JOURNAL OF ALGEBRA **84**, 285–304 (1983)

Operations in the K-Theory of Endomorphisms*

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Communicated by P. M. Cohn

Received November 2, 1981

For a commutative ring with unity A, let **End** A be the category of all pairs (P,f), where P is a finitely generated projective A-module and f an endomorphism of A. The K-group $K_0(A)$ is a direct summand and ideal of $K_0(\mathbf{End}\,A)$, and Almkvist showed that the quotient ring $W_0(A) = K_0(\mathbf{End}\,A)/K_0(A)$ is a functorial subring of the ring of the big Witt vectors $W(A)$ [1]. In this paper, I determine the ring of all continuous functorial operations on $W_0(-)$, and the semiring of all operations (and all continuous operations) liftable to **End**(A). This solves some of the open problems listed in [1].

1. INTRODUCTION, DEFINITIONS AND STATEMENT OF MAIN RESULTS

Let A be a commutative ring with unit element. With **End** A, I denote the category of pairs (P,f), where P is a finitely generated projective module over A, and f an endomorphism of P. A morphism $u: (P,f) \to (Q,g)$ is a morphism of A-modules $u: P \to Q$, such that $gu = uf$. There is an obvious notion of short exact sequence in **End** A: it is a commutative diagram with exact rows of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & 0. \end{array} \tag{1.1}$$

* During the research for and writing of this paper, the author was visiting the Inst. de Ciencias, Univ. Autonoma de Puebla, whose hospitality and support is gratefully acknowledged. I would like to thank Ton Vorst for pointing out some gaps in an earlier draft of this paper.

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1.2. DEFINITION [1, 2]. $K_0(\mathbf{End}\, A)$ is the free abelian group generated by all isomorphism classes $[P,f]$ of objects in $\mathbf{End}\, A$ modulo, the subgroup generated by all elements of the form $[Q,g]-[P,f]-[R,h]$ for all exact sequences (1.1).

The tensor product $((P,f),(Q,g)) \mapsto (P \otimes Q, f \otimes g)$ induces a ring structure on $K_0(\mathbf{End}\, A)$ for which the unit element is the class of $(A, 1)$. (All tensor products are over A.) Further, the classes of the form $(Q, 0)$ form an ideal in $K_0(\mathbf{End}\, A)$. This ideal identifies naturally with $K_0(A)$ via $P \mapsto (P, 0)$.

1.3. DEFINITION. *The ring of rational Witt vectors.* The quotient ring is denoted $K_0(\mathbf{End}\, A)/K_0(A) = W_0(A)$. I like to call the elements of $W_0(A)$ rational Witt vectors for reasons which will become obvious immediately below.

1.4. *The Big Witt Vectors*

For each ring R let $W(R)$ be the abelian group of all power series of the form $1 + r_1 t + r_2 t^2 + \cdots$, $r_i \in R$. Obviously, this functor is represented by the ring $\mathbb{Z}[X_1, X_2, \ldots]$; i.e., $\mathbf{Ring}(\mathbb{Z}[X], R) \simeq W(R)$ functorially. The group $W(R)$ also carries a multiplication which is characterized by $(1 - r_1 t) * (1 - r_2 t) = 1 - r_1 r_2 t$ for which $1 - t$ acts as a unit. This makes $W(R)$ functorially a commutative ring with unit. This functorial ring $W(R)$ admits functorial ring endomorphisms called Frobenius operators which are characterized by $F_n(1 - at) = (1 - a^n t)$.

Compare [4, Chapter 3] for a rather detailed treatment of Witt vectors.

1.5. *Almkvist's Homomorphism*

Let $(P,f) \in \mathbf{End}\, A$. Let Q be a finitely generated projective A-module such that $P \oplus Q$ is free, and consider the endomorphism $f \oplus 0$ of $P \oplus Q$. Consider $\det(1 + t(f \oplus 0))$. This is a polynomial in t which does not depend on Q. This induces a homomorphism $K_0(\mathbf{End}\, A) \to W(A)$ which is (obviously) zero on $K_0(A)$. It is also obviously additive and multiplicative, so that there results a homomorphism of rings

$$c\colon K_0(\mathbf{End}\, A)/K_0(A) = W_0(A) \to W(A), \tag{1.6}$$

which is functorial in A. In [2] Almkvist now proves:

1.7. THEOREM [2]. *The homomorphism c is injective for all A, and the image of c (for a given A) consists of all power series $1 + a_1 t + a_2 t^2 + \cdots$, which can be written in the form*

$$1 + a_1 t + a_2 t^2 + \cdots = \frac{1 + b_1 t + \cdots + b_r t^r}{1 + d_1 t + \cdots + d_n t^n}, \qquad b_i, d_j \in A.$$

(Whence the name, rational Witt vectors; the c in (1.6) *stands for characteristic polynomial.)*

1.8. *Topology on* $W_0(A)$, $W(A)$

Let $W^{(n)}(A)$ be the subgroup of all power series of the form $1 + a_{n+1}t^{n+1} + \cdots \in W(A)$. These subgroups define a topology on $W(A)$, and $W_0(A) \subset W(A)$ is given the induced topology. Let $W_0^+(A)$ be the subset of $W(A)$ consisting of all polynomials $1 + a_1 t + a_2 t^2 + \cdots a_r t^r$. Then $W_0^+(A)$ and $W_0(A)$ are dense in $W(A)$. With this definition, W_0, W, W_0^+ become functors **Ring** $\to$ **Top**, where **Top** is the category of Hausdorff topological spaces. The $W^{(n)}(A)$ are in fact ideals in $W(A)$, so that W_0, W_n can also be considered to take their values in the categories **TRng** of topological rings or **TAb** of topological abelian groups, and W_0^+ can be considered to take its values in the category of topological semigroups.

1.9. *Operations*

Let F be a functor, e.g., a functor F: **Ring** $\to$ **Set**. Then an operation for $F(-)$ is a functorial transformation $u: F \to F$. Below I shall determine all operations for the functors W_0 and W_0^+ considered as functors **Ring** $\to$ **Top**, i.e., all functorial transformations of sets $W_0(A) \to W_0(A)$, $W_0^+(A) \to W_0^+(A)$ which are continuous with respect to the topologies on $W_0(A)$, $W_0^+(A)$, and also of W_0 as a functor to **TAb** (additive operations) and as a functor to **TRng** (multiplicative operations). Here $W_0^+(A)$ is the image of $\mathbf{End}_A$ in $W_0(A)$, which via c identifies with the commutative sub-semiring of $W(A)$ consisting of all polynomials $1 + a_1 t + \cdots + a_r t^r$. (This is fairly obvious, but cf. also 2.4 below.) I shall also determine what various natural operations on **End** A, like exterior products and symmetric products, correspond to in $W(A)$. All these questions were posed as problems in [1].

1.10. *Two Topologies on the Ring* $\mathbb{Z}[X]$

Before I can describe the results I have to define two topologies on the ring $\mathbb{Z}[X_1, X_2, X_3, \ldots] = \mathbb{Z}[X]$. For each $n \in \mathbb{N}$, let I_n be the ideal of $\mathbb{Z}[X]$ generated by the elements $X_{n+1}, X_{n+2}, \ldots$. The I-topology on $\mathbb{Z}[X]$ is the one defined by this sequence of ideals. The second and more important topology is also more difficult to describe. Consider the infinite Hankel matrix

$$\begin{pmatrix} 1 & X_1 & X_2 & X_3 & \cdots \\ X_1 & X_2 & X_3 & X_4 & \cdots \\ X_2 & X_3 & X_4 & X_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{1.11}$$

Now for each $n \in \mathbb{N}$, let J_n be the ideal generated by all the $(n+1) \times (n+1)$ minors of this matrix. Let $\mathbb{Z}_I[X]$ and $\mathbb{Z}_J[X]$ denote the completions of $\mathbb{Z}[X]$ with respect to the I-topology and the J-topology.

The ring of power series in infinitely many variables $\mathbb{Z}[[X]]$ is defined as the ring of all expressions $\sum_\alpha c_\alpha X^\alpha$ where α runs through all multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, such that $\alpha_i = 0$ for all but finitely many i. Here, X^α is short for the finite monomial

$$X^\alpha = \prod_{\alpha_i \neq 0} X_i^{\alpha_i}.$$

Both $\mathbb{Z}_I[X]$ and $\mathbb{Z}_J[X]$ can be considered as subrings of $\mathbb{Z}[[X]]$. For instance, the elements of $\mathbb{Z}_I[X]$ are power series $f(X)$ in $X_1, X_2, \ldots$, with the extra property that $f(X)$ is a polynomial mod I_n for all n. Thus, e.g., $X_1X_2 + X_1X_3 + X_1X_4 + X_1X_5 + \cdots$ is in $\mathbb{Z}_I[X]$, but $1 + X_1 + X_1^2 + X_1^3 + \cdots$ is not in $\mathbb{Z}_I[X]$.

We also note that $J_n \subset I_{n-1}$, so that there is a natural inclusion $\mathbb{Z}_J[X] \to \mathbb{Z}_I[X]$.

With these notions we can state the main results as

1.12. Theorem. *The continuous operations of $W_0^+(-)$ correspond naturally to ring endomorphisms of $\mathbb{Z}[X]$ which are continuous in the I-topology (on both source and target). The (not necessarily continuous) operations of W_0^+ correspond naturally to ring endomorphisms of $\mathbb{Z}_I[X]$.*

1.13. Theorem. (i) *The continuous operations of $W_0(-)$ correspond naturally to ring endomorphisms of $\mathbb{Z}[X]$, which are continuous in the J-topology (on both source and target).*

(ii) *The additive continuous operations of $W_0(-)$ correspond to elements $1 + x_1t + x_2t^2 + \cdots \in W(\mathbb{Z}[X])$, such that $\lim_{i\to\infty} x_i = 0$ in the J-topology, and $\mu(x_n) = \sum_{i+j=n} x_i \otimes x_j$, where $\mu\colon \mathbb{Z}[X] \to \mathbb{Z}[X] \otimes \mathbb{Z}[X]$ is the coalgebra structure defined by $X_n \mapsto \sum_{i+j=n} X_i \otimes X_j$.*

(iii) *The multiplicative and unit preserving continuous operations of $W_0(-)$ are the Frobenius operations.*

2. Representing the Functor W_0^+

2.1. *Universal Examples of Endomorphisms*

For each $n \in \mathbb{N}$, let $U_n = \mathbb{Z}[X_1, \ldots, X_n]$, and consider the free module $P_n = U_n^n$ with the endomorphism f_n given by the matrix

$$f_n = \begin{pmatrix} X_1 & -1 & 0 & \cdots & 0 \\ X_2 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & -1 \\ X_n & 0 & & \cdots & 0 \end{pmatrix}. \tag{2.2}$$

Then, of course, $\det(1+tf_n) = 1 + X_1 t + \cdots + X_n t^n$. And (P_n, f_n) has the following universality property: for each polynomial of degree $\leqslant n$, $1 + a_1 t + \cdots + a_n t^n = a \in W_0^+(A)$, there is a unique homomorphism $\phi_a: U_n \to A$ such that $\phi_{a*}: W_0^+(U_n) \to W_0^+(A)$ takes $\gamma_n = [P_n, f_n]$ into a. This, of course, also shows that the image of $\mathbf{End}\, A$ in $W_0(A)$ is precisely the subsemiring of polynomials of the form $1 + a_1 t + \cdots + a_n t^n$.

The $\gamma_n = [P_n, f_n]$ fit together in the sense that if $\pi_n^{n+1}: U_{n+1} \to U_n$ is the projection $X_i \mapsto X_i$ for $i = 1, \ldots, n$, $X_{n+1} \mapsto 0$, then

$$(\pi_n^{n+1})_* \gamma_{n+1} = \gamma_n. \tag{2.3}$$

The following proposition follows immediately.

2.4. PROPOSITION. *There is a functorial isomorphism between* $W_0^+(A)$ *and* $\mathbf{TRng}(\mathbb{Z}_I[X_1, X_2, \ldots], A)$, *where* **TRng** *stands for continuous ring homomorphisms from* $\mathbb{Z}[X_1, X_2, \ldots]$ *with the I-topology, to A with the discrete topology.*

Indeed, if $\phi: \mathbb{Z}[X] \to A$ is continuous, then there is an I_n such that $\phi(I_n) = 0$, so that ϕ factors through $\pi_n: \mathbb{Z}[X] \to U_n$. Let ϕ_n be the induced homomorphism, then the element in $W_0^+(A)$ corresponding to ϕ is $\phi_{n*}\gamma_n$. And inversely, if $A(t) \in W_0^+(A)$, $a(t) = 1 + a_1 t + \cdots + a_n t^n$, let $\phi'_a: U_n \to A$ be defined by $\phi'_a(X_i) = a_i$. Then $\phi_a = \phi'_a \circ \pi_n$ is the desired continuous homomorphism $\mathbb{Z}[X] \to A$.

3. THE FATOU PROPERTY

3.1. DEFINITION. An integral domain R is said to be *Fatou* if the following property holds. For every power series $a(s^{-1}) = \sum_{i=0}^{\infty} a_i s^{-i}$ in s^{-1} with coefficients in R such that there exist polynomials $p(s)$, $q(s)$ with coefficients in the quotient field $Q(R)$ such that $a(s^{-1}) = q(s)^{-1} p(s)$, there exist also polynomials $\bar{p}(s), \bar{q}(s) \in R[s]$ such that $\bar{q}(s)$ has leading coefficient 1 which also satisfy $\bar{q}(s)^{-1} \bar{p}(s) = a(s^{-1})$. (The same property then holds obviously also with respect to Laurent series.) The following result comes out of mathematical system theory [7, 8].

3.2. PROPOSITION. *Every noetherian integral domain R is Fatou.*

Proof. Let $a(s^{-1}) = \sum_{i=0}^{\infty} a_i s^{-i}$ be a power series in s^{-1} over R. Write down the Hankel matrix of $a(s^{-1})$.

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.3}$$

Now suppose that $a(s^{-1}) = q(s)^{-1} p(s)$ for certain polynomials over the quotient field $Q(R)$ of R. This means that there is a certain recursion relation,

$$q_1 a_{n+t-1} + q_2 a_{n+t-2} + \cdots + q_t a_n = 0, \tag{3.4}$$

between the coefficients a_n for all large enough n, and in turn this means that the rank of the matrix (3.3) is finite. Let this rank be r. Now consider the A-module M generated by the columns of (3.3). This module can be seen as a submodule of some $b^{-1}R^r$ for some $b \in R$. (For b, one can take any nonzero $r \times r$ minor of (3.3)). But $b^{-1}R^r$ is a finitely generated R-module, and, as R is noetherian, it follows that M is finitely generated. Now define an endomorphism F of M by $F(a(i)) = a(i+1)$, where $a(i)$ is the column of (3.3) starting with a_i. Let $g = a(0)$, and let $h: M \to R$ be defined by $h(a(i)) = a_i$. Note that because of the structure of (3.3), the endomorphism F is well defined. We note that $hF^i g = a_i$ for all $i = 0, 1, 2, \ldots$. Now because M is finitely generated, there is a surjection of R-modules $\pi: R^m \to M$ for some m. Define $\tilde{h} = h\pi$; let $\tilde{F}$ be any lift of F, i.e., any endomorphism (matrix) of R^m such that $\pi\tilde{F} = F\pi$ and $\tilde{g}$ any element of R^m such that $\pi(\tilde{g}) = g$. Then $\tilde{h}\tilde{F}^i\tilde{g} = hF^i g = a_i$ for all $i = 0, 1, 2, \ldots$ and consequently $s\tilde{h}(sI - \tilde{F})^{-1}\tilde{g} = a(s^{-1})$, proving the proposition.

4. "REPRESENTING" THE FUNCTOR W_0

We are now in a position to represent, in a certain sense, the functor $W_0(-)$.

4.1. DEFINITION OF THE "UNIVERSAL OBJECT." Let J_n be the ideal in $\mathbb{Z}[X]$ defined in the introduction and let $V_n = \mathbb{Z}[X]/J_n$, let $\rho_n: \mathbb{Z}[X] \to V_n$ be the natural projection, let $\xi = 1 + X_1 t + X_2 t^2 + \cdots \in W(\mathbb{Z}[X])$, and let $\xi_n = (\rho_n)_*(1 + X_1 t + X_2 t^2 + \cdots) \in W(V_n)$.

4.2. *Warning and Intermezzo*

It is not clear that ξ_n is in $W_0(V_n)$. In fact, this is definitely not the case, because there are integral domains which are not Fatou. It also follows that the V_n are examples. (The V_n are integral by the Appendix.) It follows that the V_n are not noetherian. Let $\tilde{D}_n$ be the top left $n \times n$ minor of (1.11). Then, as we shall see in Sect. 6.10 below, ξ_n becomes a rational Witt vector over V_n localized at $(1, D_n, D_n^2, ...)$, where $D_n = \rho_n(\tilde{D}_n)$. It is easy to check that the map β_n of diagram (6.11) contains V_n in its image, and it follows that the localization $(V_n)_{D_n}$ is noetherian.

It is still not true, however, that ξ_n over $(V_n)_{D_n}$ is universal for rational Witt vectors of numerator degree $\leqslant n-1$ and denominator degree $\leqslant n$. To obtain universal rational Witt vectors, one needs something like a universal Fatourization construction.

4.5. THEOREM. *For each* $1 + a_1 t + \cdots = a \in W_0(A)$, *let* $\phi_a\colon \mathbb{Z}[X] \to A$ *be the ring homomorphism defined by* $X_i \mapsto a_i$. *Then* $a(t) \mapsto \phi_a$ *is a functorial and injective correspondence from* $W_0(A)$ *to ring homomorphisms* $\mathbb{Z}[X] \to A$, *which are continuous with respect to the J-topology on* $\mathbb{Z}[X]$ *and the discrete topology on A. If A is Fatou, so in particular if A is integral and noetherian, then this induces a functorial isomorphism.*

Proof. The rational Witt vector a can be written $a = (1 + c_1 t + \cdots + c_n t^n)^{-1}(1 + b_1 t + \cdots + b_{n-1} t^{n-1})$. Consider $\mathbb{Z}[Y_1, ..., Y_{n-1}; Z_1, ..., Z_n]$, and define $\psi\colon \mathbb{Z}[Y; Z] \to A$ by $\psi(Y_i) = c_i$ and $\psi(Z_j) = b_j$, $i, j = 1, ..., n$. Let δ_n be the rational Witt vector

$$\delta_n = \frac{1 + Y_1 t + \cdots + Y_{n-1} t^{n-1}}{1 + Z_1 t + \cdots + Z_n t^n} \in W_0(\mathbb{Z}[Y, Z]). \tag{4.6}$$

Then, of course, $\psi_* \delta_n = a$ (but there may be several ψ's with this property). Define $\varepsilon_n\colon \mathbb{Z}[X] \to \mathbb{Z}[Y, Z]$ by $\varepsilon_{n*}\xi = \delta_n$. Then $(\psi\varepsilon_n)_*\xi = a$, so that $\psi\varepsilon_n = \phi_a$. Now δ_n is rational, so there is a recursion relation between its coefficients $a_i(Y, Z)$ in

$$\delta_n = 1 + a_1(Y, Z)\, t + a_2(Y, Z)\, t^2 + \cdots. \tag{4.7}$$

This, in turn, means that the rank of the associated Hankel matrix (cf. (3.3)) is finite (over the quotientfield $Q(\mathbb{Z}[Y, Z])$, and because $\mathbb{Z}[Y, Z]$ is an integral domain, this means that for some n, all minors of the Hankel matrix of (4.6) vanish. Thus $\varepsilon_n(J_m) = 0$ for some m (in fact $m = n$ works), so that a fortiori $\phi_a(J_m) = 0$, i.e., ϕ_a is continuous. The injectivity of $a \mapsto \phi_a$ is obvious, because $\phi_a(X_i) = a_i$.

Now let A be Fatou (and an integral domain). Let $\psi\colon \mathbb{Z}[X] \to A$ be continuous. Let $a_i = \psi(X_i)$. Then there is an m such that $\psi(I_m) = 0$. Thus all

$(m+1)\times(m+1)$ minors of the Hankel matrix (3.3) of $a_0=1, a_1, a_2,\ldots$ vanish, so that this matrix is of finite rank. So there are $q_0,\ldots,q_m\in Q(A)$ such that $q_0 a(0)+\cdots+q_m a(m)=0$, where as before $a(i)$ is the ith column of (3.3). Hence

$$q_0 a_t + q_1 a_{t+1} + \cdots + q_m a_{t+m} = 0, \qquad t=0,1,2,\ldots, \tag{4.8}$$

so that

$$\frac{p_0 + p_1 t + \cdots + p_{m-1} t^{m-1}}{q_m + q_{m-1} t + \cdots + q_0 t^m} = 1 + a_1 t + a_2 t^2 + \cdots, \tag{4.9}$$

with $p_0 = q_m$, $p_1 = q_m a_1 + q_{m-1},\ldots, p_{m+1} = q_m a_{m-1} + \cdots + q_1$. Now write $t=s^{-1}$, multiply numerator and denominator of (4.6) with s^m, and apply the Fatou property to find an expression

$$\frac{c_n s^n + c_{n-1} s^{n-1} + \cdots + c_1 s + c_0}{s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0} = 1 + a_1 s^{-1} + a_2 s^{-2} + \cdots, \tag{4.10}$$

with $c_0,\ldots,c_n, b_0,\ldots,b_{m-1}\in A$. It follows that $n=m$ and $c_n=1$. Now write $t=s^{-1}$ again, and multiply numerator and denominator in (4.10) with t^n to find the desired expression.

5. The Operations of W_0^+

5.1. *Functorial Transformations* $W_0^+ \to W$

Consider the functor W_0^+ and W as functors **Ring** $\to$ **Set**, and let $u\colon W_0^+ \to W$ be a functorial transformation. Consider the element $\gamma_n \in W_0^+(U_n)$, cf., Section 2.1 above. Let

$$u(\gamma_n) = 1 + u_1(n)\,t + u_2(n)\,t^2 + \cdots \in W(U_n), \tag{5.2}$$

and let $\phi_n\colon \mathbb{Z}[X] \to U_n = \mathbb{Z}[X_1,\ldots,X_n]$ be the unique homomorphism of rings, such that $\phi_n(X_i) = u_i(n)$ for all i. We claim that the ϕ_n are compatible in the sense that

$$\pi_n^{n+1}\phi_{m+1} = \phi_n, \qquad n=1,2,\ldots. \tag{5.3}$$

Indeed, because u is functorial, we have $u(\gamma_n) = u((\pi_n^{n+1})_* \gamma_{n+1}) = (\pi_n^{n+1})_* u(\gamma_{n+1})$, and (5.3) follows. Thus the ϕ_n combine to define a homomorphism of rings

$$\phi_u\colon \mathbb{Z}[X] \to \mathbb{Z}_l[X] \subset \mathbb{Z}[[X]]. \tag{5.4}$$

Moreover, ϕ_u determines u uniquely. Inversely, given a ring homomorphism $\phi: \mathbb{Z}[X] \to \mathbb{Z}_l[X]$, there is an induced functorial transformation

$$u_\phi: W_0^+(A) \simeq \mathbf{Ring}(\mathbb{Z}_l[X], A) \xrightarrow{\phi^*} \mathbf{Ring}(\mathbb{Z}[X], A) \simeq W(A). \qquad (5.5)$$

Now suppose that $u: W_0^+ \to W$ is continuous. By continuity (because $W_0^+(A)$ is dense in $W(A)$), u extends to a functorial transformation $u: W \to W$. Because $W(A) = \mathbf{Ring}(\mathbb{Z}[X], A)$, u induces a ring endomorphism $\phi_u: \mathbb{Z}[X] \to \mathbb{Z}[X]$. Inversely, every ring endomorphism $\phi: \mathbb{Z}[X] \to \mathbb{Z}[X]$ obviously defines a functorial transformation $u_\phi: W(A) \simeq \mathbf{Ring}(\mathbb{Z}[X], A) \xrightarrow{\phi^*} \mathbf{Ring}(\mathbb{Z}[X], A) \simeq W(A)$. This u_ϕ is automatically continuous. Indeed, let $a \in W(A)$ and $u_\phi(a) = b$. Given m, let $n(m) \in \mathbb{N}$ be such that $\phi(X_1), \ldots, \phi(X_m)$ involve only the indeterminates $X_1, \ldots, X_{n(m)}$. Then if $a' \in W(A)$ is such that the first $n(m)$ coefficients of a' are equal to those of a, we have that the first m coefficients of $b' = u_\phi(a')$ are equal to those of b. This proves the continuity of u_ϕ.

Putting all this together we have

5.6. PROPOSITION. *Every operation $u: W_0^+ \to W$ corresponds uniquely to a ring homomorphism $\phi_u: \mathbb{Z}[X] \to \mathbb{Z}_l[X]$ and inversely. If the image of ϕ_u is in $\mathbb{Z}[X] \subset \mathbb{Z}_l[X]$, the operation is continuous and extends uniquely to an operation $W \to W$. The continuous operations $W_0^+ \to W$ and the (automatically continuous) operations $W \to W$ correspond bijectively to the ring endomorphisms $\mathbb{Z}[X] \to \mathbb{Z}[X]$.*

There are also discontinuous operations $W_0^+ \to W$ and $W_0^+ \to W_0^+$. An example is the one given by the ring homomorphism $X_1 \to X_1X_2 + X_1X_3 + X_1X_4 + \cdots$, $X_i \to 0$ for $i \geqslant 2$.

5.7. *Proof of Theorem* 1.12. *The ring of operations* $Op(W_0^+)$. Let $Op(W_0^+)$ be the ring of operations $W_0^+ \to W_0^+$, and let $u \in Op(W_0^+)$. Then $u(\gamma_n)$ (cf. (5.3) above) is a polynomial, and it follows that $\phi_n(I_t) = 0$ for t large enough (where I_t is the ideal $(X_{t+1}, X_{t+2}, \ldots) \subset \mathbb{Z}[X]$). Thus, ϕ_u satisfies $\phi_u(I_t) \subset I_n$. There is such a t for every n so that ϕ_u is continuous. Inversely, let $\phi: \mathbb{Z}[X] \to \mathbb{Z}[X]$ be continuous, and let $a \in W_0^+(A)$. Let $\phi_a: \mathbb{Z}[X] \to A$ be the classifying homomorphism of a (cf. Proposition 2.4). Then $\phi_a(I_r) = 0$ for some r. Because ϕ is continuous, there is an m such that $\phi(I_m) \subset I_r$. Now $u_\phi(a) = (\phi_a \phi)_*(\xi)$, $\xi = 1 + X_1 t + X_2 t^2 + \cdots \in W(\mathbb{Z}[X])$, and it follows that $u_\phi(a)$ is in $W_0^+(A) \subset W(A)$. This proves the second statement of Theorem 1.12. The first statement follows because for continuous operations u the homomorphism ϕ_u is such that $Im(\phi_u) \subset \mathbb{Z}[X]$ (by Proposition 5.6).

6. The Operations of W_0

6.1. *J-Continuous Endomorphisms of $\mathbb{Z}[X]$ Define Operations*

Let $u \in \mathrm{Opc}(W_0)$ be a continuous operation of W_0. Then, because W_0 is dense in W, as in Section 5.1 above, u defines uniquely an endomorphism of $\mathbb{Z}[X]$. It remains to determine what endomorphisms can arise in this way. The first step is to show that J-continuous endomorphisms indeed give rise to operations.

Let $T_n = \mathbb{Z}[Y_1,..., Y_n; Z_1,..., Z_{n-1}]$, and consider the element

$$\eta_n = \frac{1 + Z_1 t + \cdots + Z_{n-1} t^{n-1}}{1 + Y_1 t + \cdots + Y_n t^n} = 1 + v_1(Y, Z)t + \cdots \in W_0(T_n). \quad (6.2)$$

The $v_i(Y, Z) \in T_n$ are easy to calculate explicitly. The result is

$$\begin{aligned}
&v_1 + Y_1 = Z_1,\\
&v_2 + v_1 Y_1 + Y_2 = Z_2,\\
&\quad\vdots\\
&v_{n-1} + v_{n-2} Y_1 + \cdots + v_1 Y_{n-2} + Y_{n-1} = Z_{n-1},\\
&v_n + v_{n-1} Y_1 + \cdots + v_1 Y_{n-1} + Y_n = 0,\\
&\quad\vdots\\
&v_{n+r} + v_{n+r-1} Y_1 + \cdots + v_2 Y_{n-1} + v_2 Y_n = 0.\\
&\quad\vdots
\end{aligned} \quad (6.3)$$

Let $\Delta_n(X)$ be the $n \times n$ upper left-hand corner submatrix of (1.11), i.e.,

$$\Delta_n(X) = \begin{pmatrix} 1 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \\ \vdots & \vdots & & \vdots \\ X_{n-1} & X_n & \cdots & X_{2n-2} \end{pmatrix}. \quad (6.4)$$

Finally, let $d_n(Y, Z) \in T_n$ be obtained by substituting $v_i(Y, Z)$ for X_i in (6.4) and taking the determinant of the resulting matrix. It is not difficult to see that

$$0 \neq d_n(Y, Z) \in T_n. \quad (6.5)$$

Indeed, take, e.g., $Z_1 = \cdots = Z_{n-1} = 0$, $Y_1 = \cdots = Y_{n-1} = 0$, $Y_n = 1$. Then $v_1 = \cdots = v_{n-1} = 0$, $v_n = -1$, $v_{n+1} = \cdots = v_{2n-2} = 0$, so that for these values d_n becomes -1 (if $n \geqslant 2$).

Now let $\sigma_n\colon \mathbb{Z}[X] \to T_n$ be defined by

$$\sigma_n(X_i) = v_i(Y, Z). \tag{6.6}$$

Then, because the $v_i(Y, Z)$ satisfy the recurrence relations (6.3), we have that $\sigma_n(J_n) = 0$, so that

$$J_n \subset \operatorname{Ker} \sigma_n. \tag{6.7}$$

Now let $\phi\colon \mathbb{Z}[X] \to \mathbb{Z}[X]$ be continuous with respect to the J-topology. Let u_ϕ be the associated functorial transformation $W(-) \to W(-)$. Then, in particular,

$$u_\phi(\eta_n) = (\sigma_n \phi)_*(\xi). \tag{6.8}$$

Now ϕ is continuous with respect to the J-topology. So there is an $m \in \mathbb{N}$ such that $\phi(J_m) \subset J_n$, and then $(\sigma_n \phi)(J_m) = 0$. Because T_n is Fatou (Proposition 3.2), it follows that $u_\phi(\eta_n) \in W_0(T_n) \subset W(T_n)$. It follows that u_ϕ maps $W_0(A) \to W_0(A)$ for all rings A, because for every $a \in W_0(A)$ there is a ring homomorphism $\psi\colon T_n \to A$ for some n such that $\psi_*(\eta_n) = a$. So we have proved

6.9. PROPOSITION. *For every J-continuous ring endomorphism ϕ of $\mathbb{Z}[X]$, the associated functorial transformation u_ϕ: $W \to W$ maps W_0 into W_0.*

6.10. *Operations on W_0 Give Rise to J-Continuous Endomorphisms*

To obtain the inverse statement, we need the inverse inclusion of (6.7). To that end, consider the following diagram:

$$\begin{array}{ccc} & \mathbb{Z}[X] & \\ {}^{\sigma_n}\swarrow & & \searrow \\ T_n & \xleftarrow{\ \alpha_n\ } & \mathbb{Z}[X]/J_n = V_n \\ {}_{\beta_n}\searrow & & \swarrow \\ & (V_n)_{D_n} & \end{array} \tag{6.11}$$

Here, the homomorphism in the upper right-hand corner is the natural projection π_n. Because $J_n \subset \operatorname{Ker} \sigma_n$, σ_n factors through V_n to give α_n. Finally, $V_n \to (V_n)_{D_n}$ is localization with respect to the multiplicative system $(1, D_n, D_n^2, \ldots)$. This is injective because $D_n \neq 0$ (by 6.5), and because D_n is not a zero divisor, (cf. the Appendix).

Now we claim that there exists a homomorphism β_n, making the lower triangle commutative. To define β_n we try to solve

$$\frac{1+Z_1t+\cdots+Z_{n-1}t^{n-1}}{1+Y_1t+\cdots+Y_nt^n}=1+X_1t+X_2t^2+\cdots \tag{6.12}$$

for $Y_1,\ldots,Y_n$, $Z_1,\ldots,Z_{n-1}$ in terms of the X's. Substituting X_i for v_i in the Eqs. (6.3), this gives in particular

$$\begin{pmatrix} 1 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \\ \vdots & \vdots & & \vdots \\ X_{n-1} & X_n & \cdots & X_{2n-2} \end{pmatrix}\begin{pmatrix} Y_n \\ Y_{n-1} \\ \vdots \\ Y_1 \end{pmatrix}=\begin{pmatrix} -X_n \\ -X_{n+1} \\ \vdots \\ -X_{2n-1} \end{pmatrix},$$

and from this we can calculate $Y_1,\ldots,Y_n$ as a polynomial $b_i(X)$, $i=1,\ldots,n$ in $X_1,\ldots,X_{2n-1}$, and $\tilde{D}_n(X)^{-1}$, where $\tilde{D}_n(X)$ is the determinant of (6.4). Given the $Y_1,\ldots,Y_{n-1}$, the $Z_1,\ldots,Z_{n-1}$ follow directly from the first $n-1$ equations of (6.3), and are also polynomials $c_i(X)$ in $X_1,\ldots,X_{2n-1}$ and $\tilde{D}_n(X)^{-1}$.

It is now straightforward to check that the expression

$$\tilde{D}_n(X)(X_{n+r}+X_{n+r-1}Y_1+\cdots+X_{r-1}Y_{n-1}+X_rY_n), \qquad r\geqslant n,$$

is precisely equal to the minor of the Hankel matrix (1.11) obtained by taking the first $n+1$ rows and columns $1,2,\ldots,n$ and $r+1$. (Alternatively, we can use the proof of Proposition 3.2 to see that it suffices to invert D_n to be able to solve Eqs. (6.12). Thus, we can define $\beta_n: T_n\to(V_n)_{D_n}$ by $Y_i\mapsto b_i(X)$ and $Z_i\mapsto c_i(X)$. The polynomials $b_i(X)$, $c_i(X)$ are unique, and it follows that the lower triangle in (6.11) commutes. It follows that α_n is injection, so that

$$\operatorname{Ker}\sigma_n=J_n. \tag{6.13}$$

Now let $u\in\operatorname{Op}(W_0)$ be a continuous operation, and let $\phi_u\in\operatorname{End}(\mathbb{Z}[X])$ be the associated endomorphism. Consider $u(\eta_n)\in W_0(T_n)$. Because $u(\eta_n)$ is rational, there is a T_m and a homomorphism of rings $\psi: T_m\to T_n$, such that $\psi_*\eta_m=u(\eta_n)$. Both $\sigma_n\phi_u$ and $\psi\sigma_m$ take $\xi\in W(\mathbb{Z}[X])$ to $u(\eta_n)$, therefore $\sigma_n\phi_u=\psi\sigma_m$

$$\begin{array}{ccc} \mathbb{Z}[X] & \xrightarrow{\phi_u} & \mathbb{Z}[X] \\ \downarrow{\scriptstyle\sigma_m} & & \downarrow{\scriptstyle\sigma_n} \\ T_m & \xrightarrow{\psi} & T_n. \end{array} \tag{6.14}$$

follows that ϕ_u takes the kernel of $\psi\sigma_m$ into the kernel of σ_n. But the el of σ_n is J_n, and the kernel of σ_m is J_m, which is contained in the kernel σ_m. Thus $\phi_u(J_m) \subset J_n$. There is such an m for every n, which proves that s continuous, w.r.t. the J-topology. This finishes the proof of part (i) of orem 1.13.

;. *Additive Operations in* Opc(W_0)

he addition in $W_0(A)$ and $W(A)$ corresponds to a comultiplication on]. It is in fact (as is very easily verified) the comultiplication $\mu: X_n \mapsto$ $_{j=n} X_i \otimes X_j$. There is also a counit $\mathbb{Z}[X] \to \mathbb{Z}$, $X_i \mapsto 0$, and a coinverse. s turns $\mathbb{Z}[X]$ into a Hopf-algebra (with antipode). An operation Op(W_0) is additive (group structure preserving) iff its associated omorphism is a Hopf-algebra endomorphism. Now according to Moore $\mathbb{Z}[X]$ is the free Hopf-algebra on the coalgebra $\bigoplus \mathbb{Z}X_i$, $X_n \mapsto$ $_{+j=n} X_i \oplus X_j$, meaning that for every Hopf-algebra H and coalgebra nomorphism $\bigoplus \mathbb{Z}X_i \to H$, there is a unique extension $\mathbb{Z}[X] \to H$, which is Hopf-algebra endomorphism. Thus the endomorphism of an additive ration u is uniquely specified by the elements $\phi_u(X_i) = x_i$ subject to $\mu x_n =$ $_{+j=n} x_i \otimes x_j$, and inversely. This proves part (ii) of Theorem 1.13.

6. *Addendum to Theorem* 1.13(ii)

et $\phi \in \mathrm{End}\,\mathbb{Z}[X]$ be a Hopf-algebra endomorphism, and suppose it is tinuous as a morphism $\mathbb{Z}[X] \to \mathbb{Z}[X]$, with the J-topology on the source l the I-topology on the target. Then, cf. 5.1 above, the associated ration takes $W_0^+(A)$ into $W_0(A)$, and hence by additivity $W_0(A)$ into (A). It follows that ϕ also has the stronger continuity property of being a tinuous J-topology endomorphism of $\mathbb{Z}[X]$.

7. *Splitting Principle and Frobenius Operators*

Before discussing multiplicative operations we need to define the obenius operators and the splitting principle. Consider $\mathbb{Z}[X]$ as a subring $\mathbb{Z}[[\xi_1, \xi_2, \ldots]]$ by viewing X_i as $(-1)^i e_i(\xi_1, \xi_2, \ldots)$, where e_i is the ith mentary symmetric function in $\xi_1, \xi_2, \ldots$. Then we can write $\xi = 1 + X_1 t +$ $t^2 + \cdots = \prod_{i=1}^{\infty}(1 - \xi_i t)$. It follows that to specify an additive operation $W(-)$, it suffices to specify what it does to elements of the form $1 + a_1 t \in$ (A), and similarly the functorial multiplication on $W(A)$ is also characized by the equation $(1 - at) * (1 - bt) = (1 - abt)$. The Frobenius erations are now characterized by

$$F_n(1 - at) = (1 - a^n t). \tag{6.18}$$

ey are functorial endomorphisms of $W(A)$ (cf., e.g., [4, Chap. 3]). They e defined on the level of **End** A by

$$(P, f) \mapsto (P, f^n). \tag{6.19}$$

6.20. *Multiplicative Operations*

Define new coordinates for the Witt vectors by the equation

$$\prod_{i=1}^{\infty} (1 - Z_i t^i) = 1 + X_1 t + X_2 t^2 + \cdots. \tag{6.21}$$

Then the Z_i can be calculated as polynomials in the X_i, and vice versa, defining an isomorphism $\mathbb{Z}[Z] \simeq \mathbb{Z}[X]$. Some aspects of the big Witt vectors are more easily discussed using "Z coordinates" than "X coordinates." Let

$$w_n(Z) = \sum_{d|n} dZ^{n/d}. \tag{6.22}$$

Then the w_n define a functorial homomorphism of rings $w\colon W(A) \to A^{\mathbb{N}}$, where $\mathbb{N} = \{1, 2, \ldots\}$, and if A is a Q-algebra this is an isomorphism. Here $A^{\mathbb{N}}$ is a ring with component wise addition and multiplication. Now let $u\colon W \to W$ be a transformation of ring valued functors. Then, at least for Q-algebra's, this induces a transformation on $A^{\mathbb{N}}$, functorial in A. These are easy to describe and are given by an infinite matrix with precisely one 1 in each row, and zero's elsewhere. Let $\tau\colon \mathbb{N} \to \mathbb{N}$ be the corresponding mapping. Now if this transformation comes from one on $W(A)$, there must be polynomials $U_1(Z), U_2(Z), \ldots$ such that

$$w_n(U_1(Z), U_2(Z), \ldots) = w_{\tau(n)}(Z_1, Z_2, \ldots). \tag{6.23}$$

Taking $n = 1$, gives $U_1(Z) = w_{\tau(1)}(Z)$, so that this transformation takes an element $(1 - at) \in W(A)$ to $(1 - a^n t)$. But this determines, by the splitting principle, the transformation uniquely, and moreover there is a multiplicative transformation acting precisely like this. Thus the functorial ring endomorphisms of $W(A)$ are the Frobenius operators $F_1, F_2, \ldots$, and they obviously take $W_0^+(A)$ and $W_0(A)$ into themselves. This proves part (iii) of Theorem 1.13.

Note. Not all mappings $\tau\colon \mathbb{N} \to \mathbb{N}$ give rise to a functorial ring endomorphism of W. For that to happen, the polynomials $U_1(Z), U_2(Z), \ldots$ defined by (6.22) must turn out to have integral coefficients. As it turns out (and this is proved by the preceding), this is the case iff there is a number n such that $\tau(m) = nm$ for all m. This follows because the Frobenius operators F_n satisfy (and are characterized by) $w_m F_n = w_{nm}$, cf. [4, Chap. 3].

6.24. *Remark.* It is not clear (to me at least) whether the (not necessarily continuous) operations $W_0 \to W_0$ correspond bijectively to continuous ring endomorphisms $\mathbb{Z}_J[X] \to \mathbb{Z}_J[X]$. Certainly such a ring endomorphism gives rise to an operation $W_0 \to W_0$. The opposite is less clear (and in my opinion probably not true). The difficulty is of course that the canonical "representing elements" ξ_n are not in $W_0(V_n)$.

7. The Operations Λ^i and S^i

These are several operations which are naturally defined on **End** A, and the question arises as to what these correspond in $W_0(A) \subset W(A)$ [1]. On the other hand, a number of the more mysterious operations of $W(A)$ have natural interpretations on the level of **End** A which sometimes can be used to advantage, [3]. Thus, e.g., the Frobenius operator corresponds to $f \mapsto f^n$ (f composed with itself n times), and the Verschiebung operator corresponds to

$$V_n: f \mapsto \begin{pmatrix} 0 & 0 & f \\ 1 & \ddots & \\ 0 & 1 & 0 \end{pmatrix}. \tag{7.1}$$

In [1] the question was asked to what the exterior and symmetric products correspond. The answer is rather obvious.

$W(A)$ is functorially a λ-ring, with the operations λ^i defined as follows. Because in any λ-ring $\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$, it suffices by the splitting principle to specify the λ^i on elements of the form $(1-at)$. The characterizing definition is now

$$\lambda^1(1-at) = 1-at, \qquad \lambda^i(1-at) = 1 \qquad \text{for} \quad i \geqslant 2. \tag{7.2}$$

(Recall that 1 is the zero element of the abelian group $W(A)$.)

Now consider the module with endomorphism (P_n, f_n) over $U_n = \mathbb{Z}[X_1, ..., X_n]$ of Section 2.1. Write $1 + X_1 t + \cdots + X_n t^n = \prod_{i=1}^n (1 - \xi_i t)$. Then over $Q(\xi_1, ..., \xi_n)$, the module with endomorphism (P_n, f_n) is isomorphic to a free n-dimensional module with diagonal endomorphism with eigenvalues $-\xi_1, ..., -\xi_n$. Thus there is a splitting principle for **End** A also. Now $\Lambda^1 = id$ and Λ^i (one dimensional module) $= 0$ if $i \geqslant 2$, and finally if ξ_i is the endomorphism multiplication with ξ_i of A, then $c(\xi_i) = 1 + \xi_i t$. It follows that the Λ^i on **End** A correspond to the natural λ-operations on $W(A)$.

7.3. *Adams Operations*

Every λ-ring has Adams operations defined on it, which are defined by the formula

$$\frac{d}{dt} \log \lambda_t(x) = \sum_{i=0}^{\infty} (-1)^n \psi^{n+1}(x) t^n, \tag{7.4}$$

where $\lambda_t(x) = 1 + \lambda^1(x)\,t + \lambda^2(x)\,t^2 + \cdots$. Using this one easily checks that the Adams operations ψ^n on $W(A)$ coincide with the Frobenius operations F_n (Adams = Frobenius). It follows that the Adams operations corresponding to the Λ^i on **End** A are given by $(P,f) \to (P,f^n)$.

7.5. *Symmetric Powers*

For any projective module P over A, there is a well-known exact sequence of projective modules

$$\begin{aligned} 0 \to S^nP \to S^{n-1}P \otimes \Lambda^1 P \to S^{n-2}P \otimes \Lambda^2 P \to \cdots \\ \to S^1 P \otimes \Lambda^{n-1}P \to \Lambda^n P \to 0. \end{aligned} \tag{7.6}$$

It follows that the exterior product operations λ^i and the symmetric product operations s^i on $W_0(A) \subset W(A)$ are related by the formula

$$\begin{aligned} s^n(a) - s^{n-1}(a)\,\lambda^1(a) + s^{n-2}(a)\,\lambda^2(a) - \cdots \\ + (-1)^{n-1}\,s^1(a)\,\lambda^{n-1}(a) + (-1)^n\,\lambda^n(a) = 0. \end{aligned} \tag{7.7}$$

A description for the s^i similar to the one given above for the λ^i is given by

$$s^1((1+at)^{-1}) = (1+at)^{-1}, \qquad s^i((1+at)^{-1}) = 0 \qquad \text{for} \quad i \geqslant 2. \tag{7.8}$$

The s^i of the other elements are determined by this because the s^i also satisfy $s^n(a+b) = \sum_{i+j=n} s^i(a)\,s^j(b)$ (where $+$ denotes the addition in $W(A)$), and on the right-hand side we have both multiplication and addition in $W(A)$. In other words, the s^i define a different λ-ring structure (also functorial) on $W(A)$. This comes about as follows. If the X_i are the elementary symmetric functions in $-\xi_1, -\xi_2, \ldots$ so that $1 + X_1 t + X_2 t^2 + \cdots = \prod(1 - \xi_i t)$, then the complete symmetric functions h_i in the $-\xi_1, -\xi_2, \ldots$ are given by $1 + h_1 t + h_2 t^2 + \cdots = \prod(1 + \xi_i t)^{-1}$. They are (therefore) related by $\sum_{i=0}^{n}(-1)^i X_i h_{n-i} = 0$, cf. (7.7).

Now the functorial λ-ring structure on $W(A)$ is given by certain ring endomorphisms $\phi(\lambda^i)\colon \mathbb{Z}[X] \to \mathbb{Z}[X]$, or, equivalently, by certain universal polynomials, the $\phi(\lambda^i)(X_j) = \Phi_{ij}(X_1, X_2, \ldots)$. Now recoordinatize $\mathbb{Z}[X]$, and view it as $\mathbb{Z}[h]$. Write down the polynomials $\Phi_{ij}(h_1, h_2, \ldots)$, and substitute the expressions in $X_1, X_2, \ldots$ to which the h_i are equal. Then these new universal polynomials define the new functorial λ-ring structure on $W(A)$ defined by the s^i.

APPENDIX: Proof that J_n is a Prime Ideal

A.1. *Sylvester's Theorem* [10]

Let $x_1,..., x_n$ be n vectors. Denote with $\det(x_1,..., x_n)$ the determinant of the matrix consisting of the columns $x_1,..., x_n$ (in that order). Then Sylvester proved a noteworthy identity concerning products of the form

$$\det(x_1, x_2,..., x_n) \det(y_1,..., y_n). \tag{1}$$

Namely, choose any subset of r integers $i_1,..., i_r$, $1 \leqslant i_k \leqslant n$. For each r tuple $1 \leqslant j_1 < \cdots < j_r \leqslant n$, let

$$\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix} \det(x_1,..., x_n) \det(y_1,..., y_n) \tag{2}$$

denote the expression (1), with x_{i_k} interchanged with y_{j_k}, $k = 1, 2,..., r$. Then Sylvester's identity says that for any fixed set $i_1,..., i_r$

$$\det(x_1,..., x_n) \det(y_1,..., y_n) = \sum \begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix} \det(x_1,..., x_n) \det(y_1,..., y_n), \tag{3}$$

where the sum is over all $\binom{n}{r}$ possible choices for $j_1 < \cdots < j_r$.

A.2. *Proof that D_n is not a Zero Divisor in $\mathbb{Z}[X]/J_n$.* Consider the semi-infinite matrix

$$\begin{pmatrix} 1 & X_1 & X_2 & X_3 & X_4 & \cdots \\ X_1 & X_2 & X_3 & X_4 & X_5 & \cdots \\ \vdots & \vdots & & & & \\ X_n & X_{n+1} & \cdots & & & \end{pmatrix}. \tag{4}$$

Now observe that all the $(n+1) \times (n+1)$ minors of the Hankel matrix (1.11) are linear combinations (with integral coefficients) of the minors of the matrix (4). This is essentially also a result from linear system theory, more precisely realization theory, cf., e.g., Section 4 of [9]. Let $m(i_1,..., i_n; j_1,..., j_n)$ denote the determinant of the submatrix of (1.11) whose top row consists of $X_{i_1},..., X_{i_{n+1}}$ and first column consists of $X_{j_1},..., X_{j_{n+1}}$ ($i_1 = j_1$; $i_1 < \cdots < i_{n+1}$; $j_1 < \cdots < j_{n+1}$) and $m(j_1,..., j_{n+1})$ denotes the minor of (4) obtained by taking the columns starting with $X_{j_1},..., X_{j_{n+1}}$. Then, for example, $m(1, 3, 5; 1, 4, 7) = m(1, 5, 9) + m(2, 4, 9) + m(1, 6, 8) + 2m(2, 5, 8) + m(3, 4, 8) + m(2, 6, 7) + m(3, 5, 7)$. Hence, J_n is the ideal generated by all the $(n+1) \times (n+1)$ minors of (4). Recall that $\Delta_n(X)$ is the $n \times n$ upper left hand corner submatrix of (4), and that $\tilde{D}_n$ is the determinant of $\Delta_n(X)$, or, what is the same, the determinant of

$$\begin{pmatrix} 1 & X_1 & \cdots & X_{n-1} & 0 \\ \vdots & & & \vdots & \vdots \\ X_{n-1} & & \cdots & X_{2n-2} & 0 \\ X_n & & \cdots & X_{2n-1} & 1 \end{pmatrix}. \tag{5}$$

We shall from now on write D for $\tilde{D}_n$. Let the columns of (4) be numbered $0, 1, \ldots$. Let $m(j_1, \ldots, j_{n+1})$ denote the minor of (4) obtained by taking columns $j_1, \ldots, j_{n+1}$, and let m_s be short for $m(0, 1, \ldots, n-1, s)$, $s \geqslant n$. Let J denote the ideal generated by the m_r.

Then, by applying Sylvester's identity with $r = n$ and $(i_1, \ldots, i_r) = (1, \ldots, n)$ to the product of the determinant of (5), i.e., D, and $m(j_1, \ldots, j_{n+1})$, we see that

$$DJ_n \subset J. \tag{6}$$

Now suppose that $DP \in J_n$ for some polynomial P. Then we can write

$$D^2 P = \sum_{i=1}^{t} f_i m_i \tag{7}$$

for certain polynomials f_i. We can, of course, even assume that the f_i are monomials. Let f be any monomial, and let X_s be the largest X occurring in f. Then we can write, if $f = f' X_s$

$$Df = f' D X_s = m_{s-n} f' + p(X_1, \ldots, X_{s-1}) f', \tag{8}$$

where p is a polynomial in $X_1, \ldots, X_{s-1}$. Using this repeatedly, we obtain from (7) an expression of the form

$$D^k P = \sum f_{\underline{i}} m_{\underline{i}}, \tag{9}$$

where $\underline{i}$ is a multi-index, $m_{\underline{i}}$ is short for $m_{i_1} m_{i_2} \cdots m_{i_r}$ if $\underline{i} = (i_1, \ldots, i_r)$, and the $f_{\underline{i}}$ are polynomials in $X_1, \ldots, X_{2n-1}$ only.

Let k be minimal such that there exists an expression of the form (9) with the property just mentioned. If $k = 0$, we are through, so assume $k > 0$. The sum in (9) is over multi-indices $\underline{i}$ such that $n \leqslant i_1 \leqslant \cdots \leqslant i_r$. Now rewrite (9) as a sum

$$D^k P = \sum_{\underline{j}} g_{\underline{j}} m_{\underline{j}}, \tag{10}$$

where the $g_{\underline{j}}$'s are equal to

$$g_{\underline{j}} = \sum f_{\underline{i}} m_n^t, \tag{11}$$

where the sum is over all $\underline{i}$ such that $i_1 = \cdots = i_t = n < i_{t+1}$ and $\underline{j} = (i_{t+1},..., i_r)$. The $g_{\underline{j}}$ in (10) depend on $X_1,..., X_{2n}$, but the dependence on X_{2n} occurs only through polynomials in $X_1,..., X_{2n-1}$ and the product DX_{2n}. Now let $V(D)$ be the subvariety of $\mathbb{C}^{2n-2}$ of zero's of D. Let $x \in V(D)$, $x = (x_1,..., x_{2n-2})$ and x_{2n-1} be fixed, $x_{2n-1} \neq 0$. Let $m_{\underline{j}}(x)$ denote the polynomial obtained from $m_{\underline{j}}$ by substituting x_i for X_i, $i = 1,..., 2n-1$. Suppose $D_{n-1}(x) = t \neq 0$. Then the lexicographically largest term in $m_{\underline{j}}(x)$ is, $\underline{j} = (j_1,..., j_s)$, $n < j_1 \leqslant \cdots \leqslant j_s$

$$(tx_{2n-1})^s X_{n+j_1-1} X_{n+j_2-1} \cdots X_{n+j_s-1}, \tag{12}$$

and these terms are different for different $\underline{j}$. This means that by varying the $X_{2n}, X_{2n+1},...$ we can produce a nonsingular $N \times N$ matrix of $m_{\underline{j}}$ values where N is the number of terms in (10). Now because $g_{\underline{j}}$ is a polynomial in $X_1,..., X_{2n-1}$, DX_{2n}, the $g_{\underline{j}}(x)$ do not depend on $x_{2n}, x_{2n+1},...$ (as long as $x \in V(D)$). Therefore, $g_{\underline{j}}(x) = 0$ for all $x \in V(D)$ such that $D_{n-1}(x) \neq 0$. These x form an open dense subset of $V(D)$, so that $g_{\underline{j}}(x) = 0$ for all $x \in V(D)$. Hence, the $g_{\underline{j}}(X)$ in (10) are divisible by D, so that we can reduce k by 1 and we are through. (D_n is a prime element as an easy induction shows.)

A.3. *Proof that J_n is a Prime Ideal.* Consider again diagram (6.11). Because D_n is not a zero divisor, the lower right hand arrow is injective. Hence α_n is injective, so that V_n is a subring of the integral domain T_n, which proves that V_n is itself integral and that J_n is a prime ideal.

References

1. G. Almkvist, K-theory of endomorphisms, *J. Algebra* **55** (1978), 308–340.
2. G. Almkvist, The Grothendieck ring of the category of endomorphisms, *J. Algebra* **28** (1974), 375–388.
3. D. Grayson, The K-theory of endomorphisms, *J. Algebra* **48** (1977), 439–446.
4. M. Hazewinkel, Formal groups and applications, Academic Press, New York, 1978.
5. A. Liulevicius, Arrows, symmetries and functors, preprint, Univ. of Chicago, 1979.
6. J. C. Moore, Algèbres de Hopf universelles, *Sém. H. Cartan* **12** (1959/1960), exposé 10.
7. Y. Rouchaleou, B. F. Wyman, and R. E. Kalman, Algebraic structure of linear dynamical systems, III: Realization theory over a commutative ring, *Proc. Nat. Acad. Sci. USA* **69** (1972), 3404–3406.

8. E. D. SONTAG, Linear systems over commutative rings: A survey, *Richerche Automat.* 7 (1976), 1–14.
9. M. HAZEWINKEL, On the (internal) symmetry groups of linear dynamical systems, *in* "Groups, Systems and Many-body Physics" (P. Kramer and M. Dal Cin, Eds.), Vieweg, Brunswick, 1980, pp. 362–404.
10. J. J. SYLVESTER, *Phil. Mag.* **4**, No. 2 (1851), 142–145.