Xi_m$ where $\Xi, i = 1, \ldots, m$ are the maximal connected components of Ξ. Then (15) can be rewritten as

$$e^{-\lambda|S|} \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} \frac{1}{(n_1 + \cdots + n_m)!} \binom{n_1 + \cdots + n_m}{n_1, \ldots, n_m} \lambda^{n_1 + \cdots + n_m}$$

$$\prod_{j=1}^{m} \int_{K^{n_j}} \mathbf{1}\{\cup_{i=1}^{n_j}(s_i \oplus K_i) = \Xi_j\} ds_1 d\nu(K_1) \cdots ds_{n_j} d\nu(K_{n_j}) =$$

$$e^{-\lambda|S|} \prod_{j=1}^{m} \left[\sum_{n_j=1}^{\infty} \frac{1}{n_j!} \lambda^{n_j} \int \cdots \int_{K^{n_j}} \mathbf{1}\{\cup_{i=1}^{n_j}(s_i \oplus K_i) = \Xi_j\} ds_1 d\nu(K_1) \cdots ds_{n_j} d\nu(K_{n_j}) \right].$$

The numerator in (11) factorises similarly, hence (11) satisfies (M').

Regarding the area-interaction random set,

$$p(\Xi) = \alpha \gamma^{-\tau(\Xi)} = \alpha \prod_{j=1}^{m} \gamma^{-\tau(\Xi_j)}$$

giving (M').

Finally

$$p(\Xi) = \alpha\gamma^{-c(X)}\alpha \prod_{j=1}^{m} \frac{1}{\gamma}$$

hence the continuum random cluster model also satisfies (M'), completing the proof. □

Although it does not seem clear how to define a conditional intensity for random sets in general, the likelihood ratio for adding $s_{n+1} \oplus K_{n+1}$ to $\Xi = \cup_{i=1}^{n}(s_i \oplus K_i)$ depends only on the component of $s_{n+1} \oplus K_{n+1}$ in the new set and the connected components Ξ_j of Ξ intersecting the added set.

Turning to 'global' Markov properties, Matheron showed that any Boolean model X with convex mark sets satisfies the *semi-Markov property* that $X \cap E$ and $X \cap F$ are conditionally independent given $X \cap G = \emptyset$ where E, F are compact sets separated by another compact set G, i.e. any line segment joining $x \in E$ with $x' \in F$ must hit G [17]. This property does not hold for (M')–densities in general. On the other hand, spatial Markov properties as considered by Kendall [11] and Møller [21] may carry over, provided appropriate hereditariness assumptions are satisfied [5].

Acknowledgements

I am most grateful to Ilya Molchanov for raising the problem and to Dominique Jeulin for the invitation. This research has been supported by Nuffield foundation grant SCI/180/94/103 (Applications of stochastic geometry in the analysis of spatial data)

References

[1] A.J. Baddeley and M.N.M. van Lieshout. Object recognition using Markov spatial processes. In *Proceedings 11th IAPR International Conference on Pattern Recognition*, B 136–139. IEEE Computer Society Press, 1992.

[2] A.J. Baddeley and M.N.M. van Lieshout. Area-interaction point processes. *Annals of the Institute of Statistical Mathematics*, 47:601–619, 1995.

[3] A.J. Baddeley and J. Møller. Nearest-neighbour Markov point processes and random sets. *International Statistical Review*, 57:89–121, 1989.

[4] A.J. Baddeley, W.S. Kendall and M.N.M. van Lieshout. Quermass–interaction models. Research report, University of Warwick, 1996.

[5] A.J. Baddeley, M.N.M. van Lieshout and J. Møller. Markov properties of cluster processes. *Advances in Applied Probability*, 28:346–355, 1996.

[6] J.T. Chayes, L. Chayes and R. Kotecky. The analysis of the Widom–Rowlinson model by stochastic geometric methods. *Communications in Mathematical Physics*, 172:551–569, 1995.

[7] D.J. Daley and D. Vere-Jones. *An introduction to the theory of point processes*. Springer, New York, 1988.

[8] H.-O. Georgii and O. Häggström. Phase transition in continuum Potts models. *Communications in Mathematical Physics*, 1996. To appear.

[9] O. Häggström, M.N.M. van Lieshout and J. Møller. Characterisation results and Markov chain Monte Carlo algorithms including exact simulation for some spatial point processes. In preparation.

[10] F.P. Kelly and B.D. Ripley. On Strauss's model for clustering. *Biometrika*, 63:357–360, 1976.

[11] W.S. Kendall. A spatial Markov property for nearest-neighbour Markov point processes. *Journal of Applied Probability*, 28:767–778.

[12] M.N.M. van Lieshout. Stochastic annealing for nearest-neighbour point processes with application to object recognition. *Advances in Applied Probability*, 26:281–300, 1994.

[13] M.N.M. van Lieshout. *Stochastic geometry models in image analysis and spatial statistics*. CWI tract 108, CWI, Amsterdam, 1995.

[14] M.N.M. van Lieshout and I.S. Molchanov. Shot-noise-weighted processes: a new family of spatial point processes. CWI Report BS-R9527, October 1995.

[15] K.V Mardia and G.K. Kanji (Eds). *Statistics and Images. Advances in Applied Statistics*, a supplement to *Journal of Applied Statistics*, 1993.

[16] K.V. Mardia, W. Qian and D. Shah. Deformable template recognition of multiple occluded objects. Research Report STAT 96/04, University of Leeds, 1996.

[17] G. Matheron. *Random sets and integral geometry*. John Wiley and Sons, New York, 1975.

[18] I.S. Molchanov. Statistics of the Boolean model: from the estimation of means to the estimation of distributions. *Advances in Applied Probability*, 27:63–86, 1995.

[19] I.S. Molchanov. Random closed sets: results and problems. In: *Proceedings Seminaire Européen de Statistique, "Current trends in stochastic geometry with applications"*. O. Barndorff–Nielsen, W.S. Kendall, G. Letac and M.N.M. van Lieshout (Eds), Chapman and Hall, 1996. To appear.

[20] I.S. Molchanov and D. Stoyan. Asymptotic properties of estimators for parameters of the Boolean model. *Advances in Applied Probability*, 26:301–323, 1994.

[21] J. Møller. Markov chain Monte Carlo and spatial point processes. In: *Proceedings Seminaire Européen de Statistique, "Current trends in stochastic geometry with applications"*. O. Barndorff–Nielsen, W.S. Kendall, G. Letac and M.N.M. van Lieshout (Eds), Chapman and Hall, 1996. To appear.

[22] A.K. Penttinen, D. Stoyan and H.M. Henttonen. Marked point processes in forest statistics. *Forest Science*, 38:806–824, 1992.

[23] B.D. Ripley and F.P. Kelly. Markov point processes. *Journal of the London Mathematical Society*, 15:188–192, 1977.

[24] D. Ruelle. Existence of a phase transition in a continuous classical system. *Physical Review Letters*, 27:1040–1041, 1971.

[25] J. Serra. *Image analysis and mathematical morphology*. Academic Press, New York, 1982.

[26] D. Stoyan, W.S. Kendall and J. Mecke. *Stochastic geometry and its applications*, second edition. John Wiley and Sons, New York, 1995.

[27] D.J. Strauss. A model for clustering. *Biometrika*, 63:467–475, 1975.

[28] A. Trubuil, H. Stryhn and M. Hoebeke. Segmentation d'images: Approche par simulation d'un processus de diffusion et de saut. Proceedings INRA, 1995.

[29] B. Widom and J.S. Rowlinson. A new model for the study of liquid-vapor phase transitions. *Journal of Chemical Physics* 52:1670–1684, 1970.