

THE UBIQUITY OF COXETER-DYNKIN DIAGRAMS (AN INTRODUCTION TO THE A-D-E PROBLEM)

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1. PREFACE AND APOLOGY

The problem of the ubiquity of the Dynkin-diagrams A_k , D_k , E_k was formulated by V.I. ARNOLD as problem VIII in [52] as follows.

The A-D-E classifications. The Coxeter-Dynkin graphs A_k , D_k , E_k appear in many independent classification theorems. For instance

- (a) classification of the platonic solids (or finite orthogonal groups in euclidean 3-space),
- (b) classification of the categories of linear spaces and maps (representations of quivers),
- (c) classification of the singularities of algebraic hypersurfaces, with a definite intersection form of the neighboring smooth fibre,
- (d) classification of the critical points of functions having no moduli,
- (e) classification of the Coxeter groups generated by reflections, or, of Weyl groups with roots of equal length.

The problem is to find the common origin of all the A-D-E classification theorems and to substitute a priori proofs to a posteriori verifications of the parallelism of the classifications.

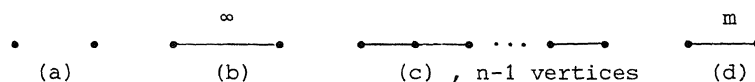
During the 13th Dutch Mathematical congress on April 6 and 7, 1977 in Rotterdam we organized a series of lectures designed to acquaint the participants with the problem mentioned above. More specifically we aimed to indicate how one obtains Coxeter-Dynkin diagrams in some of the various areas of mathematics listed in the problem. The text below is essentially a printed version of the talks

given in this series of lectures with but little editing, and with only a few extra comments, mainly of a bibliographical nature. Thus the text below is an introduction to the problem stated above; it is far too incomplete to constitute a survey of the field and it does not contain new results. The oral lectures corresponding to sections 2, 3, 4 were given by F.D. Veldkamp, the material of section 5 was presented by W. Hesselink, that of section 6 by M. Hazewinkel and that of section 7 by D. Siersma. The final redaction of this text was done by M. Hazewinkel.

2. COXETER DIAGRAMS AND GROUPS OF REFLECTIONS

2.1. Coxeter diagrams

A Coxeter diagram is a graph with all its edges labelled by an element of $\{3, 4, 5, \dots\} \cup \{\infty\}$. As a rule the label 3 is suppressed. Thus one has for example the Coxeter diagrams

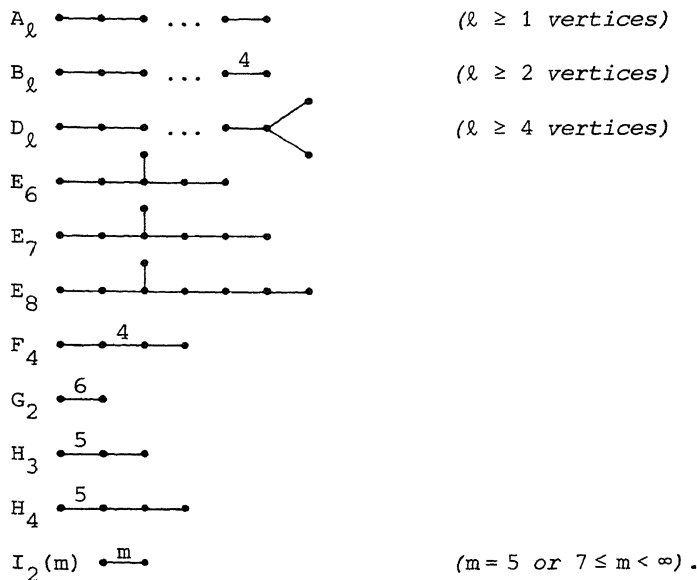


2.2. Group associated to a Coxeter diagram

Let Γ be a Coxeter diagram. Let S be its set of vertices. For all $s, s' \in S$, $s \neq s'$, define $m(s, s') = 2$ if there is no edge connecting s and s' , and $m(s, s') = \text{label of edge connecting } s \text{ and } s'$, otherwise. We now associate to Γ the group $W(\Gamma)$ generated by the symbols $s \in S$ subject to the relations $(ss')^{m(s, s')} = 1$, $s^2 = 1$ for all $s, s' \in S$, $s \neq s'$. If Γ is the disconnected union of two subgraphs Γ_1 and Γ_2 , then $W(\Gamma)$ is the direct product $W(\Gamma_1) \times W(\Gamma_2)$, because in this case $s_1 s_2 = s_2 s_1$ for all $s_1 \in \Gamma_1$, $s_2 \in \Gamma_2$.

2.3. EXAMPLES. If Γ is the graph (a) of 2.1 above then $W(\Gamma) = \mathbb{Z}/(2) \times \mathbb{Z}/(2)$, the Klein fourgroup. If Γ is the graph (b) of 2.1 then $W(\Gamma)$ is the semidirect product $\mathbb{Z}/(2) \rtimes \mathbb{Z}$, where $\mathbb{Z}/(2)$ acts on \mathbb{Z} as $\sigma x = -x$, where σ is the generator of $\mathbb{Z}/(2)$; the isomorphism is induced by $s_1 \mapsto (\sigma, 0)$, $s_2 \mapsto (\sigma, 1)$. Similarly $W(\Gamma)$ is the dihedral group $\mathbb{Z}_2 \rtimes \mathbb{Z}/(m)$ if Γ is the diagram (d) of 2.1. Finally if Γ is diagram (c) of 2.1 then $W(\Gamma) = S_n$, the permutation group on n letters. Here the isomorphism is induced by mapping the i -th vertex of Γ to the transposition $(i, i+1) \in S_n$. (Cf. [8], Ch.4, §1, exercise 4 or §2.4, example, for a proof.)

2.4. THEOREM. Let Γ be a connected Coxeter diagram. Then $W(\Gamma)$ is finite if and only if Γ is one of the following Coxeter diagrams:



2.5. Bilinear form associated to Γ

Let Γ be a Coxeter diagram with vertex set S . For each $s, s' \in S$, let $b_{s, s'}$ be the real number $b_{s, s'} = -\cos(m(s, s')^{-1}\pi)$, where we take $m(s, s') = 1$ if $s = s'$. Let E be the direct sum vector space $E = \mathbb{R}^{(S)}$ and let B_Γ be the symmetric bilinear form on E defined by the matrix $(b_{s, s'})$.

2.6. THEOREM. *The group $W(\Gamma)$ is finite if and only if B_Γ is positive nondegenerate.*

For a proof cf. [8], Ch.V, §4.8. Given this theorem (whose proof uses the realization of $W(\Gamma)$ as a group of reflections which will be discussed below), theorem 2.4 follows readily (cf. [8], Ch.VI, §4, théorème 1). E.g. $B_{\bullet \cdots \bullet}$ is positive definite iff $n \leq 5$.

2.7. Realization of $W(\Gamma)$

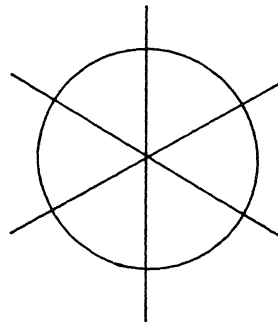
Let Γ , S , E be as in 2.5 above. Let $GL(E)$ be the group of real vector space automorphisms of E . To each $s \in S$ we associate the reflection

$$\sigma_s(x) = x - 2B_\Gamma(e_s, x)e_s,$$

where e_s is the canonical basis vector in $E = \mathbb{R}^{(S)}$ corresponding to $s \in S$.

This induces an injective embedding $W(\Gamma) \rightarrow GL(E)$, and, incidentally shows that the map $i: S \ni s \mapsto$ generator of $W(\Gamma)$ corresponding to s , is injective; the pair $(W(\Gamma), i(S))$ is a Coxeter system in the sense of [8], Ch.IV, §1. Cf. [8], Ch.V, §4 for all this.

Let Γ be one of the Coxeter diagrams listed in theorem 2.4. The reflecting hyperplanes of the σ_s then cut up \mathbb{R}^ℓ into connected pieces, the chambers. Taking the intersection of these with the unit sphere $S^{\ell-1} \subset \mathbb{R}^\ell$ we find a partition of $S^{\ell-1}$ into spherical simplices. In the case of dihedral group belonging to $I_2(3) = A_2$ the picture is ($\ell = 2$).



2.8. The crystallographic condition

Let $W(\Gamma)$ be realized as a group of reflections as in 2.7 above. Then the crystallographic condition says that there is a lattice $\mathbb{Z}^\ell \subset \mathbb{R}^\ell$ which is invariant under $W(\Gamma)$. The groups of type A, B, D, E, F, G of the list in theorem 2.4 satisfy this condition, but the groups of type H and $I_2(m)$, $m = 5$, or $m \geq 7$ do not satisfy this condition. This condition has, of course, to do with the crystallographic symmetry groups (BRAVAIS, MÖBIUS, HESSEL, 1830-1840; cf. [19], 9.3 and 4.2).

2.9. Notational remark

Instead of $\overset{4}{\bullet} \text{---} \bullet$ in a Coxeter diagram one also writes $\bullet \text{---} \bullet$ and instead of $\overset{6}{\bullet} \text{---} \bullet$ one also uses $\bullet \text{---} \bullet$. Thus $\bullet \text{---} \bullet \text{---} \bullet$ is an alternative version of F_4 .

3. LIE GROUPS, LIE ALGEBRAS AND DYNKIN-DIAGRAMS

3.1. Lie algebras

Let k be a field, e.g. $k = \mathbb{R}, \mathbb{C}$. A finite dimensional Lie algebra over k is a finite dimensional vector space L over k equipped with a bilinear multiplication $L \times L \rightarrow L$, $(x, y) \mapsto [x, y]$, such that $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. Then, of course, also $[x, y] = -[y, x]$ for all $x, y \in L$. An *ideal* $\underline{a} \subset L$ is a subvectorspace such that $[x, y] \in \underline{a}$ for all $x \in L, y \in \underline{a}$; a *subalgebra* of L is a subvectorspace \underline{h} such that $[x, y] \in \underline{h}$ for all $x, y \in \underline{h}$. A Lie algebra L is called *abelian* if $[x, y] = 0$ for all $x, y \in L$. (Then every subvectorspace is an ideal.)

A Lie algebra L is *simple* if it is not abelian and if L and $\{0\}$ are the only ideals of L . If \underline{a} is an ideal in a Lie algebra L then \underline{a} is also a Lie algebra and L/\underline{a} inherits a Lie algebra structure from L . Thus the simple Lie algebras appear as the natural building blocks for all Lie algebras. Below we shall outline the classification of the simple Lie algebras over \mathbb{C} , cf. 3.3 for the result.

One of the main reasons for the importance of Lie algebras in mathematics and physics is their intimate connection with Lie groups,

cf. 3.13 below. A basis of the Lie algebra $L(G)$ of a Lie group G is, in physicists terms, a set of infinitesimal generators for G .

3.2. EXAMPLE. Let $\mathfrak{gl}_n(k)$ be the vector space of all $n \times n$ matrices over k . We define a bracket multiplication on $\mathfrak{gl}_n(k)$ by $[X, Y] = XY - YX$. This makes $\mathfrak{gl}_n(k)$ a Lie algebra. Let $\mathfrak{sl}_n(k)$ be the subvector space of all matrices $X \in \mathfrak{gl}_n(k)$ with $\text{trace}(X) = 0$. Then $\mathfrak{sl}_n(k)$ is an ideal in $\mathfrak{gl}_n(k)$. The quotient is the abelian Lie algebra of dimension 1. Let \mathfrak{h} be the subvectorspace of $\mathfrak{sl}_n(k)$ consisting of all diagonal matrices $\text{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 + \dots + \lambda_n = 0$. Then \mathfrak{h} is an abelian subalgebra of $\mathfrak{sl}_n(k)$ of dimension $n-1$; \mathfrak{h} is not an ideal of $\mathfrak{sl}_n(k)$ if $n \geq 2$.

3.3. List of simple complex Lie algebras

There are four big families A_n , $n \geq 1$; B_n , $n \geq 2$; C_n , $n \geq 3$; D_n , $n \geq 4$ and 5 exceptional simple complex Lie algebras E_6 , E_7 , E_8 , F_4 , G_2 . The A_n , B_n , C_n , D_n are easily defined, e.g. $A_n = \mathfrak{sl}_{n+1}(\mathbb{C})$; cf. [40], section 2.7, for the remaining ones.

As we shall see it is no coincidence that we here encounter similar labels as in theorem 2.4 above. For the Dynkin diagrams A_n, \dots, G_2 cf. 3.12 below.

3.4. Real simple Lie algebras

Let L be a Lie algebra over \mathbb{R} . Then by extension of scalars one finds a natural complex Lie algebra structure on $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C}$. If now L is a complex Lie algebra then any real Lie algebra L_0 such that L is isomorphic over \mathbb{C} to $L_0 \otimes \mathbb{C}$ is called a *real form* of L . Every simple complex Lie algebra has several nonisomorphic real forms (cf. [31], Ch.III, §6), and these real forms have been classified by E. CARTAN ([18]; cf. also e.g. [1]).

3.5. We now want to indicate how one associates a Dynkin diagram (a class of objects closely related to Coxeter diagrams) to a simple Lie algebra over \mathbb{C} . This association proceeds in two steps: (i) to a simple Lie algebra there corresponds a root system, (ii) a root system gives rise to a Dynkin diagram. We now first describe in

sections 3.6-3.11 how root systems translate into Dynkin diagrams. Step (i) above is the subject of 3.12 below.

3.6. Abstract root systems

Let V be a finite dimensional vector space over a field k of characteristic zero. A root system $R \subset V$ is a subset R of V such that

- (i) R is finite, $0 \notin R$, and R generates V as a vector space over k ;
- (ii) for every $\alpha \in R$, there exists an element $\alpha^* \in V^*$, the dual space of V , such that $\alpha^*(\alpha) = 2$ and such that the reflection $s_\alpha(x) = x - \alpha^*(x)\alpha$ maps R into R ;
- (iii) if $\alpha, \beta \in R$, then $s_\alpha(\beta) - \beta$ is an integer multiple of α .

The reflection s_α , whose existence is required by condition (ii) is necessarily unique, thus (iii) makes sense (cf. [40], Ch.V, §1).

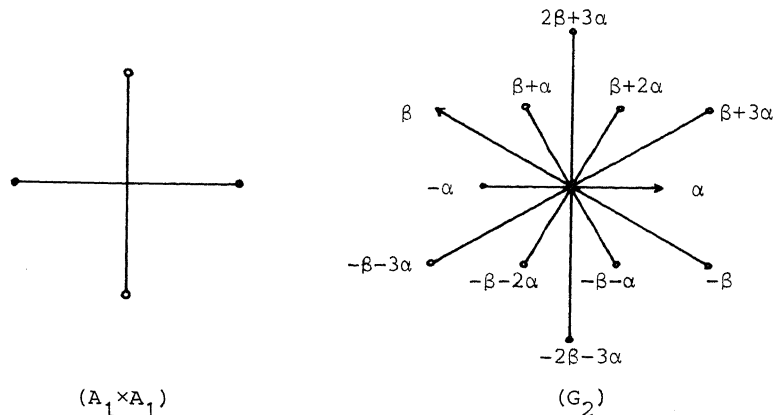
In the following we shall take $k = \mathbb{R}$ or \mathbb{C} . It does not matter much which we take. If $R \subset V$ is a complex root system, then

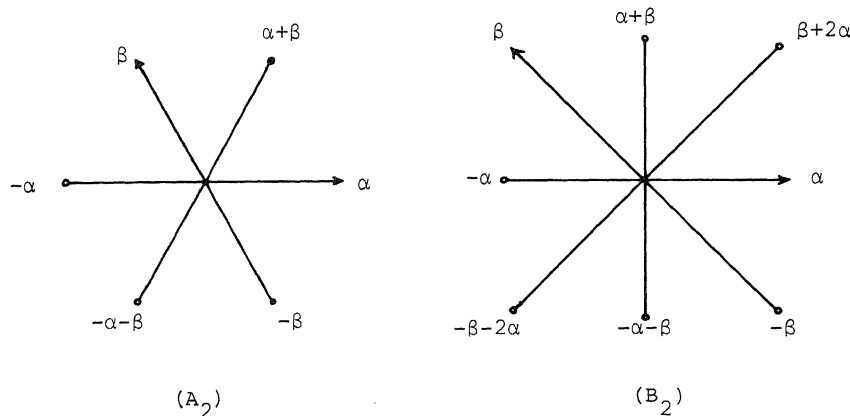
$$R \subset \sum_{\alpha \in R} \alpha \mathbb{R} \subset V$$

is a real root system in $\sum \alpha \mathbb{R}$ and this sets up bijective correspondence between real and complex root systems. Cf. also [40], Ch.VI, §1, prop.1.

The root system $R \subset V$ is called *reduced* if for all $\alpha \in R$ the only roots proportional to α are α and $-\alpha$. The *rank* of a root system $R \subset V$ is the dimension of V . Two root systems $R \subset V$ and $R' \subset V'$ are isomorphic if there exists an isomorphism $\phi: V \rightarrow V'$ of vector spaces such that $\phi(R) = R'$.

3.7. EXAMPLES. The reduced root systems of rank 2 are





3.8. Weyl group and Coxeter system of a root system

Let $R \subset V$ be a (real) root system. The Weyl group $W(R)$ is then defined as the subgroup of $GL(V)$ generated by the reflections s_α , $\alpha \in R$. Because R generates V , s_α is uniquely determined by its action on R , and because R is finite this means that $W(R)$ is a finite group.

EXAMPLES. $W(A_1 \times A_1) = \mathbb{Z}/(2) \times \mathbb{Z}/(2)$, $W(A_2) = S_3$, the permutation group on 3 letters.

Let $R \subset V$ be a root system. A basis for R (or a simple set of roots) is a subset $S \subset R$ which is a basis for V and which is such that every $\alpha \in R$ can be uniquely written in the form $\alpha = \sum m_i \alpha_i$, $m_i \in \mathbb{Z}$, $\alpha_i \in S$ with either $m_i \geq 0$ for all i (positive roots) or $m_i \leq 0$ for all i (negative roots). It is now a theorem that every root system has a basis ([40], Ch.V, §8). Let S be a basis for R and let $S' \subset W(R)$ be the set of reflexions $\{s_\alpha \mid \alpha \in S\} \subset W(R)$. Then $(W(R), S')$ is a Coxeter system in the sense of 2.7 above ([8], Ch.VI, §1.5, théorème 2).

3.9. Invariant metric

Let $R \subset V$ be a real root system. There is a symmetric positive definite bilinear form $(\ , \)$ on V which is invariant with respect to $W(R)$. This follows simply from the fact that $W(R)$ is finite; indeed,

if $(,)'$ is any positive definite symmetric form on V then

$$(x, y) = \sum_{w \in W(R)} (wx, wy)'$$

works. In terms of $(,)$ the coefficient $\alpha^*(x)$ appearing in the reflection s_α is equal to $\alpha^*(x) = (\alpha, \alpha)^{-1} s(\alpha, x)$.


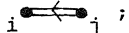
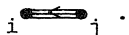
With respect to this metric $W(R)$ acts as a finite group of orthogonal transformations. The invariant bilinear form $(,)$ is by no means unique. For each $\alpha, \beta \in R$, let $n(\alpha, \beta) = (\beta^*, \alpha) = 2(\beta, \beta)^{-1}(\beta, \alpha)$. If ϕ is the angle between α and β (with respect to the invariant metric discussed above) then $4 \cos^2 \phi = n(\beta, \alpha)n(\alpha, \beta)$. Now $n(\alpha, \beta)$ is an integer by condition (iii) of the definition of a root system. Hence $4 \cos^2 \phi = 0, 1, 2, 3, 4$ which severely limits the possible values for ϕ and $n(\alpha, \beta)$, $n(\beta, \alpha)$. In fact there are only seven possibilities (for α and β non-proportional, $|\alpha| \leq |\beta|$). They are:

- (i) $n(\alpha, \beta) = 0, n(\beta, \alpha) = 0, \phi = 2^{-1}\pi,$
- (ii) $n(\alpha, \beta) = 1, n(\beta, \alpha) = 1, \phi = 3^{-1}\pi, |\alpha| = |\beta|$
- (iii) $n(\alpha, \beta) = -1, n(\beta, \alpha) = -1, \phi = 3^{-1}2\pi, |\alpha| = |\beta|$
- (iv) $n(\alpha, \beta) = 1, n(\beta, \alpha) = 2, \phi = 4^{-1}\pi, |\beta| = \sqrt{2}|\alpha|$
- (v) $n(\alpha, \beta) = -1, n(\beta, \alpha) = -2, \phi = 4^{-1}3\pi, |\beta| = \sqrt{2}|\alpha|$
- (vi) $n(\alpha, \beta) = 1, n(\beta, \alpha) = 3, \phi = 6^{-1}\pi, |\beta| = \sqrt{3}|\alpha|$
- (vii) $n(\alpha, \beta) = -1, n(\beta, \alpha) = -3, \phi = 6^{-1}5\pi, |\beta| = \sqrt{3}|\alpha|.$

3.10. Cartan matrix and Dynkin diagram of a root system

Let $S \subset R$ be a basis for the reduced root system $R \subset V$. The *Cartan matrix* (with respect to S) is the matrix $(n(\alpha, \beta))_{\alpha, \beta \in S}$. One now has the proposition that a reduced root system is determined (up to isomorphism) by its Cartan matrix ([40], Ch.V, prop.8,8' or [8], Ch.VI, §1, Prop.15, Cor.). Also if both α, β are part of a basis of R only possibilities (i), (iii), (v), (vii) of the list in 3.8 above are possible; cf. [8], Ch.VI, §1, théorème 1.

We now assign a Dynkin diagram to the root system $R \subset V$ as follows: the vertices correspond to the element of a basis $S \subset R$. Two vertices $i, j \in S$ are joined according to the following recipe:

- (i) if $n(i,j) = n(j,i) = 0$ then i and j are not joined;
(ii) if $n(i,j) = n(j,i) = -1$ ;
(iii) if $2n(i,j) = n(j,i) = -2$ ;
(iv) if $3n(i,j) = n(j,i) = -3$ .

This exhausts all possibilities. And we also see that the Dynkin diagram of $R \subset V$ (relative to S) determines the Cartan matrix of $R \subset V$ (relative to S) and hence R itself according to the theorem quoted above.

3.11. EXAMPLES. The Dynkin diagrams of the reduced root systems of example 3.7 above are respectively

$$A_1 \times A_1: \bullet \quad \bullet ; \quad G_2: \text{triple arrow} \\ A_2: \bullet \text{---} \bullet ; \quad B_2: \text{double arrow}$$

3.12. The root system of a simple Lie algebra over \mathbb{C}

We now proceed to indicate how one obtains the classification theorem 3.3, i.e., given 3.6-3.11 above, how one constructs a root system from a (semi) simple Lie algebra over \mathbb{C} . We shall outline the general theory and treat a specific example (viz. $\mathfrak{sl}_n(\mathbb{C})$) in two parallel columns. In the following L is some fixed simple Lie-algebra over \mathbb{C} , and in example of course $L = \mathfrak{sl}_n(\mathbb{C})$.

(i) Cartan subalgebra

Let $x \in L$, then $\text{ad } x: L \rightarrow L, y \mapsto [x,y]$ is a linear endomorphism of L . We say that x is semisimple if $\text{ad } x$ is diagonalizable. A Cartan subalgebra of L is maximal abelian subalgebra with the additional property that all its elements are semisimple in L . Cartan subalgebras \underline{h} always exist.

The subalgebra \underline{h} of $\mathfrak{sl}_n(\mathbb{C})$ consisting of all diagonal matrices (of trace zero) is a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$. The dimension of \underline{h} is $n - 1$.

(ii) Roots and root vectors

Let $\alpha \in \underline{h}^*$, the complex linear dual of \underline{h} . We define $L^\alpha = \{x \in L \mid [h,x] =$

Let $\omega_i(\text{diag}(\lambda_1, \dots, \lambda_n)) = \lambda_i$. Then $\omega_i - \omega_j: \underline{h} \rightarrow \mathbb{C}$ is a root

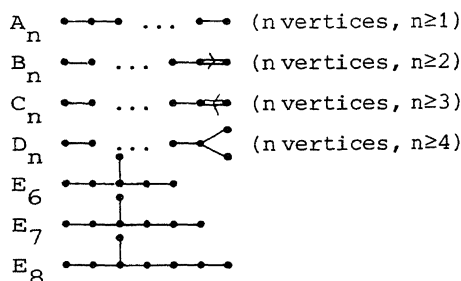
$= \alpha(h)x$ for all $h \in \underline{h}$. Then α is called a root (of L with respect to \underline{h}) if $\alpha \neq 0$ and $L^\alpha \neq 0$. One then has that $\dim L_\alpha = 1$ for all roots α and if Σ is the set of all roots then $L = \underline{h} \oplus \bigoplus_{\alpha \in \Sigma} L_\alpha$ as a vector space.

(iii) Root system and basis

Σ is a reduced root system in \underline{h}^* ([8], Ch.VI, §1, théorème 2) and hence has a basis. Moreover Σ is irreducible which means that there is no nontrivial decomposition $R = R_1 \cup R_2$ with $R_1 \subset V_1$, $R_2 \subset V_2$ root systems, $V = V_1 \times V_2$. This root system determines L up to isomorphism ([31], Ch.III, §5, theorem 5.4; [8], Ch.VI, §5, théorème 8).

(iv) Dynkin diagram

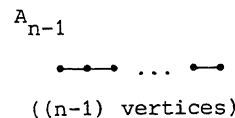
Now construct the Dynkin diagram of the root system Σ (cf. 3.10 above). This Dynkin diagram is connected because Σ is irreducible. The Dynkin diagrams which arise in this way are



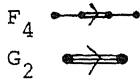
if $i \neq j$. A nonzero element of $L^{\omega_i - \omega_j}$ is E_{ij} the matrix with zero entries everywhere except a 1 at spot (i, j) .

$V = \underline{h}^* = \{ \sum_{i=1}^n \xi_i \omega_i \mid \sum \xi_i = 0 \}$. The reflection s_α associated to $\omega_i - \omega_j$ interchanges ω_i and ω_j and leaves all other ω_k invariant. Hence $(\omega_i - \omega_j)^* \in \underline{h}$ is $\text{diag}(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$ with the 1 in spot i and -1 in spot j . A basis (or set of simple roots) is e.g. $\alpha_1 = \omega_1 - \omega_2$, $\alpha_2 = \omega_2 - \omega_3, \dots, \alpha_{n-1} = \omega_{n-1} - \omega_n$.

We find $\langle \alpha_i^*, \alpha_j \rangle = 0$ if $i < j-1$ or $i > j+1$, $\langle \alpha_i^*, \alpha_i \rangle = 2$, $\langle \alpha_{i-1}^*, \alpha_i \rangle = \langle \alpha_{i+1}^*, \alpha_i \rangle = -1$. It follows that the Dynkin diagram of $\mathfrak{sl}_n(\mathbb{C})$ is



The Weyl group of $\mathfrak{sl}_n(\mathbb{C})$ is S_n .



By removing the arrows one finds the Coxeter diagram of the Weyl group $W(R)$ of L .

3.13. On the connections between Lie groups and Lie algebras

Some, first presumably largely superfluous, preliminaries on analytic manifolds. Let $k = \mathbb{R}$ or \mathbb{C} . An *analytic manifold* of dimension n over k is a Hausdorff topological space M together with an open covering $U = \{U_i \mid i \in I\}$ and homeomorphisms $\phi_i: U_i \rightarrow \phi(U_i) \subset k^n$ onto some open subset of k^n , such that for all $i, j \in I$

$$\phi_j \phi_i^{-1}: \phi_i(U_i \cap U_j) \rightarrow U_i \cap U_j \rightarrow \phi_j(U_i \cap U_j)$$

is an analytic mapping. A function $f: U \rightarrow k$ in M is *analytic* if $f \phi_i^{-1}: \phi_i(U \cap U_i) \rightarrow U \cap U_i \rightarrow k$ is analytic for all i . Let $F_M(U)$ be the ring of analytic functions on U . A mapping $\phi: M \rightarrow N$ between analytic manifolds M and N is *analytic* if for all open $V \subset N$ and analytic functions $f \in F_N(V)$ the function $f \phi$ on $\phi^{-1}(V) \subset M$ is analytic.

Let $p \in M$. We define $F_M(p)$, the k -algebra of germs of analytic functions in p , as the set of equivalence classes of analytic functions $f: U \rightarrow k$ defined on some neighbourhood U of x , under the equivalence relation $f: U \rightarrow k \sim g: V \rightarrow k$ iff there is a neighbourhood $W \subset U \cap V$ of x on which f and g agree. A *tangent vector* to M at p is a k -linear mapping $t: F_M(p) \rightarrow k$ such that $t(fg) = (tf)g(p) + f(p)(tg)$. There is an obvious k -vector space structure on M_p , the set of tangent vectors to M at p , and $\dim(M_p) = n$. An *analytic (tangent) vector field* X on an open subset $Y \subset M$ is a collection of derivations $X_U: F_M(U) \rightarrow F_M(U)$, one for each open $U \subset Y$, such that $r_{U,V} \circ X_U = X_V \circ r_{U,V}$ for all open $V \subset U$. Here $r_{U,V}: F_M(U) \rightarrow F_M(V)$ is restriction. Given a vector field X on $U \subset M$ and a point $p \in U$ one defines a tangent vector $X_p \in M_p$ by $X_p(f) = (Xf)(p)$.

If $\phi: M \rightarrow N$ is analytic and $t \in M_p$ then $(d\phi)_p(t)(g) = t(g\phi)$ defines a tangent vector $(d\phi)_p(t) \in N_{\phi(p)}$, giving us a k -linear mapping $(d\phi)_p: M_p \rightarrow N_{\phi(p)}$.

A *Lie group* is now an analytic manifold G which is equipped with analytic mappings "product": $G \times G \rightarrow G$ and "inverse": $G \rightarrow G$ and an element $e \in G$ which make G a group. Example: $G = GL_n(\mathbb{C})$, the group of invertible $n \times n$ matrices over \mathbb{C} . (Here the covering U defining the analytic structure has just one element.) Other examples are the orthogonal groups, symplectic groups, unitary groups, special linear groups, projective linear groups,

Let G be a Lie group, let $y \in G$ then $\lambda_y: G \rightarrow G, x \mapsto yx$ is an analytic mapping. A vector field X on G is said to be *left invariant* if for all open $U \subset G$, $f \in F(U)$ we have $X_{y^{-1}U}(f\lambda_y) = X_U(f)\lambda_y$. Now let $t \in G_e$ be a tangent vector at the identity element. We define a left invariant vector field $X(t)$ on G by $(X(t)f)(x) = t(f\lambda_x)$. This sets up a bijection between G_e and left invariant vector fields on G . (Easy.) Now if X, Y are any two vector fields on G then so is $[X, Y] = XY - YX$, and $[X, Y]$ is left invariant if X and Y are left invariant. This defines a Lie algebra structure on the vector space of left invariant vector fields on G and hence a Lie algebra structure on the tangent space G_e . This is the Lie algebra $L(G)$ associated to G . Locally the structure of G is determined by $L(G)$. More precisely:

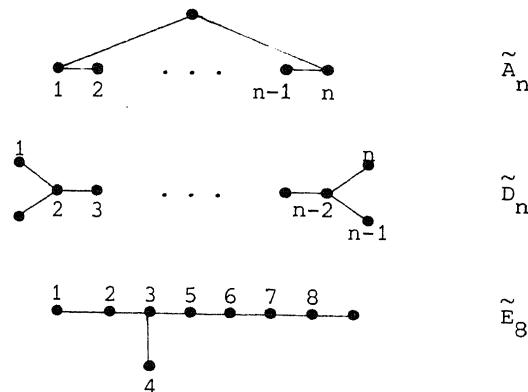
- (i) for every $m \in L(G)$ there exists a unique analytic map $e_m: k \rightarrow G$, such that $e_m(s_1)e_m(s_2) = e_m(s_1+s_2)$ and such that $(de_m)_0(1) = m$ (where we have identified the tangent space at 0 to the analytic manifold " k " with k itself);
- (ii) $\exp: L(G) \rightarrow G, m \mapsto e_m(1)$ is a local analytic isomorphism of analytic manifolds;
- (iii) locally near e the group structure of G is given by $\exp(m)\exp(m') = \exp(F(m, m'))$ where $F(m, m') = m + m' + \frac{1}{2}[m, m'] + \frac{1}{12}([m, [m, m']] + [m', [m', m]]) - \frac{1}{24}(\dots) + \dots$ is some well-defined universal expression (Campbell-Baker-Hausdorff formula);
- (iv) connected Lie subgroups of G correspond biuniquely to Lie subalgebras of $L(G)$;
- (v) connected normal Lie subgroups correspond biuniquely to ideals in $L(G)$;
- (vi) G is quasi-simple ($\Leftrightarrow G$ is connected and has only discrete proper normal subgroups) iff $L(G)$ is simple.

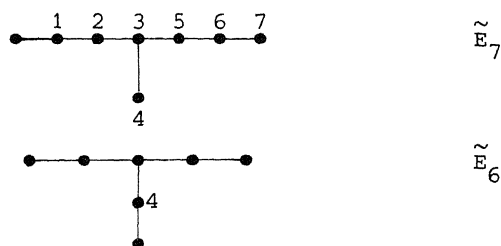
EXAMPLE. $G = GL_n(\mathbb{C})$, $L(G) = M_n(\mathbb{C})$, the Lie algebra of all $n \times n$ matrices under $[A, B] = AB - BA$. The map $\exp: M_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$ is given by $A \mapsto e^A = I + A + (2!)^{-1}A^2 + (3!)^{-1}A^3 + \dots$ (whence the notation "exp" in general).

For the proofs of all these facts, cf. any of the standard books on Lie groups and Lie algebras, e.g. [33], [31] and, in a slightly different context [35].

3.14. Extracting information from Dynkin diagrams

- (i) Let I be the set of vertices of a connected subgraph of a Dynkin diagram. Then $\sum_{i \in I} \alpha_i$ is a positive root. Every root $\sum_i m_i \alpha_i$ with $m_i = 0, 1$ is obtainable in this way. In the case of A_n one thus obtains all positive roots.
- (ii) $\text{Aut}(D) = \text{Aut}_{\text{Lie}}(G)/\text{Int}(G)$. Here D is the Dynkin diagram of the simple Lie group G , $\text{Int}(G)$ is the group of interior automorphisms of G and $\text{Aut}_{\text{Lie}}(G)$ is the group of automorphisms of G . One has $\text{Aut}(A_n) = \mathbb{Z}/(2)$, $\text{Aut}(D_4) = S_3$, $\text{Aut}(D_n) = \mathbb{Z}/(2)$ for $n \geq 5$, $\text{Aut}(E_6) = \mathbb{Z}/(2)$ and $\text{Aut}(D) = \{1\}$ for all other Dynkin diagrams D .
- (iii) The so-called *completed Dynkin diagrams* play an important role in the determining of all maximal compact subgroups of compact (real) simple Lie groups, cf. [54]. One adds a vertex corresponding to the largest root, cf. [8], Ch. VI, §4.3 for details. The completed Dynkin diagrams \tilde{A}_n , \tilde{D}_n and \tilde{E}_6 , \tilde{E}_7 , \tilde{E}_8 are

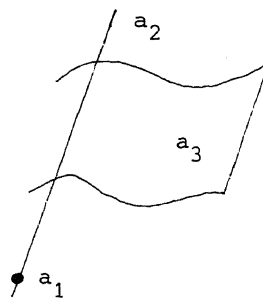




4. TITS GEOMETRIES

4.1. EXAMPLE. $\mathbb{P}^n(\mathbb{C})$ as a Tits-geometry. We start with an example. Let $\mathbb{P}^n(\mathbb{C})$ be a complex projective space of (complex) dimension n , and let $\text{PGL}_{n+1}(\mathbb{C}) = \text{PSL}_{n+1}(\mathbb{C})$ be its group of linear projective automorphisms. We show how the geometry of $\mathbb{P}^n(\mathbb{C})$, i.e. the sets of points, lines, planes, ... of $\mathbb{P}^n(\mathbb{C})$ together with their incidence relations are recoverable from the Lie group $\text{PGL}_{n+1}(\mathbb{C})$.

Let F_j be the set of all $(j-1)$ -dimensional linear subspaces of $\mathbb{P}^n(\mathbb{C})$, $j = 1, 2, \dots, n$. If $i \neq j$, $x \in F_i$, $y \in F_j$ we write $x|y$ if x and y are incident, i.e. if $x \subset y$ if $i < j$ or if $y \subset x$ if $i > j$. A *flag* is a sequence of elements (a_1, \dots, a_t) , $a_i \in F_{i_j}$, $i_1 < \dots < i_t$ such that $a_i|a_{i+1}$ for all $i = 1, \dots, t-1$. If $t=n$ the flag is *maximal*. The terminology comes from the picture of a maximal flag in \mathbb{P}^3 .



Choose a basis e_1, e_2, \dots, e_{n+1} of \mathbb{C}^{n+1} . Interpreting points of $\mathbb{P}^n(\mathbb{C})$ as lines through 0 in \mathbb{C}^{n+1} , lines in $\mathbb{P}^n(\mathbb{C})$ as planes through 0 in \mathbb{C}^{n+1} , ... we find a maximal flag

$$E = (\langle e_1 \rangle, \langle e_1, e_2 \rangle, \dots, \langle e_1, \dots, e_n \rangle).$$

The stabilizer of E in $G = \text{PGL}_{n+1}(\mathbb{C})$ is then the subgroup B of all uppertriangular matrices (with respect to the chosen basis).

$$B = \left\{ \begin{pmatrix} * & * & \cdot & \cdot & \cdot & * \\ 0 & * & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & * \end{pmatrix} \right\}$$

The subgroups conjugate to B are all stabilizers of a maximal flag, and these are precisely the maximal solvable subgroups of G , that is the *Borel subgroups*.

A *parabolic subgroup* is a subgroup of G which contains a Borel subgroup. The parabolic subgroups containing B above are the groups

$$P = \left\{ \begin{pmatrix} * & & & & & \\ \cdot & * & & & & \\ * & \cdot & * & & & \\ & \cdot & \cdot & * & & \\ & & * & \cdot & \cdot & \\ & & & * & \cdot & \\ & & & & * & \end{pmatrix} \right\}$$

(different block sizes are allowed); i.e. they are the groups consisting of all blocks upper triangular matrices for a given sequence of block sizes n_1, \dots, n_s , $n_1 + \dots + n_s = n+1$. These groups are the stabilizers of flags contained in E , e.g. if $n = 3$, then the parabolic subgroups $\neq B, G$ containing B are

$$\left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \right\} \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}$$

which are respectively the stabilizers of the flags $(\langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle)$, $(\langle e_1 \rangle, \langle e_1, e_2, e_3 \rangle)$, $(\langle e_1 \rangle, \langle e_1, e_2 \rangle)$, $(\langle e_1 \rangle)$, $(\langle e_1, e_2 \rangle)$, $(\langle e_1, e_2, e_3 \rangle)$.

Every parabolic subgroup of G is conjugate to precisely one parabolic subgroup containing B . In particular the subspaces of $\mathbb{P}^n(\mathbb{C})$, i.e. the elements of the F_j , $j = 1, \dots, n$, correspond to the maximal parabolic subgroups $\neq G$. In case $n = 3$ the last three of the parabolic subgroups listed above are maximal.

Now let P' be any parabolic subgroup, then $P' = gPg^{-1}$ with $P \supset B$ where B is the standard Borel subgroup given above. Now the normalizer of a parabolic subgroup P is P itself and it follows that $\{h | hPh^{-1} = P'\} = gP$ so that $P' \mapsto gP$ sets up a bijective correspondence between parabolic subgroups conjugate to a given $P \supset B$ and cosets of P in G . Let $P_{(i)}$ be the stabilizer of (e_1, \dots, e_i) ; then we see that

$$F_i = \{(i-1) - \dim \text{ subspaces} \} \xrightarrow{1-1} \{gP_{(i)}g^{-1} \mid g \in G\} \xrightarrow{1-1} G/P_{(i)}.$$

Furthermore we recover the incidence relations as follows:

$gP_{(i)} \mid g'P_{(j)} \iff gP_{(i)}$ and $g'P_{(j)}$ correspond to elements of the same maximal flag

$$\iff \exists g'' \text{ such that } gP_{(i)} \cap g'P_{(j)} \supset g''B$$

$$\iff gP_{(i)} \cap g'P_{(j)} \neq \emptyset.$$

4.2. The Tits geometry of a (quasi-)simple Lie group G

Let G be a quasi-simple Lie group and let \mathfrak{g} be its Lie-algebra. Let \mathfrak{h} be a Cartan subalgebra, let R be the root system of \mathfrak{g} with respect to \mathfrak{h} and let $S = \{\alpha_1, \dots, \alpha_\ell\}$ be a set of simple roots. For each $\alpha = \sum m_i \alpha_i$ we set $\text{supp}(\alpha) = \{\alpha_i \mid m_i \neq 0\}$. For each subset $I \subset S$ we set

$$\mathfrak{p}_I = \mathfrak{h} \oplus \sum_{\alpha > 0} \mathbb{C} e_\alpha \oplus \sum_{\substack{\alpha < 0 \\ \text{supp}(\alpha) \subset I}} \mathbb{C} e_\alpha,$$

where e_α is a nonzero element of \mathfrak{g}^α . In particular we have

$$\mathfrak{p}_\emptyset = \mathfrak{h} \oplus \sum_{\alpha > 0} \mathbb{C} e_\alpha.$$

Then $B = \langle \exp(\mathfrak{p}_\emptyset) \rangle$ is a Borel subgroup and the $P_I = \langle \exp \mathfrak{p}_I \rangle$ are the parabolic subgroups containing B . Every parabolic subgroup of G is conjugate with precisely one P_I .

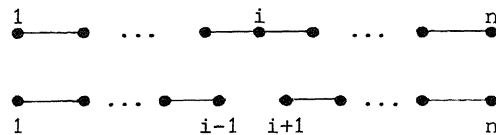
E.g. if $G = \text{PGL}_4(\mathbb{C})$ and $S = \{\alpha_1, \alpha_2, \alpha_3\}$ as in the example of 3.12 above, then the six parabolic subgroups listed in 4.1 above correspond respectively to the following subsets of S :

$$\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3\}, \{\alpha_2, \alpha_3\}, \{\alpha_1, \alpha_3\}, \{\alpha_1, \alpha_2\}.$$

For each $i \in \{1, 2, \dots, \ell\}$ let $P_{(i)}$ be the maximal parabolic subgroup $P_{(i)} = P_{I(i)}$, where $I(i) = S \setminus \{\alpha_i\}$. Now define sets of points, lines, ... by $F_i = G/P_{(i)}$ and define the incidence relations by $xP_{(i)}|yP_{(j)} \iff xP_{(i)} \cap yP_{(j)} \neq \emptyset$. This is the Tits geometry (or Tits building) of G .

4.3. Reducing Tits geometries

Let $\alpha_i \in S$ be a given vertex of the Dynkin diagram. Take any $a \in F_i = G/P_i$. The geometry of all x which are incident with this given a corresponds to the diagram one obtains by removing α_i and all edges through α_i . Thus in the case of $\mathbb{P}^n(\mathbb{C})$ if $a \in F_i$, i.e. if a is an $(i-1)$ -dimensional linear subspace we have



and the "residual geometry" of all $x|a$ consists of a $\mathbb{P}^{i-1}(\mathbb{C})$ (consisting of those $x|a$ with $\dim(x) < i-1$) and a $\mathbb{P}^{n-i}(\mathbb{C})$ (consisting of those $x|a$ with $\dim(x) > i-1$). Thus one can establish various properties of the Tits geometries by reduction to the geometries of rank 2:

A_2 : $\bullet \text{---} \bullet$, the projective plane \mathbb{P}^2

B_2 : $\bullet \text{---} \bullet \text{---} \bullet$, points and lines on a quadric in \mathbb{P}^4

G_2 : $\bullet \text{---} \bullet \text{---} \bullet$, a geometry related to the Cayley numbers.

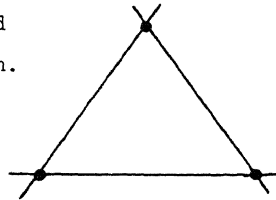
4.4. EXAMPLE 4.1 continued (The Skeleton geometry). Consider again the situation of 4.1 above. The subgroup of G which stabilizes all the $\langle e_{i_1}, e_{i_2}, \dots, e_{i_p} \rangle$ is

$$T = \left\{ \begin{pmatrix} * & * & & 0 \\ & \ddots & & \\ 0 & & \ddots & * \end{pmatrix} \right\} = \exp(\underline{h}).$$

The Weyl group W acts as permutations on the coefficients of the matrices in T ; it is the automorphism group of the skeleton $Sk = \{ \langle e_{i_1}, \dots, e_{i_p} \rangle \}$. Let W_i be the stabilizer in W of $\langle e_{i_1}, \dots, e_{i_p} \rangle$. Then $W_i = \{ s_{\alpha_1}, \dots, s_{\alpha_{i-1}}, s_{\alpha_{i+1}}, \dots, s_{\alpha_n} \}$. The $(i-1)$ -dimensional subspaces of Sk are the wW_i , $w \in W$, or, the cosets wW_i , $w \in W$. The geometry Sk is described by the W/W_i just as the geometry \mathbb{P}^n is described by the G/P_i . The Skeleton geometry Sk is an "n-dimensional projective geometry over the field of one element".

In case $n = 2$ it consists of three points and three lines with incidence relations as shown.

(An i -dimensional projective space over a finite field of q -elements has $1 + q + q^2 + \dots + q^i$ points; so an i -dimensional projective space over the field of 1 element should have $i+1$ points.)



4.5. Bibliographical notes

The reference [45] is a good first introduction to the subject of Tits geometries; [47] and [48] are useful after one has read [45], and [50] describes a number of applications. The standard reference, containing all proofs, is [49], which also contains an extensive bibliography.

5. DYNKIN CURVES AND SINGULARITIES

5.1. Introduction

Here is, how, very roughly, the Dynkin diagram of a quasi-simple Lie group G arises as the fibre of a resolution of singularities of a certain variety associated to G . Let G be a quasi-simple algebraic complex Lie group. Let $U(G)$ be the algebraic variety of its unipotent elements. This variety has singularities. Let $U_{\text{sing}}(G)$ be the subvariety of singular points. There is a more or less canonical desingularisation $\pi: V(G) \rightarrow U(G)$ and there is a single open and dense conjugacy class $C \subset U_{\text{sing}}(G)$ of so-called subregular unipotents.

For $x \in C$ the fibre $\pi^{-1}(x)$ is a connected one dimensional variety which is a union of projective lines. The intersection graph of this union of projective lines is the unfolded Dynkin diagram of G . In the following we shall try to precisize all this to some extent.

5.2. Algebraic varieties over \mathbb{C} and singular points

For the purposes of this section an *affine algebraic variety* V is the set of solutions in \mathbb{C}^r (for some r) of a collection of polynomials in r variables X_1, \dots, X_r and a *projective variety* is the set of solutions in $\mathbb{P}^r(\mathbb{C})$ (for some r) of a collection of homogeneous polynomials in $r + 1$ variables X_0, X_1, \dots, X_r .

Let $V \subset \mathbb{C}^r$ be an affine algebraic variety, $x \in V$. Let $f_1(X), \dots, f_n(X)$ be the polynomials defining V . Then we can write $f_i(X_1 - x_1, \dots, X_r - x_r) = L_i(X) + g_i(X)$ where $L_i(X)$ is homogeneous of degree 1 in X and all monomials in $g_i(X)$ have degree ≥ 2 in X . An r -vector $a \neq 0$ (starting in x) is now said to be a tangent vector to V at x if $L_i(a) = 0$, $i = 1, \dots, n$. Let $T_x(V)$ be the linear space spanned by the tangent vectors to V at x . The point $x_0 \in V$ is called *smooth* if $\dim(T_{x_0}(V))$ is constant in a neighbourhood of x_0 in V ; otherwise x_0 is called *singular*. The variety V is smooth if all its points are smooth. A projective variety $V \subset \mathbb{P}^r(\mathbb{C})$ can be seen as $r + 1$ affine varieties $V_i = V \cap U_i$ glued together where $U_i = \{x \in \mathbb{P}^r(\mathbb{C}) \mid x_i \neq 0\} \cong \mathbb{C}^r$, and a point $x \in V_i \subset V$ is smooth if it is smooth as a point of V_i . Cf. [41], Ch.II, §1 for more details.

EXAMPLE. Let $V \subset \mathbb{C}^2$ be the curve defined by $X_1^2 - X_2^3 = 0$. Then $(0,0) \in V$ and $\dim(T_{(0,0)}(V)) = 2$ and $\dim(T_x(V)) = 1$ for all $x \in V$, $x \neq (0,0)$. Hence $(0,0)$ is singular and all other points of V are smooth.

5.3. Algebraic Lie groups over \mathbb{C}

An algebraic Lie group over \mathbb{C} is (for the purpose of these lectures) a closed connected subgroup G of $GL_n(\mathbb{C})$, the group of complex invertible $n \times n$ matrices, such that the points of G are the solutions of a set of polynomials in the matrix coefficients. Now

$GL_n(\mathbb{C})$ can be identified with the variety in \mathbb{C}^{n^2+1} defined by the polynomial $\det((x_{ij}))x_0 - 1$. Hence G is an affine algebraic variety in the sense of 5.2 above.

Examples of such Lie groups are:

$$GL_n(\mathbb{C}), B_n(\mathbb{C}) = \left\{ \begin{pmatrix} * & \dots & \dots & * \\ 0 & & & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 0 & * \end{pmatrix} \right\}$$

$$U_n(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & & & \vdots \\ \vdots & \ddots & & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\}$$

$$SL_n(\mathbb{C}) = \{x \in GL_n(\mathbb{C}) \mid \det(x) = 1\},$$

$$SO_n(\mathbb{C}) = \{x \in SL_n(\mathbb{C}) \mid x^t x = I\},$$

$$SB_n(\mathbb{C}) = \{x \in B_n(\mathbb{C}) \mid \det(x) = 1\},$$

where x^t is the transposed matrix of x and I is the $n \times n$ identity matrix. In the following we shall write GL_n, \dots, SB_n instead of $GL_n(\mathbb{C}), \dots, SB_n(\mathbb{C})$.

5.4. The variety of unipotent elements

A matrix $x \in GL_n$ is said to be unipotent if all its eigenvalues are 1, or, equivalently, if $(x-I)^n = 0$. Let G be as in 5.3 above. Then $U(G) = \{g \in G \mid (g-I)^n = 0\}$ is called the unipotent variety of G . This is a closed subset of G defined by polynomial equations, hence it is an affine algebraic variety in the sense of 5.2 above.

EXAMPLE A. $G = SL_2$. Then $U(SL_2) = \left\{ \begin{pmatrix} 1+x & y \\ z & 1-x \end{pmatrix} \mid x^2 + yz = 0 \right\}$. This is isomorphic to the complex cone $\{(x,y,z) \in \mathbb{C}^3 \mid x^2 + yz = 0\}$ with top in $(0,0,0)$. This top corresponds to $I \in SL_2$. The point $I \in U(SL_2)$ is singular, all other points are smooth.

EXAMPLE B. $G = \text{SB}_n$. Then $U(G) = U_n$, which is a smooth variety.

EXAMPLE C. $G = \text{SL}_n$. Then $U(G) = \{gxg^{-1} \mid g \in \text{SL}_n, x \in U_n\} = \bigcup_{g \in \text{SL}_n} gU_n g^{-1}$.

Thus we have written $U(G)$ as a union of smooth varieties in this case. This is a general phenomenon, cf. below in 5.5.

5.5. The variety $\text{IB}(G)$ of Borel subgroups

A Lie subgroup $G \subset \text{GL}_n$ is *solvable* if it is conjugate in GL_n to a subgroup of B_n . If G is solvable then $U(G)$ is smooth (as in example B), cf. [35], 19.1. A maximal solvable Lie subgroup of G is called a *Borel subgroup* (cf. also section 4 above). Every two Borel subgroups are conjugate ([35], 21.3) and it follows that the set of Borel subgroups is the homogeneous variety G/B because the normalizer of B in G is B itself ([35], 29.3). In fact G/B is a projective variety ([35], 21.3).

THEOREM ([35], 23.4).

- (i) $\text{IB}(G)$ is a non-empty smooth connected compact variety on which G acts transitively (by $(g, B) \mapsto gBg^{-1}$; i.e. all Borel subgroups are conjugate);
- (ii) $G = \bigcup_{B \in \text{IB}(G)} B$;
- (iii) $U(G) = \bigcup_{B \in \text{IB}(G)} U(B)$, and all the $U(B)$ are smooth and connected.

In case $G = \text{GL}_n$, part (ii) is proved by the fact that every $x \in \text{GL}_n$ is triangulizable.

EXAMPLE A (continued). $\text{IB}(\text{SL}_2) = \text{SL}_2/\text{SB}_2 \simeq \mathbb{P}^1(\mathbb{C})$ as is easily checked by hand.

EXAMPLE C (continued): $\text{IB}(\text{SL}_n)$ consists of SB_n and its conjugates.

5.6. Reductive Lie groups

The intersection $\bigcap_{B \in \text{IB}(G)} U(B)$ is a normal subgroup of G and one can take the quotient of G by this subgroup without changing the singularities of $U(G)$. We shall therefore from now on suppose that

this normal subgroup is trivial, i.e. that G is *reductive*. The groups GL_n , SL_n , SO_n are reductive but B_n and U_n are not reductive if $n \geq 2$.

5.7. Conjugacy classes

Let $x \in G$. Then $C(x) = \{gxg^{-1} \mid g \in G\}$, the conjugacy class of x_1 is a connected homogeneous and smooth subvariety of G .

THEOREM (RICHARDSON-LUSZTIG [55],[44]). *The variety $U(G)$ is a disjoint union of a finite number of conjugacy classes.*

EXAMPLE C (continued). In the case of $G = SL_n$ this follows from the theory of the Jordan normal form.

5.8. Regular unipotents

THEOREM (STEINBERG [43], pp. 108, 110).

- (i) *There is precisely one conjugacy class $C_{\text{reg}} \subset U(G)$ which is open and dense in $U(G)$;*
- (ii) *the variety $U(G)$ is smooth in the points of C_{reg} ;*
- (iii) *for every $x \in C_{\text{reg}}$ there is precisely one $B \in \mathcal{B}(G)$ such that $x \in U(B)$;*
- (iv) *for every $x \in U(G) \setminus C_{\text{reg}}$ there are infinitely many $B \in \mathcal{B}(G)$ such that $x \in U(B)$.*

The elements of C_{reg} are called the *regular unipotents*. They can be characterized in various ways (cf. [43], 3.7).

EXAMPLE A (continued). The cone $U(SL_2)$ is the union $\{I\} \cup C_{\text{reg}}$ where C_{reg} is the conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Through every $x \in C_{\text{reg}}$ there passes precisely one line $U(B)$ on the cone. All these lines pass through I .

EXAMPLE C (continued). In case $G = SL_n$, C_{reg} is the conjugacy class of the "one Jordan block" matrix with eigenvalue 1. E.g. if $n = 4$ C_{reg} is the conjugacy class of

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5.9. The Springer desingularisation

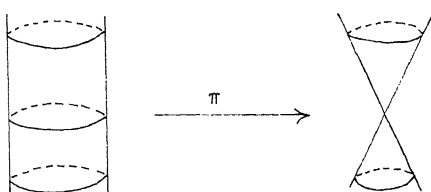
Let $V(G) = \{(B, x) \mid x \in U(B)\} \subset \mathbb{B}(G) \times G$, and $\pi: V(G) \rightarrow U(G)$ be defined by $\pi(B, x) = x$. Then $V(G)$ is a closed subvariety of $\mathbb{B}(G) \times G$. The algebraic morphism π is a desingularisation in that the following theorem holds.

THEOREM ([42],[15],[44],[43], 3.9).

- (i) $V(G)$ is smooth and connected;
- (ii) π is surjective and proper (that is $\pi^{-1}(Y)$ is compact if Y is compact);
- (iii) $\pi: \pi^{-1}(C_{\text{reg}}) \rightarrow C_{\text{reg}}$ is an isomorphism and $\pi^{-1}(C_{\text{reg}})$ is open and dense in $V(G)$ (i.e. π is a birational morphism).

The fibre $\pi^{-1}(x)$ for $x \in U(G)$ is the set of all Borel subgroups of G containing x , i.e. it is the set of fixed points of $x \in G$ acting on $\mathbb{B}(G) \simeq G/B$ as in the theorem of section 5.5 above. It follows that $\pi^{-1}(x)$ is a projective variety. This variety is also connected ([43], 3.9, prop.1).

EXAMPLE A (continued). The desingularisation of the cone $U(\text{SL}_2)$ looks as follows:



(where we have, of course, only drawn the real points of the 2-dimensional complex surfaces involved).

5.10. The parabolic lines of $\mathbb{B}(G)$

For simplicity we assume that G is quasi-simple. We have seen in section 4.2 above how to associate a parabolic subgroup P to every subset I of the set of simple roots S . For each $\alpha_i \in S$ let P_i be the parabolic subgroup corresponding to $I = \{\alpha_i\}$. These are the minimal

parabolic subgroups $\neq B$ in G . (Do not confuse them with the $P_{(i)}$, the maximal parabolic subgroups used in 4.2.) Of the six parabolic subgroups of 4.1 above the first three are minimal. They are also called *simple parabolic subgroups*, as is every parabolic subgroup conjugate to one of these.

For each P_i , $\mathbb{B}(P_i) = P_i/B$, cf. [49], 3.2.3, is isomorphic to $\mathbb{P}^1(\mathbb{C})$. We shall call $\mathbb{B}(P) \subset \mathbb{B}(G)$ a parabolic line of type i if P is conjugate to P_i .

THEOREM ([43], p.146).

- (i) Every point $B \in \mathbb{B}(G)$ lies on ℓ parabolic lines, one of each type (here ℓ is the number of vertices of the Dynkin diagram);
- (ii) two parabolic lines of different type intersect each other in at most one point.

EXAMPLE C (continued). The Dynkin diagram of SL_n is $\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet$ with vertices labeled $1, 2, \dots, n-1$. The Borel subgroup SB_n lies on the parabolic lines $\mathbb{B}(P_1), \dots, \mathbb{B}(P_{n-1})$.

If $n = 4$ then P_1, P_2, P_3 are respectively equal to

$$P_1 = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}, \quad P_2 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}, \quad P_3 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}.$$

In this case one easily checks by hand that $\mathbb{B}(P_i) \simeq P_i/B \simeq \mathbb{P}^1(\mathbb{C})$.

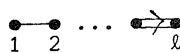
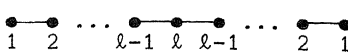
5.11. Subregular unipotents

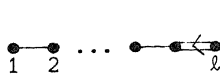
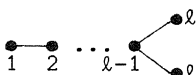
As in 5.10 above let G be quasi-simple, so that the Dynkin diagram of G is connected.

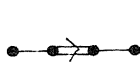
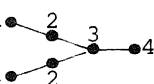
THEOREM (STEINBERG-TITS [43], p.145,153).



- (i) There is precisely one conjugacy class C_{sub} which is open and dense in $U(G) \setminus C_{\text{reg}}$.
- (ii) For $x \in U(G)$ we have $x \in C_{\text{sub}} \iff \dim(\pi^{-1}(x)) = 1$.
- (iii) If $x \in C_{\text{sub}}$, then the fibre $\pi^{-1}(x) = \{B \in \mathbb{B}(G) \mid x \in U(B)\}$ is a connected one dimensional projective variety. It is a finite union of projective lines whose intersection diagram is the unfolded Dynkin diagram of G .
Here the unfolded versions of A_n, \dots, G_2 are defined as follows:

(a) A_n, D_n, E_n remain the same

(b)  becomes 

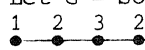
(c)  becomes 

(d)  becomes 

(e)  becomes 

Notice that, apart from the numbering of the vertices, all unfolded Dynkin diagrams are of the types A_n, D_n, E_6, E_7, E_8 .

Thus if G has Dynkin diagram B_l , then part (iii) of the theorem above says that $\pi^{-1}(x)$ consists of a union of 2 lines each of types $1, 2, \dots, l-1$ and one line of type l , which intersect as indicated by the diagram. (Two lines intersect iff the corresponding vertices are joined.)

EXAMPLE D. Let $G = SO_7$ with Dynkin diagram B_3 . The unfolded Dynkin diagram is . Thus the

Dynkin curve $\pi^{-1}(x)$ for $x \in C_{\text{sub}}$

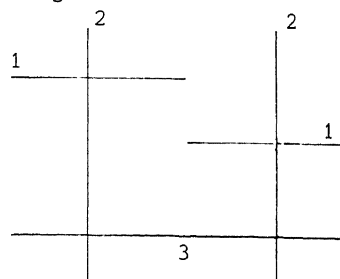
consists of 5 projective lines,


two of type 1, two of type 2, and

one of type 3, which intersect as

indicated in the picture on the

right.



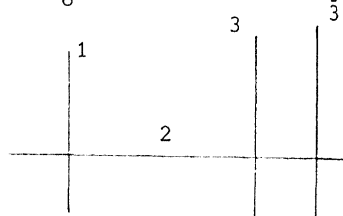
EXAMPLE E. Let $G = Sp_6$, a symplectic group. Sp_6 has the Dynkin diagram C_3 with unfolding .

Thus the Dynkin curve $\pi^{-1}(x)$ consists

of two lines of type 3 and one each

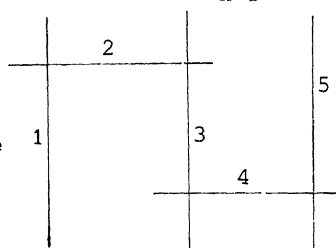
of type 1 and 2, which intersect each

other as in the picture on the right.



EXAMPLE C (continued). $G = SL_n$ has Dynkin diagram A_{n-1} with unfolding

$\bullet_1 \text{---} \bullet_2 \text{---} \dots \text{---} \bullet_{n-1}$, Thus $\pi^{-1}(x)$
 consists of $(n-1)$ lines, one each
 of type $1, 2, \dots, n-1$ which inter-
 sect each other as indicated in the
 diagram on the right for the case
 $n = 6$.



5.12. Local description of singularities with a Dynkin curve as exceptional fibre in a resolution

THEOREM (BRIESKORN [15]). *In a neighbourhood of a subregular element x $U(G)$ is isomorphic with a neighbourhood of the origin in $X_\ell \times \mathbb{C}^r$ where X_ℓ is a surface in \mathbb{C}^3 with rational singularity in $(0,0,0)$. This means that X_ℓ is one of the following surfaces with isolated singularity*

$$A_\ell: \{(x,y,z) \in \mathbb{C}^3 \mid x^{\ell+1} + yz = 0\}, \quad \ell \geq 1,$$

$$D_\ell: \{(x,y,z) \in \mathbb{C}^3 \mid x^{\ell+1} + xy^2 + z^2 = 0\}, \quad \ell \geq 3,$$

$$E_6: \{(x,y,z) \in \mathbb{C}^3 \mid x^4 + y^3 + z^2 = 0\},$$

$$E_7: \{(x,y,z) \in \mathbb{C}^3 \mid x^3y + y^3 + z^2 = 0\},$$

$$E_8: \{(x,y,z) \in \mathbb{C}^3 \mid x^5 + y^3 + z^2 = 0\}.$$

(There is a nonlinear coordinate transformation which takes D_3 into A_3 .)

5.13. Transversal sections

A different more concrete method for getting at the structure of the singularities at $x \in C_{\text{sub}}$ is as follows. Construct a smooth subvariety S of G through x such that $T_x(S) + T_x(C_{\text{sub}}) = T_x(G)$. By the implicit function theorem a neighbourhood of x in $U(G)$ is isomorphic with a neighbourhood of $(x,0)$ in $(S \cap U(G)) \times \mathbb{C}^r$ for a certain r . By choosing S cleverly one finds that $S \cap U(G) \simeq X_\ell$. Cf. [4], [15], [32].

EXAMPLE G. Let $G = GL_3$. Take $n = 3$. The matrix x_0 is then a subregular unipotent.

$$x_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1+v & 1 & 0 \\ w & 1 & z \\ y & 0 & 1+t \end{pmatrix}.$$

The variety of matrices x with $\det(x) \neq 0$ is a transversal section, and $S \cap U(\mathrm{GL}_3)$ consists of the matrices $x \in S$ which satisfy $\mathrm{trace}(x) = 3$, $\det(x) = 1$ and

$$\det \begin{pmatrix} 1+v & 1 \\ w & 1 \end{pmatrix} + \det \begin{pmatrix} 1+v & 0 \\ y & 1+t \end{pmatrix} + \det \begin{pmatrix} 1 & z \\ 0 & 1+t \end{pmatrix} = 3.$$

This gives $v = -t$, $w = -t^2$, $t^3 + yz = 0$. Hence $S \cap U(\mathrm{GL}_3)$ is the singularity A_2 , and one verifies that the Dynkin curve consists of two intersecting lines. (Remark: $U(\mathrm{GL}_3) = U(\mathrm{SL}_3)$, so whether one considers GL_3 or SL_3 does not matter much.)

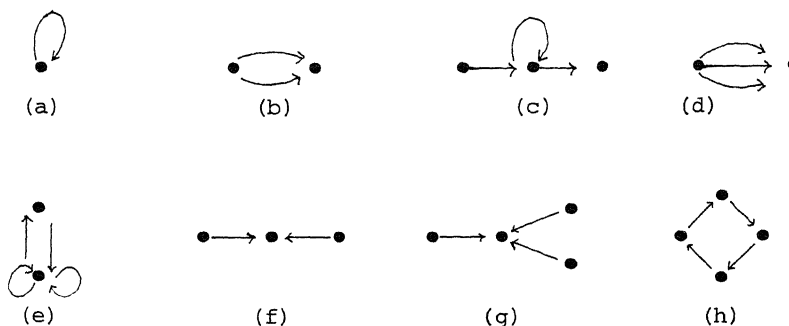
6. QUIVERS AND THEIR REPRESENTATIONS

6.1. Introduction

A *quiver* Q is a finite connected directed graph. A *representation* over a field K assigns to each vertex of the graph a vector space over K and to each arrow a homomorphism of vector spaces. It now turns out that a quiver Q has (up to isomorphism) only finitely many indecomposable representations if and only if the underlying undirected graph of Q is one of the Dynkin diagrams A_n , D_n , E_6 , E_7 , E_8 .

6.2. Quivers and representations

A *quiver* is a finite connected directed graph. Thus it consists of a finite set P_Q of vertices and a finite set A_Q of arrows between elements of P_Q . Let $s, r: A_Q \rightarrow P_Q$ be the two maps which assign to an arrow $a \in A_Q$ its initial vertex $s(a)$ and its end vertex $r(a)$. Some examples of quivers are



Let K be a field. A representation V of a quiver Q assigns to each $p \in P_Q$ a vector space $V(p)$ over k (finite dimensional) and to each arrow $a \in A_Q$ a homomorphism of vector spaces $V(a): V(s(a)) \rightarrow V(r(a))$. The zero representation assigns to each $p \in P_Q$ the zero vector space (and to each $a \in A_Q$ the zero mapping). Given two representations V_1, V_2 their direct sum is the representation $(V_1 \oplus V_2)(p) = V_1(p) \oplus V_2(p)$, $(V_1 \oplus V_2)(a) = V_1(a) \oplus V_2(a)$. A representation V is called *indecomposable* if it cannot be written as a direct sum $V = V_1 \oplus V_2$ with both V_1 and $V_2 \neq 0$.

Finally two representations V_1 and V_2 are said to be *isomorphic* if there exists for each $p \in P_Q$ an isomorphism $\phi(p): V_1(p) \rightarrow V_2(p)$ such that for all $a \in A_Q$, $\phi(r(a)) \circ V_1(a) = V_2(a) \circ \phi(s(a))$.

EXAMPLE (a). A representation of quiver (a) above consists of a vector space and an endomorphism; i.e. after choosing a basis a representation is given by a square matrix M . Two representations M, M' are isomorphic iff there is an invertible matrix S such that $M' = SMS^{-1}$. A representation M over an algebraically closed field k is indecomposable iff its Jordan canonical form consists of one Jordan block, and the indecomposables over k are classified by their sizes and the eigenvalue appearing.

EXAMPLE (b). Here a representation is given by two (not necessarily square) matrices M, N and two representations $(M, N), (M', N')$ are

isomorphic if and only if there exist invertible matrices S and T such that $SM = M'T$, $SN = N'T$. Thus the theory of the representations of quiver (b) is the theory of Kronecker pencils of matrices. Cf. [25] for the results of this theory.

6.3. Gabriel's theorem

A quiver Q is said to be of *finite type* if, up to isomorphism, there are only finitely many indecomposable representations of Q ; the quiver Q is said to be *tame* if there are infinitely many isomorphism classes of indecomposable representations but these can be parametrized by a finite set of integers together with a polynomial irreducible over k ; the quiver Q is said to be *wild* if for every finite dimensional algebra E over k there are infinitely many pairwise nonisomorphic representations of Q which have E as their endomorphism algebra. These three classes of quivers are clearly exclusive; they are, as it turns out, also exhaustive.

THEOREM (GABRIEL [23]). *A quiver Q is of finite type if and only if its underlying undirected graph is one of the Dynkin diagrams A_n , D_n , E_6 , E_7 , E_8 .*

EXAMPLES. The quivers (f) and (g) of the examples of 6.2 above are of finite type.

Let Q be a quiver. We chose a fixed ordering of P_Q . For each representation V of Q we now define $n(V)$, the dimension vector of V , as the vector $n(V) = (\dim(V(p_1)), \dots, \dim(V(p_\ell)))$.

THEOREM (GABRIEL, cf. also [7]). *Let Q be a quiver of finite type. The map $V \mapsto n(V)$ sets up a bijective correspondence between the indecomposable representations of Q and the set of positive roots of the underlying Dynkin diagram of V .*

6.4. Nazarova's extension of Gabriel's theorem

THEOREM ([38]). *The quivers of tame type are precisely the quivers whose underlying undirected graph is one of the extended Dynkin diagrams \tilde{A}_n , \tilde{D}_n , \tilde{E}_6 , \tilde{E}_7 , \tilde{E}_8 . (Cf. section 3.14 above for a description of these Dynkin diagrams.)*

EXAMPLES. The quivers (a), (b), (h) of the examples of 6.2 above are tame. The quivers (c), (d), (e) are wild.

6.5. Quadratic form of a quiver

Let Q be a quiver with ℓ vertices. We associate to Q a quadratic form in ℓ -variables as follows:

$$B_Q(x_1, \dots, x_\ell) = \sum_{i=1}^{\ell} x_i^2 - \sum_{a \in A_Q} x_{s(a)} x_{r(a)}.$$

EXAMPLES. The quadratic forms of the quivers (a), (b), (c), (d), (f), (g) of 6.2 above are respectively 0, $x_1^2 + x_2^2 - 2x_1x_2$, $x_1^2 + x_3^2 - x_1x_2 - x_2x_3$, $x_1^2 + x_2^2 - 3x_1x_2$, $x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_2x_3 - x_2x_4$.

THEOREM ([7]). A quiver Q is of finite type (resp. tame) iff B_Q is positive definite (resp. semipositive definite).

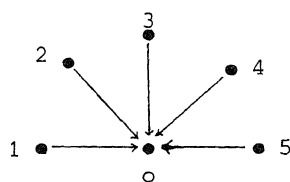
6.6. Proof of " Q is of finite type" $\Rightarrow B_Q$ is positive definite (Tits)

Let Q be a quiver of finite type and let $n = (n_1, \dots, n_\ell)$ be a fixed dimension vector. Because Q is of finite type there are only finitely many isomorphism classes of representations V such that $n(V) = n$. Now giving a representation with $n(V) = n$ is the same as specifying an $n_{r(a)} \times n_{s(a)}$ matrix for each $a \in Q_A$. This gives us a $\prod_{a \in Q_A} n_{s(a)} n_{r(a)}$ dimensional space of representations. The group $G = GL_{n_1}(k) \times \dots \times GL_{n_\ell}(k)$ acts on this space of representations by $(M_a)_{a \in A_Q} \rightarrow (T_{r(a)} M_a T_{s(a)}^{-1})_{a \in A_Q}$ and the isomorphism classes of representations V with $n(V) = n$ are precisely the orbits X/G . The subgroup $H = \{(sI_{n_1}, \dots, sI_{n_\ell}) \mid s \in k\}$ of G acts trivially. Because X/G is finite it follows (if we are working over an infinite field) that $\dim G - 1 \geq \dim(X)$. Hence $n_1^2 + \dots + n_\ell^2 - 1 \geq \sum_a n_{s(a)} n_{r(a)}$; i.e. $B_Q(n_1, \dots, n_\ell) \geq 1$. This holds for all sequences of positive integers $n = (n_1, \dots, n_\ell)$ and hence, because clearly $B_Q(x_1, \dots, x_\ell) \geq B_Q(|x_1|, |x_2|, \dots, |x_\ell|)$, it follows that B_Q is positive definite.

6.7. EXAMPLE. Let Q be a quiver with underlying Dynkin diagram A_ℓ . For all $r, s \in \mathbb{N}$ with $1 \leq r < s \leq n$. Let $V_{r,s}(i) = k$ for $r \leq i \leq s$

and $V_{r,s}(j) = 0$ for $j < r$ or $j > s$. For $a \in Q_A$ we set $V_{r,s}(a) = \text{id}$ if a joins two points in $\{i \mid r \leq i \leq s\}$ and $V_{r,s}(a) = 0$ otherwise. Then $V_{r,s}$ is an indecomposable representation of Q and all indecomposable representations of Q are isomorphic to one of these.

6.8. EXAMPLE ([24],[26]). Consider the quiver Q_5 :



with the vertices numbered as indicated. This quiver is wild. We show that every finite dimensional algebra A arises as an endomorphism algebra of Q_5 . To this end consider first Q_4 , the quiver obtained from Q_5 by removing the vertex 5 and the arrow incident with it. We now first construct a representation U of Q_4 over a field k with $\dim(U) = 2n+1$, $n = 1, 2, \dots$ such that the endomorphism algebra of U is k . To this end let E be an $n+1$ -dimensional vector space over k with basis e_1, \dots, e_{n+1} and F an n -dimensional vector space with basis f_1, \dots, f_n . We set $U(0) = E \oplus F$, $U(1) = E \oplus 0$, $U(2) = 0 \oplus F$, $U(3) = \{(\lambda(f), f) \mid f \in F\}$, $U(4) = \{(\delta(f), f) \mid f \in F\}$. Where $\lambda, \delta: F \rightarrow E$ are defined by $\lambda(f_i) = e_i$, $\delta(f_i) = e_{i+1}$. The maps associated to the arrows are the natural inclusions. An endomorphism of U is then given by an endomorphism α of $U(0) = E \oplus F$, which preserves the subspaces $U(1), \dots, U(4)$. One easily checks that this means α is multiplication with an element of k , i.e. one finds $\text{End}(U) = k$. Now let A be any finite dimensional algebra over k and let a_1, \dots, a_m be a set of generators of A (as a k -module). Let $a_0 = 1$ and see to it that m is even, $m \geq 2$. Let U be the representation of Q_4 constructed above with $\dim(U) = m+1$. We now define a representation V of Q_5 by $V(0) = A \otimes U(0)$, $V(i) = A \otimes U(i)$, $i = 1, \dots, 4$, $V(5) = \{ \sum_{i=0}^m a a_i \otimes e_i \mid a \in A \} \subset A \otimes U(0)$, where e_0, \dots, e_m is a basis for $U(0)$. An endomorphism of V is an endomorphism of $V(0)$ which preserves the five subspaces $V(j)$, $j = 1, \dots, 5$. Because $\text{End}(U) = k$

the endomorphisms of $V(0)$ which preserve $V(1), \dots, V(4)$ are necessarily of the form $\phi \otimes 1$ where ϕ is a k -vector space endomorphism of A . Now $(\phi \otimes 1)(\sum_{i=0}^m aa_i \otimes e_i) = \sum_{i=0}^m \phi(aa_i) \otimes e_i$ and it follows that if $\phi \otimes 1$ also preserves $V(5)$, there must be, for all $a \in A$, a $b(a)$ such that $\sum_{i=0}^m \phi(aa_i) \otimes e_i = \sum_{i=0}^m b(a)a_i \otimes e_i$. Now $1 \otimes e_0, \dots, 1 \otimes e_m$ is a basis for $A \otimes U(0)$ as a module over A , hence $\phi(aa_i) = b(a)a_i$ for all i . Taking $i = 0$ we find $\phi(a) = b(a)$. Hence we have for all $a \in A$ and all i that $\phi(aa_i) = \phi(a)a_i$. Let $c = \phi(1)$, then $\phi(a_i) = ca_i$ for all i and we see that ϕ is given by multiplication with $c \in A$. This shows that indeed $\text{End}(V) = A$.

7. SIMPLE SINGULARITIES AND DYNKIN DIAGRAMS

7.1. Finitely determined map germs

Let $f: U \rightarrow \mathbb{C}$, $0 \in U \subset \mathbb{C}^{n+1}$ be a holomorphic mapping with isolated critical point in 0. I.e. 0 is critical (that is $df(0) = 0$) and there is a $\delta > 0$ such that for $\|z\| < \delta$, $df(z) \neq 0$ if $z \neq 0$. A critical point 0 is *nondegenerate* if

$$\det\left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0)\right) \neq 0.$$

PROPOSITION (Morse lemma). *If f has a nondegenerate critical point in 0 then there is a biholomorphic change of coordinates ϕ such that $f\phi(z_0, \dots, z_n) = f(0) + z_0^2 + \dots + z_n^2$.*

More generally one has

THEOREM. *if 0 is an isolated critical point of f then there is a local biholomorphic change of coordinates ϕ such that $f\phi$ is equal to a finite part of the Taylor expansion of f around 0.*

A proof can e.g. be found in [16], chapter 11. It uses the Nullstellensatz for holomorphic function germs, which shows that df is a finite mapping, and next a theorem of Tougeron. One can give

a bound for the degree of the Taylor approximation in this theorem in terms of the ideal $(\partial_0 f, \dots, \partial_n f) \subset \mathbb{C} \langle\langle z_0, \dots, z_n \rangle\rangle$ generated by the partial derivatives of f . If the critical point is nondegenerate this number is 2 and one reobtains the Morse lemma.

From now on we consider *polynomials* with an isolated critical point in $0 \in \mathbb{C}^{n+1}$. (This is justified by the theorem above.)

7.2. Right equivalence and simple germs

Two germs of holomorphic mappings f, g are *right equivalent* (or *are of the same type*) if there exists a biholomorphic change of coordinates ϕ such that $g = f\phi$. A germ f is called *simple* if there is a finite list of germs such that every small perturbation of f is equivalent to a germ from this list.

THEOREM (ARNOLD [6]). $f: \mathbb{C}^{n+1} \supset U \rightarrow \mathbb{C}$ is simple if f is equivalent to a germ in the following list:

$$\begin{aligned} x^{k+1} + y^2 + z_2^2 + \dots + z_n^2 & \quad \text{type } A_k \quad (k \geq 0) \\ x^2 y + y^{k-1} + z_2^2 + \dots + z_n^2 & \quad \text{type } D_k \quad (k \geq 4) \\ x^3 + y^4 + z_2^2 + \dots + z_n^2 & \quad \text{type } E_6 \\ x^3 + xy^3 + z_2^2 + \dots + z_n^2 & \quad \text{type } E_7 \\ x^3 + y^5 + z_2^2 + \dots + z_n^2 & \quad \text{type } E_8 \quad . \end{aligned}$$

7.3. Morsifications

Let f be a polynomial with isolated critical point in $0 \in \mathbb{C}^{n+1}$. A morsification of f is a polynomial mapping $F: \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ such that $F(z, 0) = f(z)$ and $f_\lambda(z) = F(z, \lambda)$ has only nondegenerate critical points in a neighbourhood of $0 \in \mathbb{C}^{n+1}$ for small enough $\lambda \neq 0$. Morsifications always exist. In fact, one can take $F(z, \lambda) = f(z) + \sum_{i=1}^n \lambda_i z_i$ for suitable (generic) $\lambda_i = \lambda_i(\lambda)$.

7.4. Milnor number

For a small enough neighbourhood of 0 in \mathbb{C}^{n+1} and small enough $\lambda \neq 0$ the number of critical points of f_λ in this neighbourhood is constant. This number $\mu(f)$ is called the *Milnor number* of f . This definition is independent of the choice of the Morsification. In fact

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C} \langle\langle z_0, \dots, z_n \rangle\rangle}{(\partial_0 f, \dots, \partial_n f)},$$

which is finite if and only if f has an isolated critical point. For different characterisations of $\mu(f)$ cf. [39].

7.5. Examples of Morsifications

We now give a number of examples of Morsifications of polynomials $\mathbb{C}^2 \rightarrow \mathbb{C}$ with real coefficients. The Morsifications given below all have the property that all critical points are real and all saddle points have the same critical value. Let

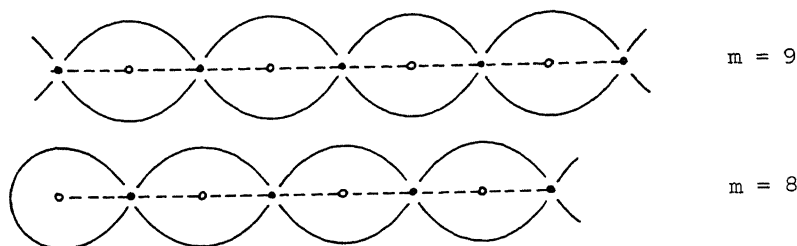
$$\phi_m(x, \lambda) = \begin{cases} (x+\lambda)^2 (x+2\lambda)^2 \dots (x+k\lambda)^2 & \text{if } m = 2k \\ (x+\lambda)^2 \dots (x+(k-1)\lambda)^2 (x+k\lambda) & \text{if } m = 2k-1. \end{cases}$$

EXAMPLE (i). Morsifications for type A_m .

Polynomial : $x^{m+1} - y^2$

Morsification: $\phi_{m+1}(x, \lambda) - y^2$

Picture of the zero level

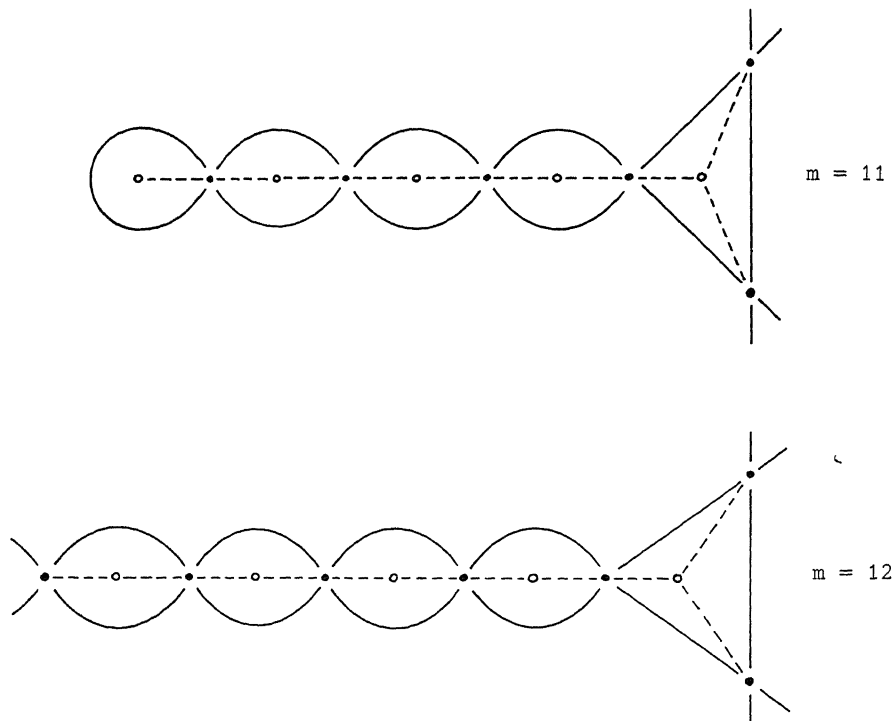


EXAMPLE (ii). Morsifications for type D_m

Polynomial : $x^{m-1} - xy^2 = x(x^{m-2} - y^2)$

Morsification: $x(\phi_{m-2}(x, \lambda) - y^2)$

Picture of the zero level



In the following three examples one first constructs a deformation f_λ with one critical point in 0 and the other critical points non-degenerate. Moreover, the lowest degree part g_λ of f_λ is a polynomial of type D_4 , which factors over \mathbb{R} in three different linear factors. Let $g_{\lambda, \mu}$ be a Morsification for g_λ . Then for μ small enough $f_{\lambda, \mu} = g_{\lambda, \mu} + (f_\lambda - g_\lambda)$ is a Morsification of f_λ since the nondegenerate critical points of f_λ survive and stay approximately in the same place. For appropriate $\mu = \mu(\lambda)$ this gives us a Morsification for f .

EXAMPLE (iii). Morsification for type E_6

Polynomial : $x^3 + y^4$

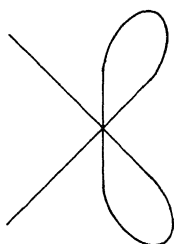
Deformation : $x(x^2 - \lambda y^2) + y^4$

Morsification: $(x - \mu)(x^2 - \lambda y^2) + y^4$

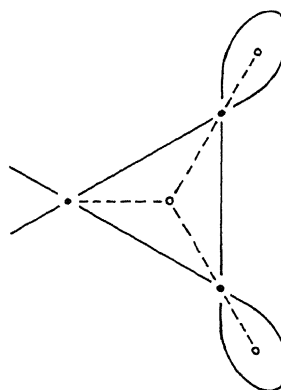
Pictures of the zero level for various λ, μ



$\lambda = 0, \mu = 0$



$\lambda > 0, \mu = 0$



$\lambda > 0, \lambda \gg \mu > 0$

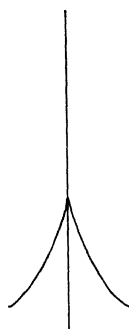
EXAMPLE (iv). Morsification for type E_7

Polynomial : $x^3 + xy^3 = x(x^2 + y^3)$

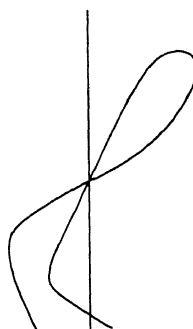
Deformation : $x(x^2 + y^3 + \lambda y^2 - 6\lambda xy)$

Morsification: $(x - \mu)(x^2 + y^3 + \lambda y^2 - 6\lambda xy)$

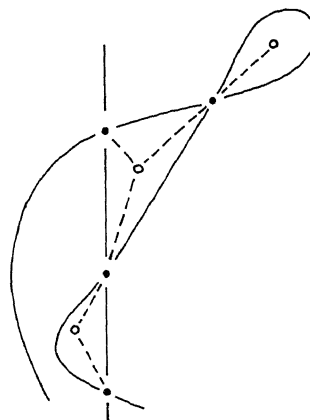
Pictures of the zero level for various λ, μ



$\lambda = 0$



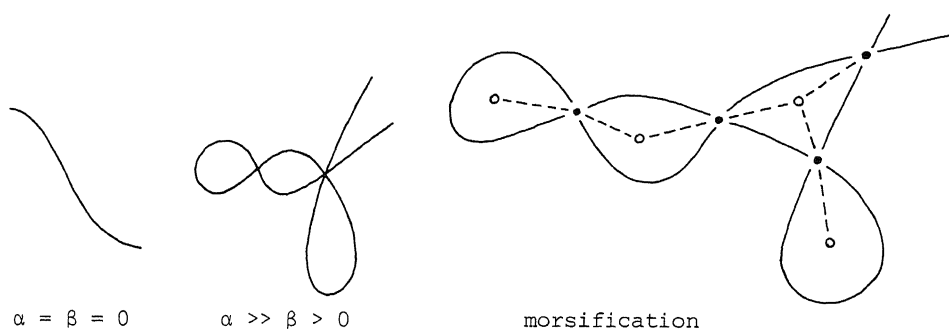
$\lambda > 0, \mu = 0$



$\lambda > 0, \lambda \gg \mu > 0$

EXAMPLE (v). Morsification for type E_8 Polynomial : $x^3 + y^5$ Deformation $x^3 + y^3(y-\beta)^2 + 3\alpha xy^2(y-\beta) + 2\alpha^2(x^2-y)$

Pictures of the zero levels

7.6. Separatrices

For the examples given above in 7.5 consider the gradient vector fields $(\frac{\partial f}{\partial x}(x,y,\lambda), \frac{\partial f}{\partial y}(x,y,\lambda))$, $\lambda \neq 0$ and construct the corresponding separatrix diagrams. These consist of a number of vertices, corresponding to the critical points of f and a number of lines, joining these vertices, where there is a line joining two given vertices if and only if there is an integral curve which joins the two corresponding critical points. An example is E_6 :



In the examples (i) - (v) of 7.5 above the separatrix diagrams of the Morsifications of the polynomials given are precisely the Coxeter-Dynkin diagrams A_n , D_n , E_6 , E_7 , E_8 .

The following example shows that Morsifications and separatrix diagrams are not unique. Consider $x^4 - xy^2 = x(x^3 - y^2)$ which is of

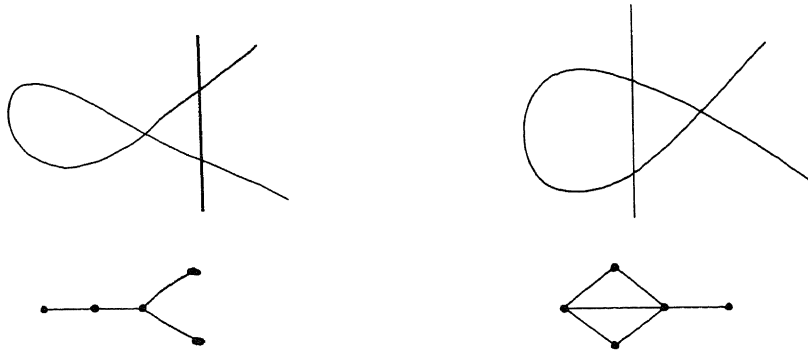
type D_5 . Two Morsifications of this polynomial are

$$x((x+\lambda)^2(x+2\lambda) - y^2)$$

and

$$(x + \frac{3}{2}\lambda)((x+\lambda)^2(x+2\lambda) - y^2)$$

with zero level pictures and separatrix diagrams



7.7. The Milnor fibration

As above we consider a polynomial $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ with $f(0) = 0$ and isolated critical point in 0.

Let $X_{\epsilon, \delta} = B_{\epsilon} \cap f^{-1}(D_{\delta} \setminus \{0\})$, where

$D_{\delta} = \{z \in \mathbb{C} \mid \|z\| \leq \delta\}$ and

$B_{\epsilon} = \{z \in \mathbb{C}^{n+1} \mid \|z\| \leq \epsilon\}$. Let

$F_{\epsilon, \delta} = B_{\epsilon} \cap f^{-1}(\delta)$. The restriction $f: X_{\epsilon, \delta} \rightarrow D_{\delta} \setminus \{0\}$ is, for

ϵ and δ sufficiently small, a locally trivial fibre bundle

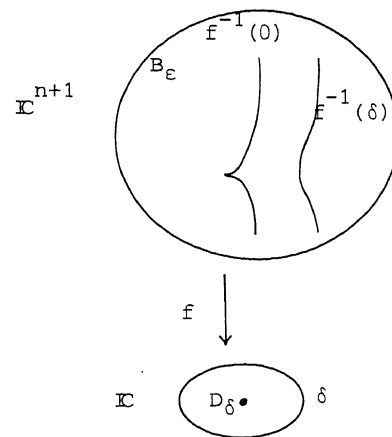
(cf. MILNOR [37]). Moreover,

in the case of an isolated critical point at 0 the fibre

$F = F_{\epsilon, \delta}$ is homotopy equivalent to a wedge of μ (the Milnor number) copies of the n -sphere S^n .

Thus $H_n(F) = \mathbb{Z}^{\mu}$, $H_0(F) = \mathbb{Z}$,

$H_i(F) = 0$ for $i \neq 0, n$.



EXAMPLE. Let $f = z_0^2 + z_1^2$. The equations for the fibre F are $|z_0^2| + |z_1^2| \leq \epsilon$, $z_0^2 + z_1^2 = \delta$. Writing $z_j = x_j + iy_j$ we find $x_0^2 + x_1^2 + y_0^2 + y_1^2 \leq \epsilon$, $x_0^2 + x_1^2 - y_0^2 - y_1^2 = \delta$, $2x_0y_0 + 2x_1y_1 = 0$. Thus $x_0^2 + x_1^2 = \delta + y_0^2 + y_1^2$ which gives

$$((\delta + y_0^2 + y_1^2)^{-\frac{1}{2}} x_0)^2 + ((\delta + y_0^2 + y_1^2)^{-\frac{1}{2}} x_1)^2 = 1,$$

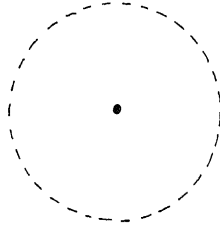
$$x_0 y_0 + x_1 y_1 = 0,$$

$$y_0^2 + y_1^2 \leq 2^{-1}(\epsilon^2 - \delta).$$

Thus F is diffeomorphic to the bundle of tangent vectors to the circle S^1 , the circle S^1 itself being obtained for $y_0 = y_1 = 0$.

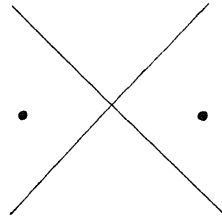
The pictures of the real points of the situation look as follows:

$$f = x_0^2 + x_1^2$$



S^1 is the level line
 $x_0^2 + x_1^2 = \delta$

$$f = x_0^2 - x_1^2$$



S^1 intersects \mathbb{R}^2 in two points
of the level lines $x_0^2 - x_1^2 = \delta$

THEOREM (TJURINA [51], BRIESKORN [13]). Let $n = 2$ and let $\tilde{F}_{\epsilon,0} \rightarrow F_{\epsilon,0}$ be the resolution of the isolated singularity at 0 of $f^{-1}(0)$. Then f is simple iff $\tilde{F}_{\epsilon,0}$ is diffeomorphic with $F_{\epsilon,0}$.

Cf. also section 5 above (especially 5.11 and 5.12) for a statement on the exceptional fibre of $\tilde{F}_{\epsilon,0} \rightarrow F_{\epsilon,0}$.

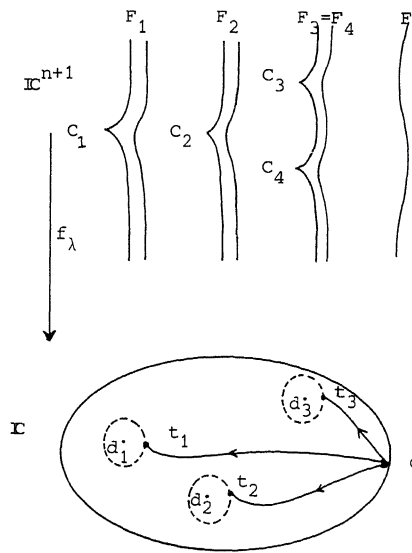
7.8. Monodromy

Using the local triviality of the fibre bundle $f: X_{\epsilon,\delta} \rightarrow D_\delta \setminus \{0\}$, every piecewise smooth path $\omega: [0,1] \rightarrow D_\delta \setminus \{0\}$ can be made to induce a diffeomorphism $F_{\omega(0)} \rightarrow F_{\omega(1)}$. (In fact one defines a so-called

connection.) Let $\omega(t) = \delta e^{2\pi i t}$. The corresponding diffeomorphism $F \rightarrow F$ is called the geometric monodromy; the induced map on homology $h: H_n(F) \rightarrow H_n(F)$ is called the algebraic monodromy.

7.9. Vanishing cycles

Now let f_λ be a given Morsification of f . Let the critical points



of f (for a given small λ) be

c_1, \dots, c_μ and let the corresponding critical values be d_1, \dots, d_μ . For

small λ we obtain a fibration over $D \setminus \{d_1, \dots, d_\mu\}$, which, over ∂D , the boundary of D , is equivalent to the Milnor fibering of f . (Cf. 7.7

above.). Near every c_i we have again a Milnor fibration. Let

t_1, \dots, t_μ be values near d_1, \dots, d_μ such that locally $f^{-1}(t_i)$ is a

Milnor fibre near c_i . Set $F_i = F_{\epsilon, t_i}$.

Since c_i is nondegenerate each fibre F_i contains an n -sphere Z_i .

And using paths (as in the picture) from δ to t_i we find embeddings

$$Z_i \hookrightarrow F_i \xrightarrow{\text{diffeo}} F.$$

In this way we find μ embedded n -spheres S_1, \dots, S_μ in F . These are called the *vanishing cycles*.

THEOREM (BRIESKORN [14]). *The homology classes $[S_1], \dots, [S_\mu]$ are a basis for $H_n(F)$.*

7.10. Intersection form

Now consider the intersection numbers $\langle S_i, S_j \rangle$ of the spheres S_i and S_j . The intersection form \langle, \rangle is defined on $H_n(F)$ and (using small deformations of representing cycles if necessary) can be computed by counting intersection points (with multiplicities).

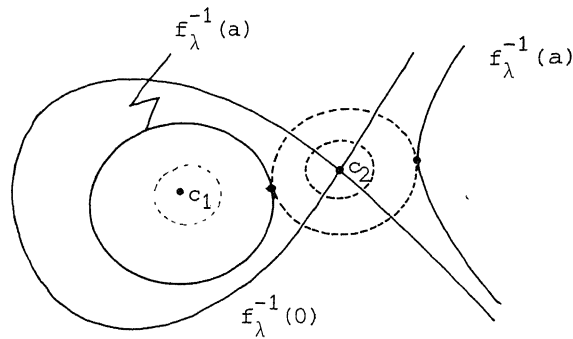
The intersection form is symmetric if n is even and antisymmetric if n is odd. If n is even then $\langle S_i, S_j \rangle = (-1)^k 2$. DURFEE [22] proved that the intersection matrix $(\langle S_i, S_j \rangle)$ determines the topology of the Milnor fibration.

THEOREM (TJURINA [51]). *Let $n \equiv 2 \pmod{4}$. Then f is simple if and only if the intersection form is negative definite.*

7.11. Separatrix diagrams (continued)

We return to real Morsifications and the separatrix diagram.

EXAMPLE. $f = x^3 - y^2$ with Morsification f_λ . The intersection numbers of the vanishing cycles can be computed from the picture of the Morsification. In this example we have two vanishing cycles (one near c_1 , the other near c_2). After transporting them to the same level curve $f_\lambda^{-1}(a)$ we see that their intersection number is one.



THEOREM (GUSEIN-ZADE [28], cf. also A'CAMPO [2] for a slightly different version). *Let f be a polynomial in two variables with real coefficients and let f_λ be a Morsification with real critical points and let all saddle points have the same critical value. Then*

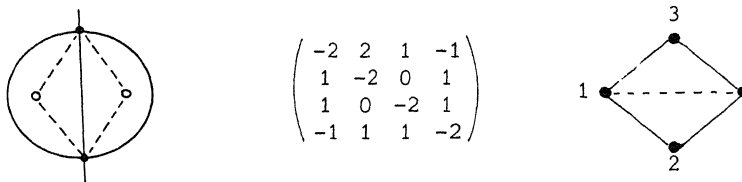
- (i) *if c_i is a saddle point and c_j a minimum, then $\langle S_i, S_j \rangle$ is equal to the number of integral curves joining c_i and c_j ;*
- (ii) *if c_i is a maximum and c_j a saddle point, then $\langle S_i, S_j \rangle$ is equal to the number of integral curves joining c_i and c_j ;*
- (iii) *if c_i is a minimum and c_j a maximum, then $\langle S_i, S_j \rangle$ is equal to the number of families of integral curves joining c_i with c_j ;*
- (iv) *in all other cases $\langle S_i, S_j \rangle = 0$.*

If $g(x, y, z) = f(x, y) + z^2$ one obtains almost the same result. In fact the critical points of g all satisfy $z = 0$ and (otherwise)

coincide with those of f . Thus $\mu(f) = \mu(g)$. The intersection numbers are equal to $-|\langle S_i, S_j \rangle|$ in the maximum-minimum case and equal to $|\langle S_i, S_j \rangle|$ in all other cases except if $i = j$ then $\langle S_i, S_i \rangle = -2$.

More or less as usual one represents the intersection matrix by a diagram of μ vertices, with two vertices joined by a number of lines equal to the intersection number of the corresponding vanishing cycles. Negative intersection numbers are represented by dotted lines (and no lines are drawn joining a vertex to itself).

EXAMPLE. $(x^2 + y^2 - \lambda)x$.



If there are only saddle points and minima we do not find negative entries off the diagonal and we obtain exactly the separatrix diagram.

THEOREM (A'CAMPO [3]). *The following are equivalent:*

- (i) f has a Morsification with two critical values;
- (ii) the diagram of the intersection matrix is a tree;
- (iii) f is simple.

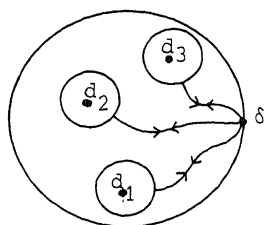
A'CAMPO [2] and GUSEIN-ZADE [29] have shown that for an arbitrary polynomial $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ one can always find a $\tilde{f}: \mathbb{C}^2 \rightarrow \mathbb{C}$ with real coefficients, the same intersection matrix and admitting a Morsification \tilde{f}_λ , which satisfies the conditions of the theorem above. In fact f and \tilde{f} can be joined by a family of constant Milnor number.

7.12. The monodromy group

The monodromy group W_f is the image of the mapping

$$\pi_1(D \setminus \{d_1, \dots, d_\mu\}) \rightarrow \text{Aut}(H_n(F)),$$

cf. 7.9 above. Given a Morsification of f one considers paths ω_i as indicated in the picture on the next page.



(First go from δ to t_i , then go around d_i , then back from t_i to δ .)
 Let $\sigma_i: H_n(F) \rightarrow H_n(F)$ be induced by the diffeomorphism corresponding to ω_i (cf. 7.8 above).

THEOREM (LAMOTKE [36]).

- (i) $\sigma_1, \dots, \sigma_\mu$ generate W_f ;
- (ii) $\sigma_i(x) = x - (-1)^{(n-1)n/2} \langle x, S_i \rangle S_i$;
- (iii) $h = \sigma_\mu \circ \sigma_{\mu-1} \circ \dots \circ \sigma_1$ is the algebraic monodromy.

Let $n \equiv 2 \pmod{4}$. Then W_f is a Coxeter group if the intersection form \langle, \rangle is negative definite.

THEOREM. f is simple if and only if W_f is finite.

7.13. Bibliographical note

General references for this section are [5], [6], [12] and the very recent survey paper [30]. These papers are suitable as introductions and summaries of the subject. Of these papers [12] also pays attention to singularities of differential equations.

8. CONCLUDING REMARKS AND ADDITIONAL BIBLIOGRAPHICAL NOTES

8.1. Systems of lines at angles of $\pi/3$ and $\pi/2$

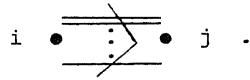
A *star* is a planar set of three lines which all make an angle of $\pi/3$ with each other. A set of lines in Euclidean n -space which mutually have the angles $\pi/3$ or $\pi/2$ is *star closed* if with any two it also contains the third line of a star. In [17] all such indecomposable (same notion as in 3.12 (iii)) sets of lines are determined. They are the root systems A_n, D_n, E_6, E_7, E_8 . These are all maximal apart from $A_8 \subset E_8, D_8 \subset E_8, A_7 \subset E_7$.

8.2. Species and their representations

If one extends the notion of quiver a bit (cf. section 6 above) the "missing" Dynkin diagrams B_n , C_n , F_4 , G_2 also appear. More precisely: let k be a field, a k -species (GABRIEL [24]), $(K_i, {}_iM_j)_{i,j \in I}$ is a finite set of fields K_i , which are finite dimensional over a common central subfield k , together with a set of $K_i - K_j$ bimodules ${}_iM_j$, such that for all $a \in k$, $m \in {}_iM_j$, $am = ma$, and such that ${}_iM_j$ is finite dimensional over k (for all i, j). The *diagram* of a species is defined as follows. The set of vertices is I , and there are

$$\dim_{K_i}({}_iM_j) \times \dim_{K_j}({}_iM_j) + \dim_{K_j}({}_jM_i) \times \dim_{K_i}({}_jM_i)$$

edges between the vertices i and j . In the special case ${}_jM_i = 0$ and $\dim_{K_i}({}_iM_j) < \dim_{K_j}({}_iM_j)$ we shall pictorially represent these facts by



A *representation* $(V_i, {}_j\phi_i)$ of the k -species $(K_i, {}_iM_j)_{i,j \in I}$ is a set of right vector spaces V_i over K_i together with a set of K_j -linear mappings

$${}_j\phi_i: V_i \otimes_{K_i} {}_iM_j \rightarrow V_j, \quad i, j \in I.$$

A homomorphism of representations $\alpha: (V_i, {}_j\phi_i) \rightarrow (V'_i, {}_j\phi'_i)$ is a set of K_i -linear mappings $\alpha_i: V_i \rightarrow V'_i$ such that

$${}_j\phi'_i(\alpha_i \otimes 1) = \alpha_j \circ {}_j\phi_i.$$

A k -species is a k -quiver if $K_i = k$ for all i . Such a quiver is completely determined by its diagram where the number of arrows going from i to j is equal to the k -dimension of ${}_jM_i$.

There is an obvious notion of direct sum and being indecomposable for representations of k -species. A k -species is of *finite type* if it has only finitely many non isomorphic indecomposable representations.

THEOREM (GABRIEL [24], DLAB-RINGEL [20]). *A k -species is of finite type if and only if its diagram is a finite disjoint union of the Dynkin diagrams $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. Moreover the number of indecomposable representations of a K -species of the type of one of these Dynkin diagrams is equal to the number of positive roots of the corresponding root system.*

8.3. Rational singularities

Let V be the germ at v of a normal two-dimensional complex analytic space with singularity at v . (For definitions cf. [53]; for example $V = f^{-1}(0)$, where $f(x,y,z)$ is the germ at 0 of a complex analytic function of 3 variables with isolated critical point at 0.) Let $\pi: M \rightarrow V$ be a resolution of the singularity. The genus p of V is the dimension of the complex vector space $H^1(M, \mathcal{O}_M)$ where \mathcal{O}_M is the sheaf of holomorphic functions on M . The analytic set V has a *rational* singularity at v if $p = 0$. There are many characterizations of rational singularities. One of them says that V has a rational singularity iff V is isomorphic (as a germ of a complex analytic space) to $f^{-1}(0)$ with $f(x,y,z)$ one of the polynomials of type A_n, D_n, E_6, E_7, E_8 discussed above; cf. 7.2. For more characterizations, cf. [21], and also [53], [15], [56].

8.4. Finite subgroups of $SU(2)$

The group $SU(2)$ acts linearly on \mathbb{C}^2 . The discrete subgroups of $SU(2)$ are the so-called binary cyclic, dihedral, tetrahedral, octahedral and icosahedral groups. (By factoring out the centre $\{\pm I\}$ one obtains the corresponding group of rotations of the sphere.) The quotient manifold $M = \mathbb{C}^2/\Gamma$ where Γ is a discrete subgroup of $SU(2)$ is an algebraic surface with singularity. The ring of polynomials in two variables invariant under Γ has 3 generators. There is one relation (syzygy) connecting these 3 generators and this equation then is the equation of M as a surface in \mathbb{C}^3 . The singularities (of polynomials) which one obtains in this way are respectively of type A_n (cyclic), D_n (dihedral), E_6 (tetrahedral), E_7 (octahedral), E_8 (icosahedral). Cf. [21], [5] and [15].

REFERENCES

- [1] ARAKI, S., *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. of Math. Osaka City Univ. 13 (1962) 1-34.
- [2] A'CAMPO, N., *Le groupe de monodromie du déploiement des singularités isolées de courbes planes I*, Math. Ann. 213 (1975) 1-32.
- [3] A'CAMPO, N., *Le groupe de monodromie du déploiement des singularités isolées de courbes planes II*, Proc. Int. Congres Math. Vancouver 1974, Vol. 1, 395-404.
- [4] ARNOL'D, V.I., *On matrices depending on parameters*, Usp. Mat. Nauk 26, 2(1971) 101-114 (Translation: Russ. Math. Surv. 26, 2(1971) 29-43).
- [5] ARNOL'D, V.I., *Critical points of smooth functions and their normal forms*, Usp. Mat. Nauk 30, 5(1975) 3-66 (Translation: Russ. Math. Surveys 30, 5(1975) 1-75).
- [6] ARNOL'D, V.I., *Critical points of smooth functions*, Proc. Int. Congress Math. Vancouver, 1974, Vol. 1, 19-39.
- [7] BERNSTEIN, I.N., GELFAND, I.M. & V.A. PONOMAREV, *Coxeter functors and Gabriel's theorem* (Russian), Usp. Mat. Nauk 28 2(1973) 19-33 (Translation: Russ. Math. Surveys 28, 2(1973) 17-32).
- [8] BOURBAKI, N., *Groupes et algèbres de Lie*, Ch. 4,5,6 (*Groupes de Coxeter et systèmes de Tits; Groupes engendrés par des réflexions; systèmes de racines*), Hermann, 1968.
- [9] BOURBAKI, N., *Groupes et algèbres de Lie*, Ch. 7,8 (*Sous-algèbres de Cartan, éléments réguliers; Algèbres de Lie semisimples déployées*), Hermann, 1975.
- [10] BRENNER, S., *Quivers with commutativity conditions and some phenomenology of forms*, Lect. Notes Math. 488, Springer, 1975, 29-53.

- [11] BRENNER, S., *Endomorphism algebras of vector spaces with distinguished set of subspaces*, J. of Algebra 6 (1967) 100-114.
- [12] BRIESKORN, E., *Singularitäten*, J.ber. Deutsche Math. Verein 78, H2(1976) 93-112.
- [13] BRIESKORN, E., *Die Auflösung der rationalen Singularitäten holomorpher Abbildungen*, Math. Ann. 178 (1968) 255-270.
- [14] BRIESKORN, E., *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. 2 (1970) 103-161.
- [15] BRIESKORN, E., *Singular elements of semisimple algebraic groups*, Actes Congr. Inst. Math. (Nice 1970), Tome 2, 279-284, Gauthier-Villars, 1971.
- [16] BRÖCKER, TH., *Differentiable germs and catastrophes*, London Math. Soc. Lect. Notes 17, Cambridge Univ. Pr., 1975.
- [17] CAMERON, P.J., GOETHALS, J.M., SEIDEL, J.L. & E.E. SHULT, *Line graphs, root systems and elliptic geometry*, J. of Algebra 43, 1(1976) 305-327.
- [18] CARTAN, E., *Les groupes réels simples finis et continus*, Ann. Ec. Norm Sup. 31 (1914) 263-355.
- [19] COXETER, H.S.M. & W.O.J. MOSER, *Generators and relations for discrete groups*, Springer, 2nd ed., 1965.
- [20] DLAB, V. & C.M. RINGEL, *On algebras of finite representation type*, J. of Algebra 33 (1975) 306-394.
- [21] DURFEE, A.H., *Fourteen characterizations of rational double points and simple critical points* (to appear).
- [22] DURFEE, A.H., *Fibered knots and algebraic singularities*, Topology 13 (1974) 47-59.
- [23] GABRIEL, P., *Unzerlegbare Darstellungen I*, Manuscripta Math. 6 (1972) 71-103.
- [24] GABRIEL, P., *Indecomposable representations II*, Symp. INDAM XI, Acad. Press, 1973, 81-104.

- [25] GANTMACHER, F.R., *Matrizenrechnung*, Vol. II, Deutsche Verlag der Wissenschaften, 1959.
- [26] GELFAND, I.M. & V.A. PONOMAREV, *Indecomposable representations of the Lorentz group* (Russian), *Usp. Mat. Nauk* 23 (1968) 3-60.
- [27] GELFAND, I.M. & V.A. PONOMAREV, *Free modular lattices and their representations* (Russian), *Usp. Mat. Nauk* 29, 6(1974) 3-58 (Translation: *Russian Math. Surv.* 29, 6(1974) 1-56).
- [28] GUSEIN-ZADE, S.M., *Intersection matrices for certain singularities of functions of two variables* (Russian), *Funk. Analiz i pril.* 8, 1(1974) 11-15 (Translation *Funct. Analysis Appl.* 8, 1(1974) 10-13).
- [29] GUSEIN-ZADE, S.M., *Dynkin diagrams for singularities of functions of two variables* (Russian), *Funk. Analiz i pril.* 8, 4(1974) 23-30).
- [30] GUSEIN-ZADE, S.M., *The Monodromy group of isolated singularities of hypersurfaces* (Russian), *Usp. Mat. Nauk* 32, 2(1977) 23-66.
- [31] HELGASON, S., *Differential geometry and symmetric spaces*, Acad. Press, 1962.
- [32] HESSELINK, W., *Singularities in the nilpotent scheme of a classical group*, *Trans. Amer. Math. Soc.* 222 (1976) 1-32.
- [33] HOCHSCHILD, G., *Structure of Lie groups*, Holden-Day, 1965.
- [34] HUMPHREYS, J.E., *Introduction to Lie-algebras and representation theory*, Springer, 1972.
- [35] HUMPHREYS, J.E., *Linear algebraic groups*, Springer, 1975.
- [36] LAMOTKE, K., *Die Homologie isolierter Singularitäten*, *Math. Z.* 143 (1975) 27-44.
- [37] MILNOR, J., *Singular points of complex hypersurfaces*, *Ann. Math. Studies* 61, Princeton Univ. Pr., 1968.

- [38] NAZAROVA, L.A., *Representations of quivers of infinite type* (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* 37 (1973) 752-791 (Translation: *Math. USSR Izv.* 7 (1973) 749-792).
- [39] ORLIK, P., *The multiplicity of a holomorphic map at an isolated critical point*, *Proc. Nordic Summer School Math.*, Oslo 1976 (to appear).
- [40] SERRE, J.P., *Algèbres de Lie-simples complexes*, Benjamin, 1966.
- [41] SHAFAREVICH, I.R., *Basic algebraic geometry*, Springer, 1974.
- [42] SPRINGER, T.A., *The unipotent variety of a semi-simple group*, *Algebraic geometry* (Int. Coll. Bombay, 1968), Oxford Univ. Pr., 1969, 373-391.
- [43] STEINBERG, R., *Conjugacy classes in algebraic groups*, *Lect. Notes in Math.* 366, Springer, 1974.
- [44] STEINBERG, R., *On the desingularization of the unipotent variety*, *Inv. Math.* 36 (1976) 209-224.
- [45] TITS, J., *Sur les analogues algébriques des groupes semisimples complexes*, *Coll. d'Alg. Sup. CBRM* (Bruxelles 1956), Ceuterick, Louvain, 1957, 261-289.
- [46] TITS, J., *Groupes semisimples complexes et géométrie projective*, *Sém. Bourbaki*, exp. 112 (Fevrier 1955).
- [47] TITS, J., *Géométries polyédriques et groupes simples*, *II^e Réunion de Math. d'Expression Latine* (Florence 1961), Ed. Cremonese, Roma (1962) 66-88.
- [48] TITS, J., *Géométries polyédriques finies*, *Simp. Geom. Finita* (Roma 1963), *Rendiconti di Mat.* 23 (1964) 156-165.
- [49] TITS, J., *Buildings of spherical type and finite BN-pairs*, *Lect. Notes Math.* 386, Springer, 1974.
- [50] TITS, J., *On buildings and their applications*, *Proc. Int. Congress Math.* 1974 (Vancouver), Vol. I, 209-220, *Can. Math. Congr.*, 1975.

- [51] TJURINA, G.N., *The topological properties of isolated singularities of complex spaces in codimension one* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968) 605-620.
(Translation: Math. USSR Izv. 2 (1968) 557-571).
- [52] *Problems of present day mathematics*, in: Mathematical Developments arising from Hilbert's Problems, F.E. BROWDER (ed.), Proc. Symp. Pure Math. 28, Amer. Math. Soc., 1976, 35-80.
- [53] LAUFER, H., *Normal two-dimensional singularities*, Ann. of Math. Studies 71, Princeton Univ. Pr., 1971.
- [54] BOREL, A. & J. DE SIEBENTHAL, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comm. Math. Helv. 23 (1949) 200-221.
- [55] LUSZTIG, G., *On the finiteness of the number of unipotent classes*, Inv. Math. 34 (1976) 201-213.
- [56] HESSELINK, W., *Cohomology and the resolution of singularities of the nilpotent variety*, Math. Ann. 223 (1976) 249-252.

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